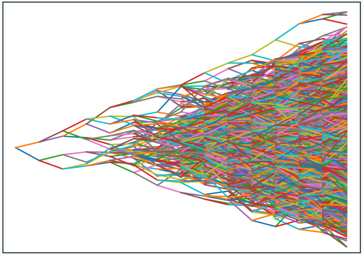
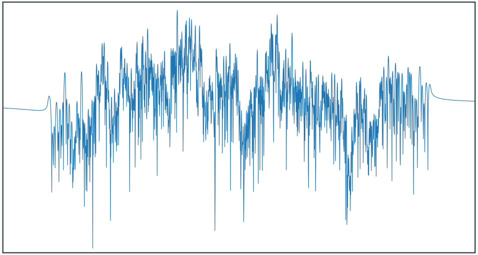
Extrema of branching random walks and log-correlated fields

Michel Pain (Courant Institute, NYU)

CMS Winter Meeting

December 6, 2020





Log-correlated fields

- **Continuous log-correlated Gaussian field** $(X(x))_{x\in D}$: centered Gaussian field with covariances

$$\mathbb{E}[X(x)X(y)] = -c \cdot \log|x-y| + \text{bounded function}.$$

X is defined as a random distribution (not defined pointwise).

- **Regularization** or **discrete approximation** $(X_N(x))_{x\in D}$ of a log-correlated field: asymptotically Gaussian with covariances

$$\mathbb{E}[X_N(x)X_N(y)] = -c \cdot \log\left(|x-y| \vee \underbrace{\frac{1}{N}}_{\text{saturation effect}}\right) + \text{bounded function}.$$

They have many properties in common.

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- **Riemann ζ function** on a short interval on the critical line:

$$\log\zeta\left(\frac{1}{2}+\mathrm{i}(x+\tau)\right), \qquad x\in[0,1],$$

  where τ is a uniform random variable in $[T,2T]$.
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- **Random unitary matrix** U_N of size N:

$$\log \det\left(I - \mathrm{e}^{-\mathrm{i}x} U_N\right) = \sum_{k=1}^{\infty} \frac{\mathrm{Tr}(U_N^k)}{k} \mathrm{e}^{-\mathrm{i}kx}, \qquad x \in [0, 2\pi].$$

  Log-correlated Gaussian behavior follows from Diaconis–Shahshahani '94 (see Bourgade '10).
- Logarithm of the characteristic polynomial of other random matrix models.
  Gustavsson '05, O'Rourke '10, Tao-Vu '11, Bourgade-Mody '19,. . .

The case of general β-ensembles in dimension 1

The model: N particles $\lambda_1 \leq \cdots \leq \lambda_N$ on the real line chosen according to

$$\prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{2} \sum_{k=1}^N V(\lambda_k)} \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N,$$

for $\beta > 0$ and a potential $V \colon \mathbb{R} \to \mathbb{R}$ smooth and with sufficient growth at infinity.

$\to$ Includes Gaussian β-ensembles with $V(x) = x^2/2$.

Logarithm of the characteristic polynomial:

$$X_N(x) = \sum_{k=1}^N \log(x - \lambda_k) - N \int \log(x - \lambda) \underbrace{\mu_{\mathrm{eq}}}_{\text{equilibrium measure}}(\mathrm{d}\lambda).$$

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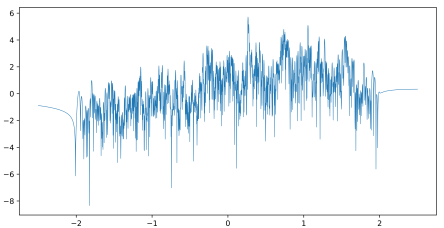
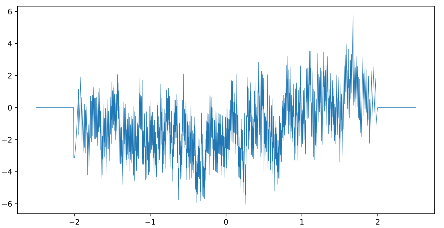
Real part



