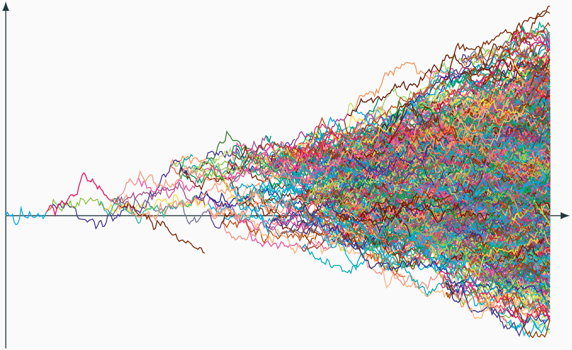
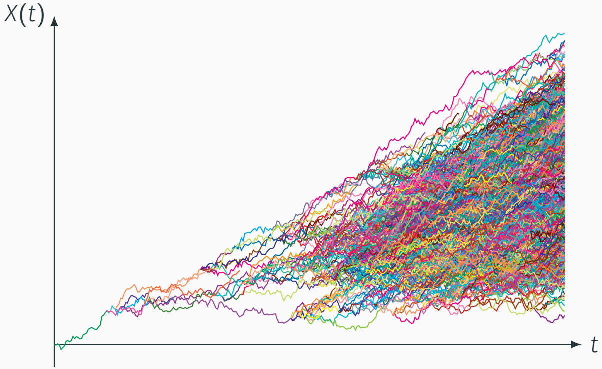
Fluctuations of branching Brownian motion at criticality

Michel Pain (ENS Paris/Sorbonne Université)
joint work with **Pascal Maillard** (Université Paris Sud)

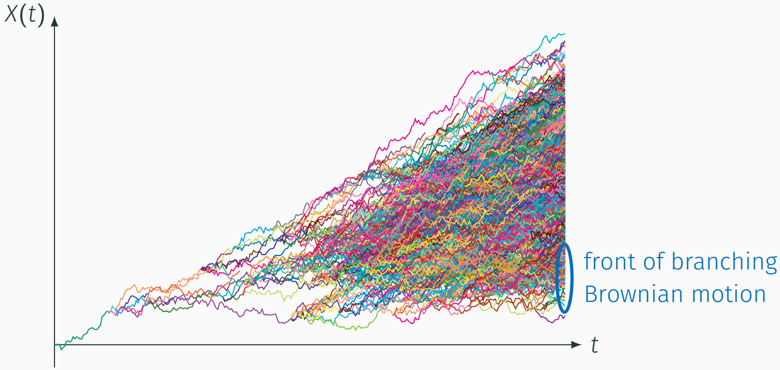
41st SPA conference
Northwestern University
9 July 2019



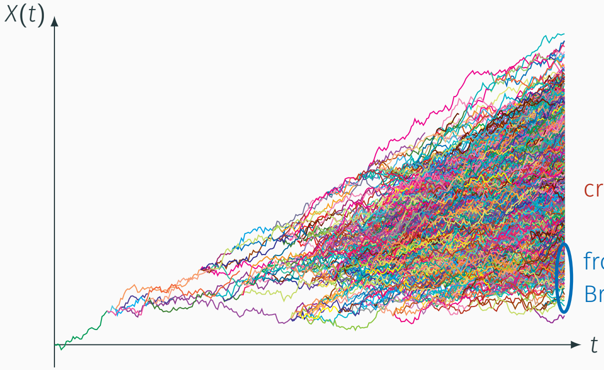
Objective of the talk



Objective of the talk

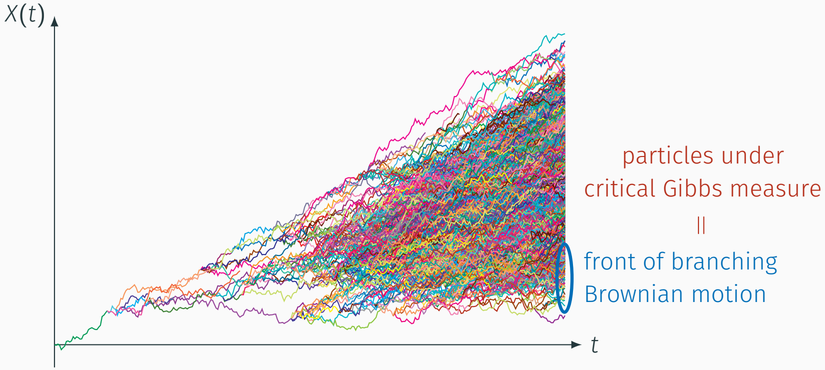


Objective of the talk



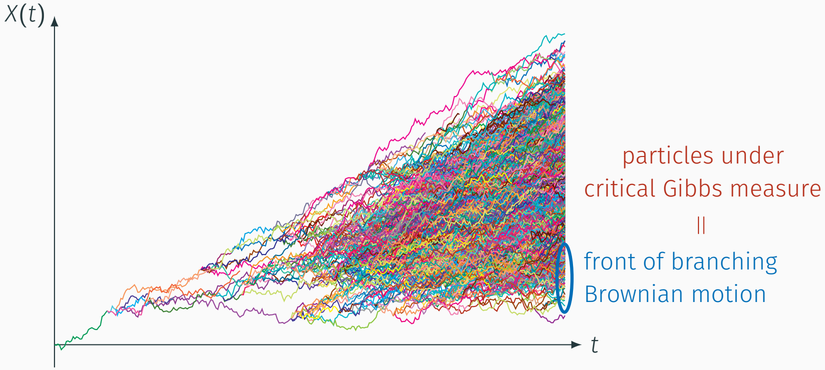
particles under
critical Gibbs measure
||
front of branching
Brownian motion

Objective of the talk



At first order, the density of particles in the front is **deterministic** up to a random **non-universal** factor Z_∞.

