LINEARIZATION OF POISSON BRACKETS

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Abstract. We review the linearization of Poisson brackets and related problems, in the formal, analytic and smooth categories.

1. Introduction

Recall that a **Poisson bracket** on a manifold M is a Lie bracket $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$, satisfying the derivation property

$$\{fg, h\} = f\{g, h\} + g\{f, h\}, \quad f, g, h \in C^\infty(M).$$

The Weinstein Splitting Theorem (see [34], Theorem 2.1) states that a neighborhood of a point $x_0 \in M$ is Poisson diffeomorphic to the direct product of a symplectic manifold and a Poisson manifold for which the bracket vanishes at x_0. Since a symplectic manifold has no local invariants (Darboux's Theorem), the local study of Poisson brackets reduces to the study of brackets vanishing at x_0.

Let $\{\cdot,\cdot\}$ be a Poisson bracket in M that vanishes at x_0. If we choose local coordinates $(x^1, \dots, x^m)$ in a neighborhood U of x_0, the Taylor series of the function $\{x^i, x^j\}$ reads

$$\{x^i, x^j\}(x) = c^{ij}_k x^k + g^{ij}_k(x)x^k,$$

where $c^{ij}_k = \frac{\partial\{x^i,x^j\}}{\partial x^k}(x_0)$, and the g^{ij}_k are smooth functions vanishing at x_0. In this paper we are interested in the following *linearization problem*:

Linearization Problem. *Are there new coordinates where the functions g^{ij}_k vanish identically, so that the bracket is linear in these coordinates?*

More generally, we will be interested in local canonical forms for the Poisson bracket.

Alternative algebraic and geometric descriptions of this problem will be given below. The constants c^{ij}_k are the structure constants of a Lie algebra $\mathfrak{g}$, called the **isotropy Lie algebra** at x_0. It can be defined independently of the choice of coordinates. Both the algebraic and the geometric descriptions will show the relevance of this Lie algebra to the linearization problem.

The linearization problem can be studied in different setups. We have just described it in the smooth category, which is the most challenging version of the problem. We are also interested in the following categories:

- *Formal category*: we consider a Poisson bracket on a formal neighborhood of the origin, and we look for a formal change of coordinates where the functions g^{ij}_k vanish to all orders at x_0.
- *Analytic category*: we consider an analytic Poisson bracket on an analytic manifold, and we look for an analytic change of coordinates where the functions g^{ij}_k vanish.

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The first results on linearization of Poisson brackets are due to V.I. Arnold ([1]) and Alan Weinstein. In the foundational paper [34], Weinstein showed that the formal linearization problem can be reduced to a cohomology question which can always be solved if the isotropy Lie algebra is *semisimple*. For the other categories, he did not give definite results, but he made the following conjectures:

(i) for a semisimple isotropy Lie algebra, analytic linearization of Poisson brackets can always be achieved.
(ii) for a semisimple isotropy Lie algebra of compact type, smooth linearization of Poisson brackets can always be achieved.

Weinstein also gave a counter-example of a smooth Poisson bracket with isotropy Lie algebra $\mathfrak{sl}(2)$ which is not linearizable.

Both Weinstein conjectures were proved later by Jack Conn in [9] and [10]. Conn's proofs are hard analysis proofs. They consist in showing that the formal power series change of coordinates obtained by Weinstein, which formally linearizes the Poisson bracket, actually yield linearization coordinates. For the analytic case, Conn uses methods due to Arnol'd and Kolmogoroff to prove convergence, while for the smooth case, he applies the Nash-Moser approximation scheme, used by Nash on his famous proof of the isometric embedding theorem, to obtain convergence. We will outline the main steps of the proofs below.

After Conn's work was completed, attention turned to other type of isotropy Lie algebras. Let us call a Lie algebra $\mathfrak{g}$ **non-degenerate relative to Poisson brackets** if any Poisson structure with isotropy Lie algebra isomorphic to $\mathfrak{g}$ is linearizable, and **degenerate** otherwise. In [35], Weinstein showed that semisimple Lie algebras of real rank greater than one are degenerate. The case of real rank 1, with the exception of $\mathfrak{sl}(2,\mathbb{R})$, remains open. In [16], Dufour has classified all non-degenerate 3-dimensional Lie algebras. More recently, in [18] Dufour and Zung have proved formal and analytic non-degeneracy for the Lie algebra of affine transformations $\mathfrak{aff}(n)$. On the other hand, in [2], examples are given of Poisson structures for which linearization can be decided only from knowledge of its higher order jets. The ultimate goal, which seems beyond our current state of knowledge, would be:

Open Problem. *Characterize the non-degenerate Lie algebras relative to Poisson brackets.*

For linearization around singular symplectic leaves, we refer the reader to the recent preprint [5].

In this paper we will survey what is known about the linearization of Poisson brackets. We will focus on comparing the analytic approach, which one can find in the literature, and the geometric approach which is the subject of recent work (see [13, 14, 39]), and which sheds much light on this problem. During the preparation of this survey we were informed by Dufour and Zung of their upcoming book [19] on normal forms for Poisson structures, where the material discussed here (and much more) will be presented in greater detail.

The problem of linearization of Poisson brackets has remarkable similarities with the problem of linearization of Lie algebra actions. In Section 2 below, we will recall the classical results of Cartan ([8]), Bochner ([3]), Hermann ([25]) and Guillemin and Sternberg ([23]), on linearization of Lie group and Lie algebra actions, in the formal, analytic and smooth categories. This easier problem sets up the stage for the study of the linearization of Poisson brackets. Again, we will see here the advantages of the geometric approach over the analytic approach.

We take up the linearization of Poisson brackets in Section 3 below. We discuss the formal, analytic and smooth linearization problems, and we sketch the known proofs (without any technical details). These are all analytic proofs. We will mention briefly the geometric proofs proposed in [13, 14, 39] (analytic case) and in [13] (smooth case). We shall also emphasize the similarity with the linearization of Lie algebra actions, though the known proofs are quite different in each case.

In section 4, we turn to a more general normal form: the so called *Levi decomposition* for a Poisson bracket. This is an analogue for Poisson brackets of the usual Levi decomposition for finite dimensional Lie algebras. After all, Poisson brackets are infinite dimensional Lie brackets, and so one should expect some kind of Levi decomposition to hold as well. This Levi decomposition may also be seen as a semi-linearization of Poisson brackets. Again, there is a formal Levi decomposition due to Wade ([32]), an analytic Levi decomposition due to Zung ([38]), and a smooth Levi decomposition due to Monnier and Zung ([29]).

Finally, in Section 5, we give a short overview of linearization and Levi decomposition in the context of Lie algebroids which allows one to unify the results for actions of Lie algebras and for Poisson brackets.

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2. Linearization of Lie algebra actions

The problem of linearization of Poisson brackets has remarkable similarities with the problem of linearization of Lie algebra actions. First of all, for group actions, we have the following well-known classical result:

Theorem 2.1 (Bochner [3]). *Let $G \times M \to M$ be a smooth action of a compact Lie group on a smooth manifold, which has a fixed point. Then, in suitably chosen local coordinates around the fixed point, the action is linear.*

For a modern version of the proof, we refer to the book by Duistermaat and Kolk ([20]). This result grew out of H. Cartan's investigations on analytic group actions, and Cartan himself proved a similar result in the analytic case ([8]). Palais and Smale suggested extending this result to non-compact Lie group actions and Hermann ([25]) explained that the corresponding formal problem can be reduced to a cohomology question which can always be solved if the group is semisimple.

If the group G is connected, trying to find a linear system of coordinates for the Lie group action, is the same as finding a linear system of coordinates for the associated Lie algebra action. Note, however, that since a Lie algebra action does not always integrate to a Lie group action, the linearization problem for Lie algebra actions is harder. Let us look then at this problem.

Consider an action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ of a Lie algebra $\mathfrak{g}$ on a manifold M, which has a fixed point $x_0 \in M$. We are interested in the following problem.

- Are there coordinates $(x^1, \ldots, x^m)$ around x_0 such that, for all $X \in \mathfrak{g}$, the vector fields $\rho(X)$ are linear in these coordinates ?

In general, if we fix a basis $\{X_1, \ldots, X_N\}$ for $\mathfrak{g}$ and an arbitrary system of coordinates $(x^1, \ldots, x^m)$ around x_0, the action will be written in the form[1]:

$$\rho(X_\alpha) = c^i_{\alpha j} x^j \frac{\partial}{\partial x^i} + O(2), \qquad (\alpha = 1, \ldots, N), \tag{2.1}$$

and so we are looking for coordinates where the higher order terms vanish.

In order to put this problem in a more geometric framework, observe that the action ρ induces a linear action ρ_L on the tangent space $T_{x_0}M$. In terms of a basis for $\mathfrak{g}$ and a system of coordinates around x_0 as above, the linear action is just given by truncating in (2.1) the higher order terms: If $(u^1, \ldots, u^m)$ denotes the linear

[1] We will make systematic use of the Einstein sum convention.

coordinates defined by the basis $\{\frac{\partial}{\partial x^1}\big|_{x_0}, \dots, \frac{\partial}{\partial x^m}\big|_{x_0}\}$ for $T_{x_0}M$, the linear action is defined by

$$\rho_L(X_\alpha) = c^i_{\alpha j} u^j \frac{\partial}{\partial u^i} \qquad (\alpha = 1, \dots, N). \tag{2.2}$$

The *geometric formulation* of the linearization problem is then:

- Is there a local diffeomorphism $\phi : M \to T_{x_0}M$, from a neighborhood of x_0 in M to a neighborhood of 0 in $T_{x_0}M$, which is $\mathfrak{g}$-equivariant?

