Lecture 10 : Two-speed BBM (2nd part) 05/02/2025

IV.3.2) Upper bound (case $\sigma_1 > \sigma_2$)

As for the usual BBM, we prove a weaker version of the result, which is :

$$M_t \le \bar{m}_t + O(\log\log t).$$

Proof strategy is similar :

- Argue that there is a barrier touched by no particle with high probability.
- Do a first moment calculation on the number of particles going above $\bar{m}_t + C\log\log t$ while staying below the barrier.

Lemma: For $L, t > 0$, we define $f_L^t : s \in [0,t] \longrightarrow \begin{cases} \lambda_c \sigma_1 s + L & \text{if } s \le bt \\ \lambda_c \sigma_1 bt + \lambda_c \sigma_2 (s - bt) + L & \text{if } s > bt \end{cases}$

and $F_L^t = \{\forall u \in \mathcal{N}_t, \forall s \in [0,t], X_u^t(s) < f_L^t(s)\}$.

Then $\mathbb{P}((F_L^t)^c) \le e^{-\lambda_c L/\sigma_1}$.

Proof: We introduce a martingale analogous to $W_t^{\lambda_c}$.

For $s \in [0,t]$, let $V_s = \begin{cases} \displaystyle\sum_{u \in \mathcal{N}_s} \exp\left(\frac{\lambda_c}{\sigma_1} X_u^t(s) - \left(m + \frac{\lambda_c^2}{2}\right)s\right) & \text{if } s \le bt \\ \displaystyle\sum_{u \in \mathcal{N}_s} \exp\left(\frac{\lambda_c}{\sigma_1} X_u^t(bt) + \frac{\lambda_c}{\sigma_2}\left(X_u^t(s) - X_u^t(bt)\right) - \left(m + \frac{\lambda_c^2}{2}\right)s\right) & \text{if } s > bt \end{cases}$

Then $(V_s)_{s \in [0,t]}$ is a martingale (Exercise 1 : check it).

Let $\tau = \inf\{s \in [0,t] : \exists u \in \mathcal{N}_s : X_u^t(s) = f_L^t(s)\}$ ($= +\infty$ if empty set).

Note that $\mathbb{P}((F_L^t)^c) = \mathbb{P}(\tau \le t)$.

Since τ is a stopping time, by the optional stopping theorem,

$$\mathbb{E}[V_{\tau \wedge t}] = \mathbb{E}[V_0] = 1.$$

Now we argue that when $\tau \le t$, V_τ is large :

- if $\tau \le bt$, then $V_\tau \underset{\substack{\text{keep only the particle that hit the barrier}}}{\ge} \exp\left(\frac{\lambda_c}{\sigma_1} f_L^t(\tau) - \left(m + \frac{\lambda_c^2}{2}\right)\tau\right) = e^{\lambda_c L/\sigma_1}$ (recall $\lambda_c = \sqrt{2m}$)
- if $\tau \in (bt, t]$, if u is the particle hitting the barrier at time τ, we have

$$\frac{\lambda_c}{\sigma_1} X_u^t(bt) + \frac{\lambda_c}{\sigma_2}\left(X_u^t(\tau) - X_u^t(bt)\right) = \frac{\lambda_c}{\sigma_2} \underbrace{X_u^t(\tau)}_{= f_L^t(\tau)} - \underbrace{\lambda_c\left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right)}_{> 0 \text{ because } \sigma_1 > \sigma_2} \underbrace{X_u^t(bt)}_{< f_L^t(bt) \text{ because } \tau > bt}$$

$$\geq \frac{\lambda_c}{\sigma_2}\left(\cancel{\lambda_c\sigma_1 bt} + \lambda_c\sigma_2(\tau - \cancel{bt}) + L\right) - \lambda_c\left(\cancel{\tfrac{1}{\sigma_2}} - \cancel{\tfrac{1}{\sigma_1}}\right)\lambda_c\sigma_1 bt$$

$$= \lambda_c^2\tau + \frac{\lambda_c}{\sigma_2}L$$

So $V_\tau \geq \exp\left(\lambda_c^2\tau + \frac{\lambda_c}{\sigma_2}L - \left(m + \frac{\lambda_c^2}{2}\right)\tau\right) = e^{\lambda_c L/\sigma_2} \geq e^{\lambda_c L/\sigma_1}$

Therefore $\mathbb{E}[V_{\tau\wedge t}] \geq \mathbb{E}[V_\tau \mathbb{1}_{\tau\leq t}] \geq e^{\lambda_c L/\sigma_1} \mathbb{P}(\tau \leq t)$

Combining with $\mathbb{E}[V_{\tau\wedge t}]$, we get $\mathbb{P}(\tau\leq t) \leq e^{-\lambda_c L/\sigma_1}$. ∎

Exercise 2: Consider a usual BBM and recall that we defined the event $E_L = \{\forall t>0, \forall u \in \mathcal{N}_t, X_u(t) \leq \lambda_c t + L\}$. Use a similar method (with the martingale W^{λ_c}) to prove $\mathbb{P}(E_L^c) \leq e^{-\lambda_c L}$.

Proof of the upper bound:

Recall that, in case $\sigma_1 > \sigma_2$, $\overline{m}_t = \lambda_c(b\sigma_1 + (1-b)\sigma_2)t - \frac{3(\sigma_1+\sigma_2)}{2\lambda_c}\log t$

$$= \sigma_1 m_{bt} + \sigma_2 m_{(1-b)t} + \frac{3\sigma_1}{2\lambda_c}\log b + \frac{3\sigma_2}{2\lambda_c}\log(1-b)$$

We want to prove $\mathbb{P}(\overline{\Pi}_t \geq \overline{m}_t + x_t) \xrightarrow[t\to\infty]{} 0$ where $x_t = a\log\log t$ with $a>0$ to be chosen later. Let $\varepsilon>0$.