Imaginary part



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Theorem (Bourgade–Mody–P. '20): *For x, y in the bulk of the spectrum such that $-\log|x - y| / \log N \to \alpha$,*

$$\sqrt{\frac{\beta}{\log N}} (\mathrm{Re}\, X_N(x), \mathrm{Re}\, X_N(y)) \xrightarrow[N \to \infty]{(law)} \mathcal{N}\left(0, \begin{pmatrix} 1 & \alpha \wedge 1 \\ \alpha \wedge 1 & 1 \end{pmatrix}\right).$$

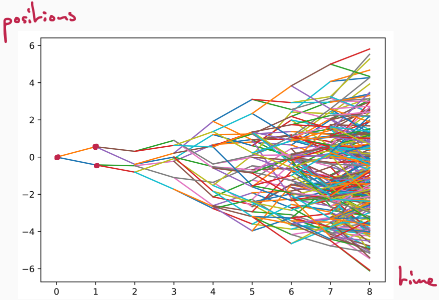
The same result holds for $\mathrm{Im}\, X_N$, which is asymptotically independent of $\mathrm{Re}\, X_N$.

See also related results for Gaussian β-ensembles by Lambert-Paquette '20, Augeri-Butez-Zeitouni '20.

Branching random walk:

- ▷ Start with one particle at 0.
- ▷ At each step, each particle has two children.
- ▷ Each child jumps from the position of its parent with law $\mathcal{N}(0,1)$.

$X(u) =$ position of particle u.



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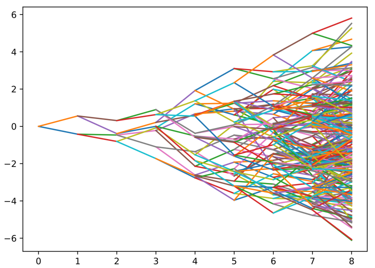
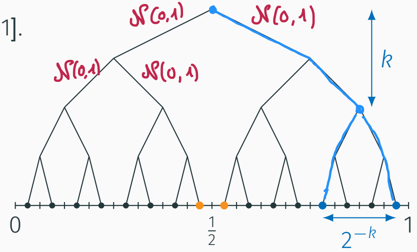


Why is it log-correlated?

- Embed particles of generation n in $[0, 1]$.
- $X(u) \sim \mathcal{N}(0, \underbrace{\log_2 N}_{n})$, $\quad N = 2^n$
- Blue particles:
  $$\mathbb{E}[X(u)X(v)] = -\log_2 d(u, v).$$
- Orange particles u and v:
  $$\mathbb{E}[X(u)X(v)] = 0.$$

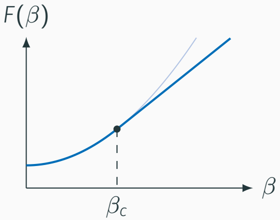


Some common properties of log-correlated fields universality class:

- Phase transition of the **free energy**:

$$F(\beta) = \lim_{N\to\infty} \frac{1}{\log N} \log \int_D e^{\beta X_N(x)}\, \mathrm{d}x$$

Freezing phenomenon in disordered systems.



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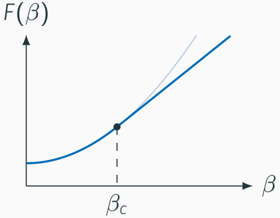
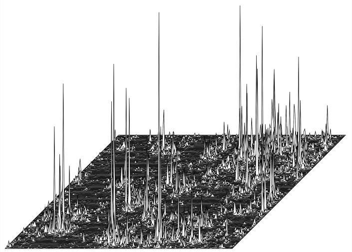


- Convergence of the measure $e^{\beta X_N(x)}\,\mathrm{d}x$ after renormalization to the **Gaussian multiplicative chaos**.



