Lecture 11: Review through exercises

12/02/2025

Exercise 5 of Lecture 8: The derivative martingale

1.| A new martingale

Let $L>0$. For $t\geq 0$, let $Z_t^{(L)} = \sum_{u\in\mathcal{N}_t} (\lambda_c t + L - X_u(t))\, e^{\lambda_c(X_u(t)-\lambda_c t)}\, \mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\leq L}$.

1.a| For $x\in\mathbb{R}$, prove $\left((\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\right)_{t\geq 0}$ is a martingale under P_x.

1.b| Deduce that $\left((\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s\leq L}\right)_{t\geq 0}$ is a martingale under $\mathbb{P}_x$.

Hint: see this process as a stopped version of the martingale in question 1.a.

1.c.| Prove that $(Z_t^{(L)})_{t\geq 0}$ is a martingale.

1.d.| Deduce that $(Z_t^{(L)})_{t\geq 0}$ converges a.s. to a limit $Z_\infty^{(L)}$.

2.| Convergence a.s. of $(Z_t)_{t\geq 0}$

Recall $Z_t = \sum_{u\in\mathcal{N}_t} (\lambda_c t - X_u(t))\, e^{\lambda_c(X_u(t)-\lambda_c t)}$.

2.a.| Prove that $Z_t + L W_t^{\lambda_c} = Z_t^{(L)}$ on $E_L = \{\forall s\geq 0,\ \Pi_s \leq \lambda_c s + L\}$

2.b.| Deduce that $Z_t \xrightarrow[t\to\infty]{a.s.} Z_\infty^{(L)}$ on E_L.

2.c.| Conclude that $(Z_t)_{t\geq 0}$ converges a.s. to a limit $Z_\infty \geq 0$ and that $Z_\infty = Z_\infty^{(L)}$ a.s. on E_L for any $L>0$. Hint: Recall $\mathbb{P}(E_L) \xrightarrow[L\to\infty]{} 1$

3.| The limit is non trivial

3.a.| Prove $(Z_t^{(L)})_{t\geq 0}$ is bounded in L^2.

Hint: This relies on the many-to-two. Try it on your own first! Some help if you are stuck: if $f_t(x) = (\lambda_c t + L - x)\, e^{\lambda_c x - \frac{\lambda_c^2}{2}t}$ you should have to compute $\mathbb{E}\left[f_t(B_t^{1,r})\, \mathbb{1}_{\max_{s\in[0,t]} B_s^{1,r} - \lambda_c s\leq L}\ f_t(B_t^{2,r})\, \mathbb{1}_{\max_{s\in[0,t]} B_s^{2,r} - \lambda_c s \leq L}\right]$.

Show this equals $\mathbb{E}\left[\mathbb{1}_{\max_{s\in[0,r]} B_s - \lambda_c s\leq L}\, f_r(B_r)^2\right]$ using question 1.b.

Then show it is $\leq C(L)\left(\frac{1}{r^{3/2}}\wedge 1\right)$ by following the argument used for lemma 2 and conclude.

3.b.| Deduce that $\mathbb{P}(Z_\infty^{(L)} > 0) > 0$.

3.c. Prove that $Z_\infty > 0$ a.s. on the survival event.

Hint: follow the same strategy as for W_∞^λ.

4. $Z_\infty \notin L^1$

4.a. Prove that $\mathbb{E}[Z_\infty^{(L)}] = L$.

4.b. Deduce that $\mathbb{E}[Z_\infty] = +\infty$.