By the previous lemma, we can choose $L>0$ such that $\mathbb{P}((F_L^t)^c) \leq \varepsilon$.

Also recall we proved $\mathbb{P}(\Pi_t \geq m_t + \frac{3}{\lambda_c}\log\log t) \xrightarrow[t\to\infty]{} 0$ for usual BBM so letting $G_t = \{\forall u\in\mathcal{N}_{bt}, X_u^t(bt) \leq \sigma_1(m_{bt} + \frac{3}{\lambda_c}\log\log t)\}$ we have $\mathbb{P}((G_t)^c) \xrightarrow[t\to\infty]{} 0$.

So it is now enough to prove $\mathbb{P}(\{\overline{\Pi}_t \geq \overline{m}_t + x_t\} \cap F_L^t \cap G^t) \xrightarrow[t\to\infty]{} 0$

$\mathbb{P}(\{\overline{\Pi}_t \geq \overline{m}_t + x_t\} \cap F_L^t \cap G^t)$

$\leq \mathbb{P}\left(\exists u\in\mathcal{N}_t : X_u^t(t) \geq \overline{m}_t + x_t, \forall s\in[0,t], X_u(s) \leq f_L^t(s), X_u(bt) \leq \sigma_1\left(m_{bt} + \frac{3}{\lambda_c}\log\log t\right)\right)$

$\leq \mathbb{E}\left[\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u^t(t)\geq \overline{m}_t + x_t,\ \forall s\in[0,t], X_u(s)\leq f_L^t(s),\ X_u(bt)\leq \sigma_1(m_{bt} + \frac{3}{\lambda_c}\log\log t)}\right]$

$= e^{mt}\,\mathbb{P}\left(\overline{B}_t^t \geq \overline{m}_t + x_t, \forall s\in[0,t], \overline{B}_s^t \leq f_L^t(s), \overline{B}_{bt}^t \leq \sigma_1\left(m_{bt} + \frac{3}{\lambda_c}\log\log t\right)\right)$

many-to-one lemma

recall $\overline{B}_s^t = \begin{cases} \sigma_1 B_s & \text{for } s\in[0,bt] \\ \sigma_1 B_{bt} + \sigma_2 B'_{s-bt} & \text{for } s\in(bt,t]\end{cases}$ with B, B' independent standard BM.

$$= e^{mt}\,\mathbb{P}\left(\begin{array}{l}\sigma_1 B_{bt} + \sigma_2 B'_{(1-b)t} \geq \bar m_t + x_t\ ,\ \forall s\in[0,bt],\ \sigma_1 B_s \leq \lambda_c \sigma_1 s + L,\\ \forall s\in[bt,t],\ \sigma_1 B_{bt} + \sigma_2 B'_{s-bt} \leq \lambda_c\sigma_1 bt + \lambda_c\sigma_2(s-bt) + L,\ \sigma_1 B_{bt} \leq \sigma_1\left(m_{bt} + \frac{3}{\lambda_c}\log\log t\right)\end{array}\right)$$

integrate w.r.t. B' first

$$= e^{mt}\,\mathbb{E}\left[\mathbb{1}_{\forall s\in[0,bt],\ B_s\leq \lambda_c s + \frac{L}{\sigma_1},\ B_{bt}\leq m_{bt} + \frac{3}{\lambda_c}\log\log t}\ \varphi(B_{bt})\right]$$

where $\varphi(x) = \mathbb{P}\left(\sigma_1 x + \sigma_2 B'_{(1-b)t} \geq \bar m_t + x_t,\ \forall s\in[0,(1-b)t],\ \sigma_1 x + \sigma_2 B'_s \leq \lambda_c\sigma_1 bt + \lambda_c\sigma_2 s + L\right)$

We now bound $\varphi(x)$ for $x \leq m_{bt} + \frac{3}{\lambda_c}\log\log t$. (because we are on the event $B_{bt} \leq m_{bt} + \frac{3}{\lambda_c}\log\log t$)

$$\varphi(x) = \mathbb{P}\left(B_{(1-b)t} - \lambda_c(1-b)t \geq \frac{1}{\sigma_2}\left(\bar m_t + x_t - \sigma_2\lambda_c(1-b)t - \sigma_1 x\right),\ \max_{s\in[0,(1-b)t]}(B_s - \lambda_c s) \leq \frac{1}{\sigma_2}\left(\lambda_c\sigma_1 bt + L - \sigma_1 x\right)\right)$$

$$= \mathbb{E}\left[\underbrace{e^{-\lambda_c B_{(1-b)t} - \frac{\lambda_c^2}{2}(1-b)t}}\ \mathbb{1}_{B_{(1-b)t}\geq \frac{1}{\sigma_2}(\bar m_t + x_t - \sigma_2\lambda_c(1-b)t - \sigma_1 x)}\ \mathbb{1}_{\max_{s\in[0,(1-b)t]} B_s \leq \frac{1}{\sigma_2}(\lambda_c\sigma_1 bt + L - \sigma_1 x)}\right]$$

$$\leq \exp\left(-\frac{\lambda_c}{\sigma_2}\left(\bar m_t + x_t - \sigma_2\lambda_c(1-b)t - \sigma_1 x\right)\right)$$

$$\leq \exp\left(-\frac{\lambda_c}{\sigma_2}x_t + \frac{3}{2}\log t - \frac{\lambda_c}{\sigma_2}\sigma_1(m_{bt} - x)\right)$$

