

# Final Exam

## Exercise 1

1.] Let  $L > 0$ ,  $t \geq 0$ . Then

$$\begin{aligned} \mathbb{E}[W_t^{1c} \mathbb{1}_{E_L}] &\leq \mathbb{E}\left[\sum_{u \in \mathcal{D}_t} e^{\lambda_c X_u(t) - (\mu + \frac{1}{2}\lambda_c^2)t} \mathbb{1}_{\max_{s \in [0,t]} X_u(s) - \lambda_c s \leq L}\right] \\ &\stackrel{\text{many-to-one}}{=} e^{-\lambda_c t} \mathbb{E}\left[e^{\lambda_c B_t - (\mu + \frac{1}{2}\lambda_c^2)t} \mathbb{1}_{\max_{s \in [0,t]} B_s - \lambda_c s \leq L}\right] \\ &\stackrel{\text{Girsanov}}{=} \mathbb{P}\left(\max_{s \in [0,t]} B_s \leq L\right) \\ &\stackrel{\text{remainder}}{\leq} \frac{L}{\sqrt{t}}. \end{aligned}$$

2.] Let  $\varepsilon > 0$ . For any  $L > 0$  and  $t \geq 0$ , we have

$$\begin{aligned} \mathbb{P}(\sqrt{t} W_t^{1c} \geq y) &\leq \mathbb{P}(E_L^c) + \underbrace{\mathbb{P}(\sqrt{t} W_t^{1c} \mathbb{1}_{E_L} \geq y)}_{\leq \frac{\mathbb{E}[W_t^{1c} \mathbb{1}_{E_L}]}{y/\sqrt{t}} \text{ by Markov's inequality}} \\ &\leq \frac{L}{y} \text{ by question 1.} \end{aligned}$$

We can choose  $L$  large enough such that  $\mathbb{P}(E_L^c) \leq \frac{\varepsilon}{2}$ , and then  $y = \frac{L}{2\varepsilon}$  and then, for any  $t \geq 0$ ,  $\mathbb{P}(\sqrt{t} W_t^{1c} \geq y) \leq \varepsilon$ .

3.] It follows from the previous question that  $W_t^{1c} \xrightarrow[t \rightarrow \infty]{P} 0$

(for any  $\delta, \varepsilon > 0$ , for  $t$  large enough  $\mathbb{P}(W_t^{1c} \geq \delta) \leq \varepsilon$ ).

But  $W_t^{1c} \xrightarrow{\text{a.s.}} W_\infty^{1c}$  so  $W_t^{1c} \xrightarrow{P} W_\infty^{1c}$ .

This implies  $W_\infty^{1c} = 0$  a.s. (by unicity of the limit in probability).

## Exercise 2

1.]  $E[K_t] = e^{-\lambda t} P(B_t \geq \lambda t)$  by the many-to-one lemma.  
 $= e^{-\lambda t} P(B_1 \geq \lambda \sqrt{t})$   
 $\sim e^{-\lambda t} \frac{1}{\lambda \sqrt{2\pi t}} e^{-(\lambda \sqrt{t})^2/2} = \frac{e^{(\lambda - \frac{1}{2})t}}{\lambda \sqrt{2\pi t}}.$

2.]  $P(\Gamma_t \geq \lambda t) = P(K_t \geq 1) \leq E[K_t]$  by Markov inequality.  
 By question 1, for  $t$  large enough,  $P(\Gamma_t \geq \lambda t) \leq \frac{1}{\lambda \sqrt{t}} e^{(\lambda - \frac{1}{2})t}$  (using  $\frac{1}{\sqrt{2\pi}} < 1$ )

3.a]  $P(B_r^0 + B_{t-r}^1 \geq \lambda t, B_r^0 + B_{t-r}^2 \geq \lambda t) = E[P(B_{t-r}^1 \geq \lambda t - B_r^0 | B_r^0)^2]$   
 $= \int_{\mathbb{R}} \left[ P(B_{t-r}^1 \geq \lambda t - x) \right]^2 \frac{e^{-x^2/2r}}{\sqrt{2\pi r}} dx$   
 $\leq \begin{cases} e^{-(\lambda t - x)^2/2(t-r)} & \text{if } \lambda t - x > 0 \Leftrightarrow x < \lambda t \\ 1 & \text{if } x > \lambda t \end{cases}$   
 $\leq \underbrace{\int_{\lambda t}^{\infty} \frac{e^{-x^2/2r}}{\sqrt{2\pi r}} dx}_{= P(B_r > \lambda t)} + \int_{-\infty}^{\lambda t} \exp\left(-\frac{(\lambda t - x)^2}{t-r} - \frac{x^2}{2r}\right) \frac{dx}{\sqrt{2\pi r}}$   
 $= P(B_r > \lambda t) = P(B_1 \geq \frac{\lambda t}{\sqrt{r}}) \leq \frac{\sqrt{r}}{\lambda t} \exp\left(-\frac{(\lambda t)^2}{r}\right)$