Universal properties

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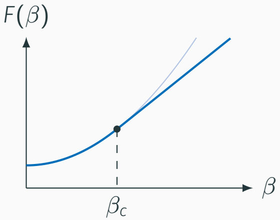

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- **Maximum** of the field:

$$\max_{x\in D} X_N(x) = \mathrm{cste}\cdot\log N - \frac{3}{2\beta_c}\log\log N + Y_N$$

and Y_n converges in distribution to a randomly shifted Gumbel.

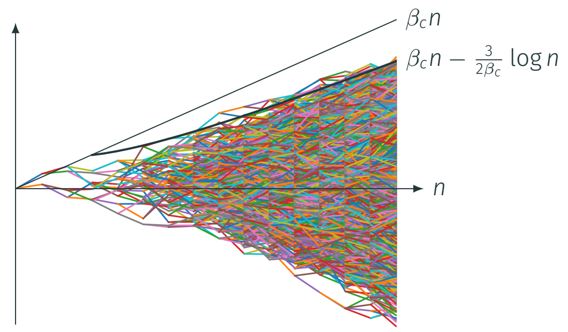
Conjectured by Fyodorov–Hiary–Keating for the logarithm of ζ and of the characteristic polynomial of U_N.
For U_N, the limit should be a sum of two independent Gumbel random variables.

For the binary **BRW** with jumps $\mathcal{N}(0,1)$: $\beta_c = \sqrt{2\log(2)}$ and

generation of u

$$\max_{|u|=n} X(u) = \beta_c n - \frac{3}{2\beta_c}\log n + \text{shifted Gumbel}.$$

Proved by Bramson '78-'83 and Lalley–Sellke '87 for branching Brownian motion and Aïdékon '13 for general BRW.

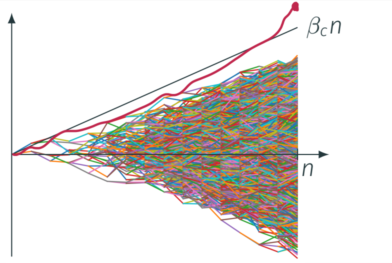


To be compared with 2^n i.i.d. random variables X_i with law $\mathcal{N}(0,n)$:

$$\max_{1\leq i\leq 2^n} X_i = \beta_c n - \frac{1}{2\beta_c}\log n + \text{Gumbel}.$$

Proof based on **first moment calculations** for the number of particles satisfying a certain property.

▷ **For the 1st order:** $\mathbb{E}\left[\sum_{|u|=n} \mathbb{1}_{X(u)>\beta_c n}\right] = 2^n \cdot \mathbb{P}(\mathcal{N}(0,n) > \beta_c n) \to 0.$



Maximum of the BRW: upper bound, 1st and 2nd orders

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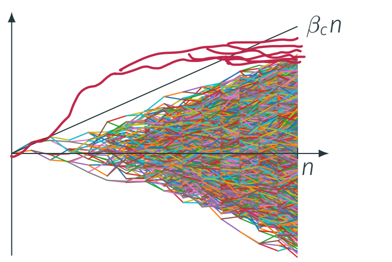

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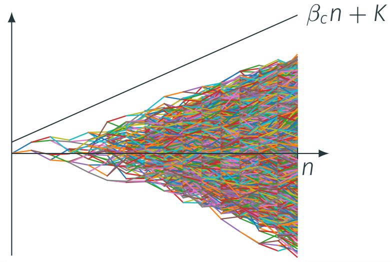
  First prove that with high probability, for any ℓ, $\max_{|u|=\ell} X(u) \leq \beta_c \ell + K$.