Let us denote by $\mathfrak{M}$ the maximal ideal formed by germs of smooth functions that vanish at x_0:

$$\mathfrak{M} = \{f \in C^\infty(M) : f(x_0) = 0\}.$$

There is also an induced linear action, denoted by ρ_L^*, of $\mathfrak{g}$ on $\mathfrak{M}$:

$$\rho_L^*(X)(f) = X(f).$$

It is easy to see that $\mathfrak{M}^2 \subset \mathfrak{M}$ is an invariant ideal, so both $\mathfrak{M}^2$ and the cotangent space $T^*_{x_0}M \simeq \mathfrak{M}/\mathfrak{M}^2$ have induced linear $\mathfrak{g}$-actions. Obviously, the $\mathfrak{g}$-actions on $T^*_{x_0}M$ and $T_{x_0}M$ are transpose to each other. The *algebraic formulation* of the linearization problem is:

- Is there a splitting σ of the short exact sequence of $\mathfrak{g}$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{M}^2 \arrow[r] & \mathfrak{M} \arrow[r] & \mathfrak{M}/\mathfrak{M}^2 \arrow[r] \arrow[l, bend left, dashed, "\sigma"] & 0 \ ? \end{tikzcd} \tag{2.3}$$

In fact, the projection $\mathfrak{M} \to \mathfrak{M}/\mathfrak{M}^2$ is just taking the differential at x_0: $f \mapsto d_{x_0}f$. Hence, if $\sigma : \mathfrak{M}/\mathfrak{M}^2 \to \mathfrak{M}$ is such a splitting and we fix a basis $\{\xi^1, \dots, \xi^n\}$ for $\mathfrak{M}/\mathfrak{M}^2$, then $\sigma(\xi^i)$ are germs of smooth functions $x^1, \dots, x^m$, for which the differentials $d_{x_0}x^i = \xi^i$ are independent. Therefore, $(x^1, \dots, x^m)$ yields a system of coordinates around x_0. On the other hand, since σ is $\mathfrak{g}$-equivariant, when we express the action in this system of coordinates we obtain a linear action.

The linearization problem we have just explained was set up in the smooth category. Similarly, we can consider formal linearization (replace $C^\infty(M)$ by the ring of power series $\mathbb{R}[[x^1, \dots, x^m]]$) or analytic linearization (replace $C^\infty(M)$ by the ring of analytic functions $C^\omega(M)$).

2.1. **Formal Linearization.** The algebraic version of the linearization problem suggests one should look at obstructions coming from the Lie algebra cohomology of $\mathfrak{g}$. These are the only obstructions for the formal problem, as we show now.

Recall that if V is any $\mathfrak{g}$-module, then we have the Chevalley-Eilenberg complex $(C^\bullet(\mathfrak{g}; V), d)$, where:

- $C^r(\mathfrak{g}; V)$ is the vector space consisting of all r-multilinear alternating forms $\omega : \mathfrak{g} \wedge \cdots \wedge \mathfrak{g} \to V$, and
- the differential $d : C^r(\mathfrak{g}; V) \to C^{r+1}(\mathfrak{g}; V)$ is defined by

$$\begin{aligned} d\omega(X_0, \dots, X_r) = &\sum_{i=0}^{r} (-1)^k X_k(\omega(X_0, \dots, \widehat{X}_i, \dots, X_r)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_r). \end{aligned} \tag{2.4}$$

The cohomology of this complex is called the **Lie algebra cohomology** of the $\mathfrak{g}$-module V, and will be denoted $H^\bullet(\mathfrak{g}; V)$.

It is a well-known fact from Homological Algebra (see, e.g., [33]) that the obstructions to the existence of a splitting of a short exact sequence of $\mathfrak{g}$-modules lie in the first Lie algebra cohomology. Let us recall this briefly. Suppose we are given a short exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow V \longrightarrow W \xrightarrow{\ \pi\ } Z \longrightarrow 0\,.$$

Let $\sigma : Z \to W$ be a vector space splitting for this sequence. The space of linear endomorphisms $\mathrm{End}(Z, V)$ is naturally a $\mathfrak{g}$-module, and we can define an element $\omega \in C^1(\mathfrak{g}; \mathrm{End}(Z, V))$ by:

$$\omega(X) : z \longmapsto \sigma(X \cdot z) - X \cdot \sigma(z) \qquad (z \in Z).$$

Notice that the left-hand side actually lies in $\mathrm{Ker}\,\pi = V$. A quick computation shows that $d\omega = 0$, so we have a well-defined class $[\omega] \in H^1(\mathfrak{g}; \mathrm{End}(Z, V))$. It is easy to check that (i) this class is independent of the splitting, and (ii) it vanishes iff there exists a splitting of the sequence, as $\mathfrak{g}$-modules.

Now, the First Whitehead Lemma states that if $\mathfrak{g}$ is a semisimple Lie algebra and V is any *finite dimensional* $\mathfrak{g}$-module, then $H^1(\mathfrak{g}; V) = 0$. Certainly, we cannot apply this directly to the short exact sequence (2.3), since the $\mathfrak{g}$-modules are infinite dimensional. Still, one has:

Theorem 2.2 (Hermann [25]). *If $\mathfrak{g}$ is a semisimple Lie algebra, then the action is formally linearizable, i.e., there exist coordinates $(x^1_\infty, \dots, x^m_\infty)$ around x_0 such that, for any basis $\{X_1, \dots, X_N\}$ for $\mathfrak{g}$, we have*

$$\rho(X_\alpha) = c^i_{\alpha j} x^j_\infty \frac{\partial}{\partial x^i_\infty} + o(\infty), \qquad (\alpha = 1, \dots, N),$$

where $c^i_{\alpha j}$ are some constants.

Let us sketch a proof of this result. Instead of considering the full ring $\mathfrak{M}$ at once, one looks at the spaces $\mathfrak{M}/\mathfrak{M}^{k+1}$ (germs mod terms of order higher than k). Notice that we still have short exact sequences of $\mathfrak{g}$-modules

$$0 \longrightarrow \mathfrak{M}^k/\mathfrak{M}^{k+1} \longrightarrow \mathfrak{M}/\mathfrak{M}^{k+1} \xrightarrow{\ \pi_k\ } \mathfrak{M}/\mathfrak{M}^k \longrightarrow 0\,. \tag{2.5}$$

In this sequence, all $\mathfrak{g}$-modules are finite dimensional. Hence, by the First Whitehead Lemma, we have splittings $\phi_k : \mathfrak{M}/\mathfrak{M}^k \to \mathfrak{M}/\mathfrak{M}^{k+1}$. It follows that one has a commutative diagram

$$\begin{tikzcd}
\mathfrak{M} \arrow[r] \arrow[drrrrrr, "\pi"'] & \cdots \arrow[r] & \mathfrak{M}/\mathfrak{M}^{k+1} \arrow[r, "\pi_k"] & \mathfrak{M}/\mathfrak{M}^k \arrow[r] \arrow[l, dashed, bend right, "\phi_k"'] & \cdots \arrow[r] & \mathfrak{M}/\mathfrak{M}^3 \arrow[r, "\pi_2"] & \mathfrak{M}/\mathfrak{M}^2 \arrow[l, dashed, bend right, "\phi_2"'] \\
&&&&&& T^*_{x_0}M \arrow[u, "\sigma_2"'] \arrow[ul, dashed, "\sigma_3"] \arrow[ulll, dashed, "\sigma_k"] \arrow[ullll, dashed, "\sigma_{k+1}"]
\end{tikzcd}$$

In this diagram, $\sigma_2 : T^*_{x_0}M \to \mathfrak{M}/\mathfrak{M}^2$ denotes the natural isomorphism, while for $k > 2$ the $\mathfrak{g}$-module homomorphisms $\sigma_k : T^*_{x_0}M \to \mathfrak{M}/\mathfrak{M}^k$ are obtained by induction:

$$\sigma_k = \phi_k \circ \sigma_{k-1}.$$

By passing to the projective limit, we obtain a $\mathfrak{g}$-module homomorphism

$$\sigma : T^*_{x_0}M \to \varprojlim \mathfrak{M}/\mathfrak{M}^k \subset \mathfrak{M},$$

and, as above, this defines the coordinate system $(x^1_\infty, \dots, x^m_\infty)$ that formally linearizes the action.

2.2. **Analytic Linearization.** The proof given above for Hermann's Linearization Theorem shows that one constructs the formal linearizing coordinates by a succession of changes of coordinates, which at each step removes terms of order k, without modifying the terms of order less than k. Hence, it is natural to ask if one can make the choices at each step so that, in the limit, one obtains a smooth or analytic coordinate system, that linearizes the action. In the analytic category this is indeed the case:

Theorem 2.3 (Guillemin and Sternberg [23])**.** *If $\mathfrak{g}$ is a semisimple Lie algebra and the action is analytic, then it is analytically linearizable, i.e., there exist analytic coordinates $(x^1_\infty, \ldots, x^m_\infty)$ around x_0 such that, for any basis $\{X_1, \ldots, X_N\}$ for $\mathfrak{g}$, we have*

$$\rho(X_\alpha) = c^i_{\alpha j} x^j_\infty \frac{\partial}{\partial x^i_\infty}, \qquad (\alpha = 1, \ldots, N),$$

for some constants $c^i_{\alpha j}$.