$\bar m_t \geq \sigma_1 m_{bt} + \sigma_2\lambda_c(1-b)t - \frac{3\sigma_2}{2\lambda_c}\log t$

$$\leq t^{3/2} e^{-\frac{\lambda_c}{\sigma_2}x_t - \frac{\lambda_c^2}{2}(1-b)t}\, e^{\lambda_c\frac{\sigma_1}{\sigma_2}(x - m_{bt})}\,\mathbb{P}\left(B_{(1-b)t} \geq \overbrace{\frac{1}{\sigma_2}\left(x_t - \frac{3\sigma_2}{2\lambda_c}\log t + \sigma_1(m_{bt} - x)\right)}^{=a-y \text{ with } y = \left(\frac{\sigma_1}{\sigma_2}+1\right)\frac{3}{2\lambda_c}\log t + \frac{L - x_t}{\sigma_2}},\ \max_{s\in[0,(1-b)t]} B_s \leq \underbrace{\frac{\sigma_1}{\sigma_2}(\lambda_c bt - x) + \frac{L}{\sigma_2}}_{=a}\right)$$

Corollary 2 of Lecture 6

$$\leq t^{3/2} e^{-\frac{\lambda_c}{\sigma_2}x_t - m(1-b)t}\, e^{\lambda_c\frac{\sigma_1}{\sigma_2}(x - m_{bt})} \times \frac{1}{((1-b)t)^{3/2}}\left(\frac{\sigma_1}{\sigma_2}(\lambda_c bt - x) + \frac{L}{\sigma_2}\right)\left(\left(\frac{\sigma_1}{\sigma_2}+1\right)\frac{3}{2\lambda_c}\log t + \frac{L - x_t}{\sigma_2}\right)^2$$

$\lambda_c bt - x \geq \frac{3}{2\lambda_c}\log t - \frac{3}{\lambda_c}\log\log t$ for our choice of x

$$\leq C\, e^{-\frac{\lambda_c}{\sigma_2}x_t - m(1-b)t}\, e^{\lambda_c\frac{\sigma_1}{\sigma_2}(x - m_{bt})}\,(\lambda_c bt - x)\,(\log t)^2$$

for t large enough (depending on L) and C depends on $\lambda_c, \sigma_1, \sigma_2, b, L$.

So $\mathbb{P}\left(\{\bar M_t \geq \bar m_t + x_t\} \cap F_L^t \cap G^t\right)$

$$\leq C e^{-\frac{\lambda_c}{\sigma_2}x_t}(\log t)^2 e^{mbt}\,\mathbb{E}\left[e^{\lambda_c\frac{\sigma_1}{\sigma_2}(B_{bt} - m_{bt})}(\lambda_c bt - B_{bt})\,\mathbb{1}_{\max_{s\in[0,bt]} B_s - \lambda_c s \leq \frac{L}{\sigma_1},\ B_{bt}\leq m_{bt} + \frac{3}{\lambda_c}\log\log t}\right]$$

$$\mathbb{E}[\dots] \overset{\text{Girsanov}}{=} \mathbb{E}\left[e^{-\lambda_c B_{bt} - \frac{\lambda_c^2}{2}bt} \times e^{\lambda_c\frac{\sigma_1}{\sigma_2}\left(B_{bt} + \frac{3}{2\lambda_c}\log bt\right)}(-B_{bt})\,\mathbb{1}_{\max_{s\in[0,bt]} B_s \leq \frac{L}{\sigma_1},\ B_{bt} \leq \underbrace{-\frac{3}{2\lambda_c}\log bt + \frac{3}{\lambda_c}\log\log t}_{=:y_t}}\right]$$

cut $(-\infty, y_t]$ into $\bigcup_{k\geq 1}(y_t - k, y_t - k + 1]$

$$\leq e^{-\frac{\lambda_c^2}{2}bt}\sum_{k\geq 1} e^{-\lambda_c(y_t - k)} \times e^{\lambda_c\frac{\sigma_1}{\sigma_2}\left(y_t - k + 1 + \frac{3}{2\lambda_c}\log bt\right)}(k - y_t)\,\mathbb{P}\left(\max_{s\in[0,bt]} B_s \leq \frac{L}{\sigma_1},\ B_{bt} \in (y_t - k, y_t - k + 1]\right)$$

Corollary 2 again

$$\le \mathbb{P}\left(\max_{s\in[0,bt]} B_s \le \frac{L}{\sigma_1},\ B_{bt} \ge y_t - k\right)$$

$$\le \frac{\frac{L}{\sigma_1}\left(\frac{L}{\sigma_1} - y_t + k\right)^2}{(bt)^{3/2}}$$

$$\le \frac{C'(k-y_t)^2}{t^{3/2}} \quad \text{for } t \text{ large enough}$$

$$\le \frac{C''}{t^{3/2}} e^{-mbt} \underbrace{e^{-\lambda_c y_t}}_{=(bt)^{3/2} e^{-\lambda_c \times \frac{3}{\lambda_c}\log\log t}} e^{\lambda_c \frac{\sigma_1}{\sigma_2}\times\frac{3}{\lambda_c}\log\log t} \underbrace{\sum_{k\ge 1} \underbrace{(k-y_t)^3}_{\le 8(k^3+|y_t|^3)} e^{\lambda_c \overbrace{\left(1-\frac{\sigma_1}{\sigma_2}\right)}^{<0} k}} \le \text{cste} + |y_t|^3 \times \text{cste}$$

$$\le C''' e^{-mbt} (\log t)^{3\left(\frac{\sigma_1}{\sigma_2}\cancel{-1}\right)} \times \cancel{(\log t)^3}.$$

