NORMAL FORMS OF VECTOR FIELDS ON POISSON MANIFOLDS

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Abstract. We study formal and analytic normal forms of radial and Hamiltonian vector fields on Poisson manifolds near a singular point.

1. Introduction

This paper is devoted to the study of normal forms à la Poincaré-Birkhoff for analytic or formal vector fields on Poisson manifolds. We will be interested in two kinds of vector fields, namely Hamiltonian vector fields, and "radial" vector fields, i.e. those vector fields X such that $[X, \Pi] = \mathcal{L}_X \Pi = -\Pi$, where Π denotes the Poisson structure, and the bracket is the Schouten bracket. Our motivation for studying radial vector fields comes from Jacobi structures [7], while of course the main motivation for studying Hamiltonian vector fields comes from Hamiltonian dynamics. We will assume that our vector field X vanishes at a point, $X(0) = 0$, and that the linear part of Π or of its transverse structure at 0 corresponds to a semisimple Lie algebra. In this case, it is well known [13, 4] that Π admits a formal or analytic linearization in a neighborhood of 0. We are interested in a simultaneous linearization or normalization of Π and X.

In Section 2, we study the problem of simultaneous linearization of couples (Π, X) where Π is a Poisson structure and X is a vector field such that $\mathcal{L}_X \Pi = -\Pi$. Such couples are called *homogeneous Poisson structures* in the sense of Dazord, Lichnerowicz and Marle [7], and they are closely related to Jacobi manifolds. More precisely, a 1-codimensional submanifold of a homogeneous Poisson manifold (M, Π, X) which is transverse to the vector field X has an induced Jacobi structure, and all Jacobi manifolds can be obtained in this way. On the other hand, a 1-codimensional submanifold of a Jacobi manifold (N, Λ, E) transverse to the structural vector field E has an induced homogeneous Poisson structure, and all homogeneous Poisson manifolds can be obtained in this way (see [7]). Our first result is the following (see Theorem 2.4):

Theorem A. *Let (Π, X) be a formal homogeneous Poisson structure on $\mathbb{K}^n$ (where $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$) such that the linear part Π_1 of Π corresponds to a semisimple Lie algebra $\mathfrak{g}$. Suppose that its linear part $(\Pi_1, X^{(1)})$ is semisimple nonresonant. Then there exists a formal diffeomorphism which sends (Π, X) to $(\Pi_1, X^{(1)})$.*

The *semisimple nonresonant* condition in the above theorem is a generic position on $X^{(1)}$: the set of $X^{(1)}$ which does not satisfy this condition is of codimension 1, and moreover if $X^{(1)} - I$ is diagonalizable and small enough, where $I = \sum x_i \frac{\partial}{\partial x_i}$

denotes the standard radial (Euler) vector field, then the semisimple nonresonant is automatically satisfied.

For analytic linearization, due to possible presence of small divisors, we need a Diophantine-type condition. Here we choose to work with a modified Bruno's ω-condition [2, 3] adapted to our case. See Definition 2.5 for the precise definition of our ω-condition. The set of $(\Pi_1, X^{(1)})$ which satisfy this ω-condition is of full measure. We have (see Theorem 2.7):

Theorem B. *Let (Π, X) be an analytic homogeneous Poisson structure on $\mathbb{K}^n$ (where $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$) such that the linear part Π_1 of Π corresponds to a semisimple Lie algebra. Suppose moreover that its linear part $(\Pi_1, X^{(1)})$ is semisimple nonresonant and satisfies the ω-condition. Then there exists a local analytic diffeomorphism which sends (Π, X) to $(\Pi_1, X^{(1)})$.*

In Section 3, we study local normal forms of Hamiltonian systems on Poisson manifolds. According to Weinstein's splitting theorem [13], our local Poisson manifold $((\mathbb{K}^n, 0), \Pi)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is a direct product $((\mathbb{K}^{2l}, 0), \Pi_{symp}) \times ((\mathbb{K}^m, 0), \Pi_{trans})$ of two Poisson manifolds, where the Poisson structure Π_{symp} is nondegenerate (symplectic), and the Poisson structure Π_{trans} (the transverse structure of Π at 0) vanishes at 0. If Π_{trans} is trivial, i.e. the Poisson structure Π is regular near 0, then the problem local normal forms of Hamiltonian vector fields near 0 is reduced to the usual problem of normal forms Hamiltonian vector fields (with parameters) on a symplectic manifold. Here we are interested in the case when Π_{trans} is not trivial. We will restrict our attention to the case when the linear part of Π_{trans} corresponds to a semisimple Lie algebra $\mathfrak{g}$. According to linearization theorems of Weinstein [13] and Conn [4], we may identify $((\mathbb{K}^m, 0), \Pi_{trans})$ with a neighborhood of 0 of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ equipped with the associated linear (Lie-Poisson) structure. In other words, there is a local system of coordinates $(x_1, y_1, \ldots, x_l, y_l, z_1, \ldots, z_m)$ on $\mathbb{K}^{2l+m}$ such that $\Pi_{symp} = \sum_{i=1}^{l} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$, $\Pi_{trans} = \Pi_{\mathfrak{g}} = \frac{1}{2} \sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ with c_{ij}^k being structural constants of $\mathfrak{g}$, and

$$\Pi = \sum_{i=1}^{l} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}. \tag{1.1}$$

Such a coordinate system will be called a *canonical coordinate system* of Π near 0. Let H be a formal or analytic function on $((\mathbb{K}^n, 0), \Pi)$. We will assume that the Hamiltonian vector field X_H of H vanishes at 0. Note that the differential of H does not necessarily vanish at 0 (for example, if $l = 0$ then we always have $X_H(0) = 0$ for any H). We may assume that $H(0) = 0$.

