FORMAL POISSON COHOMOLOGY OF QUADRATIC POISSON STRUCTURES

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Abstract. In this paper, we compute the formal Poisson cohomology of quadratic Poisson structures. We first recall that, generically, quadratic Poisson structures are diagonalizable. Then we compute the formal cohomology of diagonal Poisson structures.

Introduction

A *Poisson structure* on a manifold M is given by a 2-vector Π which satisfies

$$[\Pi,\Pi]=0$$

where $[\,,\,]:\mathcal{X}^a(M)\times\mathcal{X}^b(M)\longmapsto\mathcal{X}^{a+b}(M)$ is the Schouten bracket (see [9]). We recall that $\mathcal{X}^a(M)$ denotes the vector space of a-vectors on M, i.e. the space of sections of the vector bundle $\Lambda^a(TM)$. We will say that a Poisson structure Π on a vector space V is *quadratic* if, for any linear functions f and g, the function $\Pi(df\wedge dg)$ is a quadratic polynomial. Using coordinates $(x_1,\dots,x_n)$, this can be written

$$\Pi=\sum_{\substack{i<j\\ r\leq s}} a_{ij}^{rs}x_rx_s\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$$

where the a_{ij}^{rs} are constants.
Such structures have a particular application in mathematical physics. It is possible to construct quadratic Poisson structures from a solution of the classical Yang-Baxter equation (see [16]). Some informations on the quantization of quadratic Poisson structures may be found in [5], [11] and [14]. Some classifications in low dimension of quadratic Poisson structures have been established, for instance, in [1], [3] and [10]. Finally, for a Poisson structure which has a zero 1-jet at a point, the problem of "quadratization" (i.e. of finding a coordinate system in which the expression of the Poisson structure is quadratic) arises naturally (see [2] and [8]).

The Poisson cohomology of a Poisson structure was introduced by Lichnerowicz in [9]. It is constructed as follows. If (M,Π) is a Poisson manifold, we consider the linear maps ∂^k

$$\dots\longrightarrow\mathcal{X}^{k-1}(M)\xrightarrow{\partial^{k-1}}\mathcal{X}^k(M)\xrightarrow{\partial^k}\mathcal{X}^{k+1}(M)\longrightarrow\dots$$

defined by $\partial^k(A)=[A,\Pi]$ ($[\,,\,]$ indicates the Schouten bracket). It can be shown that $\partial^k\circ\partial^{k-1}=0$. The induced cohomology spaces $H^\bullet(M,\Pi)$ are the *Poisson cohomology* spaces of (M,Π).
These cohomology spaces are invariants of the Poisson structure and they have

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applications for instance in problems of deformation of the structure. The main feature of this cohomology is that it is particularly difficult to compute. Among the publications on this subject, the explicit results are scarce. It is fairly easy to see that if the Poisson structure is "symplectic" then its cohomology is isomorphic to the de Rham cohomology. The case of regular Poisson structures has been studied, for instance, by P. Xu ([21]) and I. Vaisman ([17], [18]). In [7], V. Ginzburg and A. Weinstein consider the Poisson cohomology of Lie-Poisson groups. Finally, in dimension two, the Poisson cohomology has been studied and computed in [12], [13] and [15].

In this paper, the aim is to compute the *formal* Poisson cohomology of quadratic Poisson structures, which means that we work with formal k-vectors instead of smooth or analytic ones. The 2-dimensional case has already been studied by N. Nakanishi in [13]. Here, we work in $\mathbb{R}^n$ (the results can be extended to $\mathbb{C}^n$) with $n > 2$.

In the first section, we recall that under a hypothesis of genericity, a quadratic Poisson structure is diagonalizable. Then, in the following two sections, we compute the Poisson cohomology of diagonal Poisson structures. We first study the 3-dimensional case (section 2). In this situation, the diagonal Poisson structures may be interpreted in terms of the geometry of $\mathbb{R}^3$. The computation of the cohomology is then reduced to an elementary problem of geometry. Moreover, we can make the cohomology spaces explicit in a relatively clear way.

Finally, in the last section, we generalize to higher dimensions.

We note that some informations on the space $H^2(\Pi)$ have been given in [4] by J.-P. Dufour and A. Wade in order to study normal forms of Poisson structures.

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1. Diagonalizable Poisson structures

We first recall the definition of the curl vector fields of an oriented Poisson manifold (M, Π). This notion of curl has been defined in [3] (also in [20] under the name of *modular vector field*) in order to classify quadratic Poisson structures in dimension 3.

Let ν be a volume form on M. We denote by $\nu^\flat$ the isomorphism $\mathcal{X}^p(M) \longrightarrow \Omega^{n-p}(M)$ (where $\Omega^{n-p}(M)$ is the vector space of the $(n-p)$-forms on M) with $\nu^\flat(u) = i_u\nu$ (the contraction of ν by u). The *curl* of Π (with respect to ν) is the vector field $D_\nu\Pi = (\nu^\flat)^{-1} \circ d \circ \nu^\flat(\Pi)$.

If Π is a quadratic Poisson structure on a vector space V, its curl (with respect to ν) is then a linear vector field whose trace is zero. Moreover, the Jordan decomposition of $D_\nu\Pi$ is an invariant of Π. Consequently, we can define the eigenvalues of Π as the eigenvalues of its curl (with respect to any volume form).

We will say that a quadratic Poisson structure Π on $\mathbb{R}^n$ is *diagonalizable* if there exists a coordinate system $(x_1, \ldots, x_n)$ in which Π can be written as

$$\Pi = \sum_{i<j} a_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

where the a_{ij} are constants.
Diagonal Poisson structures play a part, for instance, in some integrable systems (see [6]). Actually, quadratic Poisson structures are generically diagonalizable; more precisely :

Theorem 1.1. **[3]** *If the eigenvalues λ_i of a quadratic Poisson structure Π do not satisfy relations of type*

$$\lambda_i + \lambda_j = \lambda_r + \lambda_s \quad (*)$$

with $r \neq s$ and $\{i, j\} \neq \{r, s\}$, then the Poisson structure Π is diagonalizable.