Finally, we get $\mathbb{P}\left(\{\overline{M}_t \ge \overline{m}_t + x_t\} \cap F_L^t \cap G^t\right) \le C'''' (\log t)^{3\frac{\sigma_1}{\sigma_2} + 2 - \frac{\lambda_c}{\sigma_2} a} \xrightarrow[t\to\infty]{} 0$

by choosing $a > \frac{1}{\lambda_c}(3\sigma_1 + 2\sigma_2)$. ∎

IV.4) The case $\sigma_1 < \sigma_2$

IV.4.1) Upper bound

Here the upper bound is easy because the maximum behaves like $\tilde{m}_t$, which is the behavior in the iid. So it could be deduced from Slepian's lemma or from the following direct proof: for $y > 0$,

$$\mathbb{P}(\overline{M}_t \ge \tilde{m}_t + y) \le \mathbb{E}\left[\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u^t(t) \ge \tilde{m}_t + y}\right]$$

many-to-one lemma

$$= e^{mt}\, \mathbb{P}(B_t^t \ge \tilde{m}_t + y) \qquad B_t^t \overset{(d)}{=} B_t \overset{(d)}{=} \sqrt{t}\, B_1$$

$$= e^{mt}\, \mathbb{P}\left(\mathcal{N}(0,1) \ge \frac{\tilde{m}_t + y}{\sqrt{t}}\right)$$

$$\underset{t\to\infty}{\sim} e^{mt} \times \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t}}{\tilde{m}_t + y} \exp\left(-\frac{1}{2}\left(\frac{\tilde{m}_t + y}{\sqrt{t}}\right)^2\right) \qquad \tilde{m}_t = \lambda_c t - \frac{1}{2\lambda_c}\log t$$

$$\sim \cancel{e^{mt}} \frac{1}{\lambda_c\sqrt{2\pi \cancel{t}}} \exp\left(-\frac{1}{2t}\left(\cancel{\lambda_c^2 t^2} - \cancel{t\log t} - 2\lambda_c y t\right)\right)$$

$\lambda_c^2 = 2m$

$$= \frac{e^{-\lambda_c y}}{\lambda_c\sqrt{2\pi}}$$

$$\xrightarrow[y\to\infty]{} 0.$$

IV.4.2) Lower bound

We use a strategy similar to the one for the usual BBM but with different constraints on trajectories. Here the 1^{st} moment calculation gives the correct behavior without truncation (see upper bound), so we truncate only so that the 2^{nd} moment does not explode.

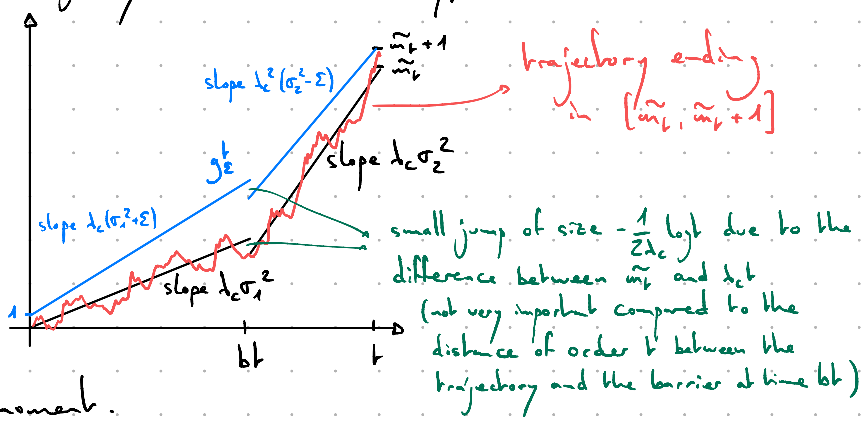
We choose $K_t = \sum_{u \in \mathcal{N}_t} \mathbb{1}_{X_u^t(t) \geq \tilde{m}_t,\ \forall s \in [0,t],\ X_u^t(s) \leq g_\varepsilon^t(s)}$

where $g_\varepsilon^t(s) = \begin{cases} \lambda_c(\sigma_1^2+\varepsilon)s + 1 & \text{if } s \leq bt \\ \tilde{m}_t + 1 - \lambda_c(\sigma_2^2-\varepsilon)(t-s) & \text{if } s \in (bt,t] \end{cases}$ with $\varepsilon > 0$ to be chosen small enough later.

Idea behind this choice: We argued in Lecture 9 that a trajectory going above $\tilde{m}_t$ at time t typically has a drift $\lambda_c \sigma_1^2$ on $[0,bt]$ and $\lambda_c^2\sigma_2^2$ on $[bt,t]$.

We choose the barrier g_ε^t to be slightly above this typical behavior with slopes different by a term ε.

The fact that the slope is different (even by a small term) ensures that adding this barrier is not changing the order of magnitude of the 1^{st} moment.



Lemma 1: For any $\varepsilon > 0$, $0 < \liminf_{t\to\infty} \mathbb{E}[K_t] \leq \limsup_{t\to\infty} \mathbb{E}[K_t] \leq \frac{1}{\lambda_c\sqrt{2\pi}}$.

Proof: For the upper bound: $\mathbb{E}[K_t] \leq \mathbb{E}\Big[\sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u^t(t) \geq \tilde{m}_t}\Big] \xrightarrow[t\to\infty]{} \frac{1}{\lambda_c\sqrt{2\pi}}$ by the calculation done previously (end of previous page).

For the lower bound: by the many-to-one lemma,

$$\mathbb{E}[K_t] = e^{mt}\, \mathbb{P}\big(B_t^t \geq \tilde{m}_t,\ \forall s \in [0,t],\ B_s^t \leq g_\varepsilon^t(s)\big)$$

recall $(B_s^t)_{s\in[0,t]} \overset{(d)}{=} (B_{\varphi(s)})_{s\in[0,t]}$ with $\varphi(s) = \begin{cases} \sigma_1^2 s & \text{if } s \leq bt \\ \underbrace{\sigma_1^2 bt + \sigma_2^2(s-bt)}_{= t - \sigma_2^2(t-s)} & \text{if } s \geq bt \end{cases}$

$$= e^{mt}\, \mathbb{P}\big(B_t \geq \tilde{m}_t,\ \forall s\in[0,t],\ B_{\varphi(s)} \leq g_\varepsilon^t(s)\big)$$

$$= \cancel{e^{mt}}\, \mathbb{E}\Big[\underline{e^{-\lambda_c B_t - \cancel{\frac{\lambda_c^2}{2}t}}}\ \mathbb{1}_{B_t + \lambda_c t \geq \tilde{m}_t,\ \forall s\in[0,t],\ \underline{B_{\varphi(s)} + \lambda_c\varphi(s) \leq g_\varepsilon^t(s)}}\Big]$$

$$\geq e^{\frac{1}{2}\log t - \lambda_c}$$

$$\Rightarrow B_t + \lambda_c t \le \tilde{m}_t + 1 \iff B_t \le -\frac{1}{2\lambda_c}\log t + 1$$