We have the following generalization of Birkhoff normal form [1] (see Theorem 3.1):

Theorem C. *With the above notations and assumptions, there is a formal canonical coordinate system $(\hat{x}_i, \hat{y}_i, \hat{z}_j)$, in which H satisfies the following equation:*

$$\{H, H_{ss}\} = 0 \ ,$$

where H_{ss} is a (nonhomogeneous quadratic) function such that its Hamiltonian vector field $X_{H_{ss}}$ is linear and is the semisimple part of the linear part $X_H^{(1)}$ of X_H (in this coordinate system). In particular, the semisimple part of the linear part of X_H is a Hamiltonian vector field.

Note that the normalizing canonical coordinates given in the above theorem are only formal in general. The problem of existence of a local analytic normalization for a Hamiltonian vector field (even in the symplectic case) is much more delicate than for a general vector field, due to "auto-resonances" (e.g, if λ is an eigenvalue of a Hamiltonian vector field then $-\lambda$ also is). However, there is one particular situation where one knows that a local analytic normalization always exists, namely when the Hamiltonian vector field is analytically integrable. See [16] for the case of integrable Hamiltonian vector fields on symplectic manifolds. Here we can generalize the main result of [16] to our situation (see Theorem 3.8):

Theorem D. *Assume that $\mathbb{K} = \mathbb{C}$, the Hamiltonian function H in Theorem C is locally analytic, and is analytically integrable in the generalized Liouville sense. Then the normalizing canonical coordinate system $(\hat{x}_i, \hat{y}_i, \hat{z}_j)$ can be chosen locally analytic.*

We conjecture that the above theorem remains true in the real case ($\mathbb{K} = \mathbb{R}$). Recall (see, e.g., [15] and references therein) that a Hamiltonian vector field X_H on a Poisson manifold (M, Π) of dimension n is called *integrable in generalized Liouville sense* if there are nonnegative integers p, q with $p + q = n$, p pairwise commuting Hamiltonian functions $H_1, \dots, H_p$ ($\{H_i, H_j\} = 0\ \forall i, j$) with $H_1 = H$ and q first integrals $F_1, \dots, F_p$, such that $X_{H_i}(F_j) = 0\ \forall\, i, j$, and $dF_1 \wedge \dots \wedge dF_q \neq 0$ and $X_{H_1} \wedge \dots \wedge X_{H_p} \neq 0$ almost everywhere. (The Liouville case corresponds to $p = q = n/2$ and $F_i = H_i$). *Analytic* integrability means that all Hamiltonian functions and vector fields in question are analytic.

2. Homogeneous Poisson structures

Following [7], we will use the following terminology: a *homogeneous Poisson structure* on a manifold M is a couple (Π, X) where Π is a Poisson structure and X a vector field which satisfies the relation

$$[X, \Pi] = -\Pi\,, \tag{2.1}$$

where the bracket is the Schouten bracket.

Remark 2.1. Poisson structures which satisfy the above condition are also called *exact*, in the sense that the Poisson tensor is a coboundary in the associated Lichnerowicz complex which defines Poisson cohomology. They have nothing to do with another kind of homogeneous spaces, namely those which admit a transitive group action.

An analog of Weinstein's splitting theorem for homogeneous Poisson structures is given in [7], and it reduces the study of normal forms of homogeneous Poisson structures to the case when both Π and X vanish at a point. So we will assume that (Π, X) is a homogeneous Poisson structure defined in a neighborhood of 0 in $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, such that

$$\Pi(0) = 0 \quad \text{and} \quad X(0) = 0\,. \tag{2.2}$$

We are interested in the linearization of these structures, i.e. simultaneous linearization of Π and X. Denote by Π_1 and $X^{(1)}$ the linear parts of Π and X respectively. Then the terms of degree 1 of Equation (2.1) imply that $(\Pi_1, X^{(1)})$ is again a homogeneous Poisson structure.

In this paper, we will assume that the linear Poisson structure Π_1 corresponds to a semisimple Lie algebra, which we denote by $\mathfrak{g}$. Then, according to linearization results of Weinstein [13] (for the formal case) and Conn [4] (for the analytic case), the Poisson structure Π can be linearized. In other words, there is a local coordinate system $(x_1, \ldots, x_n)$ on $(\mathbb{K}^n, 0)$, in which

$$\Pi = \Pi_1 = \frac{1}{2}\sum_{ijk} c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\,, \text{ or} \tag{2.3}$$

$$\{x_i, x_j\} = \sum_k c_{ij}^k x_k, \tag{2.4}$$

where c_{ij}^k are structural constants of $\mathfrak{g}$. In order to linearize $(\Pi, X) = (\Pi_1, X)$, it remains to linearize X by local (formal or analytic) diffeomorphisms which preserve the linear Poisson structure Π_1.