Objective of the talk



At first order, the density of particles in the front is **deterministic** up to a random **non-universal** factor Z_∞.

Goal: beyond this, universal fluctuations appear.

The front of branching Brownian motion

Definition of branching Brownian motion

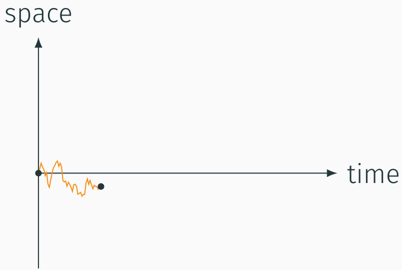
Branching Brownian motion (BBM):

- ▷ Starts with a particle at 0.

Definition of branching Brownian motion

Branching Brownian motion (BBM):

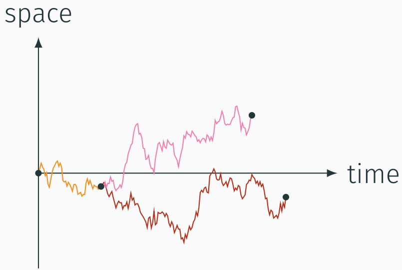
- Starts with a particle at 0.
- Particles move as a Brownian motion with drift ρ during a lifetime $\mathrm{Exp}(\lambda)$.



Definition of branching Brownian motion

Branching Brownian motion (BBM):

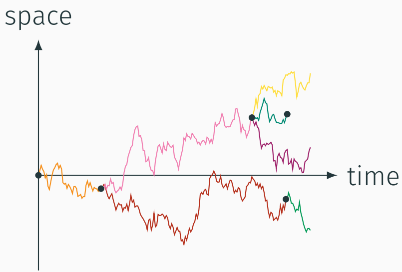
- Starts with a particle at 0.
- Particles move as a Brownian motion with drift ρ during a lifetime $\mathrm{Exp}(\lambda)$.
- When it dies, a particle has a random number of children, according to the distribution of a r.v. L.



Definition of branching Brownian motion

Branching Brownian motion (BBM):

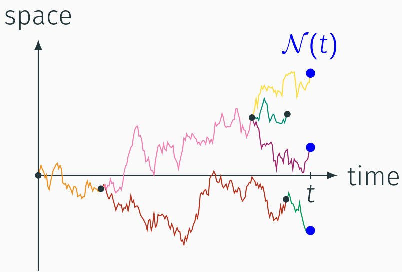
- Starts with a particle at 0.
- Particles move as a Brownian motion with drift ρ during a lifetime $\mathrm{Exp}(\lambda)$.
- When it dies, a particle has a random number of children, according to the distribution of a r.v. L.



Definition of branching Brownian motion

Branching Brownian motion (BBM):

- Starts with a particle at 0.
- Particles move as a Brownian motion with drift ρ during a lifetime $\mathrm{Exp}(\lambda)$.
- When it dies, a particle has a random number of children, according to the distribution of a r.v. L.



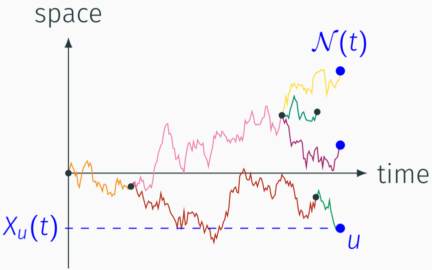
Notation:

- $\mathcal{N}(t)$ = set of particles alive at time t.

Definition of branching Brownian motion

Branching Brownian motion (BBM):

- Starts with a particle at 0.
- Particles move as a Brownian motion with drift ρ during a lifetime $\mathrm{Exp}(\lambda)$.
- When it dies, a particle has a random number of children, according to the distribution of a r.v. L.



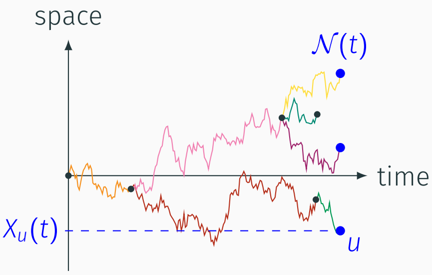
Notation:

- $\mathcal{N}(t)$ = set of particles alive at time t.
- $X_u(t)$ = position of particle u at time t.

Definition of branching Brownian motion

Branching Brownian motion (BBM):

- ▷ Starts with a particle at 0.
- ▷ Particles move as a Brownian motion with drift ρ during a lifetime $\mathrm{Exp}(\lambda)$.
- ▷ When it dies, a particle has a random number of children, according to the distribution of a r.v. L.



Notation:

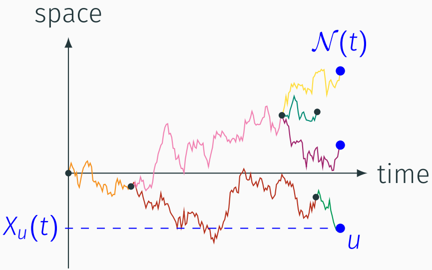
- ▷ $\mathcal{N}(t)$ = set of particles alive at time t.
- ▷ $X_u(t)$ = position of particle u at time t.

But: is it even a spin glass model?

Definition of branching Brownian motion

Branching Brownian motion (BBM):

- Starts with a particle at 0.
- Particles move as a Brownian motion with drift ρ during a lifetime $\mathrm{Exp}(\lambda)$.
- When it dies, a particle has a random number of children, according to the distribution of a r.v. L.