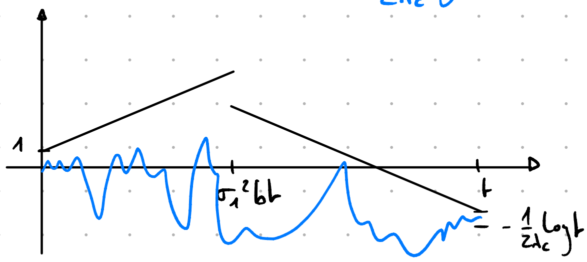
$$\geq \sqrt{t}\, e^{-\lambda_c}\, \mathbb{P}\Big(B_t \geq -\frac{1}{2\lambda_c}\log t,\ \forall s\in[0,t],\ B_{\varphi(s)} \le g_\varepsilon^t(s) - \lambda_c \varphi(s)\Big)$$

$$\iff \begin{cases} \forall s\in[0,bt],\ B_{\sigma_1^2 s} \le \lambda_c(\cancel{\sigma_1^2}+\varepsilon)s + 1 - \cancel{\lambda_c\sigma_1^2 s} \\ \forall s\in(bt,t],\ B_{t-\sigma_2^2(t-s)} \le \underbrace{\tilde{m}_t + 1 - \lambda_c(\cancel{\sigma_2^2}-\varepsilon)(t-s) - \lambda_c(t-\cancel{\sigma_2^2(t-s)})}_{= -\frac{1}{2\lambda_c}\log t + 1 + \varepsilon\lambda_c(t-s)} \end{cases}$$

setting $r=\sigma_1^2 s$ in the 1^{st} case and $r = t-\sigma_2^2(t-s)$ in the 2^{nd}.

$$\iff \begin{cases} \forall r\in[0,\sigma_1^2 bt],\ B_r \le 1 + \frac{\lambda_c\varepsilon}{\sigma_1^2} r \\ \forall r\in(\sigma_1^2 bt, t],\ B_r \le -\frac{1}{2\lambda_c}\log t + 1 + \frac{\lambda_c\varepsilon}{\sigma_2^2}(t-r) \end{cases}$$

The event we consider for B:



We have $\liminf_{t\to\infty} \sqrt{t}\, \mathbb{P}\Big(B_t \geq -\frac{1}{2\lambda_c}\log t,\ \forall s\in[0,t],\ B_{\varphi(s)} \le g_\varepsilon^t(s) - \lambda_c\varphi(s)\Big) > 0$.

This is a bit technical to prove so we admit it!

Heuristically, we have $\mathbb{P}\Big(B_t \in \Big[-\frac{1}{2\lambda_c}\log t, -\frac{1}{2\lambda_c}\log t + 1\Big]\Big) \sim \frac{1}{\sqrt{2\pi t}}$ (prove it!) and then the barrier has an effect only for times s of the form $s = O(1)$ or $s = t - O(1)$ because it grows linearly whereas $B_s = O(\sqrt{s})$. So adding the barrier to the event only reduces the probability by a constant factor $c(\varepsilon) > 0$.

With this fact in hand, we get the desired result. ■

<u>Lemma 2</u>: There exists $\varepsilon, C > 0$ such that $\limsup_{t\to\infty} \mathbb{E}[K_t^2] \le C$.

<u>Proof</u>: We write $\mathbb{E}[K_t^2] = \mathbb{E}[K_t(K_t-1)] + \mathbb{E}[K_t]$. By Lemma 1, $\mathbb{E}[K_t]$ is OK.

$$\mathbb{E}[K_t(K_t-1)] = \mathbb{E}\Big[\sum_{\substack{u,v\in\mathcal{N}_t \\ u\neq v}} \mathbb{1}_{X_u^t(t)\geq \tilde{m}_t,\ \forall s\in[0,t],\ X_u^t(s)\le g_\varepsilon^t(s)}\ \mathbb{1}_{X_v^t(t)\geq\tilde{m}_t,\ \forall s\in[0,t],\ X_v^t(s)\le g_\varepsilon^t(s)}\Big]$$

many-to-two

$$= \mathbb{E}[L(L-1)] \int_0^t e^{2mt-mr}\, \underbrace{\mathbb{P}\Big(\bigcap_{i=1}^2 \big\{\bar{B}_t^{t,i,r} \geq \tilde{m}_t,\ \forall s\in[0,t],\ \bar{B}_s^{t,i,r} \le g_\varepsilon^t(s)\big\}\Big)}\, dr$$

we keep the information of the barrier only at time r ← $\le \mathbb{P}\Big(\bigcap_{i=1}^2 \big\{\bar{B}_t^{t,i,r} \geq \tilde{m}_t,\ \bar{B}_r^{t,i,r} \le g_\varepsilon^t(r)\big\}\Big)$

$\bar{B}_r^{t,2,r} = \bar{B}_r^{t,1,r}$ ← $= \mathbb{P}\Big(\bar{B}_r^{t,1,r} \le g_\varepsilon^t(r),\ (\bar{B}_t^{t,1,r} - \bar{B}_r^{t,1,r}) + \bar{B}_r^{t,1,r} \geq \tilde{m}_t,\ (\bar{B}_t^{t,2,r} - \bar{B}_r^{t,1,r}) + \bar{B}_r^{t,1,r} \geq \tilde{m}_t$