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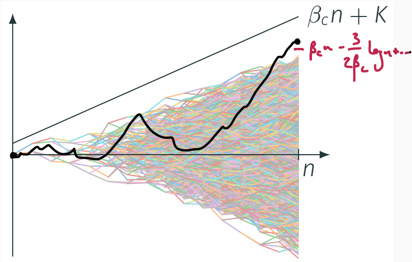
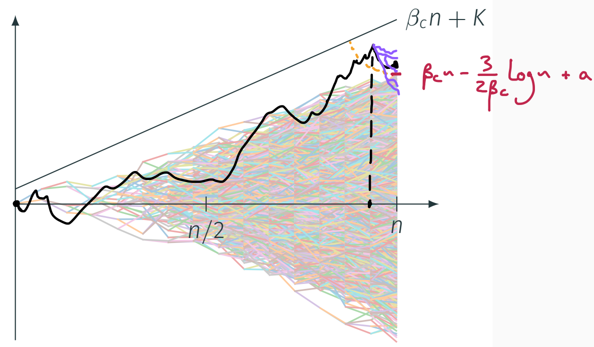
  Then, the first moment of the number of particles u at generation n such that $X(u) \geq \beta_c n - \frac{3}{2\beta_c}\log n + \cdots$ $\frac{2}{\beta_c}\log\log n$ and whose trajectory stays below the barrier tends to zero.



Goal: $\mathbb{P}\left(\max_{|u|=n} X(u) \geq \beta_c n - \frac{3}{2\beta_c}\log n + a\right) \leq Cae^{-\beta_c a}$

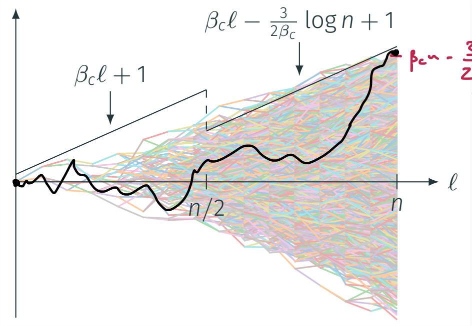
- ▷ Keep the barrier at level $\beta_c \ell + K$ at any time ℓ.
- ▷ Distinguish according to the time where the trajectory of the particle u is the closest from the barrier between times $n/2$ and n.
- ▷ Annoying particles: those getting too close to the barrier at the time close to n.
  $\to$ Deal with them in probability, before taking the first moment.



Goal: Prove $\max_{|u|=n} X(u) \geq \beta_c n - \frac{3}{2\beta_c} \log n - a$ with high probability.

- **Step 1:** Prove $\max_{|u|=n} X(u) \geq \beta_c n - \frac{3}{2\beta_c} \log n$ with positive probability.

  → First and second moments calculation on the number of such particles whose trajectory stays below the following barrier:



$\beta_c n - \frac{3}{2\beta_c} \log n$

- **Step 2:** Use the branching property to conclude.

Maximum of log-correlated fields: known results

- **Discrete Gaussian log-correlated fields:** the whole convergence is known.
  - 2D Gaussian free field: Bramson–Ding–Zeitouni ’16, Biskup–Louidor ’16.
  - Lattice approximations of general fields: Ding–Roy–Zeitouni ’17.
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- **Random unitary matrices:**
  - 1^{st} order: Arguin–Bourgade–Belius ’15.
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  - 3^{rd} order: Chhaibi–Madaule–Najnudel ’16 (for all circular β-ensembles).

  The 1^{st} order has been obtained for **other random matrix models**: Ginibre (Lambert ’19), unitarily invariant Hermitian ensembles (Lambert–Paquette ’19), permutation matrices (Cook–Zeitouni ’20).

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- **Cover times in 2D:** Dembo–Peres–Rosen–Zeitouni '04, Belius–Kistler '17, Belius–Rosen–Zeitouni '20.
- Height of **weighted recursive trees** and preferential attachment trees: P.–Sénizergues '20.
- **Riemann ζ function:** Arguin–Belius–Bourgade–Radziwiłł–Soundararajan '19, Najnudel '18, Harper '19, Arguin–Bourgade–Radziwiłł '20 $\rightarrow$ **see next talk**.
  Random model for ζ: Arguin–Belius–Harper '19.

Thanks