Notation:

- $\mathcal{N}(t)$ = set of particles alive at time t.
- $X_u(t)$ = position of particle u at time t.

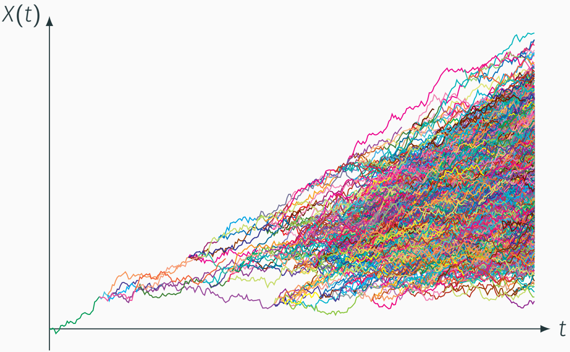
But: is it even a spin glass model? Yes, it belongs to the family of models with an explicit hierarchical structure such as the REM, GREM and CREM.

Choice of parameters

Our parameters:

- drift $\rho = 1$.
- branching rate

$$\lambda = \frac{1}{2\mathbb{E}[L-1]}.$$



Choice of parameters

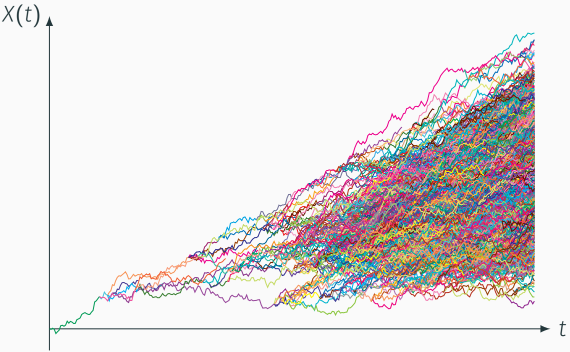
Our parameters:

- ▷ drift $\rho = 1$.
- ▷ branching rate

$$\lambda = \frac{1}{2\mathbb{E}[L-1]}.$$

Simplifying assumption:

$$\mathbb{E}\left[L^2\right] < \infty.$$



Choice of parameters

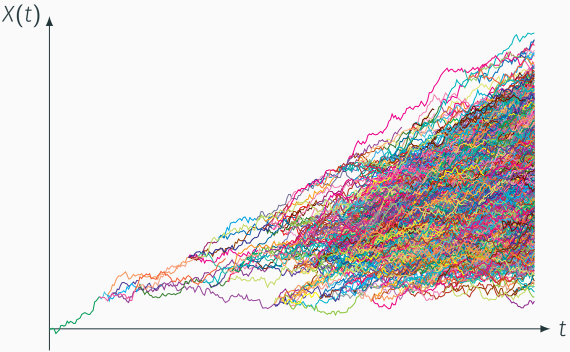
Our parameters:

- drift $\rho = 1$.
- branching rate

$$\lambda = \frac{1}{2\mathbb{E}[L-1]}.$$

Simplifying assumption:

$$\mathbb{E}\left[L^2\right] < \infty.$$



Comments:

- $\mathbb{E}[L] > 1$: supercritical Galton–Watson tree.

Choice of parameters

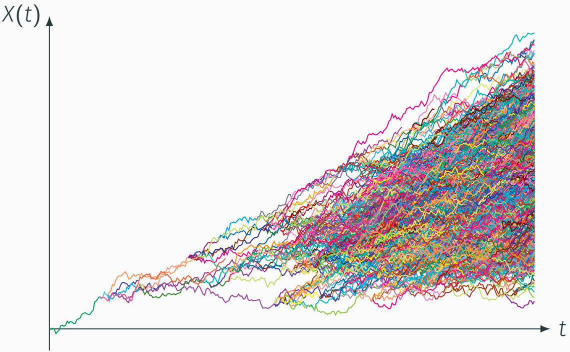
Our parameters:

- ▷ drift $\rho = 1$.
- ▷ branching rate

$$\lambda = \frac{1}{2\mathbb{E}[L-1]}.$$

Simplifying assumption:

$$\mathbb{E}\left[L^2\right] < \infty.$$



Comments:

- ▷ $\mathbb{E}[L] > 1$: supercritical Galton–Watson tree.
- ▷ Minimal velocity: $\displaystyle \frac{1}{t} \min_{u \in \mathcal{N}(t)} X_u(t) \xrightarrow[t\to\infty]{\text{a.s.}} 0.$

Additive martingales

For $\beta \geq 0$,

$$W_t(\beta) := \sum_{u \in \mathcal{N}(t)} \mathrm{e}^{-\beta X_u(t) - \frac{(\beta-1)^2}{2} t}, \qquad t \geq 0.$$

Additive martingales

For $\beta \geq 0$,

$$W_t(\beta) := \sum_{u \in \mathcal{N}(t)} \mathrm{e}^{-\beta X_u(t) - \frac{(\beta-1)^2}{2} t}, \qquad t \geq 0.$$

It is a nonnegative martingale, so

$$W_t(\beta) \xrightarrow[t\to\infty]{\text{a.s.}} W_\infty(\beta).$$

Additive martingales

For $\beta \geq 0$,

$$W_t(\beta) := \sum_{u \in \mathcal{N}(t)} \mathrm{e}^{-\beta X_u(t) - \frac{(\beta-1)^2}{2} t}, \qquad t \geq 0.$$