Remark 1.2. In [19], A. Wade showed (it is not obvious) that if a quadratic Poisson structure is diagonalizable, then its diagonal form is unique up to a permutation of the coordinates.

2. Computation of the cohomology in dimension 3

In this section, we work in $\mathbb{R}^3$ with the coordinates (x, y, z).
We will use the following notation (already introduced in [4]):

$$X = x\frac{\partial}{\partial x} \qquad Y = y\frac{\partial}{\partial y} \qquad Z = z\frac{\partial}{\partial z}\,.$$

We consider a diagonal Poisson structure Π on $\mathbb{R}^3$ of the form

$$\Pi = aY \wedge Z + bZ \wedge X + cX \wedge Y$$

where a, b and c are in $\mathbb{R}$.

2.1. **Notation.** We are going to adopt the following notation:
$\mathcal{X}^0(\mathbb{R}^3)$ is the vector space of formal series (i.e. $\mathbb{R}[[x, y, z]]$),
$\mathcal{X}^1(\mathbb{R}^3)$ is the vector space of formal vector fields on $\mathbb{R}^3$,
$\mathcal{X}^2(\mathbb{R}^3)$ is the vector space of formal 2-vectors on $\mathbb{R}^3$,
$\mathcal{X}^3(\mathbb{R}^3)$ is the vector space of formal 3-vectors on $\mathbb{R}^3$.

Let us express the elements in these spaces in terms of X, Y and Z, rather than $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ (allowing expressions such as $\frac{\partial}{\partial x} = x^{-1}X$).
Remark : In this paper, $\mathbb{N}$ is the set of non-negative integers.

- Every element f in $\mathcal{X}^0(\mathbb{R}^3)$ may be written as

$$f = \sum_{I\in\mathbb{N}^3} \lambda_I f^{(I)}$$

where the λ_I are in $\mathbb{R}$ and $f^{(I)} = x^{i_1}y^{i_2}z^{i_3}$ if $I = (i_1, i_2, i_3)$.

- In the same way, every element V in $\mathcal{X}^1(\mathbb{R}^3)$ may be written as

$$V = \sum_{I\in(\mathbb{N}\cup\{-1\})^3} f^{(I)} V_I$$

with $f^{(I)} = x^{i_1}y^{i_2}z^{i_3}$ $(I = (i_1, i_2, i_3))$ and $V_I = \alpha_I X + \beta_I Y + \gamma_I Z$ where α_I, β_I, and γ_I are real numbers which are zero if I has two or three negative components. If I has exactly one negative component then two real numbers among α_I, β_I and γ_I are zero; for instance, if $I = (-1, i_2, i_3)$ then $\beta_I = \gamma_I = 0$.

• Every element Λ in $\mathcal{X}^2(\mathbb{R}^3)$ may be written as

$$\Lambda = \sum_{I\in(\mathbb{N}\cup\{-1\})^3} f^{(I)}\Lambda_I$$

with $\Lambda_I = \alpha_I Y\wedge Z + \beta_I Z\wedge X + \gamma_I X\wedge Y$ where α_I, β_I, and γ_I are real numbers which are zero if I has three negative components. If I has exactly two negative components then two real numbers among α_I, β_I and γ_I are zero; for instance, if $I=(-1,-1,i_3)$ then $\alpha_I=\beta_I=0$. If I has exactly one negative component then one real number among α_I, β_I and γ_I is zero; for instance, if $I=(-1,i_2,i_3)$ then $\alpha_I=0$.

• Every element Γ of $\mathcal{X}^3(\mathbb{R}^3)$ may be written as

$$\Gamma = \sum_{I\in(\mathbb{N}\cup\{-1\})^3} \lambda_I f^{(I)} X\wedge Y\wedge Z$$

where λ_I are in $\mathbb{R}$ and $f^{(I)} = x^{i_1}y^{i_2}z^{i_3}$ $(I=(i_1,i_2,i_3))$.

We can then set, for $I\in(\mathbb{N}\cup\{-1\})^3$,

$$\begin{aligned}
\mathcal{X}^0_I(\mathbb{R}^3) &= \{\lambda f^{(I)}\,;\,\lambda\in\mathbb{R}\}\\
\mathcal{X}^1_I(\mathbb{R}^3) &= \{f^{(I)}(\alpha X+\beta Y+\gamma Z)\,;\,\alpha,\beta,\gamma\in\mathbb{R}\}\\
\mathcal{X}^2_I(\mathbb{R}^3) &= \{f^{(I)}(\alpha Y\wedge Z+\beta Z\wedge X+\gamma X\wedge Y)\,;\,\alpha,\beta,\gamma\in\mathbb{R}\}\\
\mathcal{X}^3_I(\mathbb{R}^3) &= \{\lambda f^{(I)} X\wedge Y\wedge Z\,;\,\lambda\in\mathbb{R}\}
\end{aligned}$$

with the convention
$\mathcal{X}^0_I(\mathbb{R}^3)=\{0\}$ if I has at least one negative component
$\mathcal{X}^1_I(\mathbb{R}^3)=\{0\}$ if I has at least two negative components
$\mathcal{X}^2_I(\mathbb{R}^3)=\{0\}$ if $I=(-1,-1,-1)$.