Solution:

1.a We argue this by a direct calculation: let $t, s \geq 0$

$$\mathbb{E}_x\left[(\lambda_c(s+t) + L - B_{s+t})\, e^{\lambda_c B_{s+t} - \frac{\lambda_c^2}{2}(s+t)} \,\middle|\, \mathcal{F}_s\right]$$

$$= (\lambda_c s + L - B_s)\, e^{\lambda_c B_s - \frac{\lambda_c^2}{2}s}\; \mathbb{E}_x\left[e^{\lambda_c(B_{s+t}-B_s) - \frac{\lambda_c^2}{2}t} \,\middle|\, \mathcal{F}_s\right] + \mathbb{E}_x\left[(\lambda_c t - (B_{t+s} - B_s))\, e^{\lambda_c(B_{s+t}-B_s) - \frac{\lambda_c^2}{2}t} \,\middle|\, \mathcal{F}_s\right]$$

$B_{s+t} - B_s$ is indep of $\mathcal{F}_s$ and $\overset{(d)}{=} B_t$ under $\mathbb{P}$

$$= (\lambda_c s + L - B_s)\, e^{\lambda_c B_s - \frac{\lambda_c^2}{2}s}\; \underbrace{\mathbb{E}\left[e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\right]}_{= \mathbb{E}[1] = 1} + \underbrace{\mathbb{E}\left[(\lambda_c t - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\right]}_{= \mathbb{E}[-B_t] = 0}$$

$$= (\lambda_c s + L - B_s)\, e^{\lambda_c B_s - \frac{\lambda_c^2}{2}s}$$ by Girsanov.

So $\left((\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\right)_{t\geq 0}$ is a martingale under $\mathbb{P}_x$.

1.b Write $\Pi_t = (\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}$ and $\tau = \inf\{s \geq 0 : B_s = \lambda_c s + L\}$

Then τ is a stopping time so $(\Pi_{t\wedge\tau})_{t\geq 0}$ is a martingale.

But $$\Pi_{t\wedge\tau} = \underbrace{\Pi_\tau \mathbb{1}_{\tau \leq t}}_{\substack{=0 \\ \text{because } \lambda_c\tau + L - B_\tau = 0}} + \Pi_t \underbrace{\mathbb{1}_{\tau > t}}_{= \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s < L} = \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L} \text{ a.s.}} = (\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L} \quad \text{a.s.}$$

So $\left((\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L}\right)_{t\geq 0}$ is a martingale under $\mathbb{P}_x$.

1.c. This can be seen as a consequence of a Lemma of Lecture 3 together with 1.b.

Let's do a direct proof. First note that, for $t \geq 0$, by the many-to-one

$$\mathbb{E}[Z_t^{(L)}] = e^{mt}\, \mathbb{E}\left[(\lambda_c t + L - B_t)\, e^{\lambda_c(B_t - \lambda_c t)}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L}\right] \qquad m = \frac{\lambda_c^2}{2}$$

$$= \mathbb{E}\left[(\lambda_c t + L - B_t)\, e^{\lambda_c B_t - \frac{\lambda_c^2}{2}t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s - \lambda_c s \leq L}\right]$$

$= L$ using that it is a martingale.

$$Z^{(L)}_{s+t} = \sum_{u\in\mathcal{N}_{s+t}} \big(\lambda_c(s+t)+L-X_u(s+t)\big)\, e^{\lambda_c(X_u(s+t)-\lambda_c(s+t))}\, \mathbb{1}_{\max_{r\in[0,s+t]} X_u(r)-\lambda_c r\le L}$$

$$= \sum_{v\in\mathcal{N}_s} e^{\lambda_c(X_v(s)-\lambda_c s)}\, \mathbb{1}_{\max_{r\in[0,s]} X_v(r)-\lambda_c r\le L} \underbrace{\sum_{\substack{u\in\mathcal{N}_{s+t}\\ u\geq v}} \Big(\underbrace{\lambda_c s+L-X_v(s)}_{=:L_v(s)} + \lambda_c t - \big(X_u(t+s)-X_v(s)\big)\Big) e^{\lambda_c(X_u(s+t)-X_v(s)-\lambda_c t)} \times \mathbb{1}_{\max_{r\in[0,t]} X_u(r)-X_v(s)-\lambda_c r \le \underbrace{L+\lambda_c s-X_v(s)}_{=L_v(s)}}}$$

by the branching property, given $\mathcal{F}_s$, these are independent r.v. with the same distribution as $Z_t^{(L_v(s))}$

So $$\mathbb{E}\big[Z^{(L)}_{s+t}\,\big|\,\mathcal{F}_s\big] = \sum_{v\in\mathcal{N}_s} e^{\lambda_c(X_v(s)-\lambda_c s)}\, \mathbb{1}_{\max_{r\in[0,s]} X_v(r)-\lambda_c r\le L}\, \underbrace{\mathbb{E}\big[Z_t^{(L_v(s))}\big]}_{=L_v(s)=\lambda_c s+L-X_v(s)} = Z_s^{(L)}$$

and $(Z_t^{(L)})_{t\geq 0}$ is a martingale.