It is a nonnegative martingale, so

$$W_t(\beta) \xrightarrow[t\to\infty]{\text{a.s.}} W_\infty(\beta).$$

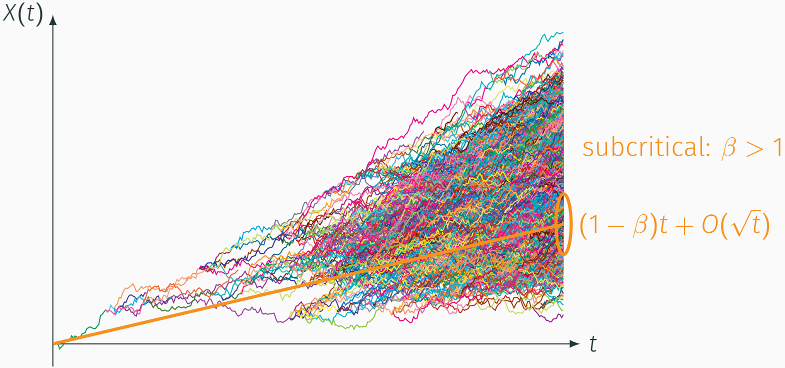
Theorem (Kahane–Peyrière 1976, Biggins 1977): *The following phase transition occurs:*

$$\begin{cases} W_\infty(\beta) > 0 \text{ a.s. on the survival} & \text{if } \beta < \beta_c = 1, \\ W_\infty(\beta) = 0 \text{ a.s.} & \text{if } \beta \geq \beta_c = 1. \end{cases}$$

Support of the Gibbs measures

Recall that $W_t(\beta) := \sum_{u \in \mathcal{N}(t)} e^{-\beta X_u(t) - \frac{(\beta-1)^2}{2} t}$.

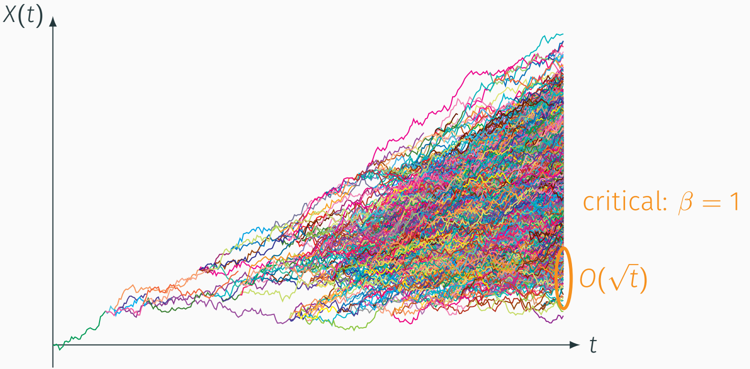
For $\beta < 1$, $W_t(\beta)$ describes the size of BBM around $(1-\beta)t + O(\sqrt{t})$.



Support of the Gibbs measures

Recall that $W_t(\beta) \coloneqq \sum_{u\in\mathcal{N}(t)} \mathrm{e}^{-\beta X_u(t) - \frac{(\beta-1)^2}{2}t}$.

For $\beta < 1$, $W_t(\beta)$ describes the size of BBM around $(1-\beta)t + O(\sqrt{t})$.



The **critical** Gibbs measure is supported by particles that are at a distance of order $\sqrt{t}$ from the minimum.

The derivative martingale

From now on, we work at criticality with $\beta = 1$:

$$W_t := \sum_{u \in \mathcal{N}(t)} \mathrm{e}^{-X_u(t)} \xrightarrow[t \to \infty]{\text{a.s.}} 0.$$

The derivative martingale

From now on, we work at criticality with $\beta = 1$:

$$W_t := \sum_{u \in \mathcal{N}(t)} \mathrm{e}^{-X_u(t)} \xrightarrow[t \to \infty]{\text{a.s.}} 0.$$

The derivative martingale is

$$Z_t := \sum_{u \in \mathcal{N}(t)} X_u(t) \mathrm{e}^{-X_u(t)}, \qquad t \geq 0.$$

The derivative martingale

From now on, we work at criticality with $\beta = 1$:

$$W_t := \sum_{u \in \mathcal{N}(t)} \mathrm{e}^{-X_u(t)} \xrightarrow[t\to\infty]{\text{a.s.}} 0.$$

The derivative martingale is

$$Z_t := \sum_{u \in \mathcal{N}(t)} X_u(t) \mathrm{e}^{-X_u(t)}, \qquad t \geq 0.$$

Theorem (Lalley–Sellke 1987): *$Z_t \longrightarrow Z_\infty$ a.s. and $Z_\infty > 0$ a.s. on the survival event.*

Importance of the derivative martingale

▷ **Minimal position** (Bramson 1983, Lalley–Sellke 1987):

$$\min_{u \in \mathcal{N}(t)} X_u(t) - \frac{3}{2} \log t \xrightarrow[t \to \infty]{\text{law}} -G - \log Z_\infty,$$

with G a Gumbel independent of Z_∞.

