Final exam

February 21, 2:00pm–5:00pm

Instructions: All electronic equipment is forbidden. The only authorized material is a handwritten double-sided A4 sheet.

Total: 20 points

Reminder: If $(B_t)_{t\geq 1}$ denotes a Brownian motion, the following holds: for any $x \geq 0$,

$$\mathbb{P}(B_1 \geq x) \leq \left(1 \wedge \frac{1}{x}\right) e^{-x^2/2} \qquad \text{and} \qquad \mathbb{P}(B_1 \geq x) \underset{x\to\infty}{\sim} \frac{e^{-x^2/2}}{x\sqrt{2\pi}},$$

and, for any $a, y \geq 0$ and $t > 0$,

$$\mathbb{P}\left(\max_{s\in[0,t]} B_s \leq a\right) \leq \frac{a}{\sqrt{t}} \qquad \text{and} \qquad \mathbb{P}\left(\max_{s\in[0,t]} B_s \leq a, B_t \geq a - y\right) \leq \frac{ay^2}{t^{3/2}}.$$

Exercice 1. (5 points) We consider a standard BBM whose reproduction law has a finite second moment. Recall that, for $\lambda \in \mathbb{R}$ and $t \geq 0$,

$$W_t^\lambda = \sum_{u\in\mathcal{N}_t} e^{\lambda X_u(t) - (m + \frac{\lambda^2}{2})t}.$$

Moreover, for $L > 0$, $E_L = \{\forall s \geq 0, \forall u \in \mathcal{N}_s, X_u(s) \leq \lambda_c s + L\}$ with $\lambda_c = \sqrt{2m}$ and we have proved that $\mathbb{P}(E_L) \to 1$ as $L \to \infty$.

1. (2 pts) Prove that, for any $L > 0$ and $t \geq 0$, $\mathbb{E}[W_t^{\lambda_c} \mathbb{1}_{E_L}] \leq L/\sqrt{t}$.
2. (2 pts) Prove that $(\sqrt{t} W_t^{\lambda_c})_{t\geq 0}$ is tight, that is: for any $\varepsilon > 0$, there exists $y > 0$ such that, for any $t \geq 0$,
$$\mathbb{P}\left(\sqrt{t} W_t^{\lambda_c} \geq y\right) \leq \varepsilon.$$
3. (1 pt) Explain why this provides a new proof of the fact that $W_\infty^{\lambda_c} = 0$ a.s.

Exercice 2. (9 points) We consider a standard BBM whose reproduction law has a finite second moment. Recall that $M_t = \max_{u\in\mathcal{N}_t} X_u(t)$ and $\lambda_c = \sqrt{2m}$. Let $\lambda > \lambda_c$. In this exercise, we are interested in the following result: there exists $c, C > 0$ such that, for t large enough,

$$\frac{c}{\sqrt{t}} e^{(m - \frac{\lambda^2}{2})t} \leq \mathbb{P}(M_t \geq \lambda t) \leq \frac{C}{\sqrt{t}} e^{(m - \frac{\lambda^2}{2})t}. \tag{1}$$

1. (1 pt) Let $K_t = \sum_{u\in\mathcal{N}_t} \mathbb{1}_{X_u(t)\geq\lambda t}$. Find an asymptotic equivalent for $\mathbb{E}[K_t]$ as $t \to \infty$.
2. (1 pt) Prove the upper bound in (1).

3.(a) (3 pt) Let B^0, B^1, B^2 be independent Brownian motions. Prove that there exist $C_1, C_2 > 0$ such that for any $t > 0$ and $r \in [0, t]$,
$$\mathbb{P}\left(B_r^0 + B_{t-r}^1 \geq \lambda t, B_r^0 + B_{t-r}^2 \geq \lambda t\right) \leq \frac{1}{\sqrt{t}}\left(C_1 + C_2\sqrt{t-r}\right) \exp\left(-\frac{(\lambda t)^2}{t+r}\right).$$

Hint: You can distinguish between the case $B_r^0 \geq \lambda t$ and $B_r^0 < \lambda t$.