Remark 2.1. It is important to note that, unless the $\mathcal{X}^j_I$ vanish, we can identify in an obvious way $\mathcal{X}^0_I$ with $\mathbb{R}$, $\mathcal{X}^3_I$ with $\mathbb{R}$, $\mathcal{X}^1_I$ with a subspace E^1_I of $\mathbb{R}^3$ and $\mathcal{X}^2_I$ with a subspace E^2_I of $\mathbb{R}^3$, where $E^1_I=\mathbb{R}^3$ and $E^2_I=\mathbb{R}^3$ if $I\in\mathbb{N}^3$ and, for instance,

$$\begin{aligned}
E^1_{(-1,i_2,i_3)} &= \{(\alpha,0,0)\in\mathbb{R}^3;\ \alpha\in\mathbb{R}\}\\
E^2_{(-1,i_2,i_3)} &= \{(0,\beta,\gamma)\in\mathbb{R}^3;\ \beta,\gamma\in\mathbb{R}\}\\
E^2_{(-1,-1,i_3)} &= \{(0,0,\gamma)\in\mathbb{R}^3;\ \gamma\in\mathbb{R}\}
\end{aligned}$$

2.2. **Description of the Poisson complex.** In our case, the complex defining the Poisson cohomology of Π is

$$0\longrightarrow \mathcal{X}^0(\mathbb{R}^3)\xrightarrow{\partial^0}\mathcal{X}^1(\mathbb{R}^3)\xrightarrow{\partial^1}\mathcal{X}^2(\mathbb{R}^3)\xrightarrow{\partial^2}\mathcal{X}^3(\mathbb{R}^3)\longrightarrow 0$$

with, for $T\in\mathcal{X}^i$, $\partial^i(T)=[T,\Pi]$ ([,] indicates the Schouten bracket).

i) <u>Computation of $\partial^0(\mathcal{X}^0_I(\mathbb{R}^3))$:</u>
Take $I\in\mathbb{N}^3$ and $f^{(I)}=x^{i_1}y^{i_2}z^{i_3}\in\mathcal{X}^0_I(\mathbb{R}^3)$. A short calculation gives

$$\partial^0(f^{(I)}) = f^{(I)}\big((ci_2-bi_3)X+(ai_3-ci_1)Y+(bi_1-ai_2)Z\big)\quad (*_0)\,.$$

ii) Computation of $\partial^1(\mathcal{X}^1_I(\mathbb{R}^3))$:
Take $V = f^{(I)} V_I \in \mathcal{X}^1_I(\mathbb{K}^3)$ with $V_I = \alpha X + \beta Y + \gamma Z$.
Let us suppose that $I \in \mathbb{N}^3$: we then have

$$\begin{aligned} \partial^1(V) &= [f^{(I)} V_I, \Pi] \\ &= -[\Pi, f^{(I)}] \wedge V_I - f^{(I)}[\Pi, V_I] \\ &= -\partial^0(f^{(I)}) \wedge V_I + 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \partial^1(V) = \; & f^{(I)}\big(\beta(bi_1 - ai_2) - \gamma(ai_3 - ci_1)\big) Y \wedge Z \\ & + f^{(I)}\big(\gamma(ci_2 - bi_3) - \alpha(bi_1 - ai_2)\big) Z \wedge X \\ & + f^{(I)}\big(\alpha(ai_3 - ci_1) - \beta(ci_2 - bi_3)\big) X \wedge Y \quad (*_1)\,. \end{aligned}$$

Now, if we assume, for instance, that $I = (-1, i_2, i_3)$ with $i_2, i_3 \in \mathbb{N}$, then we can write $V = \alpha f^{(I)} X = \alpha g \frac{\partial}{\partial x}$ where $g = y^{i_2} z^{i_3}$. Consequently, it is possible to show that

$$\partial^1(V) = f^{(I)}\big(\alpha(ai_2 - b(-1)) Z \wedge X + \alpha(ai_3 - c(-1)) X \wedge Y\big)$$

which is the same expression as $(*_1)$ with $\beta = \gamma = 0$.

iii) Computation of $\partial^2(\mathcal{X}^2_I(\mathbb{R}^3))$:
Take $\Lambda = f^{(I)} \Lambda_I \in \mathcal{X}^2_I(\mathbb{R}^3)$ with $\Lambda_I = \alpha Y \wedge Z + \beta Z \wedge X + \gamma X \wedge Y$.
We first suppose that $I \in \mathbb{N}^3$: we then have

$$\partial^2(\Lambda) = [\Pi, f^{(I)}] \wedge \Lambda_I + f^{(I)}[\Pi, \Lambda_I]$$

which implies that

$$\partial^2(\Lambda) = f^{(I)}\big(\alpha(ci_2 - bi_3) + \beta(ai_3 - ci_1) + \gamma(bi_1 - ai_2)\big) X \wedge Y \wedge Z \quad (*_2)\,.$$

Now, if we suppose, for instance, that $I = (-1, i_2, i_3)$ with $i_2, i_3 \in \mathbb{N}$, then we can write $\Lambda = f^{(I)}(\beta Z \wedge X + \gamma X \wedge Y)$ and it is possible to show that

$$\partial^2(\Lambda) = f^{(I)}\big(\beta(ai_3 - c(-1)) + \gamma(b(-1) - ai_2)\big) X \wedge Y \wedge Z\,,$$

which is the same expression as $(*_2)$ with $\alpha = 0$.

Finally, if we suppose that $I = (-1, -1, i_3)$ with $i_3 \in \mathbb{N}$, we can write Λ as $\Lambda = \gamma f^{(I)} X \wedge Y$ and, in the same way, it is possible to show that

$$\partial^2(\Lambda) = f^{(I)}\big(\gamma(b(-1) - a(-1))\big) X \wedge Y \wedge Z\,,$$

which is the same expression as $(*_2)$ with $\alpha = \beta = 0$.