Now recall that $\left(\bar B_r^{t,1,r},\ \bar B_t^{t,1,r} - \bar B_r^{t,1,r},\ \bar B_t^{t,2,r} - \bar B_r^{t,1,r}\right) \overset{(d)}{=} (Z_0, Z_1, Z_2)$ where Z_0, Z_1, Z_2 are independent and $Z_i \overset{(d)}{=} \mathcal{N}(0, \Sigma_i^2)$ with

$$\Sigma_0^2 = \begin{cases} \sigma_1^2 r & \text{if } r \le bt \\ \sigma_1^2 bt + \sigma_2^2 (r - bt) & \text{if } r > bt \end{cases}, \qquad \Sigma_1^2 = \Sigma_2^2 = t - \Sigma_0^2 .$$

Therefore, integrating first w.r.t. Z_0, we probability above equals

$$\int_{-\infty}^{g_\varepsilon^t(r)} \underbrace{\mathbb{P}\left(Z_1 + x \ge \tilde m_t\right)^2}_{} \exp\left(-\frac{x^2}{2\Sigma_0^2}\right) \frac{dx}{\Sigma_0 \sqrt{2\pi}} .$$

$$= \mathbb{P}\left(\mathcal{N}(0,1) \ge \frac{\tilde m_t - x}{\Sigma_1}\right) \le \frac{1}{\sqrt{2\pi}} \frac{\Sigma_1}{\tilde m_t - x} \exp\left(-\frac{(\tilde m_t - x)^2}{2\Sigma_1^2}\right) \quad \text{if } \underline{\tilde m_t - x > 0}$$

So we first restrict ourselves to the case $\underline{r \in [0, t - \frac{2}{\lambda_c}]}$ to ensure that $\tilde m_t - x > 0$. More precisely we have for some $c = c(\sigma_1, \sigma_2, b, \lambda_c)$ small enough

- If $r \le bt$, $\tilde m_t - g_\varepsilon^t(r) \ge \lambda_c t - \lambda_c(\sigma_1^2 + \varepsilon) bt + O(\log t) \ge ct \ge c(t-r)$
- If $r \in (bt, t - \frac{2}{\lambda_c}]$, $\tilde m_t - g_\varepsilon^t(r) = \lambda_c(\sigma_2^2 - \varepsilon)(t-r) - 1 \ge \frac{\lambda_c(\sigma_2^2 - \varepsilon)}{2}(t-r) \ge c(t-r)$

if ε is small enough $\sigma_2^2 - \varepsilon \ge 1$ so $\frac{\lambda_c(\sigma_2^2 - \varepsilon)}{2}(t-r) \ge \frac{\lambda_c}{2}(t-r) \ge 1$

So we get (in that case)

$$\mathbb{P}\left(\bigcap_{i=1}^{2} \left\{ \bar B_t^{t,i,r} \ge \tilde m_t,\ \forall s \in [0,t],\ \bar B_s^{t,i,r} \le g_\varepsilon^t(s) \right\}\right) \le \frac{C}{\Sigma_0} \left(\frac{\Sigma_1}{t-r}\right)^2 \int_{-\infty}^{g_\varepsilon^t(r)} \exp\left(-\frac{(\tilde m_t - x)^2}{\Sigma_1^2} - \frac{x^2}{2\Sigma_0^2}\right) dx$$

$$= \exp\left(-x^2 \underbrace{\left(\frac{1}{\Sigma_1^2} + \frac{1}{2\Sigma_0^2}\right)}_{} - \frac{\tilde m_t^2}{\Sigma_1^2} + \frac{2\tilde m_t x}{\Sigma_1^2}\right) = \exp\left(-\frac{\Sigma_0^2 + t}{2\Sigma_1^2 \Sigma_0^2}\left(x^2 - 2\tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2 + t} x\right) - \frac{\tilde m_t^2}{\Sigma_1^2}\right)$$

$$= \frac{2\Sigma_0^2 + \Sigma_1^2}{2\Sigma_1^2 \Sigma_0^2} = \frac{\Sigma_0^2 + t}{2\Sigma_1^2 \Sigma_0^2}$$

$$= \exp\left(-\frac{\Sigma_0^2 + t}{2\Sigma_1^2 \Sigma_0^2}\left(x - \tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2 + t}\right)^2 + \underbrace{\tilde m_t^2 \frac{2\Sigma_0^2}{\Sigma_1^2(\Sigma_0^2 + t)} - \frac{\tilde m_t^2}{\Sigma_1^2}}_{}\right)$$

$$= \frac{\tilde m_t^2}{\Sigma_1^2}\left(\frac{2\Sigma_0^2}{\Sigma_0^2 + t} - 1\right) = -\frac{\tilde m_t^2 \left(\cancel{t - \Sigma_0^2}\right)}{\cancel{\Sigma_1^2}(\Sigma_0^2 + t)} = -\frac{\tilde m_t^2}{\Sigma_0^2 + t}$$

$$\le \frac{C}{\Sigma_0}\left(\frac{\Sigma_1}{t-r}\right)^2 \exp\left(-\frac{\tilde m_t^2}{\Sigma_0^2 + t}\right) \int_{-\infty}^{g_\varepsilon^t(r)} \exp\left(-\frac{\Sigma_0^2 + t}{2\Sigma_1^2 \Sigma_0^2}\left(x - \tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2 + t}\right)^2\right) dx$$

$$y = \left(x - \tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2 + t}\right) \times \left(\frac{\Sigma_0^2 + t}{\Sigma_1^2 \Sigma_0^2}\right)^{1/2}$$