1.d. $(Z_t^{(L)})_{t\geq0}$ is a non-negative martingale so it converges a.s. to a limit $Z_\infty^{(L)}$.

2.a. Note that $E_L = \{\forall s\geq 0, \forall u\in\mathcal{N}(s), X_u(s)\leq \lambda_c s+L\}$, so, on E_L,

$$Z_t^{(L)} = \sum_{u\in\mathcal{N}_t}\big(\lambda_c t+L-X_u(t)\big)e^{\lambda_c(X_u(t)-\lambda_c t)} = Z_t + L W_t^{\lambda_c}.$$

→ Recall $W_t^{\lambda_c} = \sum_{u\in\mathcal{N}_t} e^{\lambda_c X_u(t)-(m+\frac{\lambda_c^2}{2})t}$

$\lambda_c^2 = 2m \Rightarrow = \sum_{u\in\mathcal{N}_t} e^{\lambda_c(X_u(t)-\lambda_c t)}$

2.b. We know that $Z_t^{(L)} \xrightarrow{a.s.} Z_\infty^{(L)}$ and $W_t^{\lambda_c} \xrightarrow{a.s.} 0$ so, on E_L,

$$Z_t = Z_t^{(L)} - L W_t^{\lambda_c} \xrightarrow{a.s.} Z_\infty^{(L)}.$$

2.c. The family of events $(E_L)_{L>0}$ is increasing and $\mathbb{P}(E_L)\longrightarrow 1$ so $\mathbb{P}\big(\bigcup_{L>0} E_L\big)=1$.

But, on any E_L, $(Z_t)_{t\geq0}$ converges a.s. so it converges a.s. on the full probability space. Let Z_∞ be its limit.

By 2.b., on E_L, $Z_\infty = Z_\infty^{(L)}$ a.s. for any $L>0$. But $Z_\infty^{(L)}\geq 0$.

So $Z_\infty \geq 0$ a.s. (using again that $\mathbb{P}\big(\bigcup_{L>0}E_L\big)=1$).

3.a. Write $f_t^{(L)}(x) = (\lambda_c t + L - x)e^{\lambda_c(x-\lambda_c t)}$

$$\big(Z_t^{(L)}\big)^2 = \sum_{u\in\mathcal{N}_t} f_t^{(L)}(X_u(t))^2\, \mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\le L}$$

$$+ \sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} f_t^{(L)}(X_u(t))\, \mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\le L}\, f_t^{(L)}(X_v(t))\, \mathbb{1}_{\max_{s\in[0,t]} X_v(s)-\lambda_c s\le L}$$

For the 1st term, by the many-to-one, we get

$$\mathbb{E}\left[\sum_{u\in\mathcal{N}_t} f_t^{(L)}(X_u(t))^2\, \mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\le L}\right] = e^{mt}\,\mathbb{E}\left[f_t^{(L)}(B_t)^2\, \mathbb{1}_{\max_{s\in[0,t]} B_s-\lambda_c s\le L}\right]$$

$$\le e^{mt}\,\mathbb{E}\left[(\lambda_c t+L-B_t)^2 e^{2\lambda_c(B_t-\lambda_c t)}\, \mathbb{1}_{\max_{s\in[0,t]} B_s-\lambda_c s\le L}\right]$$

$$= \cancel{e^{mt}}\,\mathbb{E}\left[e^{-\lambda_c B_t-\cancel{\frac{\lambda_c^2}{2}t}} (L-B_t)^2\, e^{2\lambda_c B_t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s\le L}\right]$$