2.1. **Formal linearization.** First consider the complex case ($\mathbb{K} = \mathbb{C}$). Let X be a formal vector field on $\mathbb{C}^n$ such that (Π_1, X) forms a homogeneous Poisson structure on $\mathbb{C}^n$. Denote by

$$I = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \tag{2.5}$$

the Euler vector field written in coordinates $(x_1, \ldots, x_n)$. Since this vector field satisfies the relation $[I, \Pi_1] = -\Pi_1$, we can write X as

$$X = I + Y\,, \tag{2.6}$$

where Y is a Poisson vector field with respect to Π_1, i.e., $[Y, \Pi_1] = 0$. It is well-known that, since the complex Lie algebra $\mathfrak{g}$ is semisimple by assumptions, the first formal Poisson cohomology space of Π_1 is trivial (see, e.g., [5]), i.e. any formal Poisson vector field is Hamiltonian. In particular, we have

$$Y = X_h = -[h, \Pi_1] \tag{2.7}$$

for some formal function h. Writing the Taylor expansion $h = h_1 + h_2 + h_3 + \cdots$ where each h_r is a polynomial of degree r, we have

$$X = I + X_{h_1} + X_{h_2} + X_{h_3} + \cdots \tag{2.8}$$

Denote by $X^{(1)} = I + X_{h_1}$ the linear part of X. In order to linearize X (while preserving the linearity of $\Pi = \Pi_1$), we want to kill all the terms X_{h_r} with $r \geq 2$, using a sequence of changes of coordinates defined by flows of Hamiltonian vector fields with respect to Π_1. Working degree by degree, we want to find for each r a homogeneous polynomial g_r of degree r such that

$$[X^{(1)}, X_{g_r}] = X_{h_r}\,. \tag{2.9}$$

Note that $[X_{h_1}, X_{g_r}] = X_{\{h_1, g_r\}}$, and $[I, X_{g_r}] = (r-1)X_{g_r}$ because X_{g_r} is homogeneous of degree r. Hence Relation (2.9) will be satisfied if g_r satisfies the following relation:

$$(r-1)g_r + \{h_1, g_r\} = h_r\,. \tag{2.10}$$

Remark that, h_1 can be viewed as an element of $\mathfrak{g}$, and h_r, g_r may be identified with elements of the symmetric power $S^r(\mathfrak{g})$ of $\mathfrak{g}$. Under this identification, $\{h_1, g_r\}$ is nothing but the result of the adjoint action of $h_1 \in \mathfrak{g}$ on $g_r \in S^r(\mathfrak{g})$.

We will suppose that h_1 is a semisimple element of $\mathfrak{g}$, and denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ which contains h_1. According to the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$, we can choose a basis $(x_1, \ldots, x_n)$ of $\mathfrak{g}$, and elements $\alpha_1, \ldots, \alpha_n$ of $\mathfrak{h}^*$, such that

$$[y, x_i] = \langle \alpha_i, y\rangle x_i \quad \forall\, y \in \mathfrak{h},\ \forall i = 1, \ldots, n. \tag{2.11}$$

Each α_i is either 0 (in which case $x_i \in \mathfrak{h}$) or a root of $\mathfrak{g}$ (in which case x_i belongs to the root subspace $\mathfrak{g}_{\alpha_i}$ of $\mathfrak{g}$).

We define for each $r \geq 2$ the linear operator

$$\begin{array}{rcl} \Theta_r : S^r(\mathfrak{g}) & \longrightarrow & S^r(\mathfrak{g}) \\ a & \longmapsto & (r-1)a + \{h_1, a\}\ . \end{array}$$

Each monomial $\prod_i x_i^{\lambda_i}$ of degree $|\lambda| = \sum \lambda_i = r$ is an eigenvector of this linear operator:

$$\Theta_r(\prod_i x_i^{\lambda_i}) = \big(r - 1 + \sum_{i=1}^{n} \lambda_i \langle \alpha_i, h_1\rangle\big) \prod_i x_i^{\lambda_i}\ . \tag{2.12}$$

Definition 2.2. With the above notations, we will say that $(\Pi_1, X^{(1)})$ is *semisimple nonresonant* if h_1 is a semisimple element of $\mathfrak{g}$ and the eigenvalues of Θ_r don't vanish, i.e., for any $r \geq 2$ and any $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n$ such that $\sum \lambda_i = r$ we have $r - 1 + \sum_{i=1}^n \lambda_i \langle \alpha_i, h_1\rangle \neq 0$.

Remark 2.3. It is easy to see that the above nonresonance condition is a generic position condition, and the subset of elements which do not satisfy this condition is of codimension 1. In fact, if the Cartan subalgebra $\mathfrak{h}$ is fixed, then the set of elements $h_1 \in \mathfrak{h}$ such that $(\Pi_1, I + X_{h_1})$ is resonant is a countable union of affine hyperplanes in $\mathfrak{h}$ which do not contain the origin, and there is a neighborhood of 0 in $\mathfrak{h}$ such that if h_1 belongs to this neighborhood then $(\Pi_1, X^{(1)})$ is automatically nonresonant.