2.3. **Computation of the cohomology.** It follows from the previous section that the computation of our cohomology may be done "degree by degree", that is to say that it is sufficient to study, for each $I \in (\mathbb{N} \cup \{-1\})^3$, the cohomology of the complex

$$0 \longrightarrow \mathcal{X}^0_I(\mathbb{R}^3) \xrightarrow{\partial^0} \mathcal{X}^1_I(\mathbb{R}^3) \xrightarrow{\partial^1} \mathcal{X}^2_I(\mathbb{R}^3) \xrightarrow{\partial^2} \mathcal{X}^3_I(\mathbb{R}^3) \longrightarrow 0$$

Now, using the identifications made in remark 2.1, we see that the computation is reduced to an elementary problem of geometry in $\mathbb{R}^3$. Indeed, if we denote by P

the vector of $\mathbb{R}^3$ of coordinates (a,b,c), our problem reduces to the study of the complex $(\mathcal{K}_I)$

$$0 \longrightarrow \mathbb{R} \xrightarrow{\delta_I^0} E_I^1 \xrightarrow{\delta_I^1} E_I^2 \xrightarrow{\delta_I^2} \mathbb{R} \longrightarrow 0 \qquad (\mathcal{K}_I)$$

with

$$\begin{aligned} \delta_I^0(\lambda) &= \lambda I \times P \\ \delta_I^1(V) &= V \times (I \times P) \\ \delta_I^2(W) &= W.(I \times P) \end{aligned}$$

where $\times$ indicates the cross product on $\mathbb{R}^3$ and $.$ is the dot product on $\mathbb{R}^3$.
We will denote by $H_I^0(\Pi)$, $H_I^1(\Pi)$, $H_I^2(\Pi)$ and $H_I^3(\Pi)$ the cohomology spaces of this complex.
The following proposition is clear

Proposition 2.2. *If $I \times P = 0$ then we have $H_I^0(\Pi) \simeq \mathbb{R}$, $H_I^1(\Pi) \simeq E_I^1$, $H_I^2(\Pi) \simeq E_I^2$ and $H_I^3(\Pi) \simeq \mathbb{R}$.*

In the sequel, we assume that $\underline{I \times P \neq 0}$.
We clearly have $\underline{H_I^0(\Pi) = \{0\}}$. The computation of the spaces $H_I^1(\Pi)$, $H_I^2(\Pi)$ and $H_I^3(\Pi)$ depends on the vector I, more precisely, on the number of negative components of I. We are going to distinguish three cases.

First case : We suppose that $I \in \mathbb{N}^3$.
• Let V be in E_I^1 with $\delta_I^1(V) = 0$. Since $V \times (I \times P) = 0$, the vectors V and $I \times P$ are collinear, i.e. there exists a real number λ such that $V = \lambda I \times P$. Consequently, $V = \delta_I^0(\lambda)$.
We deduce that $\underline{H_I^1(\Pi) = \{0\}}$.

• Let W be in E_I^2 with $\delta_I^2(W) = 0$. Since $W.(I \times P) = 0$, the vectors W and $I \times P$ are orthogonal. Therefore, the vectors W, I and P are coplanar.
If we put $V = \frac{-1}{||I \times P||^2} W \times (I \times P)$, we get $\delta_I^1(V) = W$.
We deduce that $\underline{H_I^2(\Pi) = \{0\}}$.

• Finally, it is clear that $\underline{H_I^3(\Pi) = \{0\}}$.

Second case : We suppose, for instance, that $I = (-1, i_2, i_3)$ with $i_2, i_3 \in \mathbb{N}$.
The complex is then reduced to

$$0 \longrightarrow E_I^1 \xrightarrow{\delta_I^1} E_I^2 \xrightarrow{\delta_I^2} \mathbb{R} \longrightarrow 0$$

with

$$\begin{aligned} E_I^1 &= \{(\alpha, 0, 0) \in \mathbb{R}^3;\ \alpha \in \mathbb{R}\} \\ E_I^2 &= \{(0, \beta, \gamma) \in \mathbb{R}^3;\ \beta, \gamma \in \mathbb{R}\} \end{aligned}$$

• Let V in E_I^1 be such that $\delta_I^1(V) = 0$. We suppose that $V \neq 0$. Since V is collinear with $I \times P$, it is in particular orthogonal to I. Consequently, the vector I is in the plane which is orthogonal to V. This plane is $V^\perp = \{(0, \beta, \gamma);\ \beta, \gamma \in \mathbb{R}\}$. The vector I cannot be in this plane. Therefore, $V = 0$.

We deduce that $\underline{H^1_I(\Pi) = \{0\}}$.

• Let W in $E^2_I \setminus \{0\}$ be such that $\delta^2_I(W) = 0$. The vector W is then orthogonal to $I \times P$ and so the vectors W, I and P are coplanar. Now, if we denote by $Vect(I, P)$ the plane spanned by I and P, since $E^2_I \neq Vect(I, P)$ ($I \notin E^2_I$), the vector W is a generator of the line $E^2_I \cap Vect(I, P)$.
Now, we consider $V \in E^1_I$ such that $\delta^1_I(V) = V \times (I \times P) \neq 0$. Such a vector exists because I is not orthogonal to E^1_I. Since the vectors $V \times (I \times P)$ and $I \times P$ are orthogonal, the vector $V \times (I \times P)$ is in $Vect(I, P)$. On the other hand, we have $V \times (I \times P) \in (E^1_I)^{\perp} = E^2_I$. Consequently, the vector $V \times (I \times P)$ is in $E^2_I \cap Vect(I, P)$ and is different from zero. Therefore, it is possible to find a real number λ such that $\delta^1_I(\lambda V) = W$.
We deduce that $\underline{H^2_I(\Pi) = \{0\}}$.