$$= \mathbb{E}\left[(L-B_t)^2 e^{\lambda_c B_t}\, \mathbb{1}_{\max_{s\in[0,t]} B_s\le L}\right]$$

$$\le \sum_{k\ge 1} k^2 e^{\lambda_c(L-k+1)} \underbrace{\mathbb{P}\left(B_t\in[L-k,L-k+1],\ \max_{s\in[0,t]} B_s\le L\right)}$$

$$\le \left(\frac{L}{t^{3/2}}\wedge 1\right) e^{\lambda_c(L+1)} \underbrace{\sum_{k\ge 1} k^4 e^{-\lambda_c k}}_{<\infty} \qquad \le \mathbb{P}\left(B_t\ge L-k,\ \max B_s\le L\right)\le \frac{Lk^2}{t^{3/2}}\wedge 1 \text{ by Corollary 2 Lecture 6}$$

$$\le \left(\frac{L}{t^{3/2}}\wedge 1\right)k^2$$

So this is bounded.

We now deal with the 2^{nd} term. By the many-to-two lemma,

$$\mathbb{E}\left[\sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} f_t^{(L)}(X_u(t))\, \mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\le L}\ f_t^{(L)}(X_v(t))\, \mathbb{1}_{\max_{s\in[0,t]} X_v(s)-\lambda_c s\le L}\right]$$

$$= \mathbb{E}[L(L-1)] \int_0^t e^{2mt-mr}\, \mathbb{E}\left[f_t^{(L)}(B_t^{1,r})\, \mathbb{1}_{\max_{s\in[0,t]} B_s^{1,r}-\lambda_c s\le L}\ f_t^{(L)}(B_t^{2,r})\, \mathbb{1}_{\max_{s\in[0,t]} B_s^{2,r}-\lambda_c s\le L}\right] dr$$

We first focus on the expectation in the integral:

$$\mathbb{E}\left[f_t^{(L)}(B_t^{1,r})\, \mathbb{1}_{\max_{s\in[0,t]} B_s^{1,r}-\lambda_c s\le L}\ f_t^{(L)}(B_t^{2,r})\, \mathbb{1}_{\max_{s\in[0,t]} B_s^{2,r}-\lambda_c s\le L}\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\max_{s\in[0,r]} B_s^0-\lambda_c s\le L}\, f_t^{(L)}(B_r^0+B_{t-r}^1)\, \mathbb{1}_{\max_{s\in[0,t-r]} B_r^0+B_s^1-\lambda_c(r+s)\le L}\, f_t^{(L)}(B_r^0+B_{t-r}^2)\, \mathbb{1}_{\max_{s\in[0,t-r]} B_r^0+B_s^2-\lambda_c(r+s)\le L}\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\max_{s\in[0,r]} B_s^0-\lambda_c s\le L}\, \varphi(B_r^0)^2\right]$$

with $\varphi(x) = \mathbb{E}\left[f_t^{(L)}(x+B_{t-r}^1)\, \mathbb{1}_{\max_{s\in[0,t-r]} x+B_s^1-\lambda_c(r+s)\le L}\right]$

$$= \mathbb{E}\left[(\lambda_c t+L-B_{t-r}-x)\, e^{\lambda_c(B_{t-r}+x-\lambda_c t)}\, \mathbb{1}_{\max_{s\in[0,t-r]} B_s+x-\lambda_c r-\lambda_c s\le L}\right]$$

$$= \mathbb{E}_{x-\lambda_c r}\left[(\lambda_c(t-r)+L-B_{t-r})\, e^{\lambda_c(B_{t-r}-\lambda_c(t-r))}\, \mathbb{1}_{\max_{s\in[0,t-r]} B_s-\lambda_c s\le L}\right] \quad \text{martingale (1.b.)}$$