The algorithm of formal linearization. We now show how to linearize (Π_1, X), by killing the nonlinear terms of h step by step, provided that $(\Pi_1, X^{(1)})$ is nonresonant. Actually, at each step, we will kill not just one term h_d, but a whole block of 2^d consecutive terms. This "block killing" will be important in the next section when we want to show that, under some Diophantine-type condition, our formal linearization process actually yields a local analytic linearization.

For each $q \geq 0$, denote by $\hat{\mathcal{O}}_q$ the space of formal power series on $\mathbb{C}^n$ of order greater or equal to q, i.e. without terms of degree $< q$.

We begin with $X = X^{(1)} \mod \hat{\mathcal{O}}_2$, and will construct a sequence of formal vector fields $(X_d)_d$ and diffeomorphisms $(\varphi_d)_d$, such that $X_0 = X$ and, for all $d \geq 0$,

$$X_d = X^{(1)} \mod \hat{\mathcal{O}}_{2^d+1}\ , \tag{2.13}$$

$$X_{d+1} = \varphi_{d*} X_d. \tag{2.14}$$

Assuming that we already have X_d for some $d \geq 0$, we will construct φ_d (and $X_{d+1} = \varphi_{d*} X_d$). We write

$$X_d = X^{(1)} + X_{H_d} \mod \hat{\mathcal{O}}_{2^{d+1}+1}\,, \tag{2.15}$$

where H_d is a polynomial of degree $\leq 2^{d+1}$ in $\hat{\mathcal{O}}_{2^d+1}$, i.e. H_d is a sum of homogeneous polynomials of degrees between $2^d + 1$ and 2^{d+1} (we also could write abusively that H_d is in $\hat{\mathcal{O}}_{2^d+1}/\hat{\mathcal{O}}_{2^{d+1}+1}$). Under the nonresonance condition, there exists a polynomial G_d of order $\geq 2^d+1$ and of degree $\leq 2^{d+1}$ such that if we write $G_d = G_d^{(2^d+1)} + \ldots + G_d^{(2^{d+1})}$ (where $G_d^{(u)}$ is homogeneous of degree u) and the same for H_d, we have

$$\Theta_u(G_d^{(u)}) = (u-1)G_d^{(u)} + \{h_1, G_d^{(u)}\} = H_d^{(u)}\,, \tag{2.16}$$

for every $u \in \{2^d + 1, \ldots, 2^{d+1}\}$. It implies that we have

$$[X^{(1)}, X_{G_d}] = X_{H_d}\,, \tag{2.17}$$

where X_{G_d} denotes the Hamiltonian vector field of G_d with respect to Π_1 as usual. Now, we define the diffeomorphism $\varphi_d = \exp X_{G_d}$ to be the time-1 flow of X_{G_d}. We then have

$$X_{d+1} := \varphi_{d*} X = X^{(1)} + X_{H_{d+1}} \mod \hat{\mathcal{O}}_{2^{d+2}+1}\,, \tag{2.18}$$

where H_{d+1} is a polynomial of degree 2^{d+2} in $\hat{\mathcal{O}}_{2^{d+1}+1}$.

Constructed in this way, it is clear that the successive compositions of the diffeomorphisms φ_d converge in the formal category to a formal diffeomorphism Φ_∞ which satisfies $\Phi_{\infty *} X = X^{(1)}$ and which preserves the linear Poisson structure Π_1.

Consider now the real case ($\mathbb{K} = \mathbb{R}$, and $\mathfrak{g}$ is a real semisimple Lie algebra). By complexification, we can view real objects as holomorphic objects with real coefficients, and then repeat the above algorithm. In particular, under the nonresonance condition, we will find homogeneous polynomials $G_d^{(u)}$ which satisfy Equation (2.16), i.e.,

$$(u-1)G_d^{(u)} + \{h_1, G_d^{(u)}\} = H_d^{(u)}. \tag{2.19}$$

Remark that, in the real case, the operator $\Theta_u : G^{(u)} \mapsto (u-1)G^{(u)} + \{h_1, G^{(u)}\}$ is real (and is invertible under the nonresonance condition), and $H_d^{(u)}$ is real, so $G_d^{(u)}$ is also real. This means that the coordinate transformations constructed above are real in the real case.

We have proved the following:

Theorem 2.4. *Let (Π, X) be a formal homogeneous Poisson structure on $\mathbb{K}^n$ (where $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$) such that the linear part Π_1 of Π corresponds to a semisimple Lie algebra. Assume that its linear part $(\Pi_1, X^{(1)})$ is semisimple nonresonant. Then there exists a formal diffeomorphism which sends (Π, X) to $(\Pi_1, X^{(1)})$.*

2.2. **Analytic linearization.** Now we work in the local analytic context, i.e. the vector field X is supposed to be analytic on $(\mathbb{K}^n, 0)$. In order to show that the algorithm given in the previous subsection leads to a local analytic linearization, in addition to the nonresonance condition we will need a Diophantine-type condition, similar to Bruno's ω-condition for the analytic linearization of vector fields [2, 3].