Importance of the derivative martingale

- **Minimal position** (Bramson 1983, Lalley–Sellke 1987):

$$\min_{u \in \mathcal{N}(t)} X_u(t) - \frac{3}{2} \log t \xrightarrow[t\to\infty]{\text{law}} -G - \log Z_\infty,$$

  with G a Gumbel independent of Z_∞.
- **Scaling of the critical additive martingale** (Aïdékon–Shi 2014):

$$\sqrt{t} W_t \xrightarrow[t\to\infty]{\text{probability}} \sqrt{\frac{2}{\pi}} Z_\infty.$$

Importance of the derivative martingale

- **Minimal position** (Bramson 1983, Lalley–Sellke 1987):

$$\min_{u\in\mathcal{N}(t)} X_u(t) - \frac{3}{2}\log t \xrightarrow[t\to\infty]{\text{law}} -G - \log Z_\infty,$$

  with G a Gumbel independent of Z_∞.
- **Scaling of the critical additive martingale** (Aïdékon–Shi 2014):

$$\sqrt{t}W_t \xrightarrow[t\to\infty]{\text{probability}} \sqrt{\frac{2}{\pi}}Z_\infty.$$

- For $f : \mathbb{R} \to \mathbb{R}$ continuous bounded,

$$Z_t(f) \coloneqq \sum_{u\in\mathcal{N}(t)} X_u(t)\mathrm{e}^{-X_u(t)} f\left(\frac{X_u(t)}{\sqrt{t}}\right).$$

Importance of the derivative martingale

- **Minimal position** (Bramson 1983, Lalley–Sellke 1987):

$$\min_{u\in\mathcal{N}(t)} X_u(t) - \frac{3}{2}\log t \xrightarrow[t\to\infty]{\text{law}} -G - \log Z_\infty,$$

  with G a Gumbel independent of Z_∞.
- **Scaling of the critical additive martingale** (Aïdékon–Shi 2014):

$$\sqrt{t}W_t \xrightarrow[t\to\infty]{\text{probability}} \sqrt{\frac{2}{\pi}}Z_\infty.$$

- For $f:\mathbb{R}\to\mathbb{R}$ continuous bounded,

$$Z_t(f) := \sum_{u\in\mathcal{N}(t)} X_u(t)\mathrm{e}^{-X_u(t)} f\left(\frac{X_u(t)}{\sqrt{t}}\right).$$

  Convergence of the front (Madaule 2016):

$$Z_t(f) \xrightarrow[t\to\infty]{\text{probability}} \mu(f)Z_\infty,$$

  where $\mu(\mathrm{d}x) := \mathbb{1}_{x>0}\sqrt{\frac{2}{\pi}}x^2\mathrm{e}^{-x^2/2}\,\mathrm{d}x$.

Our results

Questions

What are the rates of convergence and the fluctuations in the following convergences ?

$$Z_t \xrightarrow[t\to\infty]{\text{a.s.}} Z_\infty,$$

$$\sqrt{t}W_t \xrightarrow[t\to\infty]{\text{probability}} \sqrt{\frac{2}{\pi}}Z_\infty,$$

$$Z_t(f) \xrightarrow[t\to\infty]{\text{probability}} \mu(f)Z_\infty.$$

Questions

What are the rates of convergence and the fluctuations in the following convergences ?

$$
\begin{aligned}
Z_t &\xrightarrow[t\to\infty]{\text{a.s.}} Z_\infty, \\
\sqrt{t}W_t &\xrightarrow[t\to\infty]{\text{probability}} \sqrt{\frac{2}{\pi}}Z_\infty, \\
Z_t(f) &\xrightarrow[t\to\infty]{\text{probability}} \mu(f)Z_\infty.
\end{aligned}
$$

Motivation: Beyond the non-universal quantity Z_∞, universal fluctuations should appear.

Questions

What are the rates of convergence and the fluctuations in the following convergences ?

$$Z_t \xrightarrow[t\to\infty]{\text{a.s.}} Z_\infty,$$

$$\sqrt{t} W_t \xrightarrow[t\to\infty]{\text{probability}} \sqrt{\frac{2}{\pi}} Z_\infty,$$

$$Z_t(f) \xrightarrow[t\to\infty]{\text{probability}} \mu(f) Z_\infty.$$

Motivation: Beyond the non-universal quantity Z_∞, universal fluctuations should appear.

$\rightarrow$ Link with the conjecture of Ebert–van Saarloos (2000) for the position of the front of solutions of reaction-diffusion equations.

Fluctuations of the derivative martingale

Conjecture in the physics literature (Mueller–Munier 2014):

$$Z_t = Z_\infty +$$

Fluctuations of the derivative martingale

Conjecture in the physics literature (Mueller–Munier 2014):

$$Z_t = Z_\infty + \frac{\log t}{\sqrt{2\pi t}} Z_\infty +$$

Fluctuations of the derivative martingale

Conjecture in the physics literature (Mueller–Munier 2014):

$$Z_t = Z_\infty + \frac{\log t}{\sqrt{2\pi t}} Z_\infty + \underbrace{O\left(\frac{1}{\sqrt{t}}\right)}_{\text{fluctuations}}.$$

Fluctuations of the derivative martingale

Conjecture in the physics literature (Mueller–Munier 2014):

$$Z_t = Z_\infty + \frac{\log t}{\sqrt{2\pi t}} Z_\infty + \underbrace{O\left(\frac{1}{\sqrt{t}}\right)}_{\text{fluctuations}}.$$

Theorem (Maillard–P. 2018): *There exists a spectrally positive 1-stable Lévy process $(S_r)_{r\geq 1}$ independent of Z_∞ such that*

$$\left(\sqrt{t}\left(Z_\infty\left(1+\frac{\log t}{\sqrt{2\pi a t}}\right) - Z_{at}\right)\right)_{a\geq 1} \xrightarrow[t\to\infty]{law} (S_{Z_\infty/\sqrt{a}})_{a\geq 1},$$

in finite-dimensional distributions.