Analytic proof. One proves, directly, convergence of the formal linearizing coordinates. The estimates are rather involved, and so we will omit them. The interested reader can obtain them by specializing the proof of the analytic Levi decomposition for Lie algebroids, given recently in [38], to the case of the action Lie algebroid. □

Geometric proof. This is the proof given by Guillemin and Sternberg in [23]. It avoids the question of convergence of the formal linearizing coordinates, through the use of complexification and analytic continuation into the complex plane, to obtain an action of the complexified Lie algebra. Restricting this action to the compact real form, one obtains an action which integrates to an action of a compact Lie group. This was proved by Guillemin and Sternberg using the assumption that the action is analytic, but in the next section we will show that this is already true for smooth actions. We can then apply Bochner's Theorem to linearize the action. □

Remark 2.4. Independently of Guillemin and Sternberg, but using similar techniques, Kushnirenko ([26]) proved linearization of real-analytic or complex-analytic actions of local real or complex semisimple Lie groups.

Remark 2.5. A completely different proof was obtained a few years after [23], by Flato, Pinczon and Simon ([22]), as a corollary of their study of nonlinear actions of Lie groups and Lie algebras having a fixed point. Their main objective was the infinite dimensional case needed to study the linearizability of nonlinear evolution equations covariant under a Lie group action. However, the Lie theory that was developed (the Lie group-Lie algebra analytic nonlinear representations correspondence), combined with Weyl's unitary trick to bring the problem to compact forms, and a variant of Bochner's theorem above, leads to a simple proof of 2.3.

2.3. **Smooth Linearization.** The following example of a smooth action of $\mathfrak{sl}(2,\mathbb{R})$ which is not linearizable is due to Guillemin and Sternberg [23].

Example 2.6. *Consider the basis $\{X, Y, Z\}$ of $\mathfrak{sl}(2,\mathbb{R})$ satisfying the relations:*

$$[X, Y] = -Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$$

We have a linear action defined by:

$$\rho_L(X) = y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}, \quad \rho_L(Y) = x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x},$$
$$\rho_L(Z) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

For this action the orbits are the level sets of the quadratic form $x^2+y^2-z^2 = r^2-z^2$. *On the other hand, we can perturb this action to the non-linear action:*

$$\rho(X) = \rho_L(X) + \frac{xz}{r^2}g(r^2-z^2)V, \quad \rho(Y) = \rho_L(Y) - \frac{yz}{r^2}g(r^2-z^2)V,$$
$$\rho(Z) = \rho_L(Z) + g(r^2-z^2)V,$$

where $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ *is the radial vector field, and* $g \in C^\infty(\mathbb{R})$ *is such that* $g(x) > 0$, *if* $x > 0$, *and* $g(x) = 0$, *if* $x \leq 0$.

The orbits of ρ *coincide with the orbits of* ρ *inside the cone* $r = z$. *Outside this cone, the orbits of* $\rho(Z)$ *spiral towards the origin while the orbits of* $\rho_L(Z)$ *are circles. Hence,* ρ *is not linearizable.*

This shows that, in the smooth case, we need to restrict the class of semisimple Lie algebras. Recall that a finite dimensional Lie algebra $\mathfrak{g}$ is called **semisimple of compact type** if any of the following equivalent conditions are satisfied (see, e.g., [20], Section 3.6):

(i) The Killing form of $\mathfrak{g}$ is negative definite;
(ii) The simply connected Lie group integrating $\mathfrak{g}$ is compact;
(iii) Any Lie group integrating $\mathfrak{g}$ is compact;

For such Lie algebras we have the following linearization result:

Theorem 2.7. *If* $\mathfrak{g}$ *is a semisimple Lie algebra of compact type and the action is smooth, then it is smoothly linearizable, i.e., there exist smooth coordinates* $(x^1_\infty, \dots, x^m_\infty)$ *around* x_0 *such that, for any basis* $\{X_1, \dots, X_N\}$ *for* $\mathfrak{g}$, *we have*

$$\rho(X_\alpha) = c^i_{\alpha j} x^j_\infty \frac{\partial}{\partial x^i_\infty}, \qquad (\alpha = 1, \dots, N),$$

for some constants $c^i_{\alpha j}$.

To our surprise, in spite of Guillemin and Sternberg work being more that 30 years old, we could not find this result in the literature.

Analytic proof. Again, one proves, directly, convergence of the formal linearizing coordinates, and the estimates are even more involved than in the analytic case. Once more, they can also be obtained by specializing the proof of the smooth Levi decomposition for Lie algebroids, given recently in [29], to the case of the action Lie algebroid (see Section 4). □

Geometric proof. The proof consists in showing that the $\mathfrak{g}$-action can be integrated to a G-action, where G is the compact simply connected Lie group integrating $\mathfrak{g}$, so we can apply Bochner's Linearization Theorem for compact Lie group actions.

A Lie algebra action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ in general will not integrate to a Lie group action. Our approach is to search for a Lie group action by looking at its graph. This graph should be obtained by integrating the graph of the Lie algebra action. Since the graph of a Lie group (respectively, Lie algebra) action is in fact the associated action Lie groupoid (resp. Lie algebroid) what is involved here is the integration of Lie algebroids to Lie groupoids.

For *any* Lie algebra action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ the associated action Lie algebroid always integrate to a Lie groupoid. This fact was first proved by Dazord ([27] and [7], Section 16.4) and follows also easily from the general integrability result in [12]. Let $\mathbf{s}, \mathbf{t} : \mathcal{G} \rightrightarrows M$ be the Lie groupoid integrating the action Lie algebroid. Since x_0 is a fixed point for the action, the source and target satisfy $\mathbf{s}^{-1}(x_0) = \mathbf{t}^{-1}(x_0) = G$, a simply connected Lie group integrating $\mathfrak{g}$. So this fiber is compact, and by stability, we conclude that $\mathcal{G}$ trivializes over an neighborhood U of x_0 with compact closure. Since this neighborhood is invariant for the action, this shows that the action in U integrates. □

Remark 2.8. In general, a Lie groupoid is not assumed to be Hausdorff, but its s-fibers are. It can be shown that, in our case, the Lie groupoid that integrates the action is Hausdorff, so one can apply the usual Reeb stability result. Alternatively, one can use the stability result for non-Hausdorff manifolds due to Mrčun (see [31]).

Of course, just like in the Poisson case, one can ask for which Lie algebras $\mathfrak{g}$ is it true that every $\mathfrak{g}$-action is linearizable around a fixed point. Call such a Lie algebra non-degenerate relative to actions.

Open Problem. *Characterize the Lie algebras that are non-degenerate relative to actions.*

The $\mathfrak{sl}(2,\mathbb{R})$-action on $\mathbb{R}^3$ given by Gullemin and Sternberg, does not integrate to an $SL(2,\mathbb{R})$-action. In [6], Cairns and Ghys modify this example to obtain an $SL(2,\mathbb{R})$-action on $\mathbb{R}^3$ that is not linearizable. On the other hand, they show that a smooth $SL(n,\mathbb{R})$-action on $\mathbb{R}^n$ is always linearizable. This suggests one should look for $\mathfrak{g}$ non-degeneracy for a fixed dimension of the manifold, or even specifying the linear part of the action.

It is an intriguing question how this problem is related to the question of non-degeneracy of the Lie algebra relative to Poisson brackets.

3. Linearization of Poisson brackets

Let us now turn to Poisson geometry. Here we view a Poisson manifold as an infinitesimal object. The corresponding global object (if it exists) is a *symplectic groupoid* $\Sigma \rightrightarrows M$, and we can ask for the analogue of Bochner's theorem. There is, in fact, such a theorem:

Theorem 3.1. *Let $\Sigma \rightrightarrows M$ be a symplectic groupoid with a fixed point $x_0 \in M$. Assume that the isotropy group $G_{x_0} = s^{-1}(x_0) = t^{-1}(x_0)$ is compact. Then Σ is locally isomorphic to the symplectic groupoid $T^*G_{x_0} \rightrightarrows T_{x_0}M$.*

This result follows from a more general result valid for proper Lie groupoids (see Weinstein [37], Zung [40]). The proof given in Zung [40] is analytic, and uses a fixed point theorem. However, a much more geometric proof of this result, using averaging (in fact, vanishing of cohomology [11]) can be found in [13, 14]. Note that the isomorphism given by the theorem preserves both the Lie groupoid structure and the symplectic structure. If one ignores the Lie groupoid structure, then the symplectic isomorphism follows from the usual Lagrangian Neighborhood theorem of Weinstein, since the fiber $s^{-1}(x_0) = t^{-1}(x_0)$ over a fixed point is always a Lagrangian submanifold.

Note, also, that the local isomorphism $\Sigma \simeq T^*G$ covers a local Poisson diffeomorphism $M \simeq T_{x_0}M = \mathfrak{g}^*$ which maps a neighborhood of $x_0 \in M$ onto a neighborhood of $0 \in \mathfrak{g}^*$. In this way, trying to find a local isomorphism $\Sigma \simeq T^*G$ is the same as trying to linearize the Poisson bracket. However, Poisson manifolds do not always integrate to symplectic groupoids, and so the problem of linearizing Poisson brackets is harder. Let us look then at this problem.

We consider a Poisson manifold $(M, \{\ ,\ \})$ and we assume that the bracket vanishes at $x_0 \in M$. If we fix local coordinates $(x^1, \dots, x^m)$ around x_0, we have

$$\{x^i, x^j\} = c_k^{ij} x^k + O(2). \tag{3.1}$$

Here the c_k^{ij} are structure constants for a Lie algebra, thus the linear terms define a linear Poisson bracket. The linearization problem we are interested in, is then:

- Are there coordinates $(x^1, \dots, x^m)$ around x_0 such that the Poisson bracket is linear in these coordinates?