• Finally, we have $\delta^2_I(E^2_I) = \{0\}$ or $\mathbb{R}$. If $\delta^2_I(W) = 0$ for every $W \in E^2_I$, then the vector $I \times P$ is orthogonal to E^2_I i.e. $E^2_I = Vect(I, P)$. Thus I is in E^2_I which is false. Consequently, $\delta^2_I(E^2_I) = \mathbb{R}$ which implies that $\underline{H^3_I(\Pi) = \{0\}}$.

Third case : We suppose, for instance, that $I = (-1, -1, i_3)$ with $i_3 \in \mathbb{N}$.
The complex is then reduced to

$$0 \longrightarrow E^2_I \xrightarrow{\delta^2_I} \mathbb{R} \longrightarrow 0$$

with

$$E^2_I = \{(0, 0, \gamma) \in \mathbb{R}^3;\, \gamma \in \mathbb{R}\}\,.$$

Here, it is sufficient to work with the formula $(*_2)$.
If $b \neq a$ then it is clear that $\underline{H^2_I(\Pi) = \{0\}}$ and $\underline{H^3_I(\Pi) = \{0\}}$.
If $b = a$ then we see that $\underline{H^2_I(\Pi) \simeq \mathbb{R}\,.\big(z^{i_3}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\big)}$ and $\underline{H^3_I(\Pi) \simeq \mathbb{R}\,.\big(z^{i_3}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge Z\big)}$.

Fourth case : We suppose that $I = (-1, -1, -1)$.
In this case, it is clear that $\underline{H^3_I(\Pi) \simeq \mathbb{R}.\big(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\big)}$.

We can now sum up these results in the following proposition.

Proposition 2.3. *We suppose that $I \times P \neq 0$.*
1- If I has at most 1 negative component then the complex $\mathcal{K}_I$ is acyclic.
2- If I has 2 negative components, for instance $I = (-1, -1, i_3)$, then
if $a \neq b$, the complex $\mathcal{K}_I$ is acyclic.
if $a = b$, we have $H^1_I(\Pi) = \{0\}$, $H^2_I(\Pi) \simeq \mathbb{R}\,.\big(z^{i_3}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\big)$ and $H^3_I(\Pi) \simeq \mathbb{R}\,.\big(z^{i_3}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge Z\big)$. 3- If $I = (-1, -1, -1)$ then $H^k_I(\Pi) = \{0\}$ for $k < 3$ and $H^3_I(\Pi) \simeq \mathbb{R}.\big(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\big)$.

We deduce the cohomology of Π in the case when a, b and c are pairwise distinct.

Corollary 2.4. *Let Π be a Poisson structure on $\mathbb{R}^3$ of type*

$$\Pi = aY \wedge Z + bZ \wedge X + cX \wedge Y\,,$$

with a, b and c pairwise distinct.
Then the formal cohomology spaces of Π *are*

$$H^0(\Pi) \simeq \left\{ \sum_{\substack{I\in\mathbb{N}^3\\ I\times P=0}} \lambda_I x^{i_1}y^{i_2}z^{i_3} \ ; \ \lambda_I \ \textit{real numbers} \right\}$$

$$H^1(\Pi) \simeq \left\{ \sum_{\substack{I\in(\mathbb{N}\cup\{-1\})^3\\ I\times P=0\\ I \text{ has at most}\\ 1 \text{ negative component}}} x^{i_1}y^{i_2}z^{i_3}(\alpha_I X+\beta_I Y+\gamma_I Z) \ ; \ \alpha_I,\beta_I,\gamma_I \ \textit{real numbers} \right\}$$

$$H^2(\Pi) \simeq \left\{ \sum_{\substack{I\in(\mathbb{N}\cup\{-1\})^3\\ I\times P=0\\ I \text{ has at most}\\ 2 \text{ negative components}}} x^{i_1}y^{i_2}z^{i_3}(\alpha_I Y\wedge Z+\beta_I Z\wedge X+\gamma_I X\wedge Y) \ ; \ \alpha_I,\beta_I,\gamma_I \ \textit{real numbers} \right\}$$

$$H^3(\Pi) \simeq \left\{ \sum_{\substack{I\in(\mathbb{N}\cup\{-1\})^3\\ I\times P=0}} \lambda_I x^{i_1}y^{i_2}z^{i_3} X\wedge Y\wedge Z \ ; \ \alpha_I \ \textit{real numbers} \right\} \oplus \mathbb{R}\,.\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\,.$$

Let us note that the relation $I\times P=0$ means that the vectors I and P are collinear, which implies the existence of a real number ξ such that $(\xi a,\xi b,\xi c)\in(\mathbb{N}\cup\{-1\})^3$.

2.4. **Examples.** In the examples *i)* and *ii)*, we describe the cohomology spaces in the case when a, b and c are not pairwise distinct. The third example is just an illustration of the corollary 2.4.