$$= \frac{C}{\Sigma_0}\left(\frac{\Sigma_1}{t-r}\right)^2 \exp\left(-\frac{\tilde m_t^2}{\Sigma_0^2 + t}\right) \int_{-\infty}^{\left(g_\varepsilon^t(r) - \tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2 + t}\right) \times \left(\frac{\Sigma_0^2 + t}{\Sigma_1^2 \Sigma_0^2}\right)^{1/2}} \exp\left(-\frac{y^2}{2}\right) \left(\frac{\Sigma_1^2 \Sigma_0^2}{\Sigma_0^2 + t}\right)^{1/2} dy$$

$$\le \frac{C}{\Sigma_0}\left(\frac{\Sigma_1}{t-r}\right)^2 \exp\left(-\frac{\tilde m_t^2}{\Sigma_0^2 + t}\right) \left(\frac{\Sigma_1^2 \Sigma_0^2}{\Sigma_0^2 + t}\right)^{1/2} \mathbb{P}\left(\mathcal{N}(0,1) \ge \left(\frac{\Sigma_0^2 + t}{\Sigma_1^2 \Sigma_0^2}\right)^{1/2} \underline{\left(\tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2 + t} - g_\varepsilon^t(r)\right)}\right)$$

now we study this to show it's > 0

• If $r\in[0,bt]$, writing $r=ut$, we have $\Sigma_0^2=\sigma_1^2 ut$ and

$$\tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2+t} - g_\varepsilon^t(r) = \left(\lambda_c t - \frac{1}{2\lambda_c}\log t\right)\frac{2\sigma_1^2 u}{\sigma_1^2 u+1} - \lambda_c(\sigma_1^2+\varepsilon)ut - 1 = \lambda_c ut\left(\underbrace{\frac{2\sigma_1^2}{\sigma_1^2u+1}-\sigma_1^2}_{=\frac{\sigma_1^2(1-\sigma_1^2u)}{\sigma_1^2u+1} > \frac{\sigma_1^2}{2}(1-\sigma_1^2 b)} - \varepsilon + O\left(\frac{\log t}{t}\right)\right)$$

• If $r\in(bt,t]$, writing $r=t-vt$, we have $\Sigma_0^2 = t-\sigma_2^2(t-r) = t(1-\sigma_2^2 v)$ and

$$\tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2+t} - g^t_\varepsilon(r) = \left(\lambda_c t - \frac{1}{2\lambda_c}\log t\right)\left(\frac{2(1-\sigma_2^2 v)}{2-\sigma_2^2 v} - 1\right) + \lambda_c(\sigma_2^2-\varepsilon)vt - 1 = \lambda_c vt\left(\underbrace{\frac{-\sigma_2^2}{2-\sigma_2^2 v}+\sigma_2^2}_{=\frac{\sigma_2^2(1-\sigma_2^2 v)}{2-\sigma_2^2 v}>\frac{\sigma_2^2}{2}(1-\sigma_2^2(1-b))} - \varepsilon + O\left(\frac{\log t}{vt}\right)\right)$$

In both cases, choosing $\varepsilon < \frac{\sigma_1^2}{4}(1-\sigma_1^2 b) = \frac{\sigma_2^2}{4}(1-\sigma_2^2(1-b))$, we have for t large enough $\tilde m_t \frac{2\Sigma_0^2}{\Sigma_0^2+t} - g_\varepsilon^t(r) > \begin{cases} cut \\ cvt \end{cases} > 0$ with $c = \frac{\sigma_1^2\sigma_2^2}{8}$ so we can apply the Gaussian tail bound to get

$$P\left(\bigcap_{i=1}^{2}\left\{\bar B_t^{t,i,r} \geq \tilde m_t,\ \forall s\in[0,t],\ \bar B_s^{t,i,r} \leq g_\varepsilon^t(s)\right\}\right)$$

$$\leq \frac{C}{\Sigma_0}\left(\frac{\Sigma_1}{t-r}\right)^2 \exp\left(-\frac{\tilde m_t^2}{\Sigma_0^2+t}\right)\cdot\frac{\Sigma_1^2\Sigma_0^2}{\Sigma_0^2+t}\cdot\left(\tilde m_t\frac{2\Sigma_0^2}{\Sigma_0^2+t}-g_\varepsilon^t(r)\right)^{-1}\exp\left(-\frac12\left(\frac{\Sigma_0^2+t}{\Sigma_1^2\Sigma_0^2}\right)\left(\tilde m_t\frac{2\Sigma_0^2}{\Sigma_0^2+t}-g_\varepsilon^t(r)\right)^2\right)$$

Part $r\in[0,bt]$: The constant C below can change between occurrences and depends only on $\sigma_1^2, \sigma_2^2, b, \lambda_c$.

$$\leq \underbrace{\frac{C}{\sqrt{\sigma_1^2ut}}}_{\leq\frac{C}{\sqrt r}}\ \underbrace{\frac{1-\sigma_1^2u}{(1-u)t}}_{\leq\frac{C}{t}}\ \underbrace{\frac{(1-\sigma_1^2u)\sigma_1^2\cancel{ut}}{\sigma_1^2u+1}}_{\leq C}\cdot\frac{1}{c\cancel{ut}}\exp\left(-\frac{\tilde m_t^2}{(\sigma_1^2u+1)t}-\frac12\frac{\sigma_1^2u+1}{(1-\sigma_1^2u)\sigma_1^2ut}(\lambda_cut)^2\left(\frac{\sigma_1^2(1-\sigma_1^2u)}{\sigma_1^2u+1}-\varepsilon+O\left(\frac{\log t}{t}\right)\right)^2\right)$$

$$\tilde m_t^2 = \lambda_c^2t^2 - t\log t + O((\log t)^2)$$

$$\left(\frac{\sigma_1^2(1-\sigma_1^2u)}{\sigma_1^2u+1}-\varepsilon+O\left(\frac{\log t}{t}\right)\right)^2 \geq \left(\frac{\sigma_1^2(1-\sigma_1^2u)}{1+\sigma_1^2u}-2\varepsilon\right)^2 \geq \left(\frac{\sigma_1^2(1-\sigma_1^2u)}{1+\sigma_1^2u}\right)^2-4\varepsilon\frac{\sigma_1^2(1-\sigma_1^2u)}{1+\sigma_1^2u}$$

$$\leq \frac{C}{t\sqrt r}\exp\left(-\underbrace{\lambda_c^2}_{2m}t\left(\frac{1}{\sigma_1^2u+1}+\frac u2\frac{\sigma_1^2(1-\sigma_1^2u)}{\sigma_1^2u+1}-2u\varepsilon\right)+\frac{\log t}{\sigma_1^2u+1}\right)$$