In order to put this problem in a more geometric framework, recall that the Poisson bracket gives rise to a Lie bracket $[\cdot,\cdot]$ on the space $\Omega^1(M)$ of 1-forms on M, which is given by:

$$[\alpha,\beta]=\mathcal{L}_{\#\alpha}\beta-\mathcal{L}_{\#\beta}\alpha-d\Pi(\alpha,\beta),\quad \alpha,\beta\in\Omega^1(M). \tag{3.2}$$

Here $\Pi\in\Gamma(\Lambda^2TM)$ denotes the Poisson 2-tensor which is associated to the Poisson bracket by $\Pi(df,dg)=\{f,g\}$, and $\#:T^*M\to TM$ denotes contraction by Π. This bracket satisfies (in fact, it is uniquely determined by) the following two basic properties:

(i) for exact 1-forms it coincides with the Poisson bracket:

$$[df,dg]=d\{f,g\},\quad f,g\in C^\infty(M);$$

(ii) it satisfies the Leibniz identity:

$$[\alpha,f\beta]=f[\alpha,\beta]+\#\beta(f)\alpha,\quad \alpha,\beta\in\Omega^1(M),\ f\in C^\infty(M).$$

The triple $(T^*M,[\cdot,\cdot],\#)$ is a Lie algebroid, called the **cotangent Lie algebroid** of the Poisson manifold M.

For each $x\in M$, the kernel $\mathfrak{g}_x=\operatorname{Ker}\#_x\subset T^*_xM$ is a Lie algebra, called the **isotropy Lie algebra** at x. At x_0, we have $\mathfrak{g}_{x_0}=T^*_{x_0}M$, so the dual $T_{x_0}M$ carries a Poisson structure called the **linear approximation** at x_0. In local coordinates $(x^1,\dots,x^m)$ around x_0, the numbers c_k^{ij} in (3.1) are just the structure constants for $\mathfrak{g}_{x_0}$ relative to the basis $\{\,dx^1\big|_{x_0},\dots,dx^m|_{x_0}\}$. The geometric reformulation of the linearization problem is then:

- Is there a local Poisson diffeomorphism $\phi:M\to T_{x_0}M$, from a neighborhood of x_0 in M to a neighborhood of 0 in $T_{x_0}M$?

Let us denote again by $\mathfrak{M}$ the germs of smooth functions that vanish at x_0. Then $\mathfrak{M}$ is a maximal ideal of the Lie algebra $(C^\infty(M),\{\ ,\ \})$[2]. The quotient $\mathfrak{M}/\mathfrak{M}^2=T^*_{x_0}M$ is just the isotropy Lie algebra $\mathfrak{g}_{x_0}$, so we can reformulate the linearization problem in the following algebraic form:

- Is there a splitting σ of the short exact sequence of Lie algebras

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{M}^2 \arrow[r] & \mathfrak{M} \arrow[r] & \mathfrak{g}_{x_0} \arrow[r] \arrow[l, bend left, dashed, "\sigma"] & 0 \end{tikzcd}\ ? \tag{3.3}$$

Again, the projection $\mathfrak{M}\to\mathfrak{g}_{x_0}$ is just taking the differential: $f\mapsto d_{x_0}f$. Hence, if $\sigma:\mathfrak{g}_{x_0}\to\mathfrak{M}$ is such a splitting and we fix a basis $\{\xi^1,\dots,\xi^n\}$ for $\mathfrak{g}_{x_0}$, then $\sigma(\xi^i)$ are germs of smooth functions $x^1,\dots,x^m$, for which the differentials $d_{x_0}x^i=\xi^i$ are independent. Therefore, $(x^1,\dots,x^m)$ yields a system of coordinates around x_0. On the other hand, since σ is a homomorphism of Lie algebras, when we express the Poisson bracket in this coordinate system we obtain a linear Poisson bracket.

Like the Lie algebra case, besides the smooth category, the linearization problem for Poisson brackets can also be considered in the formal category (replace $C^\infty(M)$ by the ring of power series $\mathbb{R}[[x^1,\dots,x^m]]$) or the analytic category (replace $C^\infty(M)$ by the ring of analytic functions $C^\omega(M)$).

3.1. **Formal Linearization.** Just like the case of Lie algebra actions, the algebraic version of the linearization problem for Poisson brackets leads to obstructions to linearization lying in Lie algebra cohomology.

In this case, we shall need to look at Lie algebra cohomology in degree 2. In fact, it is a well-known fact in Homological Algebra (see, e.g., [33]) that the obstructions

[2]Notice that, for Lie algebra actions, $\mathfrak{M}$ was a $\mathfrak{g}$-*module*, while now it is a *Lie algebra*.

to the existence of central extensions lies in H^2. More exactly, let us consider a short exact sequence of Lie algebras

$$0 \longrightarrow V \longrightarrow \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0 ,$$

where V is abelian (a vector space). If we fix a splitting $\sigma : \mathfrak{g} \to \mathfrak{h}$ of this sequence as vector spaces, then V becomes a $\mathfrak{g}$-module for the action:

$$X \cdot v \equiv [\sigma(X), v], \qquad X \in \mathfrak{g}, v \in V.$$

Here we identify V with its image in $\mathfrak{h}$. This action does not depend on the splitting, and we can define an element $\omega \in C^2(\mathfrak{g}; V)$ by:

$$\omega(X, Y) = \sigma([X, Y]) - [\sigma(X), \sigma(Y)], \qquad X, Y \in \mathfrak{g}.$$

Notice that the left-hand side actually lies in $\operatorname{Ker} \pi = V$. An easy computation shows that $d\omega = 0$, so we have a well-defined class $[\omega] \in H^2(\mathfrak{g}; V)$. Again, it is not hard to check that this class is independent of the splitting and that it vanishes iff there exists a splitting of the sequence (as Lie algebras).

Now, the Second Whitehead Lemma states that if $\mathfrak{g}$ is a semisimple Lie algebra and V is any *finite dimensional* $\mathfrak{g}$-module, then $H^2(\mathfrak{g}; V) = 0$. Certainly, we cannot apply this directly to the short exact sequence (3.3), since the the Lie algebra $\mathfrak{M}^2$ is neither finite dimensional nor abelian. Still, just like in the case of Lie algebra actions, we have:

Theorem 3.2 (Weinstein [34]). *If $\mathfrak{g}$ is a semisimple Lie algebra, then the Poisson bracket is formally linearizable, i.e., there exists coordinates $(x^1_\infty, \dots, x^m_\infty)$ around x_0 such that*

$$\{x^i_\infty, x^j_\infty\} = c^{ij}_k x^k_\infty + O(\infty), \qquad (i, j = 1, \dots, m).$$

For the proof, one looks again at the spaces $\mathfrak{M}/\mathfrak{M}^{k+1}$ of germs mod terms of order higher than k. Let us look once more at the diagram (but now of Lie algebras, rather than modules):

$$\begin{tikzcd}
\mathfrak{M} \arrow[r] \arrow[drrrrrr, "\pi"'] & \cdots \arrow[r] & \mathfrak{M}/\mathfrak{M}^{k+1} \arrow[r, "\pi_k"] & \mathfrak{M}/\mathfrak{M}^k \arrow[r] & \cdots \arrow[r] & \mathfrak{M}/\mathfrak{M}^3 \arrow[r, "\pi_2"] & \mathfrak{M}/\mathfrak{M}^2 \\
& & & & & & T^*_{x_0}M = \mathfrak{g}_{x_0} \arrow[u, "\phi_2"'] \arrow[ullll, dashed, "\phi_{k+1}"] \arrow[ulll, dashed, "\phi_k"] \arrow[ul, dashed, "\phi_3"]
\end{tikzcd}$$

We claim that one can construct the injective homomorphisms of Lie algebras $\phi_k : \mathfrak{g}_{x_0} \to \mathfrak{M}/\mathfrak{M}^k$ by induction.

The case $k = 2$ is trivial, so assume we have constructed ϕ_k. Then, we restrict the short exact sequence of (finite dimensional) Lie algebras

$$0 \longrightarrow \mathfrak{M}^k/\mathfrak{M}^{k+1} \longrightarrow \mathfrak{M}/\mathfrak{M}^{k+1} \xrightarrow{\pi_k} \mathfrak{M}/\mathfrak{M}^k \longrightarrow 0 ,$$

to the image $\phi_k(\mathfrak{g})$. We obtain a new short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{M}^k/\mathfrak{M}^{k+1} \longrightarrow \pi_k^{-1}(\phi_k(\mathfrak{g}))/\mathfrak{M}^{k+1} \longrightarrow \phi_k(\mathfrak{g}) \longrightarrow 0 ,$$

with $\mathfrak{M}^k/\mathfrak{M}^{k+1}$ abelian. Since $\mathfrak{g}$, and hence $\phi_k(\mathfrak{g})$, is semisimple, by the Second Whitehead Lemma there exists a splitting $\sigma : \phi_k(\mathfrak{g}) \to \pi_k^{-1}(\phi_k(\mathfrak{g}))/\mathfrak{M}^{k+1}$. Composing with ϕ_k we obtain ϕ_{k+1}.

By passing to the projective limit, we obtain a homomorphism of Lie algebras

$$\sigma : \mathfrak{g}_{x_0} \to \varprojlim \mathfrak{M}/\mathfrak{M}^k,$$

and, as above, this defines the coordinate system $(x^1_\infty, \dots, x^m_\infty)$ that formally linearizes the Poisson bracket.