Keeping the notations of the previous subsection, for each $d \geq 1$, put
(2.20)

$$\omega_d = \min\left\{\frac{1}{2d}, \min\left\{\left||\lambda| - 1 + \sum_{i=1}^n \lambda_i \langle \alpha_i, h_1\rangle\right| ;\ \lambda \in \mathbb{Z}_+^n \text{ and } 2 \leq |\lambda| \leq 2^{d+1}\right\}\right\}.$$

Definition 2.5. We will say that $X^{(1)}$, or more precisely that a semisimple nonresonant linear homogeneous Poisson structure $(\Pi_1, X^{(1)})$ satisfies the *ω-condition* if

$$\sum_{d=1}^{\infty} \frac{-\log \omega_d}{2^d} < \infty\,. \tag{2.21}$$

Remark that, similarly to other situations, the set of $X^{(1)}$ which satisfy the about ω-condition is of full measure. More precisely, we have:

Proposition 2.6. *The set of elements h of a given Cartan subalgebra $\mathfrak{h}$ such that $X^{(1)} = I + X_h$ does not satisfy the ω-condition (2.21) is of measure 0 in $\mathfrak{h}$.*

See the Appendix for a straightforward proof of the above proposition.

Using the same analytical tools as in the proof of Bruno's theorems about linearization of analytic vector fields [2, 3], we will show the following theorem:

Theorem 2.7. *Let (Π, X) be an analytic homogeneous Poisson structure on $(\mathbb{K}^n, 0)$ (where $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$) such that the linear part Π_1 of Π corresponds to a semisimple Lie algebra. Suppose that its linear part $(\Pi_1, X^{(1)})$ is semisimple nonresonant and satisfies the ω-condition. Then there exists a local analytic diffeomorphism which sends (Π, X) to $(\Pi_1, X^{(1)})$.*

Proof. Due to Conn's theorem [4], we can assume that $\Pi = \Pi_1$ is already linear. The process to linearize the vector field X is the same as in the formal case, noting that if we start with an analytic vector field, the diffeomorphisms φ_d that we constructed will be analytic too (as is the vector fields X_d). We just have to check the convergence of the sequence $\Phi_d = \varphi_d \circ \ldots \circ \varphi_1$ in the analytic setup.

We will assume that $\mathbb{K} = \mathbb{C}$ (the real case can be reduced to the complex case by the same argument as given in the previous subsection). Denote by $\mathcal{O}_q$ the vector space of local analytic functions of $(\mathbb{K}^n, 0)$ of order greater or equal to q (i.e. without terms of degree $< q$).

For each positive real number $\rho > 0$, denote by D_ρ the ball $\{x = (x_1, \ldots, x_n) \in \mathbb{C}^n\,;\, |x_i| < \rho\}$ and if $f = \sum_{\lambda \in \mathbb{N}^n} a_\lambda x^\lambda$ is an analytic function on D_ρ we define the following norms:

$$|f|_\rho \ := \ \sum_\lambda |a_\lambda| \rho^{|\lambda|}\ , \tag{2.22}$$

$$\|f\|_\rho \ := \ \sup_{z \in D_\rho} |f(z)|\ . \tag{2.23}$$

In the same way, if $F = (F_1, \ldots, F_n)$ is a vector-valued local map then we put $|F|_\rho := \max\{|F_1|_\rho, \ldots, |F_n|_\rho\}$ and similarly for $||F||_\rho$. These norms satisfy the following properties.

Lemma 2.8. *Let ρ and ρ' be two real numbers such that $0 < \rho' < \rho$. If $f \in \mathcal{O}_q$ is an analytic function on D_ρ, then*
a)

$$\|f\|_\rho \leq |f|_\rho \quad \text{and} \quad |f|_{\rho'} \leq \frac{1}{1-(\rho'/\rho)}\|f\|_\rho \,. \tag{2.24}$$

b)

$$|f|_{\rho'} < \big(\frac{\rho'}{\rho}\big)^q |f|_\rho \,. \tag{2.25}$$

c) Let $R > 0$ be a positive constant. Then there is a natural number N such that for any $d > N$, if $q = 2^d + 1$, $\big(\frac{1}{(2d)(2^d)}\big)^{1/(2^d+1)}\rho = \rho' \geq R$, and $f \in \mathcal{O}_q$ is an analytic function on D_ρ, then we have

$$|df|_{\rho'} \leq |f|_\rho \,. \tag{2.26}$$

The proof of the above lemma is elementary (see the Appendix).