i) We suppose that $\Pi = yz\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+xz\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}+xy\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$.
We have $P=(1,1,1)$; therefore, for I in $(\mathbb{N}\cup\{-1\})^3$, the relation $I\times P=0$ is equivalent to the statement that $I=(k,k,k)$ with k in $\mathbb{N}\cup\{-1\}$.
We deduce that

$$\begin{aligned}
H^0(\Pi) &\simeq \mathbb{R}[[(xyz)]]\\
H^1(\Pi) &\simeq \mathbb{R}[[(xyz)]].X\oplus\mathbb{R}[[(xyz)]].Y\oplus\mathbb{R}[[(xyz)]].Z\\
H^2(\Pi) &\simeq \mathbb{R}[[(xyz)]].Y\wedge Z\oplus\mathbb{R}[[(xyz)]].Z\wedge X\oplus\mathbb{R}[[(xyz)]].X\wedge Y\\
&\quad \oplus\mathbb{R}[[x]].\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\oplus\mathbb{R}[[y]].\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}\oplus\mathbb{R}[[z]].\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\\
H^3(\Pi) &\simeq \mathbb{R}[[(xyz)]].\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\oplus\mathbb{R}[[x]].X\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\\
&\quad \oplus\mathbb{R}[[y]].\frac{\partial}{\partial x}\wedge Y\wedge\frac{\partial}{\partial z}\oplus\mathbb{R}[[z]].\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge Z
\end{aligned}$$

ii) We suppose that $\Pi = yz\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+xz\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}-2xy\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$.
Since $P=(1,1,-2)$, the sets I in $(\mathbb{N}\cup\{-1\})^3$ satisfying the relation $I\times P=0$ are $(-1,-1,2)$ and $(0,0,0)$.

We deduce that

$$\begin{array}{rcl} H^0(\Pi) & \simeq & \mathbb{R} \\ H^1(\Pi) & \simeq & \mathbb{R}.X \oplus \mathbb{R}.Y \oplus \mathbb{R}.Z \\ H^2(\Pi) & \simeq & \mathbb{R}.Y\wedge Z \oplus \mathbb{R}.Z\wedge X \oplus \mathbb{R}.X\wedge Y \oplus \mathbb{R}[[z]].\dfrac{\partial}{\partial x}\wedge\dfrac{\partial}{\partial y} \\ H^3(\Pi) & \simeq & \mathbb{R}[[z]].\dfrac{\partial}{\partial x}\wedge\dfrac{\partial}{\partial y}\wedge\dfrac{\partial}{\partial z} \oplus \mathbb{R}.X\wedge Y\wedge Z \end{array}$$

iii) We suppose that $\Pi = ayz\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + xz\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} - xy\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$ with a a nonrational real number. We have $P = (a, 1, -1)$. Let I in $(\mathbb{N}\cup\{-1\})^3$ be such that $I \times P = 0$. Therefore there exists a real number λ satisfying $\lambda P \in \mathbb{Z}^3$, which is possible only when λ is zero. Consequently, $I = (0, 0, 0)$.
We deduce that

$$\begin{array}{rcl} H^0(\Pi) & \simeq & \mathbb{R} \\ H^1(\Pi) & \simeq & \mathbb{R}.X \oplus \mathbb{R}.Y \oplus \mathbb{R}.Z \\ H^2(\Pi) & \simeq & \mathbb{R}.Y\wedge Z \oplus \mathbb{R}.Z\wedge X \oplus \mathbb{R}.X\wedge Y \\ H^3(\Pi) & \simeq & \mathbb{R}.\dfrac{\partial}{\partial x}\wedge\dfrac{\partial}{\partial y}\wedge\dfrac{\partial}{\partial z} \oplus \mathbb{R}.X\wedge Y\wedge Z \end{array}$$

Remark 2.5. Unfortunately, the cohomology spaces do not enable us to distinguish the diagonal Poisson structures, up to isomorphism. Indeed, we can consider the Poisson structure Π defined in example *iii)* above, and the Poisson structure Λ given by

$$\Lambda = byz\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + xz\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x} - xy\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\,,$$

where b is an nonrational real number different from a and $-a$.
The Poisson structures Π and Λ have cohomologies which are isomorphic. However, they are not equivalent because their curls are not isomorphic.

3. Generalization

Of course, we can recover the results of the previous section by setting $n = 3$ in the results of this section. The purpose of Section 2 was to clarify the geometrical meaning of the diagonal Poisson structures and of their cohomology.

For each k in $\{1, \ldots, n\}$, let Y_k denote the vector field $x_k\frac{\partial}{\partial x_k}$. We adopt the convention $x_k^{-1}Y_k = \frac{\partial}{\partial x_k}$.
Consider a diagonal Poisson structure Π on $\mathbb{R}^n$ $(n \geq 3)$, written

$$\Pi = \sum_{i<j} a_{ij}x_ix_j\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_i} = \sum_{i<j} a_{ij}Y_i\wedge Y_j\,.$$

The advantage of using the vector fields Y_k is that, in some computations, we can consider that we are working on the exterior algebra of a vector space spanned by $Y_1, \ldots, Y_n$.

In the same way as in the previous section, we denote by $\mathcal{X}^r(\mathbb{R}^n)$ the vector space of formal r-vectors on $\mathbb{R}^n$ and, for $I=(i_1,\ldots,i_n)\in(\mathbb{N}\cup\{-1\})^n$,

$$\mathcal{X}^r_I(\mathbb{R}^n)=\{x^I\sum_{\substack{K\in\mathbb{N}^r\\ k_1<\ldots<k_r}}\lambda_K Y_{k_1}\wedge\ldots\wedge Y_{k_r}\}\,.$$

where $x^I=x^{i_1}\ldots x^{i_n}$.
If I has $r+1$ or more negative components, we set $\mathcal{X}^r_I(\mathbb{R}^n)=\{0\}$.
As above, we have

$$\mathcal{X}^r(\mathbb{R}^n)=\oplus_{I\in(\mathbb{N}\cup\{-1\})^n}\mathcal{X}^r_I(\mathbb{R}^n)\,.$$

For I in $(\mathbb{N}\cup\{-1\})^n$, we denote by $A.I$ the vector field

$$A.I=\sum_{j=1}^n(\sum_{k=1}^n a_{jk}i_k)Y_j=\sum_{j=1}^n\alpha_j Y_j$$

with $\alpha_j=\sum_{k=1}^n a_{jk}i_k$.