3.2. **Analytic Linearization.** The proof given above for the formal linearization shows that one constructs the formal linearizing coordinates by a succession of changes of coordinates, which at each step removes terms of order k, without modifying the terms of order less than k. Hence, it is natural to ask if one can make the choices at each step so that, in the limit, one obtains a smooth or analytic coordinate system, that linearizes the action. In the analytic category this is indeed the case:

Theorem 3.3 (Conn [9]). *Let $\{\cdot,\cdot\}$ be an analytic Poisson structure which vanishes at x_0. If the isotropy Lie algebra $\mathfrak{g}_{x_0}$ at x_0 is semisimple, then there exist a local analytic coordinate system $(x^1_\infty, \cdots, x^n_\infty)$ around x_0 in which the Poisson structure is linear:*

$$\{x^i_\infty, x^j_\infty\} = c^{ij}_k x^k_\infty. \tag{3.4}$$

This is the analogue of the Guillemin and Sternberg linearization result for Lie algebra actions. Again, we will sketch analytic and geometric proofs. To simplify, we will denote by $\mathfrak{g}$ the isotropy Lie algebra $\mathfrak{g}_{x_0}$.

Analytic proof. The analytic proof, due to Conn, uses a fast convergence method due to Kolmogorov. To sketch a proof along these lines, let us look closer at the formal linearization. We observe that the bracket relation

$$[\mathfrak{M}^k, \mathfrak{M}^k] \subset \mathfrak{M}^{2k-1},$$

implies that one can construct the sequence of linearizing coordinates so that, at each step, we remove terms of order 2^k.

To see this, let $\{X_1, \dots, X_n\}$ be a basis for the Lie algebra $\mathfrak{g}$. We assume that we have constructed an injective homomorphism $\phi_\nu : \mathfrak{g} \to \mathfrak{M}/\mathfrak{M}^{2^\nu}$ ([3]), so that we have coordinates $(x^1_\nu, \cdots, x^n_\nu)$ where each x^i_ν represents the element $\phi_\nu(X_i)$ and, moreover, the Poisson bracket satisfies:

$$\{x^i_\nu, x^j_\nu\} = c^{ij}_k x^k_\nu + O(|x_\nu|^{2^\nu}).$$

We want to construct the next iteration $(x^1_{\nu+1}, \cdots, x^n_{\nu+1})$ by the method we described above.

For that, we note that $\mathfrak{g}$ acts on on the spaces $\mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$ by Hamiltonian diffeomorphisms:

$$X_i \cdot f = c^{ij}_k x^k_\nu \frac{\partial f}{\partial x^j_\nu} = \{x^i_\nu, f\} \mod \mathfrak{M}^{2^{\nu+1}}. \tag{3.5}$$

We consider the remainder term $R_\nu : \wedge^2\mathfrak{g} \to \mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$ which is given by:

$$R_\nu(X_i \wedge X_j) = \{x^i_\nu, x^j_\nu\} - c^{ij}_k x^k_\nu \mod \mathfrak{M}^{2^{\nu+1}}.$$

The Jacobi identity shows that R_ν is a cocycle and so defines a cohomology class in $H^2(\mathfrak{g}; \mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}})$. Since $\mathfrak{g}$ is semisimple, by the second Whitehead lemma, this class must vanish. Let us denote by $\sigma_{\nu+1} : \mathfrak{g} \to \mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$ a primitive. We can choose local analytic functions $f^i_{\nu+1}$ which represent the elements $\sigma_{\nu+1}(X_i) \in \mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$. The new coordinates are then defined by

$$x^i_{\nu+1} \equiv x^i_\nu - f^i_{\nu+1}\,. \tag{3.6}$$

and it is easy to verify that:

$$\{x^i_{\nu+1}, x^j_{\nu+1}\} = c^{ij}_k x^k_{\nu+1} + O(|x_\nu|^{2^{\nu+1}}).$$

[3]Note the change in the indices!

This establishes the claim.

Now we need to look at the formal limit of these systems of coordinates, which we denote by $(x^1_\infty, \cdots, x^n_\infty)$. To complete the proof of Theorem 3.3, we "just" have to show that these formal coordinates are local analytic coordinates.

In order to make estimates in the algorithm above, we need a normed version of the second Whitehead lemma. For this, we denote by $\mathfrak{g}_\mathbb{C}$ the complexification of $\mathfrak{g}$. This complex Lie algebra is semisimple, and we denote by $\mathfrak{g}_0$ its compact real form. We consider a (finite dimensional) Hermitian complex vector space which is a Hermitian $\mathfrak{g}_0$-module (i.e., $\mathfrak{g}_0$ acts by Hermitian transformations), and we let $\| \ \|$ denote the norm corresponding to the Hermitian structure. Then:

Proposition 3.4 (Conn [9])**.** *There exists a linear map $h : \wedge^2\mathfrak{g}_\mathbb{C}\otimes W \to \mathfrak{g}_\mathbb{C}\otimes W$ and a positive constant D (which depends only on $\mathfrak{g}$) such that, for every $R \in \wedge^2\mathfrak{g}_\mathbb{C}\otimes W$,*

$$\|h(R)\| \leq D\|R\|,$$

and

$$d(R) = 0 \quad \Longrightarrow \quad R = d\big(h(R)\big).$$

where d is the differential of the complex $(C^\bullet(\mathfrak{g}_0; W), d)$. The homotopy operator h is real, i.e., maps real cochains to real cochains.

We would like to apply this proposition to the spaces $\mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$, so we must furnish them with Hermitian metrics which are preserved by the $\mathfrak{g}_0$-action. This can be done as follows:

The Lie algebra $\mathfrak{g}$ acts on $T_{x_0}M = \mathfrak{g}^*$ by the coadjoint action. We can complexify this action obtaining an action of $\mathfrak{g}_\mathbb{C}$: we choose our original local coordinates so that their complexification $(z_1, \ldots, z_n)$ determined a basis for $\mathfrak{g}^*_\mathbb{C}$ which is orthonormal with respect to the opposite of the Killing form of $\mathfrak{g}_0$ (which is a Hermitian metric of $\mathfrak{g}^*_\mathbb{C}$). This Hermitian metric is preserved by the coadjoint action of the compact Lie algebra $\mathfrak{g}_0$.

Let r be a positive number. For each integer ν, for which we have constructed analytic coordinates $(z^1_\nu, \cdots, z^n_\nu)$, we define the (deformed) balls

$$B_{\nu,r} = \{w \in \mathbb{C}^n \mid \sqrt{\sum_i |z^i_\nu(w)|^2} \leq r\}$$

In this ball we take the standard volume form $d\mu_\nu$ relative to the coordinates $(z^1_\nu, \cdots, z^n_\nu)$ and we denote by V_r the volume of the ball. This ball is analytically diffeomorphic to the standard ball of radius r via the coordinate system $(z^1_\nu, \cdots, z^n_\nu)$, and we define the L^2-norm on local analytic functions:

$$\|f\|_{\nu,r} = \sqrt{\frac{1}{V_r}\int_{B_{\nu,r}} |f(w)|^2 d\mu_\nu}$$

If we identify the quotients $\mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$ with space of polynomials functions, these norms determined corresponding norms on the quotient. One can show that the norms obtained in this way are Hermitian. For example, the norm $\|f\|_{0,r}$ is given by the following classic Hermitian metric: if $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha \prod (z^i)^{\alpha_i}$ and $g = \sum_{\alpha\in\mathbb{N}^n} b_\alpha \prod (z^i)^{\alpha_i}$, then we have

$$\langle f, g\rangle_{0,r} = \sum_{\alpha\in\mathbb{N}^n} \frac{\alpha! n!}{(|\alpha| + n)!} a_\alpha \bar{b}_\alpha r^{2|\alpha|},$$

where $\alpha! = \prod_i \alpha_i!$ and $|\alpha| = \sum \alpha_i$. The others norms are also given by similar Hermitian metrics.

Therefore, the spaces $\mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$ equipped with the norms $\| \ \|_{\nu,r}$ are Hermitian spaces. Now remark that, since the action of $\mathfrak{g}_0$ on $\mathbb{C}^n$ preserves the Hermitian metric of $\mathbb{C}^n$, its action on $\mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$ preserves the norm $\| \ \|_{\nu,r}$. We can then apply the proposition above to conclude that:

$$\max_i \|z^i_{\nu+1} - z^i_\nu\|_{\nu,r} \leq D \max_{i,j} \|\{z^i_\nu, z^j_\nu\} - c^{ij}_k z^k_\nu\|_{\nu,r}$$

Using this estimates, one constructs a decreasing sequence of radii $(r_\nu)_\nu$ such that the balls B_{ν,r_ν} are well-defined and there exists an integer K and open neighborhood U of the origin, such that:

$$U \subset \bigcap_{\nu \geq K} B_{\nu,r_\nu}.$$

Moreover, for every $w \in U$ and $\nu \geq K$, we have uniform bounds on the norms of the corrections:

$$\sup_{w \in B_{\nu,r_\nu}} |z^i_\nu(w) - z^i_{\nu+1}(w)|,$$

and on the Poisson structure. We conclude that the limit coordinate system $(z^1_\infty, \cdots, z^n_\infty)$ is analytic in U, and therefore furnishes analytic linearizing coordinates. □

Geometric proof. In [39], Zung sketches a geometric proof of the theorem. This proof consists of the following steps:

(i) Construct a basis of Casimirs functions $\{\tilde{C}_1, \ldots, \tilde{C}_r\}$ of the Poisson bracket $\{\ ,\ \}$ by perturbing a basis of Casimir functions $\{C_1, \ldots, C_r\}$ of the linear bracket. This is possible because the Casimirs of the linear bracket are given as certain invariant polynomials of integrals of the leafwise symplectic form over the periods (2-cycles) of the coadjoint orbits. By analyticity, one can perturb these 2-cycles to 2-cycles of the symplectic leaves of the non-linear bracket.

