Exercise: The derivative martingale

1. Definition

Let $Z_t = \sum_{u \in \mathcal{N}_t} (\lambda_c t - \sqrt{2} t) e^{\lambda_c (X_u(t) - \lambda_c t)}$ for $t \geq 0$.

1.a For $x \in \mathbb{R}$, prove that $\left((\lambda_c t + L - B_t) e^{\lambda_c B_t - \frac{\lambda_c^2}{2} t} \right)_{t \geq 0}$ is a martingale under $\mathbb{P}_x$.

1.b Prove that $(Z_t)_{t \geq 0}$ is a martingale.

2. A new martingale

Let $L > 0$. For $t \geq 0$, let $Z_t^{(L)} = \sum_{u \in \mathcal{N}_t} (\lambda_c t + L - X_u(t)) e^{\lambda_c (X_u(t) - \lambda_c t)} \mathbb{1}_{\max_{s \in [0,t]} X_u(s) - \lambda_c s \leq L}$.

2.a Prove that $\left((\lambda_c t + L - B_t) e^{\lambda_c B_t - \frac{\lambda_c^2}{2} t} \mathbb{1}_{\max_{s \in [0,t]} B_s - \lambda_c s \leq L} \right)_{t \geq 0}$ is a martingale under $\mathbb{P}_x$.

Hint: see this process as a stopped version of the martingale in question 1.a.

2.b. Prove that $(Z_t^{(L)})_{t \geq 0}$ is a martingale.

2.c. Deduce that $(Z_t^{(L)})_{t \geq 0}$ converges a.s. to a limit $Z_\infty^{(L)}$.

3. Convergence a.s. of $(Z_t)_{t \geq 0}$

3.a Prove that $Z_t + L W_t^{\lambda_c} = Z_t^{(L)}$ on $E_L = \{ \forall s \geq 0, M_s \leq \lambda_c s + L \}$

3.b. Deduce that $Z_t \xrightarrow[t \to \infty]{a.s.} Z_\infty^{(L)}$ on E_L.

3.c. Conclude that $(Z_t)_{t \geq 0}$ converges a.s. to a limit $Z_\infty \geq 0$ and that $Z_\infty = Z_\infty^{(L)}$ a.s. on E_L for any $L > 0$. Hint: Recall $\mathbb{P}(E_L) \xrightarrow[L \to \infty]{} 1$

4. The limit is non trivial

4.a. Prove $(Z_t^{(L)})_{t \geq 0}$ is bounded in L^2.

Hint: This relies on the many-to-two. Try it on your own first! Some help if you are stuck: if $f_t(x) = (\lambda_c t + L - x) e^{\lambda_c x - \frac{\lambda_c^2}{2} t}$ you should have to compute $\mathbb{E}\left[f_t(B_t^{1,r}) \mathbb{1}_{\max_{s \in [0,t]} B_s^{1,r} - \lambda_c s \leq L} f_t(B_t^{2,r}) \mathbb{1}_{\max_{s \in [0,t]} B_s^{2,r} - \lambda_c s \leq L} \right]$.

Show this equals $\mathbb{E}\left[\mathbb{1}_{\max_{s \in [0,r]} B_s - \lambda_c s \leq L} f_r(B_r)^2 \right]$ using question 1.b.

Then show it is $\leq C(L) \left(\frac{1}{r^{3/2}} \wedge 1 \right)$ by following the argument used for the bound of $\mathbb{E}[K_t^2]$ in Lecture 9 and conclude.

3.b. Deduce that $\mathbb{P}(Z_\infty^{(L)} > 0) > 0$.

3.c. Prove that $Z_\infty > 0$ a.s. on the survival event.

<u>Hint</u>: follow the same strategy as for W_∞^{d}.

4. $Z_\infty \notin L^1$

4.a. Prove that $\mathbb{E}[Z_\infty^{(L)}] = L$.

4.b. Deduce that $\mathbb{E}[Z_\infty] = +\infty$.