For the 2<sup>nd</sup> integral, we rewrite

$$\exp\left(-\frac{(\lambda t - x)^2}{t-r} - \frac{x^2}{2r}\right) = \exp\left(-x^2 \underbrace{\left(\frac{1}{t-r} + \frac{1}{2r}\right)}_{=\frac{t+r}{2r(t-r)}} + \frac{2\lambda t x}{t-r} - \frac{(\lambda t)^2}{t-r}\right)$$

$$= \exp\left(-\frac{t+r}{2r(t-r)} \left(x^2 - 2x\lambda t \frac{2r}{t+r}\right) - \frac{(\lambda t)^2}{t-r}\right)$$

$$= \exp\left(-\frac{t+r}{2r(t-r)} \left(x - \lambda t \frac{2r}{t+r}\right)^2 + \frac{t+r}{2r(t-r)} \left(\lambda t \frac{2r}{t+r}\right)^2 - \frac{(\lambda t)^2}{t-r}\right)$$

$$= \frac{(\lambda t)^2}{(t-r)} \times \left(\frac{2r}{t+r} - 1\right) = \frac{(\lambda t)^2}{(t-r)} \times \frac{r-t}{t+r} = -\frac{(\lambda t)^2}{t+r}$$

$$\int_{-\infty}^{\lambda t} \exp\left(-\frac{(\lambda t - x)^2}{t-r} - \frac{x^2}{2r}\right) \frac{dx}{\sqrt{2\pi r}} \leq \exp\left(-\frac{(\lambda t)^2}{t+r}\right) \int_{\mathbb{R}} \exp\left(-\frac{t+r}{2r(t-r)} \left(x - \lambda t \frac{2r}{t+r}\right)^2\right) \frac{dx}{\sqrt{2\pi r}}$$

$$= \exp\left(-\frac{(\lambda t)^2}{t+r}\right) \sqrt{\frac{t-r}{t+r}}$$

So  $P(B_t^1 \geq \lambda t, B_t^2 \geq \lambda t) \leq \frac{\sqrt{r}}{\lambda t} \exp\left(-\frac{\lambda^2 t^2}{r}\right) + \exp\left(-\frac{(\lambda t)^2}{t+r}\right) \sqrt{\frac{t-r}{t+r}}$

$$\leq \frac{1}{\sqrt{t}} \exp\left(-\frac{(\lambda t)^2}{t+r}\right) + \exp\left(-\frac{(\lambda t)^2}{t+r}\right) \sqrt{\frac{t-r}{t}} \leq \frac{1}{\sqrt{t}} \left(\frac{1}{\lambda} + \sqrt{t-r}\right) \exp\left(-\frac{(\lambda t)^2}{t+r}\right).$$

3.b  $E[K_t^2] = E[K_t] + E[K_t(K_t-1)]$   
 $\leq \frac{e^{(m-\frac{\lambda^2}{2})t}}{\sqrt{t}}$  for  $t$  large enough by question 1.

By the many-to-two,

$$E[K_t(K_t-1)] = E[L(L-1)] \int_0^t e^{2mb-mr} P(B_t^{1,r} \geq \lambda t, B_t^{2,r} \geq \lambda t) dr$$

$$= P(B_r^0 + B_{t-r}^1 \geq \lambda t, B_r^0 + B_{t-r}^2 \geq \lambda t)$$

$$\leq \frac{1}{\sqrt{t}} \left(\frac{1}{\lambda} + \sqrt{t-r}\right) \exp\left(-\frac{(\lambda t)^2}{t+r}\right) \text{ by 3.a}$$

$$\sqrt{t} e^{(\frac{\lambda^2}{2}-m)t} E[K_t(K_t-1)] = E[L(L-1)] \int_0^t \exp\left(\frac{\lambda^2}{2}t + mb - mr - \frac{(\lambda t)^2}{t+r}\right) \left(\frac{1}{\lambda} + \sqrt{t-r}\right) dr$$

$$= \lambda^2 t \left(\frac{1}{2} - \frac{t}{t+r}\right) + m(t-r) = \lambda^2 t \frac{r-t}{2(t+r)} + m(t-r)$$

$$= -(t-r) \left(\frac{\lambda^2}{2} \frac{t}{t+r} - m\right) \leq -(t-r) \left(\frac{\lambda^2}{4} - m\right)$$