3.1. **Description of the cobord operator.** Let I be in $\mathbb{N}^n$ and take Λ in $\mathcal{X}^r_I(\mathbb{R}^n)$ of type $x^I\Lambda_I$ with

$$\begin{aligned} x^I &= x^{i_1}\ldots x^{i_n}\\ \Lambda_I &= \sum_{\substack{K\in\mathbb{N}^r\\ k_1<\ldots<k_r}}\lambda_K Y_{k_1}\wedge\ldots\wedge Y_{k_r}\end{aligned}$$

We have

$$\begin{aligned}{}[\Lambda,\Pi] &= (-1)^r[\Pi,x^I]\wedge\Lambda_I+(-1)^r x^I[\Pi,\Lambda_I]\\ &= (-1)^r x^I\big(\sum_{u<v}a_{uv}(i_vY_u-i_uY_v)\big)\wedge\Lambda_I+0\\ &= (-1)^r x^I\big(\sum_{u=1}^n(\sum_{v=1}^n a_{uv}i_v)Y_u\big)\wedge\Lambda_I\end{aligned}$$

Consequently,

$$\partial^r(\Lambda)=[\Lambda,\Pi]=x^I\Lambda_I\wedge(A.I)=\underline{\Lambda\wedge(A.I)}\,.$$

In the same way, it is possible to show that if I has negative components, we obtain the same expression.

We note, as in the previous section, that the computation of the cohomology can be done "degree by degree" i.e. we need only study the complex

$$0\longrightarrow\mathcal{X}^0_I(\mathbb{R}^n)\longrightarrow\ldots\longrightarrow\mathcal{X}^n_I(\mathbb{R}^n)\longrightarrow 0$$

for each $I\in(\mathbb{N}\cup\{-1\})^n$. If we denote by $H^r_I(\Pi)$ the cohomology spaces of these complexes, we will have $H^r(\Pi)=\oplus_I H^r_I(\Pi)$.

3.2. **The computation of the cohomology.** Let I be in $(\mathbb{N}\cup\{-1\})^n$. We set

$$\begin{aligned} Z^r_I(\Pi) &= \{\Lambda\in\mathcal{X}^r_I(\mathbb{R}^n)\,;\,\partial^r(A)=0\}\,,\\ B^r_I(\Pi) &= \{\partial^{r-1}(B)\,;\,B\in\mathcal{X}^{r-1}_I(\mathbb{R}^n)\}\,,\\ \text{and }\ H^r_I(\Pi) &= Z^r_I/B^r_I\,.\end{aligned}$$

First case : We suppose that $\underline{A.I=0}$.
In this case, it is clear that $Z^r_I(\Pi)=\mathcal{X}^r_I(\mathbb{R}^n)$ and $B^r_I(\Pi)=\{0\}$.

Proposition 3.1. *If $A.I = 0$, then $H^r_I(\Pi) = \mathcal{X}^r_I(\mathbb{R}^n)$.*

Second case : We suppose that $\underline{A.I \neq 0}$.
Here again, the expression of the cohomology spaces $H^r_I(\Pi)$ depends on the number of negative components of I.

1- We first assume that I **does not have negative components**, i.e. $I \in \mathbb{N}^n$. We are going to state the following proposition.

Proposition 3.2. *If I does not have negative components, then $H^r_I(\Pi) = \{0\}$. In particular, we have $H^0(\Pi) = \oplus_{A.I=0}\mathcal{X}^0_I(\mathbb{R}^n)$.*

Proof : Let Λ in $Z^r_I(\Pi)$. We can write $\Lambda = x^I \Lambda_I$. We then have

$$0 = \partial^r(\Lambda) = x^I \Lambda_I \wedge (A.I)\,.$$

Therefore, there exists Γ_I in $\mathcal{X}^{r-1}_I(\mathbb{R}^n)$, with

$$\Gamma_I = \sum_{\substack{K\in\mathbb{N}^{r-1}\\ k_1<\ldots<k_{r-1}}} \gamma_K Y_{k_1} \wedge \ldots \wedge Y_{k_{r-1}}$$

which satisfies

$$\Lambda_I = \Gamma_I \wedge (A.I)\,.$$

We deduce that $\Lambda = \partial^{r-1}(x^I \Gamma_I)$. ■

2- Now, we assume that the n-tuple I **has exactly s negative components with $s < r$.**
We recall that $A.I = \sum_{j=1}^n \alpha_j Y_j$ with $\alpha_j = \sum_{k=1}^n a_{jk} i_k$.

Proposition 3.3. *Suppose that I has exactly s negative components with $s < r$, for instance $i_{u_1} = \ldots = i_{u_s} = -1$.*
1- If $A.I \neq \alpha_{u_1} Y_{u_1} + \ldots + \alpha_{u_s} Y_{u_s}$ then $H^r_I(\Pi) = \{0\}$.
2- If $A.I = \alpha_{u_1} Y_{u_1} + \ldots + \alpha_{u_s} Y_{u_s}$ then $H^r_I(\Pi) = \mathcal{X}^r_I(\mathbb{R}^n)$.