Fluctuations of the derivative Gibbs measure

Recall that

$$Z_t(f) \coloneqq \sum_{u \in \mathcal{N}(t)} X_u(t) \mathrm{e}^{-X_u(t)} f\left(\frac{X_u(t)}{\sqrt{t}}\right) \xrightarrow[t \to \infty]{\text{probability}} \mu(f) Z_\infty .$$

Fluctuations of the derivative Gibbs measure

Recall that

$$Z_t(f) := \sum_{u\in\mathcal{N}(t)} X_u(t)\mathrm{e}^{-X_u(t)} f\left(\frac{X_u(t)}{\sqrt{t}}\right) \xrightarrow[t\to\infty]{\text{probability}} \mu(f) Z_\infty.$$

Theorem (Maillard–P. 2019): *If f is $\mathcal{C}^2$ on $(0,\infty)$ and $(xf(x))'' \le Ce^{Cx}$ for $x>0$, then, there exists a 1-stable Lévy process $(S^f_r)_{r\ge 1}$ independent of Z_∞ such that*

$$\sqrt{t}\left(Z_\infty\left(\mu(f)+c(f)\frac{\log t}{\sqrt{t}}\right) - Z_t(f)\right) \xrightarrow[t\to\infty]{law} S^f_{Z_\infty}.$$

Fluctuations of the derivative Gibbs measure

Recall that

$$Z_t(f) := \sum_{u \in \mathcal{N}(t)} X_u(t) \mathrm{e}^{-X_u(t)} f\left(\frac{X_u(t)}{\sqrt{t}}\right) \xrightarrow[t\to\infty]{\text{probability}} \mu(f) Z_\infty .$$

Theorem (Maillard–P. 2019): *If f is $\mathcal{C}^2$ on $(0,\infty)$ and $(xf(x))'' \leq Ce^{Cx}$ for $x > 0$, then, there exists a 1-stable Lévy process $(S_r^f)_{r\geq 1}$ independent of Z_∞ such that*

$$\sqrt{t}\left(Z_\infty\left(\mu(f) + c(f)\frac{\log t}{\sqrt{t}}\right) - Z_t(f)\right) \xrightarrow[t\to\infty]{\textit{law}} S^f_{Z_\infty} .$$

Corollary: *Applying this with $f(x) = 1/x$, it follows:*

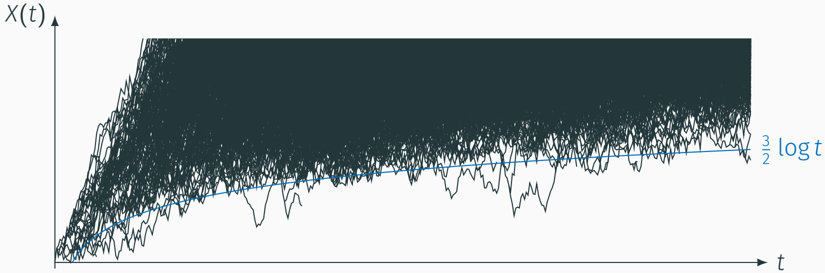
$$\sqrt{t}\left(\sqrt{\frac{2}{\pi}} Z_\infty - \sqrt{t} W_t\right) \xrightarrow[t\to\infty]{\textit{law}} S'_{Z_\infty} ,$$

where $(S'_r)_{r\geq 1}$ is a Cauchy process independent of Z_∞.

Particles contributing to the fluctuations

Recall that

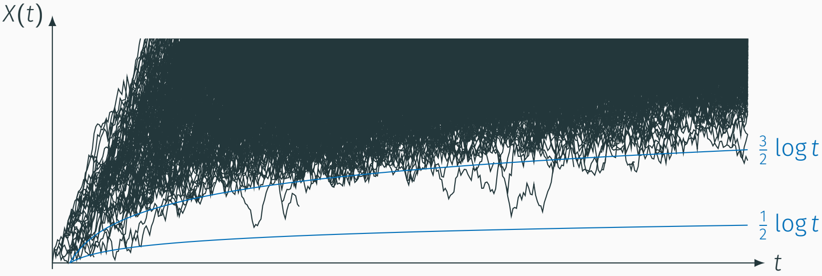
$$\frac{1}{\log t} \min_{u \in \mathcal{N}(t)} X_u(t) \xrightarrow[t\to\infty]{\text{probability}} \frac{3}{2}.$$



Particles contributing to the fluctuations

Recall that

$$\frac{1}{\log t} \min_{u \in \mathcal{N}(t)} X_u(t) \xrightarrow[t\to\infty]{\text{probability}} \frac{3}{2}.$$



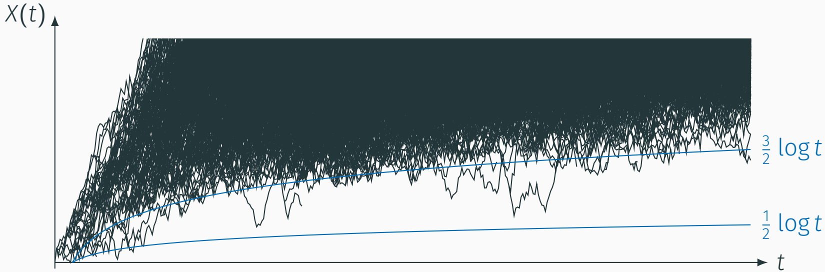
But, some particles come to a much lower position (Hu–Shi 2009):

$$\liminf_{t\to\infty} \frac{1}{\log t} \min_{u \in \mathcal{N}(t)} X_u(t) = \frac{1}{2} \text{ a.s.}$$

Particles contributing to the fluctuations

Recall that

$$\frac{1}{\log t} \min_{u \in \mathcal{N}(t)} X_u(t) \xrightarrow[t\to\infty]{\text{probability}} \frac{3}{2}.$$



But, some particles come to a much lower position (Hu–Shi 2009):

$$\liminf_{t\to\infty} \frac{1}{\log t} \min_{u \in \mathcal{N}(t)} X_u(t) = \frac{1}{2} \text{ a.s.}$$

Particles contributing to the fluctuations are those coming down to $\frac{1}{2}\log t + O(1)$ at a time of order t.