$s = t-r$   $\frac{t}{t+r} \leq \frac{t}{2t} = \frac{1}{2}$

$$\leq E[L(L-1)] \int_0^t \left(\frac{1}{\lambda} + \sqrt{s}\right) e^{-s(\frac{\lambda^2}{4}-m)} ds$$

$$\leq E[L(L-1)] \int_0^\infty \left(\frac{1}{\lambda} + \sqrt{s}\right) e^{-s(\frac{\lambda^2}{4}-m)} ds = C_3$$

So we proved  $E[K_t^2] \leq \left(\frac{1}{\lambda} + C_3\right) \frac{1}{\sqrt{t}} e^{(m-\frac{\lambda^2}{2})t}$

3.c  $P(\Gamma_t \geq \lambda t) = P(K_t \geq 1) \geq \frac{E[K_t]^2}{E[K_t^2]}$  by Cauchy-Schwarz (because  $K_t \in \mathbb{N}$ )

$$\geq \frac{c}{\sqrt{t}} e^{(m-\frac{\lambda^2}{2})t} \text{ by combining questions 1 and 3.b.}$$

4. If  $\lambda \in (\lambda_c, \sqrt{2}\lambda_c]$ , then  $E[K_t^2]$  becomes much larger than  $\frac{1}{\sqrt{t}} e^{(m-\frac{\lambda^2}{2})t}$

We need to truncate by considering

$$K'_t = \sum_{u \in \mathcal{U}_t} \mathbb{1}_{X_u(t) \geq \lambda t} \mathbb{1}_{\forall s \in [0,t], X_u(s) \leq g(s)} \text{ for some well-chosen barrier } g$$

such that: ①  $E[K'_t]$  is of the same order as  $E[K_t]$  (ie  $\geq \frac{c''}{\sqrt{t}} e^{(m-\frac{\lambda^2}{2})t}$ )

②  $E[(K'_t)^2] \leq \frac{C''}{\sqrt{t}} e^{(m-\frac{\lambda^2}{2})t}$

### Exercise 3

$$\begin{aligned}
 1. \quad V_t^A &= \sum_{v \in \mathcal{NP}_{bt}} e^{\lambda X_v^t(bt) - \left(m + \frac{\lambda^2}{2}\right)bt} \sum_{\substack{u \in \mathcal{NP}_t \\ u \geq v}} e^{\lambda X_u^t(t) - \left(m + \frac{\lambda^2}{2}\right)t} = \left(m + \frac{\lambda^2 \sigma_1^2}{2}\right)bt + \left(m + \frac{\lambda^2 \sigma_2^2}{2}\right)(1-b)t \\
 &\quad \text{using } \sigma_1^2 b + \sigma_2^2 (1-b) = 1 \\
 &= \sum_{v \in \mathcal{NP}_{bt}} e^{\lambda X_v^t(bt) - \left(m + \frac{\lambda^2 \sigma_1^2}{2}\right)bt} \sum_{\substack{u \in \mathcal{NP}_t \\ u \geq v}} e^{\lambda (X_u^t(t) - X_v^t(bt)) - \left(m + \frac{(\lambda \sigma_2)^2}{2}\right)(1-b)t} \\
 &\quad \quad \quad =: W_{(1-b)t}^{\lambda \sigma_2}(v)
 \end{aligned}$$

By the branching property, given  $\mathcal{F}_t$ ,  $(W_{(1-b)t}^{\lambda \sigma_2}(v), v \in \mathcal{NP}_{bt})$  are independent with the same law as  $\sum_{u \in \mathcal{NP}_{(1-b)t}} e^{\lambda \sigma_2 X_u((1-b)t) - \left(m + \frac{(\lambda \sigma_2)^2}{2}\right)(1-b)t} = W_{(1-b)t}^{\lambda \sigma_2}$ ,

where we used that the BST born at particle  $v$  at time  $bt$  has variance  $\sigma_2^2$  until time  $t$  so it can be written like  $\sigma_2 \times$  usual BST.