(ii) Use the Casimir functions, to conclude that the symplectic foliation of the original Poisson bracket is analytically diffeomorphic to the symplectic foliation of the linear bracket. Hence, one can assume that they are diffeomorphic.

(iii) Use averaging and Moser's path method to deform the Poisson bracket to the linear bracket along an isotopy which is the time-1 flow of a certain time-dependent vector field.

Not every detail is given in [39], but it is likely that a complete proof can be given along these lines. □

3.3. **Smooth Linearization.** Let us now turn to the linearization problem of smooth Poisson brackets. In the smooth setting it is not enough to assume that the $\mathfrak{g}_{x_0}$ is semisimple as one sees from the following example, due to Weinstein (see [34]).

Example 3.5. *Consider the Lie algebra $\mathfrak{sl}(2)$ and identify its dual with $\mathbb{R}^3$, with coordinates $\{x, y, z\}$. Then we have the linear Poisson bracket:*

$$\{x, y\} = -z, \quad \{y, z\} = x, \quad \{z, x\} = y.$$

The leaves of this Poisson bracket are the level sets of the quadratic form $x^2 + y^2 - z^2 = r^2 - z^2$. On the other hand, we can perturb this linear bracket to the non-linear bracket:

$$\begin{aligned}
\{x, y\} &= -z, \\
\{y, z\} &= x - \frac{y}{r^2} g(r^2 - z^2), \\
\{z, x\} &= y + \frac{x}{r^2} g(r^2 - z^2),
\end{aligned}$$

where $g \in C^\infty(\mathbb{R})$ is such that $g(x) > 0$, if $x > 0$, and $g(x) = 0$, if $x \leq 0$. Now, observe that the hamiltonian vector field for the function $h(x, y, z) = z$ is given by:

$$X_h = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - \frac{g(r^2 - z^2)}{r^2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),$$

and has integral curves outside the cone $r = z$ which spiral towards the cone. Since no hamiltonian vector field in $\mathfrak{sl}(2)^*$ has this behavior, this Poisson bracket is not linearizable.

Again, like the Lie algebra case, we have:

Theorem 3.6 (Conn [10])**.** *Let $(M, \{\ ,\ \})$ be a smooth Poisson manifold vanishing at x_0. If the isotropy Lie algebra $\mathfrak{g}_{x_0}$ is semisimple of compact type, then there exists smooth local coordinates $(x^1_\infty, \dots, x^m_\infty)$ around x_0 such that*

$$\{x^i_\infty, x^j_\infty\} = c^{ij}_k x^k_\infty \qquad (i, j = 1, \dots, m).$$

There is also a C^p version of this linearization theorem (see [29]). Again, we shall sketch an analytic and a geometric proof of this result.

Analytic proof. As in the analytic case, we would like to construct a sequence of smooth local coordinates systems $(x^1_\nu, \cdots, x^n_\nu)$ which converges to a smooth local coordinates system $(x^1_\infty, \cdots, x^n_\infty)$ in which the Poisson structure is linear. In the proof of the analytic case, at each step, we used a homotopy operator to correct the remainder terms:

$$R_\nu : \wedge^2\mathfrak{g} \to \mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}.$$

We could do this because the R_ν were 2-cocycles in the Chevalley-Eilenberg complex $C^\bullet(\mathfrak{g}; \mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}})$. Here we see the first major difference from the analytic case: the bracket maybe non-linear, while all such remainder terms vanish! Hence, we are now forced to work on the $\mathfrak{g}$-modules $\mathfrak{M}^{2^\nu}$, and put aside the quotients $\mathfrak{M}^{2^\nu}/\mathfrak{M}^{2^{\nu+1}}$.

So let us consider now the remainder terms as maps:

$$R_\nu : \wedge^2\mathfrak{g} \to \mathfrak{M}^{2^\nu}.$$

These are not cocycles anymore. However, we will think of them as *almost cocycles* and we apply the homotopy operator, so we now must control the size of the resulting *almost coboundary* $h(R_\nu)$. Here again, we use a normed version of the Whitehead lemma for a certain orthogonal $\mathfrak{g}$-module (of infinite dimension).

So we denote by $(x^1, \cdots, x^n)$ a coordinate system centered at x_0, such that $\{d_{x_0}x^1, \cdots, d_{x_0}x^n\}$ is an orthonormal basis of $\mathfrak{g}$, with respect to the negative of the Killing form (since $\mathfrak{g}$ is compact, this is positive definite). We have

$$[d_{x_0}x^i, d_{x_0}x^j] = c^{ij}_k d_{x_0}x^k.$$

Let r be a positive real number and denote by B_r the closed ball of radius r and centered at 0, with respect to the coordinates $(x^1, \cdots, x^n)$. We denote by $\mathcal{C}_r$ the space of smooth functions on B_r, vanishing at 0 and whose first derivatives also vanish at 0. The Lie algebra $\mathfrak{g}$ acts on B_r via the coadjoint action, and this induces an action on $\mathcal{C}_r$. On the space $\mathcal{C}_r$ the Sobolev H_k-metrics are defined by:

$$\langle f, g\rangle_{k,r} := \sum_{|\alpha|\leq k}\int_{B_r}\Big(\frac{|\alpha|!}{\alpha!}\Big)\Big(\frac{\partial^{|\alpha|}f}{\partial x^\alpha}(z)\Big)\mathrm{d}\mu(z)$$

and we denote by $\|\ ,\ \|_{k,r}$ the corresponding norm. The action of $\mathfrak{g}$ on $\mathcal{C}_r$ preserves these norms. The $\mathfrak{g}$-module $(\mathcal{C}_r, \langle\ ,\ \rangle_{k,r})$ is of infinite dimension, but Conn showed that the complex associated to this $\mathfrak{g}$-module shares the same properties as the finite dimensional $\mathfrak{g}$-modules, that we have consider in Proposition 3.4. More precisely, there exists a homotopy operator h for the truncated Chevalley-Eilenberg complex:

$$\mathcal{C}_r \otimes \wedge^1 \mathfrak{g}^* \xleftarrow{h} \mathcal{C}_r \otimes \wedge^2 \mathfrak{g}^* \xleftarrow{h} \mathcal{C}_r \otimes \wedge^3 \mathfrak{g}^*$$

such that

$$d \circ h + h \circ d = \mathrm{Id}_{\mathcal{C}_r \otimes \wedge^2 \mathfrak{g}^*},$$

and a positive constant C, which is independent of the radius r, such that

$$\|h(u)\|^H_{k,r} \leq C \|u\|^H_{k,r}, \quad \forall k \geq 0. \tag{3.7}$$

(the same property holds for every infinite direct sum of finite-dimensional orthogonal $\mathfrak{g}$-modules).

The homotopy operator is well-controlled by Sobolev norms. However, in order to control the differentiability at each step, we have to use the C^k-norms :

$$\|F\|_{k,r} := \sup_{|\alpha| \leq k} \sup_{z \in B_r} |D^\alpha F(z)| .$$

So applying Sobolev's lemma, the inequality (3.7) leads to the estimate:

$$\|h(u)\|_{k,r} \leq M \|u\|_{k+s,r} \tag{3.8}$$

where M is a positive constant and s a fix positive integer. So now we are facing a loss of differentiability: to control the C^k-differentiability of $h(u)$, we need to control the C^{k+s}-differentiability of u.

There is a standard way to get around difficulty using *smoothing operators* on $\mathcal{C}_r$: One can construct, a family of linear operators $S(t) = S_r(t)$ $(t > 1)$ from $\mathcal{C}_r$ into itself, satisfying the following properties:

(i) $\|S(t)f\|_{p,r} \leq C_{p,q} t^{(p-q)} \|f\|_{q,r}$ and,
(ii) $\|(I - S(t))f\|_{q,r} \leq C_{p,q} t^{(q-p)} \|f\|_{p,r}$.

for any $f \in \mathcal{C}_r$. Here p, q are any non-negative integers such that $p \geq q$, I denotes the identity map, and $C_{p,q}$ denotes a constant which depends on p and q. Note that (ii) means that $S(t)$ is close to identity and converges to the identity when $t \to \infty$. On the other hand, (i) means that f becomes "smoother" when we apply $S(t)$ to it.