It is important to remark that, with the same notations as in the formal case, for $\rho > 0$, we have, by (2.12):

$$|X_{G_d}|_\rho \leq \frac{1}{\omega_d}|X_{H_d}|_\rho \,. \tag{2.27}$$

Put $\rho_0 = 1$, and define the following two decreasing sequences of radii $(r_d)_d$ and $(\rho_d)_d$ by

$$r_d \;:=\; \big(\frac{\omega_d}{2^d}\big)^{1/(2^d+1)}\rho_{d-1} \,, \tag{2.28}$$

$$\rho_d \;:=\; (1-\frac{1}{d^2})r_d \,. \tag{2.29}$$

We have

$$\ldots < \rho_{d+1} < r_{d+1} < \rho_d < r_d < \rho_{d-1} < \ldots, \tag{2.30}$$

and it is clear, by the ω-condition (2.21), that the sequences $(r_d)_d$ and $(\rho_d)_d$ converge to a strictly positive limit $R > 0$. Moreover, they satisfy the following properties:

Lemma 2.9. *For d sufficiently large, we have*

a) $r_d - \rho_d > \frac{1}{2^d}$,

b) $\rho_d - r_{d+1} > \frac{1}{2^d}$.

The proof of Lemma 2.9 is elementary (see the Appendix).

Lemma 2.10. *For d sufficiently large, if $|X_d - X^{(1)}|_{\rho_{d-1}} < 1$, then*

$$D_{r_{d+1}} \subset \varphi_d(D_{\rho_d}) \subset D_{r_d} \,, \tag{2.31}$$

and moreover, we have $|\varphi_{d*}X_d - X^{(1)}|_{\rho_d} < 1$.

Proof . • We first prove the second inclusion : $\varphi_d(D_{\rho_d}) \subset D_{r_d}$.

We have

$$X_d = X^{(1)} + X_{H_d} \mod \mathcal{O}_{2^{d+1}+1} \tag{2.32}$$

where H_d is a polynomial formed by homogenous terms of degree between 2^d+1 and 2^{d+1}. By (2.27), we write

$$|X_{G_d}|_{\rho_{d-1}} < \frac{1}{\omega_d}|X_{H_d}|_{\rho_{d-1}}\,. \tag{2.33}$$

Then, by (2.25), we get

$$|X_{G_d}|_{r_d} < \frac{1}{\omega_d}\big(\frac{r_d}{\rho_{d-1}}\big)^{2^d+1}|X_{H_d}|_{\rho_{d-1}} \tag{2.34}$$

And, using the assumption $|X_d - X^{(1)}|_{\rho_{d-1}} < 1$, we obtain

$$|X_{G_d}|_{r_d} < \frac{1}{2^d}\,. \tag{2.35}$$

Finally, Lemma 2.9 gives

$$\|X_{G_d}\|_{r_d} \leq |X_{G_d}|_{r_d} < r_d - \rho_d\,, \tag{2.36}$$

which implies the inclusion $\varphi_d(D_{\rho_d}) \subset D_{r_d}$.

• Now, we prove the first inclusion $D_{r_{d+1}} \subset \varphi_d(D_{\rho_d})$. For any x on the boundary S_{ρ_d} of D_{ρ_d}, we define $x_1 := \frac{r_d+\rho_d}{2}\frac{x}{|x|}$ and $x_2 := r_d\frac{x}{|x|}$. We construct a map $\hat{\phi}_d : D_{r_d} \longrightarrow D_{r_d}$ which is φ_d on D_{ρ_d} and defined on $D_{r_d}\backslash D_{\rho_d}$ by the following : for $\mu \in [0,1]$ and $x \in S_{\rho_d}$ we put

$$\begin{array}{rcl}\hat{\phi}_d(\mu x + (1-\mu)x_1) & = & \mu\varphi_d(x) + (1-\mu)x \\ \hat{\phi}_d(\mu x_1 + (1-\mu)x_2) & = & \mu x + (1-\mu)x_2\end{array}$$

This map is continuous and is the identity on the boundary of D_{r_d} thus, by Brouwer's theorem, $\hat{\phi}_d(D_{r_d}) = D_{r_d}$.

Let x be an element of the boundary S_{ρ_d} of D_{ρ_d}.

If $z = \mu x + (1-\mu)x_1$ (for $\mu \in [0,1]$) then we have

$$\begin{array}{rcl}|\hat{\phi}_d(z)| & = & |x + \mu(\varphi_d(x) - x)| \\ & \geq & |x| - \mu|\varphi_d(x) - x|\end{array}$$

Now, we write $|\varphi_d(x) - x| \leq \big|\int_0^1 X_{G_d}\big(\varphi_d^t(x)\big)dt\big|$ where φ_d^t is the flow of X_{G_d}. As above, according to (2.36), $\varphi_d^t(x)$ is in D_{r_d} for all $t \in [0,1]$ and then, we get $|\varphi_d(x) - x| \leq \|X_{G_d}\|_{r_d} < \frac{1}{2^d}$. Therefore, by Lemma 2.9, we get

$$|\hat{\phi}_d(z)| > r_{d+1}\,. \tag{2.37}$$

Now, if $z = \mu x_1 + (1-\mu)x_2$ ($\mu \in [0,1]$) then we have

$$|\hat{\phi}_d(z)| = \big(\mu + (1-\mu)\frac{r_d}{|x|}\big)|x| \geq |x| > r_{d+1}\,. \tag{2.38}$$

As a conclusion, if y is in $D_{r_{d+1}}$ then, by the surjectivity of $\hat{\phi}_d$, $y = \hat{\phi}_d(z)$ with, a priori, z in D_{r_d}. We saw above that in fact z cannot be in D_{r_d}/D_{ρ_d}. Therefore, since $\hat{\phi}_d = \varphi_d$ on D_{ρ_d}, we get $y = \varphi_d(z)$ with z in D_{ρ_d}.