(b) (2 pts) Assume that $\lambda > \sqrt{2}\lambda_c$. Prove that there exists $C' > 0$ such that for t large enough,

$$\mathbb{E}\Big[K_t^2\Big] \leq \frac{C'}{\sqrt{t}} e^{(m-\frac{\lambda^2}{2})t}.$$

(c) (1 pt) Assume that $\lambda > \sqrt{2}\lambda_c$. Prove the lower bound in (1).

4. (1 pt) Sketch in a few lines how we could modify the lower bound argument so that it works for any $\lambda > \lambda_c$.

Exercice 3. (6 points) We consider a two-speed BBM whose reproduction law has a finite second moment and with variance σ_1^2 on $[0, bt]$ and σ_2^2 on $[bt, t]$, satisfying the relation

$$\sigma_1^2 b + \sigma_2^2(1-b) = 1.$$

Recall that the two-speed BBM is constructed from the marked tree $(\mathcal{T}, (\sigma_u)_{u\in\mathcal{T}}, (Y_u)_{u\in\mathcal{T}})$ by setting, for a time horizon $t > 0$, for $s \in [0,t]$ and $u \in \mathcal{N}_s$ (with $\varrho(u)$ denoting the parent of u),

- ▷ if $s \leq bt$, $X_u^t(s) = \begin{cases} \sigma_1 Y_\emptyset(s) & \text{if } u = \emptyset, \\ X_{\varrho(u)}^t(b_u-) + \sigma_1 Y_u(s-b_u) & \text{if } u \neq \emptyset, \end{cases}$
- ▷ if $s > bt$, $X_u^t(s) = \begin{cases} X_u^t(bt) + \sigma_2(Y_u(s-b_u) - Y_u(bt-b_u)) & \text{if } b_u \leq bt, \\ X_{\varrho(u)}^t(b_u-) + \sigma_2 Y_u(s-b_u) & \text{if } b_u > bt. \end{cases}$

For $\lambda \in \mathbb{R}$, we define

$$V_t^\lambda = \sum_{u\in\mathcal{N}_t} e^{\lambda X_u^t(t) - \left(m+\frac{\lambda^2}{2}\right)t}$$

We also write $(W_t^\lambda)_{t\geq 0}$ for the additive martingale of the usual BBM. Recall that, for $|\lambda| < \lambda_c$, $(W_t^\lambda)_{t\geq 0}$ is bounded in L^p for any $p \in (1,2] \cap (1, \lambda_c^2/\lambda^2)$.

1. (2 pts) For any $\lambda \in \mathbb{R}$ and $t \geq 0$, prove that

$$V_t^\lambda = \sum_{v\in\mathcal{N}_{bt}} e^{\lambda X_v^t(bt) - \left(m+\frac{\lambda^2}{2}\sigma_1^2\right)bt} W_{(1-b)t}^{\lambda\sigma_2}(v),$$

where, given $\mathcal{F}_{bt}$, the random variables $W_{(1-b)t}^{\lambda\sigma_2}(v)$ for $v \in \mathcal{N}_{bt}$ are independent of each other and of $\mathcal{F}_{bt}$ and have the same law as $W_{(1-b)t}^{\lambda\sigma_2}$.

2. (1 pt) For any $\lambda \in \mathbb{R}$, prove that

$$\mathbb{E}\Big[V_t^\lambda \Big| \mathcal{F}_{bt}\Big] \xrightarrow[t\to\infty]{\text{a.s.}} W_\infty^{\lambda\sigma_1}.$$

3. (3 pts) Assume now that $|\lambda| < \frac{\lambda_c}{\sigma_1} \wedge \frac{\lambda_c}{\sigma_2}$. Prove that

$$V_t^\lambda \xrightarrow[t\to\infty]{\mathbb{P}} W_\infty^{\lambda\sigma_1}.$$

Hint: You can prove that $V_t^\lambda - \mathbb{E}[V_t^\lambda | \mathcal{F}_{bt}] \to 0$ in L^p as $t \to \infty$ for $p > 1$ close enough to 1.