We are now ready to proceed with the iteration. Assume we have constructed coordinates $(x^1_\nu, \cdots, x^n_\nu)$, so that the Poisson bracket satisfies:

$$\{x^i_\nu, x^j_\nu\} = c^{ij}_k x^k_\nu + O(|x_\nu|^{2^\nu}).$$

We want to construct the next iteration $(x^1_{\nu+1}, \cdots, x^n_{\nu+1})$. We consider the remainder term $R_\nu : \wedge^2 \mathfrak{g} \to \mathfrak{M}^{2^\nu}$ which is given by:

$$R_\nu(X_i \wedge X_j) = \{x^i_\nu, x^j_\nu\} - c^{ij}_k x^k_\nu.$$

Then R_ν is an almost cocycle: dR_ν is a quadratic function in R_ν i.e., if R_ν is "ε-small" then dR_ν is "ε^2-small". We define the next change of coordinates to be given by the diffeomorphism $\phi_{\nu+1}$ defined by

$$\phi_{\nu+1} = \mathrm{Id} - S(t_\nu)(h(R_\nu), \tag{3.9}$$

where the real numbers t_ν are defined by $t_{\nu+1} = t_\nu^{3/2}$ with $t_0 > 1$ (this choice of smoothing parameter is standard). Now we are left with the question of convergence, and here arrives the most technical part. Ones constructs a decreasing sequence of radii $\{r_\nu\}$, which converges to $r > 0$, such that the corrected Poisson bracket $\{\,,\,\}_\nu$ is well defined on B_{r_ν}, and such that the $\|R_\nu\|_{k,r_\nu}$ and the $\|\phi_\nu - Id\|_{k,r_\nu}$ converge to 0 exponentially fast. This is proved using some hard estimates that follow from (3.8) and the properties of the smoothing operators. □

Remark 3.7. This method of the proof is known as the "Nash-Moser method". A sketch of a more conceptual proof, using the Nash-Moser Inverse Function Theorem on tame Fréchet spaces (see [24]) is given by Desolneux-Moulis in [15]. However, this proof seems to be incomplete.

Geometric proof. In [13], Crainic and Fernandes propose a soft geometric proof of the theorem, along the lines of the infinitesimal version of Bochner's Theorem (Theorem 2.7). The proof has two major steps:

(i) Integrate (a neighborhood) of the Poisson manifold to a symplectic groupoid.
(ii) The resulting symplectic groupoid has compact isotropy group G_{x_0} so we can apply Theorem 3.1 to linearize it.

We refer to [13] for more details. Also, in [14] this result is interpreted in the context of deformation cohomology. □

Remark 3.8. It is natural to try and use the approach of Flato, Pinczon and Simon ([22]) mentioned in Remark 2.5, to simplify Conn's theorems. Unfortunately, so far, no such simplification was obtained (see the remarks in [36]).

As we have mentioned in the introduction, an important open problem is to characterize which Lie algebras are non-degenerate relative to Poisson structures. In dimension 2, there are two Lie algebras (up to isomorphism): the 2-dimensional abelian Lie algebra (as any abelian Lie algebra) is degenerate, while Arnold has shown in [1] that the non-abelian 2-dimensional Lie algebra is non-degenerate. The case of dimension 3 is solved completely by Dufour in [16], while a discussion of dimension 4 is given by Molinier in [28]. However, in general, this problem is completely open.

Remark 3.9. In the formal case, the (non)degeneracy of a Lie algebra should be related to its (non)rigidity (recall that $H^2(\mathfrak{g};\mathfrak{g}) = 0$ implies $\mathfrak{g}$ rigid). in [4], Bordemann, Makhlouf and Petit show that, if the universal enveloping algebra $\mathcal{U}(\mathfrak{l})$ is rigid as an associative algebra, then $\mathfrak{l}$ is formally non-degenerate.

4. Levi decomposition

A Poisson bracket defines a Lie algebra structure on $C^\infty(M)$. This infinite dimensional Lie algebra retains a finite dimensional flavor, due to the Leibniz identity:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Therefore, it is natural to seek a Levi decomposition for this Lie algebra. We study in this section this decomposition, which may also be seen as a semi-linearization generalizing the linearization results of the previous section.

As we saw in the previous sections, the linearization of Lie algebra actions involves the first Whitehead lemma, whereas the linearization of Poisson structures involves the second Whitehead lemma. We will see that the Levi decomposition of Poisson structures involves both of them so, somehow, it combines both the linearization of Lie algebra actions and the linearization of Poisson structures.

Let us first recall the Levi decomposition for finite dimensional Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and denote by $\mathfrak{r}$ its radical (i.e., its maximal solvable radical). The quotient Lie algebra $\mathfrak{g}/\mathfrak{r}$ is then semisimple and we can write the following exact sequence:

$$0 \longrightarrow \mathfrak{r} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{r} \longrightarrow 0\ .$$

The Levi-Malcev theorem states that this exact sequence of Lie algebras admits a splitting $\sigma : \mathfrak{g}/\mathfrak{r} \to \mathfrak{l}$. If we denote by $\mathfrak{s}$ the image $\sigma(\mathfrak{g}/\mathfrak{r})$, then we can write $\mathfrak{g}$ as a semi-direct product:

$$\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r},$$

which is called the **Levi decomposition** of $\mathfrak{g}$. In general, for infinite dimensional Lie algebras, there exists no Levi-Malcev decomposition. However, the Levi-Malcev decomposition does hold for filtered, pro-finite, Lie algebras (details are given in the upcoming book [19]).

Let us turn now to the case of a Poisson manifold. Let $\{\ ,\ \}$ be a Poisson bracket on a manifold M vanishing at $x_0 \in M$. Denote by $\mathfrak{g} = \mathfrak{g}_{x_0} = T^*_{x_0}M$ the isotropy Lie algebra at x_0. Now we choose $\mathfrak{s} \subset \mathfrak{g}$ to be:

- a Levi factor of $\mathfrak{g}$, in the formal or analytic case, or
- a maximal compact semisimple Lie subalgebra of $\mathfrak{g}$ (a *compact Levi factor*) in the smooth case.

As before, we denote by $\mathfrak{M}$ the maximal ideal formed by functions that vanish at x_0. The Poisson bracket defines a Lie algebra structure on $\mathfrak{M}$ and we let $\mathfrak{R} \subset \mathfrak{M}$ be defined by:

$$\mathfrak{R} = \{f \in \mathfrak{M} : d_{x_0}f \in \mathfrak{s}\}.$$

We should think of $\mathfrak{R}$ as the **radical** of the Lie algebra $\mathfrak{M}$. Now we have a short exact sequence of Lie algebras:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{R} \arrow[r] & \mathfrak{M} \arrow[r] & \mathfrak{s} \arrow[r] \arrow[l, dashed, bend left, "\sigma"] & 0 \end{tikzcd} \tag{4.1}$$

We shall call a splitting σ of this short exact sequence of Lie algebras a **Levi decomposition for the Poisson bracket**.

Notice that when $\mathfrak{g} = \mathfrak{s}$ we obtain the standard linearization problem for Poisson brackets that we have studied in the previous section.

Let us give the coordinate version of the Levi decomposition for Poisson brackets. Under the assumptions above we can choose a complement $\mathfrak{r}$ to $\mathfrak{s}$ in $\mathfrak{g}$, such that

$$[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{r}.$$

Therefore, we can choose coordinates $(x^1, \cdots, x^m, y^1, \cdots, y^{n-m})$ around x_0 such that $(d_{x_0}x^1, \cdots, d_{x_0}x^m)$ spans $\mathfrak{s}$ and $(d_{x_0}y^1, \cdots, d_{x_0}y^{n-m})$ spans $\mathfrak{r}$. In terms of the Poisson bracket this means that:

$$\begin{aligned}
\{x^i, x^j\} &= c^{ij}_k x^k + O(2),\\
\{x^i, y^j\} &= a^{ij}_k y^k + O(2),\\
\{y^i, y^j\} &= O(1).
\end{aligned}$$

Now the existence of a (formal, analytic, smooth) Levi decomposition for the Poisson brackets amounts to the existence of a (formal, analytic, smooth) coordinate system $(x^1_\infty, \cdots, x^m_\infty, y^1_\infty, \cdots, y^{n-m}_\infty)$, for which the higher order terms in the first and second pair of brackets vanish:

$$\begin{aligned}
\{x^i_\infty, x^j_\infty\} &= c^{ij}_k x^k_\infty,\\
\{x^i_\infty, y^j_\infty\} &= a^{ij}_k y^k_\infty,
\end{aligned}$$

(note that nothing is said about the third pair of brackets)

Now, exactly as before, we have the following results:

Theorem 4.1 (Wade [32])**.** *Any Poisson bracket that vanishes at x_0 admits a formal Levi decomposition.*

Theorem 4.2 (Zung [38])**.** *Any analytic Poisson bracket that vanishes at x_0 admits an analytic Levi decomposition*

Theorem 4.3 (Monnier and Zung [29])**.** *Any smooth Poisson structure that vanishes at x_0 admits a smooth Levi decomposition.*

If in the above theorems the full isotropy Lie algebra $\mathfrak{g}$ is semisimple (in the formal or analytic case) or compact semisimple (in the smooth case), we rediscover the linearization theorems given in the previous section. Therefore, it should be no surprise that the proofs of the Levi decompositions are higher order versions of the proofs of the linearization results for Poisson brackets.

For example, for the analytic proofs, one constructs a sequence of (formal, analytic, smooth) coordinates systems $(x^1_\nu,\cdots,x^m_\nu,y^1_\nu,\cdots,y^{n-m}_\nu)$ which converges (formal, analytic, smooth) to the coordinate system $(x^1_\infty,\cdots,x^m_\infty,y^1_\infty,\cdots,y^{n-m}_\infty)$. The additional complication is that, at each step, we have to deal with the $\{x^i,x^j\}$-terms and the $\{x^i,y^\alpha\}$-terms together. The algorithm to linearize the $\{x^i,x^j\}$-terms is the similar to the one described in Section 3, and involves the vanishing of the second Lie algebra cohomology of $\mathfrak{s}$. To linearize the $\{x^i,y^\alpha\}$-terms, one uses a procedure similar to the one use in Section 2, and which involves, now, the vanishing of the first Lie algebra cohomology of $\mathfrak{s}$. In the analytic case or the smooth case, the estimations get more involved.