• Finally, we check that $|\varphi_{d*}X_d - X^{(1)}|_{\rho_d} < 1$. We write the obvious inequality

$$|\varphi_{d*}X_d - X^{(1)}|_{\rho_d} \leq |\varphi_{d*}X_d - X_d|_{\rho_d} + |X_d - X^{(1)}|_{\rho_d}\,. \tag{2.39}$$

By (2.25), we have

$$|X_d - X^{(1)}|_{\rho_d} < \Big(\frac{\rho_d}{\rho_{d-1}}\Big)^{2^d+1} < \frac{\omega_d}{2^d}\Big(1-\frac{1}{d^2}\Big)^{2^d+1} . \tag{2.40}$$

Now, we just have to estimate the term $|\varphi_{d*}X_d - X_d|_{\rho_d}$. To do that, we use the inequalities of Lemma 2.8. The drawback of these inequalities is that they sometimes induce a change of radius. Therefore, we define the following intermediar radii (between ρ_d and r_d) :

$$\begin{aligned} \rho_d^{(1)} &= \rho_d(1+\frac{1}{5d^2}) \\ \rho_d^{(2)} &= \rho_d^{(1)} + \frac{3}{2}\frac{1}{2^d} \\ \rho_d^{(3)} &= \rho_d^{(2)}\big((2d)(2^d)\big)^{\frac{1}{2^d+1}} \\ \rho_d^{(4)} &= \rho_d^{(3)}(1+\frac{1}{5d^2}) \end{aligned}$$

Let us explain a little bit the definitions of these radii :

- $\rho_d^{(1)}$ (resp. $\rho_d^{(4)}$) is defined from ρ_d (resp. $\rho_d^{(3)}$) in order to use inequality (2.24) and have
$$\frac{1}{1-\frac{\rho_d}{\rho_d^{(1)}}} \sim 5d^2$$
which does not grow too quickly.
- $\rho_d^{(2)}$ is defined in order to have (recall (2.35))

$$\rho_d^{(2)} - \rho_d^{(1)} > \frac{1}{2^d} > \| X_{G_d} \|_{r_d} \tag{2.41}$$

- $\rho_d^{(3)}$ is defined in order to use inequality (2.26).
- Finally, *if d is sufficiently large*, the differences $\rho_d^{(1)} - \rho_d$, $\rho_d^{(2)} - \rho_d^{(1)}$, $\rho_d^{(3)} - \rho_d^{(2)}$ and $\rho_d^{(4)} - \rho_d^{(3)}$ are strictly smaller than $\frac{r_d}{5d^2}$ and then,

$$r_d - \rho_d^{(4)} > \frac{r_d}{d^2} - \frac{4r_d}{5d^2} > \frac{r_d}{5d^2} > \frac{1}{2^d} > \| X_{G_d} \|_{r_d} . \tag{2.42}$$

We have, by (2.24),

$$|\varphi_{d*}X_d - X_d|_{\rho_d} \le \frac{1}{1-\frac{\rho_d}{\rho_d^{(1)}}}\|\varphi_{d*}X_d - X_d\|_{\rho_d^{(1)}} = (5d^2+1)\|\varphi_{d*}X_d - X_d\|_{\rho_d^{(1)}} . \tag{2.43}$$

If x is in $D_{\rho_d^{(1)}}$ then we have

$$\begin{aligned} \big|\big(\varphi_{d*}X_d - X_d\big)(x)\big| &= \Big|\int_0^1 \varphi^t_{d*}[X_{G_d}, X_d](x)dt\Big| \\ &= \Big|\int_0^1 \Big(d\varphi_d^t([X_{G_d}, X_d])\Big)\big(\varphi_d^{-t}(x)\big)dt\Big| . \end{aligned} \tag{2.44}$$

Since $\| X_{G_d} \|_{\rho_d^{(2)}} \le \| X_{G_d} \|_{r_d} < \rho_d^{(2)} - \rho_d^{(1)}$ (by (2.41)), $\varphi_d^{-t}(x)$ belongs to $D_{\rho_d^{(2)}}$ for all $t \in [0,1]$. We then get

$$\|\varphi_{d*}X_d - X_d\|_{\rho_d^{(1)}} \le \int_0^1 \|d\varphi_d^t([X_{G_d}, X_d])\|_{\rho_d^{(2)}} dt . \tag{2.45}$$

We can write $\varphi_d^t = Id + \xi_d^t$ where the n components of ξ_d^t are functions in $\mathcal{O}_{2^d+1}$. We have the estimates

$$\begin{aligned} \|d\xi_d^t\|_{\rho_d^{(2)}} &\le |d\xi_d^t|_{\rho_d^{(2)}} \quad \text{by (2.24)} \\ &\le |\xi_d^t|_{\rho_d^{(3)}} \quad \text{by (2.26)} \\ &\le (5d^2+1)\|\xi_d^t\|_{\rho_d^{(4)}} \quad \text{by (2.24)}\,. \end{aligned}$$