So $\displaystyle\int_0^{bt} e^{2mt-mr}\, P\left(\bigcap_{i=1}^{2}\left\{\bar B_t^{t,i,r} \geq \tilde m_t,\ \forall s\in[0,t],\ \bar B_s^{t,i,r} \leq g_\varepsilon^t(s)\right\}\right) dr$

$$\leq \int_0^{bt}\frac{C}{\cancel{t}\sqrt r}\exp\left(mt\left(\underbrace{2-u-\frac{2}{\sigma_1^2u+1}-\frac{\sigma_1^2u(1-\sigma_1^2u)}{\sigma_1^2u+1}}+4u\varepsilon\right)+\cancel{\log t}\right)dr$$

$$= \frac{\cancel{2}\sigma_1^2u+\cancel{2}-\sigma_1^2u^2-u-\cancel{2}-\cancel{\sigma_1^2u}+\sigma_1^4u^2}{1+\sigma_1^2u} = -\frac{u(1-\sigma_1^2)(1+\sigma_1^2u)}{1+\sigma_1^2u} = -(1-\sigma_1^2)u$$

$$= \int_0^{bt}\frac{C}{\sqrt r}\exp\left(-mr(1-\sigma_1^2-4\varepsilon)\right)dr \leq C \qquad \text{by choosing } \varepsilon\leq\frac{1-\sigma_1^2}{8}.$$

Part $r\in(bt, t-\frac{2}{\lambda_c}]$: Proceeding similarly we get, if ε is chosen small enough,

$$\int_{bt}^{t-\frac{2}{\lambda_c}} e^{2mt-mr}\, P\left(\bigcap_{i=1}^{2}\left\{\bar B_t^{t,i,r} \geq \tilde m_t,\ \forall s\in[0,t],\ \bar B_s^{t,i,r} \leq g_\varepsilon^t(s)\right\}\right) dr \leq C$$

Part $r \in (t - \frac{2}{d_c}, t]$: For this part we proceed differently from the beginning by bounding $\mathbb{P}\left(\bigcap_{i=1}^{2} \left\{ \bar{B}_t^{t,i,r} \geq \tilde{m}_t,\ \forall s \in [0,t],\ \bar{B}_s^{t,i,r} \leq g_2^t(s) \right\}\right) \leq \mathbb{P}\left(\bar{B}_t^{t,1,r} \geq \tilde{m}_t\right) = \mathbb{P}(B_t \geq \tilde{m}_t)$.

By the calculation in IV 4.1, we have $\mathbb{P}(B_t \geq \tilde{m}_t) \sim \frac{e^{-mt}}{d_c\sqrt{2\pi}}$ and so

$$\int_{t-\frac{2}{d_c}}^{t} e^{2mt - mr}\, \mathbb{P}\left(\bigcap_{i=1}^{2} \left\{ \bar{B}_t^{t,i,r} \geq \tilde{m}_t,\ \forall s \in [0,t],\ \bar{B}_s^{t,i,r} \leq g_2^t(s) \right\}\right) dr \leq C \int_{t-\frac{2}{d_c}}^{t} e^{mt - mr}\, dr \leq C.$$

Combining the three parts yields the desired result. $\blacksquare$

Proof of the lower bound: This is very similar to the proof for the usual BBM in Lecture 7 where we wait until time t_1 such that there is at least n particles with absolute value bounded by $\sigma_1 d_c t_1$, and then argue that each of them has a positive probability to reach a high level at time t.

One subtelty: the BBM's starting from particles at time t_1 and with time horizon t are two-speed BBMs but with a different parameter $\tilde{b}$ instead of b. The time horizon is now $t - t_1$ and variances are changing at time $bt - t_1$ so

$$\tilde{b} = \frac{bt - t_1}{t - t_1} = b + O\left(\frac{1}{t}\right) < b.$$

Moreover, we have $\sigma_1^2 \tilde{b} + \sigma_2^2 (1 - \tilde{b}) > 1$ so we are not under our previous assumption. We can divide all positions in this BBM by $\alpha = \sqrt{\sigma_1^2 \tilde{b} + \sigma_2^2 (1 - \tilde{b})}$ to get a two-speed BBM with parameters $(\tilde{b}, \tilde{\sigma}_1, \tilde{\sigma}_2)$ where $\tilde{\sigma}_i = \sigma_i / \alpha$ so that $\tilde{\sigma}_1^2 \tilde{b} + \tilde{\sigma}_2^2 (1 - \tilde{b}) = \frac{\sigma_1^2 \tilde{b} + \sigma_2^2 (1 - \tilde{b})}{\alpha^2} = 1$. Note that $\alpha = 1 + O\left(\frac{1}{t}\right)$ so dividing positions by α changes the maximum only up to a $O(1)$ term which is in the order of precision we need.

We can finally apply Lemmas 1&2 to the BBM with parameters $(\tilde{b}, \tilde{\sigma}_1, \tilde{\sigma}_2)$ to lower bound it's probability to reach $\tilde{m}_{t-t_1}$ by a positive constant c (to be precise we should check that Lemmas 1&2 hold uniformly in (b, σ_1, σ_2) in some compact subset of $(0,1) \times (0,1) \times (1, \infty)$ but this is fine by double checking their proofs).