Ideas of proof

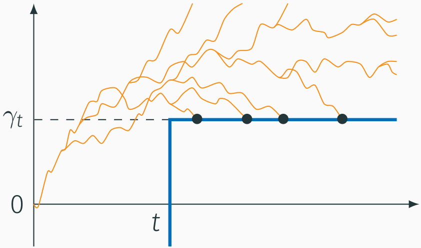
Proof for the derivative martingale

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_\infty\left(1+\frac{\log t}{\sqrt{2\pi t}}\right)-Z_t\right) \longrightarrow S_{Z_\infty}}$$

Proof for the derivative martingale

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_\infty\left(1+\frac{\log t}{\sqrt{2\pi t}}\right)-Z_t\right)\longrightarrow S_{Z_\infty}}$$

- We introduce a killing barrier from time t at level $\gamma_t = \frac{1}{2}\log t + \beta_t$, where $\beta_t \to \infty$ slowly.

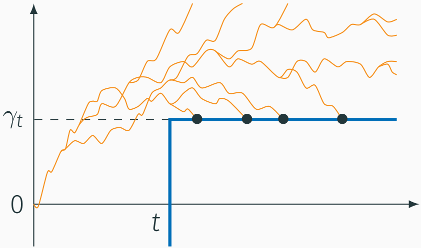


Proof for the derivative martingale

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_\infty\left(1+\frac{\log t}{\sqrt{2\pi t}}\right)-Z_t\right)\longrightarrow S_{Z_\infty}}$$

- We introduce a killing barrier from time t at level $\gamma_t = \frac{1}{2}\log t + \beta_t$, where $\beta_t \to \infty$ slowly.
- We work instead with

$$Z_s^t := \sum_{u\in\mathcal{N}(s)} (X_u(s)-\gamma_t)e^{-X_u(s)}.$$

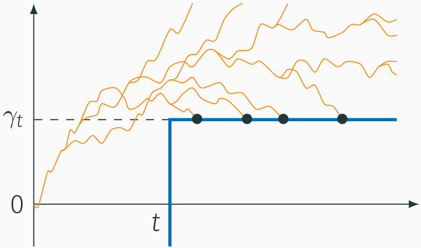


Proof for the derivative martingale

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_\infty\left(1+\frac{\log t}{\sqrt{2\pi t}}\right)-Z_t\right) \longrightarrow S_{Z_\infty}}$$

- We introduce a killing barrier from time t at level $\gamma_t = \frac{1}{2}\log t + \beta_t$, where $\beta_t \to \infty$ slowly.
- We work instead with

$$Z_s^t := \sum_{u \in \mathcal{N}(s)} (X_u(s) - \gamma_t)e^{-X_u(s)}.$$



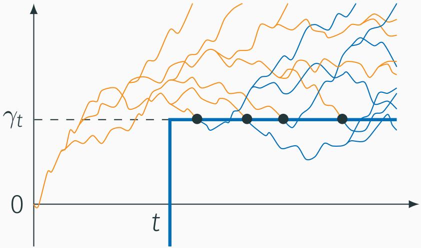
- With the barrier: Z_s^t does not vary too much on $[t, \infty)$.

Proof for the derivative martingale

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_\infty\left(1+\frac{\log t}{\sqrt{2\pi t}}\right)-Z_t\right) \longrightarrow S_{Z_\infty}}$$

- We introduce a killing barrier from time t at level $\gamma_t = \frac{1}{2}\log t + \beta_t$, where $\beta_t \to \infty$ slowly.
- We work instead with

$$Z_s^t := \sum_{u \in \mathcal{N}(s)} (X_u(s)-\gamma_t)e^{-X_u(s)}.$$



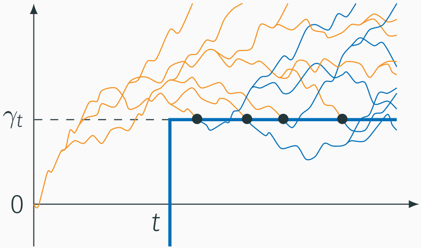
- With the barrier: Z_s^t does not vary too much on $[t,\infty)$.
- Contributions of killed particles to Z_∞: sum of i.i.d. copies of $e^{-\gamma_t}Z_\infty$, with approximately $e^{\beta_t}\sqrt{t}W_t$ terms.

Proof for the derivative martingale

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_\infty\left(1+\frac{\log t}{\sqrt{2\pi t}}\right) - Z_t\right) \longrightarrow S_{Z_\infty}}$$

- We introduce a killing barrier from time t at level $\gamma_t = \frac{1}{2}\log t + \beta_t$, where $\beta_t \to \infty$ slowly.
- We work instead with

$$Z_s^t := \sum_{u \in \mathcal{N}(s)} (X_u(s) - \gamma_t)e^{-X_u(s)}.$$



- With the barrier: Z_s^t does not vary too much on $[t, \infty)$.
- Contributions of killed particles to Z_∞: sum of i.i.d. copies of $e^{-\gamma_t} Z_\infty$, with approximately $e^{\beta_t}\sqrt{t}W_t$ terms.
  And Z_∞ is in the domain of attraction of a 1-stable law (Berestycki–Berestycki–Schweinsberg 2013).