$$= e^{-\frac{\lambda_c^2}{2}(t-r)} \times \mathbb{E}_{x-\lambda_c r}\left[(\lambda_c\times 0+L-B_0)\, e^{\lambda_c B_0-\frac{\lambda_c^2}{2}\times 0}\, \mathbb{1}_{\max_{s\in[0,0]} B_s-\lambda_c s\le L}\right]$$

$$= e^{-\frac{\lambda_c^2}{2}(t-r)} (\lambda_c r+L-x)\, e^{\lambda_c(x-\lambda_c r)}\, \mathbb{1}_{x-\lambda_c r\le L}$$

$$= e^{-\lambda_c^2(t-r)}\, \underbrace{\mathbb{E}\left[\mathbb{1}_{\max_{s\in[0,r]} B_s-\lambda_c s\le L} (\lambda_c r+L-B_r)^2 e^{2\lambda_c(B_r-\lambda_c r)}\right]}$$

$$\le e^{-mr}\left(\frac{L}{r^{3/2}}\wedge 1\right) C \quad \text{with } C \text{ depending on } L \text{ and } \lambda_c.$$

$$\le C e^{-2mt+mr}\left(\frac{L}{r^{3/2}}\wedge 1\right)$$

So finally we get

$$\mathbb{E}\Big[\sum_{\substack{u,v\in\mathcal{N}_t\\ u\neq v}} f_t^{(L)}(X_u(t))\,\mathbb{1}_{\max_{s\in[0,t]} X_u(s)-\lambda_c s\leq L}\ f_t^{(L)}(X_v(t))\,\mathbb{1}_{\max_{s\in[0,t]} X_v(s)-\lambda_c s\leq L}\Big]$$

$$\leq \mathbb{E}[L(L-1)]\int_0^t C\left(\frac{L}{r^{3/2}}\wedge 1\right)dr \leq C'.$$

So $(Z_t^{(L)})_{t\geq 0}$ is bounded in L^2.

3.b. In particular, we deduce that $(Z_t^{(L)})_{t\geq 0}$ converges in L^1 and so $\mathbb{E}[Z_\infty^{(L)}] = \mathbb{E}[Z_0^{(L)}] = L$ and therefore $\mathbb{P}(Z_\infty^{(L)}>0)>0$.

3.c. On $\{\tau_\emptyset\leq t\}$, we decompose

$$Z_t = \sum_{i=1}^{L_\emptyset}\sum_{\substack{u\in\mathcal{N}_t\\ u\geq i}}(\lambda_c t - X_u(t))\,e^{\lambda_c(X_u(t)-\lambda_c t)}$$

$\lambda_c t - X_u(t) = \lambda_c\tau_\emptyset - X_\emptyset(\tau_\emptyset) + \lambda_c(t-\tau_\emptyset) - (X_u(t)-X_\emptyset(\tau_\emptyset))$

$$= (\lambda_c\tau_\emptyset - X_\emptyset(\tau_\emptyset))\,e^{\lambda_c(X_\emptyset(\tau_\emptyset)-\lambda_c\tau_\emptyset)}\sum_{i=1}^{L_\emptyset}\underbrace{\sum_{\substack{u\in\mathcal{N}_t\\ u\geq i}} e^{\lambda_c(X_u(t)-X_\emptyset(\tau_\emptyset)-\lambda_c(t-\tau_\emptyset))}}$$

this is the critical additive martingale of the BBM rooted at i at time $t-\tau_\emptyset$: tends to 0 a.s as $t\longrightarrow\infty$.

$$+\, e^{\lambda_c(X_\emptyset(\tau_\emptyset)-\lambda_c\tau_\emptyset)}\sum_{i=1}^{L_\emptyset}\underbrace{\sum_{\substack{u\in\mathcal{N}_t\\ u\geq i}}\Big(\lambda_c(t-\tau_\emptyset)-(X_u(t)-X_\emptyset(\tau_\emptyset))\Big)e^{\lambda_c(X_u(t)-X_\emptyset(\tau_\emptyset)-\lambda_c(t-\tau_\emptyset))}}$$

this is the derivative martingale of the BBM rooted at i at time $t-\tau_\emptyset$: tends to Z_∞^i a.s. as $t\longrightarrow\infty$ where $(Z_\infty^i)_{i\geq 1}$ are iid $\overset{(d)}{=} Z_\infty$ and indep. of $\mathcal{F}_{\tau_\emptyset}$