$$\begin{aligned}
 2. \quad \text{It follows that } E[V_t^A | \mathcal{F}_{bt}] &= \sum_{v \in \mathcal{NP}_{bt}} e^{\lambda X_v^t(bt) - \left(m + \frac{(\lambda \sigma_1)^2}{2}\right)bt} \underbrace{E\left[W_{(1-b)t}^{\lambda \sigma_2}\right]}_{=1} \\
 &\quad \text{by construction, for any } s \leq bt \text{ and } u \in \mathcal{NP}_s, X_u^t(s) = \sigma_1 X_u(s) \quad \leftarrow \sum_{v \in \mathcal{NP}_{bt}} e^{\lambda \sigma_1 X_v(bt) - \left(m + \frac{(\lambda \sigma_1)^2}{2}\right)bt} \\
 &\quad \text{(where } X_u(s), u \in \mathcal{NP}_s, s \geq 0 \text{ are defined from the same marked tree)} \quad = W_{bt}^{\lambda \sigma_1} \xrightarrow[t \rightarrow \infty]{\text{a.s.}} W_{\infty}^{\lambda \sigma_1}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad V_t^A - E[V_t^A | \mathcal{F}_{bt}] &= \sum_{v \in \mathcal{NP}_{bt}} e^{\lambda X_v^t(bt) - \left(m + \frac{(\lambda \sigma_1)^2}{2}\right)bt} \underbrace{\left(W_{(1-b)t}^{\lambda \sigma_2}(v) - 1\right)}_{\text{given } \mathcal{F}_{bt}, \text{ these are independent and centered}}
 \end{aligned}$$

Let  $p \in (1, 2]$ . By von Bahr-Esseen inequality,

$$E[|V_t^A - E[V_t^A | \mathcal{F}_{bt}]|^p | \mathcal{F}_{bt}] \leq 2 \sum_{v \in \mathcal{NP}_{bt}} e^{p\lambda X_v^t(bt) - p\left(m + \frac{(\lambda \sigma_1)^2}{2}\right)bt} E[|W_{(1-b)t}^{\lambda \sigma_2} - 1|^p]$$

If we choose  $p \in (1, \frac{1}{(\lambda \sigma_2)^2})$  (we use here  $|\lambda| < \frac{1}{\sigma_2}$ ) then  $(W_s^{\lambda \sigma_2})_{s \geq 0}$  is bounded in  $L^p$ , so  $2 E[|W_{(1-b)t}^{\lambda \sigma_2} - 1|^p] \leq C$  for some constant  $C = C(\lambda, \sigma_2, p)$ .

$$\begin{aligned}
 \text{Also } \sum_{v \in \mathcal{NP}_{bt}} e^{p\lambda X_v^t(bt) - p\left(m + \frac{(\lambda \sigma_1)^2}{2}\right)bt} &= \sum_{v \in \mathcal{NP}_{bt}} e^{p\lambda \sigma_1 X_v(bt) - p\left(m + \frac{(\lambda \sigma_1)^2}{2}\right)bt} \\
 &= W_{bt}^{p\lambda \sigma_1} \exp\left(\left(m + \frac{(p\lambda \sigma_1)^2}{2}\right)bt - p\left(m + \frac{(\lambda \sigma_1)^2}{2}\right)bt\right)
 \end{aligned}$$

$$= bt \left( m(1-p) + \frac{(\lambda\sigma_1)^2}{2} (p^2 - p) \right)$$

$$= bt (p-1) \left( \frac{(\lambda\sigma_1)^2}{2} p - m \right)$$

But we can choose  $p < \frac{2m}{(\lambda\sigma_1)^2} = \frac{\lambda_c^2}{(\lambda\sigma_1)^2}$  (because  $\lambda < \frac{\lambda_c}{\sigma_1}$  so  $\frac{\lambda_c^2}{(\lambda\sigma_1)^2} > 1$ ) so

$$\mathbb{E}[|V_t^\lambda - \mathbb{E}[V_t^\lambda | \mathcal{F}_{bt}]|^p] \leq C \underbrace{\mathbb{E}[W_{bt}^{p\lambda\sigma_1}]}_{=1} \exp\left(bt \underbrace{(p-1)}_{>0} \underbrace{\left(\frac{(\lambda\sigma_1)^2}{2} p - m\right)}_{<0}\right) \xrightarrow{t \rightarrow \infty} 0$$

In particular,  $V_t^\lambda - \mathbb{E}[V_t^\lambda | \mathcal{F}_{bt}] \xrightarrow{P} 0$ . Combining with question 2, we get  $V_t^\lambda \xrightarrow{P} W_\infty^{\lambda\sigma_1}$ .