Remark 4.4. At least in the formal case, instead of treating both $\{x^i,x^j\}$-terms and the $\{x^i,y^\alpha\}$-terms at each step, one can first linearize the $\{x^i,x^j\}$-terms. This leads to an action of $\mathfrak{s}$ on $\mathfrak{M}$ which can then be linearized, thus taking care of the $\{x^i,y^\alpha\}$-terms.

Example 4.5. *As we pointed out above, the Levi decomposition can be thought of as a semi-linearization and may be seen as a first step in the tentative of linearizing Poisson structures whose linear part does not satisfy the assumptions of the linearization results of Section 3.*

For example, let $\mathfrak{g}=\mathfrak{aff}(n)$ be the Lie algebra of affine transformations of $\mathbb{R}^n$. This Lie algebra has radical $\mathfrak{r}=\mathbb{R}\ltimes\mathbb{R}^n$, where 1 acts on $\mathbb{R}^n$ by the identity map, and it admits the Levi factor $\mathfrak{s}=\mathfrak{sl}(n)$. Given an analytic Poisson bracket Π with a fixed point x_0 and isotropy $\mathfrak{g}_{x_0}=\mathfrak{aff}(n)$, we can apply the Levi decomposition to construct semi-linearizing coordinates $x_1,\dots,x_{n^2-1},y_0,\dots,y_n$. In [18]*, it is shown that one can choose the coordinates so that one also has:*

$$\{y_0,x_i\}=0,\qquad \{y_0,y_j\}=y_j.$$

Denoting by $F_1(x),\dots,F_n(x)$ the elementary symmetric polynomials in $\mathfrak{gl}(n)$, with a bit more work, one can use these relations to show that there exist analytic functions $f_i=f_i(x)$ so that the vector field

$$Y=\sum_{i=1}^n f_i X_{F_i},$$

satisfies:

$$\mathcal{L}_Y(\Pi)=\Pi_0-\Pi.$$

where Π_0 is the linear part of Π. This is explain in detail in [18].

Now consider the path of Poisson structures:

$$\Pi_t(x,y)\equiv\frac{1}{t}\Pi(t(x,y)),$$

which joins Π to Π_0. The time-dependent vector field $Y_t(x,y)=\frac{1}{t^2}Y(t(x,y))$ satisfies

$$\mathcal{L}_{Y_t}\Pi_t=\frac{d\Pi_t}{dt},$$

so we conclude that the time-1 flow of Y_t is a Poisson diffeomorphism between Π_0 and Π defined in a neighborhood of x_0. In other words, $\mathfrak{aff}(n)$ is analytically non-degenerate.

5. Linearization and Levi decomposition of Lie algebroids

The similarity between the linearization of Lie algebra actions and Lie algebroids, that we have seen before, can be fully understood in the context of Lie algebroids. This was first observed by Weinstein in [36]. In this section, we will consider only the smooth case. The formal and the analytic case are similar.

Let $\pi : A \to M$ be a Lie algebroid with Lie bracket $[\ ,\ ]$ on $\Gamma(A)$ and anchor $\# : A \to TM$. For background on Lie algebroids we refer to the book by Cannas da Silva and Weinstein [7]. The splitting theorem for Lie algebroids (see [17, 21]) states that, for every $x_0 \in M$, there exists a neighborhood U, such that $A|_U$ is a direct sum of a Lie algebroid with constant rank anchor and a Lie algebroid whose anchor vanishes at x_0. So, henceforth, we assume that $x_0 \in M$ is a fixed point, i.e., $\#_{x_0} \equiv 0$.

If we choose a basis $\{e_1, \cdots, e_r\}$ of local sections of A and a system of coordinates $(x^1, \cdots, x^n)$ around x_0, we have:

$$[e_i, e_j] = (c_{ij}^k + O(1))e_k,$$
$$\#e_i = (b_{ij}^k x^j + O(2))\frac{\partial}{\partial x^k}.$$

where b_{ij}^k and c_{ij}^k are certain structure constants. So we have now the following **linearization problem for Lie algebroids**:

- Is there a choice of basis $\{e_1^\infty, \cdots, e_r^\infty\}$ and a choice of local coordinates $(x_\infty^1, \cdots, x_\infty^n)$ where the higher order terms vanish?

The linear part of the Lie algebroid defined by

$$[e_i, e_j] = c_{ij}^k e_k,$$
$$\#e_i = b_{ij}^k x^j \frac{\partial}{\partial x^k}.$$

is a special kind of Lie algebroid, called an **action Lie algebroid**. In fact, the first set of relations defines a Lie algebra $\mathfrak{g}_{x_0}$, called the isotropy Lie algebra at x_0, while the second set of relations defines a linear action of $\mathfrak{g}_{x_0}$ on $T_{x_0}M$. This Lie algebra and this action are, in fact, independent of the choice of basis and of the choice of coordinates. We recall that *any* Lie algebra action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ defines an action Lie algebroid: the bundle $A = M \times \mathfrak{g}$ is the trivial vector bundle with fiber $\mathfrak{g}$, the anchor $\# : A \to TM$ is given by $\#_x v = \rho(x) \cdot v$, and the Lie bracket is defined fiber-wise on constant sections, and extended to any sections so that it satisfies the Leibniz identity. The linear part is an example of a linear action Lie algebroid, which we denote by $A^L = \mathfrak{g} \ltimes T_{x_0}M$.

Note that if we eliminate the higher order terms in the Lie bracket, then the second set of equations defines a (non-linear) Lie algebra action of $\mathfrak{g}_{x_0}$. The remaining issue is to linearize the action, which, under good assumptions (semisimplicity, compact type) on the Lie algebra $\mathfrak{g}_{x_0}$, can be done using the results of Section 2.

Now, one has linearization results entirely analogous to what we have seen in the Lie algebra and Poisson cases. For example, in the smooth case we have:

Theorem 5.1 (Monnier and Zung [29]). *Any smooth Lie algebroid with a fixed point x_0 for which the isotropy $\mathfrak{g}_{x_0}$ is semisimple of compact type is smoothly linearizable.*

Similarly, if $\mathfrak{g}_{x_0}$ is semisimple, we have formal linearization (Weinstein [36], Dufour [17]) and analytic linearization (Zung [38]).

Example 5.2. *It is easy to see that, for any Lie algebra $\mathfrak{g}$, to linearize an action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ around a fixed point $x_0 \in M$ is equivalent to linearize the action Lie algebroid $\mathfrak{g} \ltimes_\rho TM$ around x_0. Similarly, for a manifold M, to linearize a Poisson bracket $\{\ ,\ \}$ around a zero $x_0 \in M$ one can show that is equivalent to linearize* *the cotangent Lie algebroid T^*M. Therefore, these results imply the linearization results of the previous sections.*

One can also look for a **Levi decomposition for Lie algebroids**. In the smooth case, we choose a decomposition

$$\mathfrak{g}_{x_0} = \mathfrak{s} + \mathfrak{r},$$

where $\mathfrak{s}$ is a compact Levi factor and $\mathfrak{r}$ is a subspace invariant under the adjoint action.

Theorem 5.3 (Monnier and Zung [29]). *Let A be a smooth Lie algebroid with anchor $\# : A \to TM$ with $\#_{x_0} = 0$, and fix a decomposition of the isotropy $\mathfrak{g}_{x_0} = \mathfrak{s} + \mathfrak{r}$, as above. Then, there exists a local basis of sections $(e^1_\infty, \cdots, e^m_\infty, f^1_\infty, \cdots, f^{r-m}_\infty)$ ($m = \dim \mathfrak{s}$) and a local system of coordinates $(x^1_\infty, \cdots, x^n_\infty)$ around x_0, such that:*

$$\begin{aligned}
[e_i^\infty, e_j^\infty] &= c_{ij}^k e_\infty^k, \\
[e_i^\infty, f_j^\infty] &= a_{ij}^k v_\infty^k, \\
\# e_i^\infty &= b_{ij}^k x_\infty^l \frac{\partial}{\partial x_\infty^k},
\end{aligned}$$

where $c_k^{ij}, a_k^{ij}, b_k^{ij}$ are constants, with c_k^{ij} being the structural constants of the compact semisimple Lie algebra $\mathfrak{s}$.

Similarly, if $\mathfrak{s}$ is semisimple, we have formal and analytic Levi decompositions (see [38] for details).

The proofs of these results can be reduced to the proofs of the corresponding statements for Poisson structures. Indeed, remind that any Lie algebroid A induces (and is, in fact, determined by) a fiber-wise linear Poisson structure on the dual bundle A^*. More precisely, if $(x^1, \cdots, x^n)$ is a local coordinate system and $(e_1, \cdots, e_r)$ is a local basis of sections, then we can think of $(x^1, \cdots, x^n, e_1, \cdots, e_r)$ has a coordinate system for A^*, which is linear on the fibers. The Poisson structure on A^* is given by

$$\begin{aligned}
\{e_i, e_j\} &= [e_i, e_j], \\
\{e_i, x^j\} &= \# e_i(x^j), \\
\{x^i, x^j\} &= 0.
\end{aligned}$$

This Poisson structure is fiber-wise linear in the following sense:

(i) the bracket of two fiber-wise linear functions is again a fiber-wise linear function,
(ii) the bracket of a fiber-wise linear function with a basis function is a basic function, and
(iii) the bracket of two basic functions is zero.

The proof of the theorems above consist in showing that the Levi decomposition for the fiber-wise linear Poisson structure on A^* yields the Levi decomposition of the Lie algebroid A, around x_0. The proof is similar to the Poisson case but one must modify the $\mathfrak{g}$-modules in order to preserve the “fiber-wise linear” feature.

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