Proof for the derivative Gibbs measure

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_t(f) - \mu(f)Z_\infty - c(f)\frac{\log t}{\sqrt{t}}Z_\infty\right) \to -S^f_{Z_\infty}.}$$

Proof for the derivative Gibbs measure

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_t(f) - \mu(f)Z_\infty - c(f)\frac{\log t}{\sqrt{t}}Z_\infty\right) \to -S^f_{Z_\infty}.}$$

- We can replace Z_∞ by Z_t.

Proof for the derivative Gibbs measure

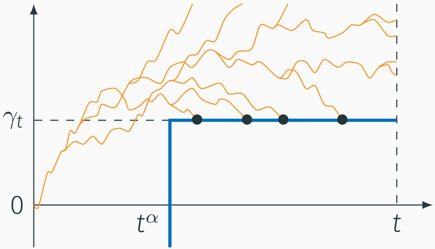
$$\boxed{\text{Goal: } \sqrt{t}\left(Z_t(f) - \mu(f)Z_t - \tilde{c}(f)\frac{\log t}{\sqrt{t}}Z_t\right) \to -\tilde{S}^f_{Z_\infty}.}$$

- ▷ We can replace Z_∞ by Z_t.

Proof for the derivative Gibbs measure

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_t(f) - \mu(f)Z_t - \tilde{c}(f)\frac{\log t}{\sqrt{t}}Z_t\right) \to -\tilde{S}^f_{Z_\infty}.}$$

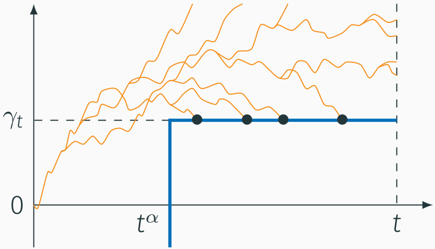
- We can replace Z_∞ by Z_t.
- We introduce the killing barrier at level γ_t, but between times t^α and t for some $\alpha \in (0,1)$.



Proof for the derivative Gibbs measure

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_t(f) - \mu(f)Z_t - \tilde{c}(f)\frac{\log t}{\sqrt{t}}Z_t\right) \to -\tilde{S}^f_{Z_\infty}.}$$

- We can replace Z_∞ by Z_t.
- We introduce the killing barrier at level γ_t, but between times t^α and t for some $\alpha \in (0,1)$.

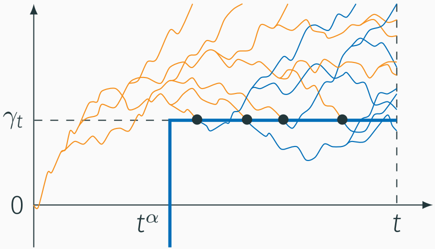


- With the barrier: $Z_t(f) - \mu(f)Z_t = o\left(\frac{1}{\sqrt{t}}\right)$ (after shifting by γ_t).

Proof for the derivative Gibbs measure

$$\boxed{\text{Goal: } \sqrt{t}\left(Z_t(f) - \mu(f)Z_t - \tilde{c}(f)\frac{\log t}{\sqrt{t}}Z_t\right) \to -\tilde{S}^f_{Z_\infty}.}$$

- We can replace Z_∞ by Z_t.
- We introduce the killing barrier at level γ_t, but between times t^α and t for some $\alpha \in (0,1)$.

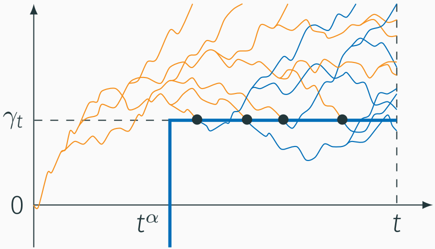


- With the barrier: $Z_t(f) - \mu(f)Z_t = o\left(\frac{1}{\sqrt{t}}\right)$ (after shifting by γ_t).
- Contributions of killed particles: their distribution depends on the time at which the particle hits the barrier.

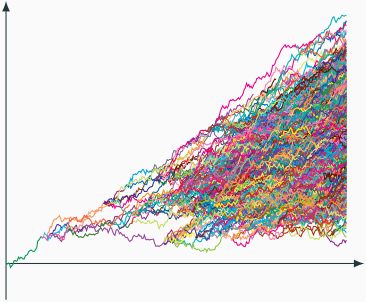
Proof for the derivative Gibbs measure

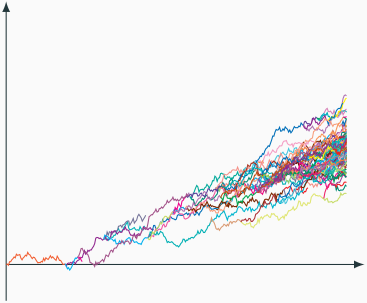
$$\boxed{\text{Goal: } \sqrt{t}\left(Z_t(f) - \mu(f)Z_t - \tilde{c}(f)\frac{\log t}{\sqrt{t}}Z_t\right) \to -\tilde{S}^f_{Z_\infty}.}$$

- We can replace Z_∞ by Z_t.
- We introduce the killing barrier at level γ_t, but between times t^α and t for some $\alpha \in (0,1)$.



- With the barrier: $Z_t(f) - \mu(f)Z_t = o\left(\frac{1}{\sqrt{t}}\right)$ (after shifting by γ_t).
- Contributions of killed particles: their distribution depends on the time at which the particle hits the barrier.
  $\to$ we want to approach them by their limit: for this we need a concentration result for $Z_t(f)$ aroud its limit!





Thank you for your attention!