So letting $t\to\infty$, we get $Z_\infty = e^{\lambda_c(X_\emptyset(\tau_\emptyset)-\lambda_c\tau_\emptyset)}\sum_{i=1}^{L_\emptyset} Z_\infty^i$

in particular $\mathbb{P}(Z_\infty=0) = \mathbb{P}(\forall i\in\{1,\dots,L_\emptyset\},\ Z_\infty^i=0)$

$$= \mathbb{E}\left[\mathbb{P}(Z_\infty=0)^{L_\emptyset}\right] = f(\mathbb{P}(Z_\infty=0))$$

where $f(s)=\mathbb{E}[s^L]$.

But f has 2 fixed points in $[0,1]$: 1 and $q=\mathbb{P}(\text{extinction})$.

Since $\mathbb{P}(Z_\infty=0)\neq 1$ by 3.b. (and the fact that $Z_\infty\geq Z_\infty^{(L)}$, see 4.b. below), we get $\mathbb{P}(Z_\infty=0)=q$.

Moreover $\{\text{extinction}\}\subset\{Z_\infty=0\}$ so $\mathbb{P}(\{Z_\infty=0\}\cap\{\text{survival}\})=0$.

Hence $Z_\infty>0$ a.s. on the survival event.

4.a. The fact $E[Z_\infty^{(L)}] = L$ has been seen in 3.b.

4.b. Recall that, on E_L, $Z_\infty = Z_\infty^{(L)}$ a.s. and that $P(\bigcup_{L>0}^{\uparrow} E_L) = 1$.

Moreover note that $L \longmapsto Z_\infty^{(L)}$ is non-decreasing.

It follows that $Z_\infty \geq Z_\infty^{(L)}$ a.s. for any $L>0$ and so $E[Z_\infty] \geq L$.

Therefore $E[Z_\infty] = +\infty$.

Exercise 2 of Lecture 5 : Let $\tilde{M}_t = \max_{1 \leq i \leq \lfloor e^{mt} \rfloor} B_t^i$ with $(B^i)_{i\geq 0}$ independent BMs.

Prove that $\frac{\tilde{M}_t}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \lambda_c = \sqrt{2m}$.

Help with intermediary questions :

1. Prove that, for any $\varepsilon > 0$, there exists $C, c > 0$ such that for t large enough

$$P\left(\left|\frac{\tilde{M}_t}{t} - \lambda_c\right| \geq \varepsilon\right) \leq C e^{-ct}.$$

2. Deduce that $\frac{\tilde{M}_k}{k} \xrightarrow[\substack{k\to\infty \\ k\in\mathbb{N}}]{\text{a.s.}} \lambda_c$.

3. Prove that $\max_{1 \leq i \leq \lfloor e^{mk} \rfloor} \max_{s \in [k-1,k]} \frac{1}{k} |B_s^i - B_k^i| \xrightarrow[\substack{k\to\infty \\ k\in\mathbb{N}}]{\text{a.s.}} 0$.

4. Conclude.

Additional hints :

1. Treat separately $P(\tilde{M}_t \leq (\lambda_c - \varepsilon)t)$ and $P(\tilde{M}_t \geq (\lambda_c + \varepsilon)t)$.

2. Borel Cantelli

3. Use that $\max_{r\in[0,1]} B_r \overset{(d)}{=} \max_{r\in[0,1]} -B_r \overset{(d)}{=} |B_1| \overset{(d)}{=} \frac{|B_k|}{\sqrt{k}}$ and use question 2.

(Direct bound + Borel Cantelli can work as well).

4. For $s \in [k-1,k]$, upper or lower bound $\tilde{M}_s$ using $\tilde{M}_k$ or $\tilde{M}_{k-1}$, and question 3.