Precise height asymptotics of weighted recursive trees (and affine preferential attachment trees)

Michel Pain (Courant Institute, NYU)
joint work with **Delphin Sénizergues** (University of British Columbia)
available soon on arXiv

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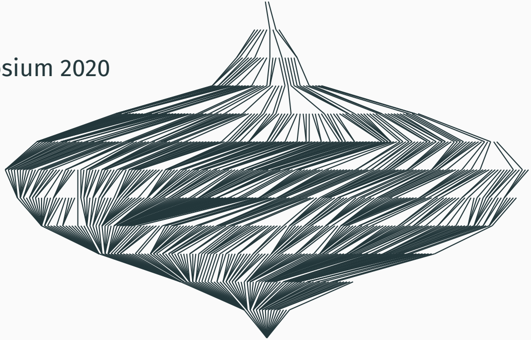

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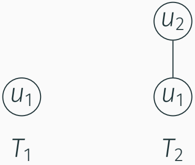

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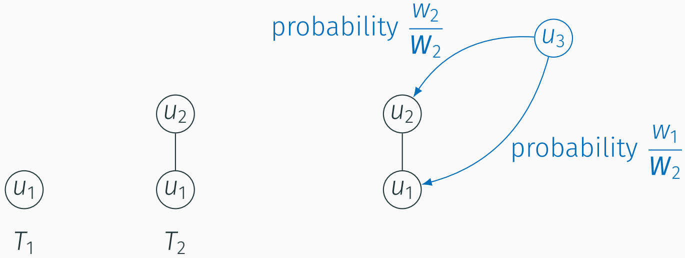

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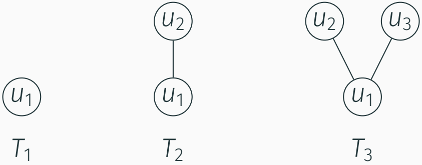

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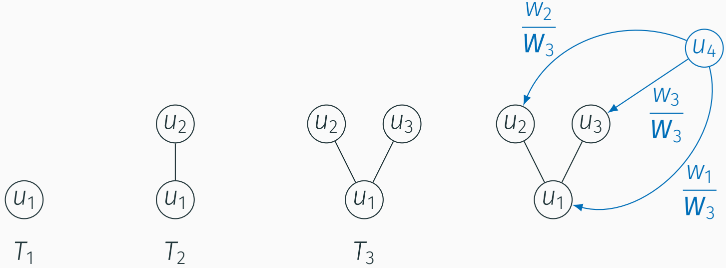

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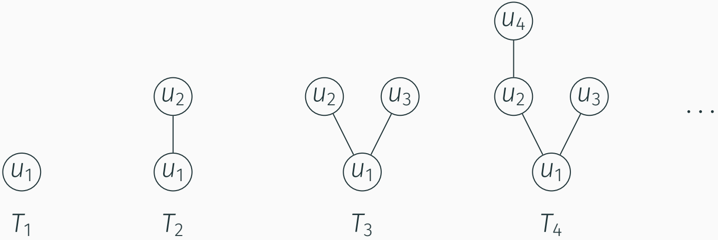

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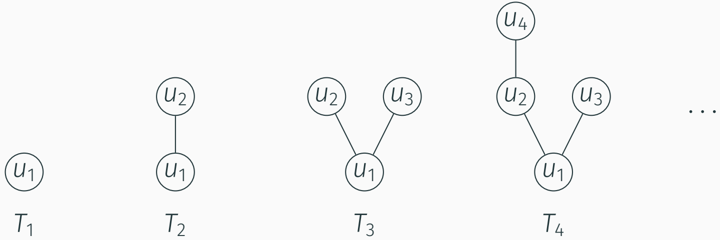

- An **affine preferential attachment tree** can be seen as a weighted recursive tree with random weights (Sénizergues 2020).

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- ▷ Same result for WRT with $w_i = 1$ for i large enough: Addario-Berry – Ford 2013, Leckey – Neininger 2013, Hiesmayr – Işlak 2020.
- ▷ Behavior similar to the maximum of branching random walks (Aïdékon 2013).

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- **Many-to-two:** the second moment can be expressed in terms of two time-inhomogeneous random walks, coinciding up to a random time and **weakly interacting** afterwards.
- **Difficulties:** no branching property and inhomogeneity.

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- *For the BRW:* Conclusion with the branching property.
- *Here:* We create a new WRT by merging $u_1, \ldots, u_N$. For this new tree, we have directly
$$\frac{\mathbb{E}[Q_n]^2}{\mathbb{E}[Q_n^2]} \simeq 1.$$


Thanks