Proof : 1- We first suppose that $A.I \neq \alpha_{u_1} Y_{u_1} + \ldots + \alpha_{u_s} Y_{u_s}$. We consider Λ in $Z^r_I(\Pi)$, and we write $\Lambda = x^I \Lambda_I$ with

$$\Lambda_I = (Y_{u_1} \wedge \ldots \wedge Y_{u_s}) \wedge \Theta$$

where Θ is an $(r-s)$-vector which does not depend on $Y_{u_1}, \ldots, Y_{u_s}$ and which can be written as

$$\sum_{\substack{K\in\mathbb{N}^{r-s}\\ k_1<\ldots<k_{r-s}}} \theta_K Y_{k_1} \wedge \ldots \wedge Y_{k_{r-s}}\,.$$

In order to simplify the notation, let us suppose that $s = 2$ and $i_1 = i_2 = -1$ (the proof in the general case can be done in the same way).
We then have $\Lambda_I = Y_1 \wedge Y_2 \wedge \Theta$ (Θ does not depend on Y_1 and Y_2).
Since Λ is an r-cocycle, we must have

$$Y_1 \wedge Y_2 \wedge \Theta \wedge (\alpha_1 Y_1) + Y_1 \wedge Y_2 \wedge \Theta \wedge (\alpha_2 Y_2) + \sum_{j=3}^n Y_1 \wedge Y_2 \wedge \Theta \wedge (\alpha_j Y_j) = 0$$

i.e.

$$Y_1 \wedge Y_2 \wedge \big(\sum_{j=3}^{n} \alpha_j \Theta \wedge Y_j \big) = 0\,.$$

We deduce that (because Θ does not depend on Y_1 and Y_2)

$$\sum_{j=3}^{n} \alpha_j \Theta \wedge Y_j = 0$$

i.e.

$$\Theta \wedge \big(A.I - \alpha_1 Y_1 - \alpha_2 Y_2 \big) = 0\,.$$

Now, since $A.I \neq \alpha_1 Y_1 + \alpha_2 Y_2$, there exists an $(r-3)$-vector Δ_I of type

$$\Delta_I = \sum_{\substack{K \in \mathbb{N}^{r-3} \\ k_1 < \ldots < k_{r-3}}} \delta_K Y_{k_1} \wedge \ldots \wedge Y_{k_{r-3}}$$

such that

$$\Theta = \Delta_I \wedge \big(A.I - \alpha_1 Y_1 - \alpha_2 Y_2 \big)\,.$$

Therefore, we can write

$$\Lambda_I = Y_1 \wedge Y_2 \wedge \Delta_I \wedge (A.I)\,.$$

Now, if we set $\Gamma = x^I Y_1 \wedge Y_2 \wedge \Delta_I$, we have $\partial^{r-1}(\Gamma) = \Lambda$.

2- Now, we suppose that $A.I = \alpha_{u_1} Y_{u_1} + \ldots + \alpha_{u_s} Y_{u_s}$. In this case, since $Y_{u_1} \wedge \ldots \wedge Y_{u_s}$ divides Λ, it is clear that, if Λ is in $\mathcal{X}_I^r$, then $\partial^r(\Lambda) = 0$. Consequently, $Z_I^r(\Pi) = \mathcal{X}_I^r(\mathbb{R}^n)$.
Now, let Λ be in $B_I^r(\Pi)$. There exists Γ in $\mathcal{X}_I^{r-1}(\mathbb{R}^n)$ such that $\Lambda = \Gamma \wedge (A.I)$.
Since $i_{u_1} = \ldots = i_{u_s} = -1$, the term $Y_{u_1} \wedge \ldots \wedge Y_{u_s}$ divides Γ.
We deduce that $\Gamma \wedge (A.I) = 0$. ■

Remark 3.4. If I has only one negative component (for instance i_k), then $A.I$ is not collinear with Y_k.
Indeed, if $A.I = \alpha_k Y_k$, we have $\alpha_u = 0$ for every $u \neq k$, which can be interpreted as

$$a_{uk} = \sum_{v \neq k} a_{uv} i_v \quad \text{for each } u \neq k$$

hence,

$$-a_{ku} i_u = \sum_{v \neq k} a_{uv} i_u i_v \quad \text{for each } u \neq k\,.$$

Therefore, we get

$$-\alpha_k = -\sum_{u} a_{ku} i_u = \sum_{u \neq k, v \neq k} a_{uv} i_u i_v\,.$$

Since the matrix $(a_{uv})_{1 \leq u,v \leq n}$ is skewsymmetric, this last sum is zero. This implies that $A.I = 0$, which is not compatible with our hypothesis.
We deduce that, in this case,

$$H_I^r(\Pi) = \{0\}\,.$$

3- Finally, we assume that I **has** r **negative components**.
We then show the following result.

Proposition 3.5. *We suppose that I has r negative components, for instance $i_{u_1} = \ldots = i_{u_r} = -1$.*
1- If there exists $k \notin \{u_1, \ldots, u_r\}$ such that $\alpha_k \neq 0$ then $H^r_I(\Pi) = \{0\}$.
2- If not, we have $H^r_I(\Pi) = \mathcal{X}^r_I(\mathbb{R}^n)$.

Proof : In this case, it is easy to see that $B^r_I(\Pi) = \{0\}$.
Now, let us describe $Z^r_I(\Pi)$. Consider an element Λ in $Z^r_I(\Pi)$.
We can write $\Lambda = \lambda \frac{\partial}{\partial x_{u_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{u_r}}$ where λ is a real number.
Consequently,

$$\partial^r(\Lambda) = \lambda \sum_{k \notin \{u_1, \ldots, u_r\}} \alpha_k x_k \frac{\partial}{\partial x_{u_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{u_r}} \wedge \frac{\partial}{\partial x_{u_k}} .$$

We deduce that, if there exists $k \notin \{u_1, \ldots, u_r\}$ such that $\alpha_k \neq 0$ then $Z^r_I(\Pi) = \{0\}$.
If not, then we have $Z^r_I(\Pi) = \mathcal{X}^r_I(\mathbb{R}^n)$. ■

Corollary 3.6. *The space $H^1(\Pi)$ is given by $H^1(\Pi) = \oplus_{A.I=0} \mathcal{X}^1_I(\mathbb{R}^n)$.*

Proof : According to remark 3.4 and case *1-* of the previous proposition, we have $H^1_I(\Pi) = \{0\}$ whenever I has one negative component. ■

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