If x is in $D_{\rho_d^{(4)}}$ then we can write

$$\xi_d^t(x) = \varphi_d^t(x) - x = \int_0^t X_{G_d}(\varphi_d^u(x))du\,. \tag{2.46}$$

Since $\|X_{G_d}\|_{r_d} < \frac{1}{2^d} < r_d - \rho_d^{(4)}$ (see (2.42)), we have $\varphi_d^u(x) \in D_{r_d}$ for all u in $[0,t]$. Thus $\|\xi_d^t\|_{\rho_d^{(4)}} \le \|X_{G_d}\|_{r_d} < \frac{1}{2^d}$ which gives

$$\|d\xi_d^t\|_{\rho_d^{(2)}} \le \frac{5d^2+1}{2^d}\,, \tag{2.47}$$

and then, by (2.45),

$$\|\varphi_{d*}X_d - X_d\|_{\rho_d^{(1)}} \le \big(1 + \frac{5d^2+1}{2^d}\big)\|[X_{G_d}, X_d]\|_{\rho_d^{(2)}}\,. \tag{2.48}$$

We then deduce by (2.43) that

$$|\varphi_{d*}X_d - X_d|_{\rho_d} \le (5d^2+1)\big(1 + \frac{5d^2+1}{2^d}\big)\|[X_{G_d}, X_d]\|_{r_d}\,. \tag{2.49}$$

Finally, we just have to estimate $\|[X_{G_d}, X_d]\|_{r_d}$. We first have by (2.24),

$$\|[X_{G_d}, X_d]\|_{r_d} \le |[X_{G_d}, X_d]|_{r_d}\,. \tag{2.50}$$

Now, we write

$$[X_{G_d}, X_d] = [X_{G_d}, X^{(1)}] + [X_{G_d}, X_d - X^{(1)}] = -X_{H_d} + [X_{G_d}, X_d - X^{(1)}]\,, \tag{2.51}$$

which gives, by (2.26), recalling that $\omega_d \le \frac{1}{2d}$,

$$\begin{aligned} |[X_{G_d}, X_d]|_{r_d} \le\ & |X_{H_d}|_{r_d} + |X_{G_d}|_{r_d}|X_d - X^{(1)}|_{\rho_{d-1}} \\ & + |X_{G_d}|_{\rho_{d-1}}|X_d - X^{(1)}|_{r_d}\,. \end{aligned}$$

Using (2.27) and (2.25), we get

$$|X_{G_d}|_{r_d}|X_d - X^{(1)}|_{\rho_{d-1}} < \frac{1}{\omega_d}\big(\frac{r_d}{\rho_{d-1}}\big)^{2^d+1}|X_{H_d}|_{\rho_{d-1}}|X_d - X^{(1)}|_{\rho_{d-1}}\,, \tag{2.52}$$

and then

$$|X_{G_d}|_{r_d}|X_d - X^{(1)}|_{\rho_{d-1}} < \frac{1}{2^d}\,. \tag{2.53}$$

In the same way, one can prove that

$$|X_{G_d}|_{\rho_{d-1}}|X_d - X^{(1)}|_{r_d} < \frac{1}{2^d}\,. \tag{2.54}$$

In addition, by (2.25), we get

$$|X_{H_d}|_{r_d} \le \big(\frac{r_d}{\rho_{d-1}}\big)^{2^d+1}|X_{H_d}|_{\rho_{d-1}} < \frac{\omega_d}{2^d}\,. \tag{2.55}$$

We deduce finally that

$$|\varphi_{d*}X_d - X_d|_{\rho_d} < (5d^2+1)\big(1+\frac{5d^2+1}{2^d}\big)\big(\frac{\omega_d}{2^d}+\frac{2}{2^d}\big)\,. \tag{2.56}$$

This gives the following estimate

$$|\varphi_{d*}X_d - X^{(1)}|_{\rho_d} < (5d^2+1)\big(1+\frac{5d^2+1}{2^d}\big)\big(\frac{\omega_d}{2^d}+\frac{2}{2^d}\big)+\frac{\omega_d}{2^d}\big(1-\frac{1}{2^d}\big)^{2^d+1}\,, \tag{2.57}$$

and the conclusion follows. □

End of the proof of Theorem 2.7. Let d_0 be a positive integer such that Lemmas 2.9 and 2.10 are satisfied for $d \geq d_0$. By the homothety trick (dilate a given coordinate system by appropriate linear transformations), we can assume that $|X_{d_0} - X^{(1)}|_{\rho_{d_0-1}} < 1$.

By recurrence, for all $d \geq d_0$, we have

$$D_{r_{d+1}} \subset \varphi_d(D_{\rho_d}) \subset D_{r_d}\,,$$

which give

$${\varphi_d}^{-1}(D_{r_{d+1}}) \subset D_{\rho_d}\,, \tag{2.58}$$

for all $d \geq d_0$.

We consider the sequence $(\Psi_d)_d$ given by

$$\Psi_d := \varphi_0^{-1}\circ\varphi_1^{-1}\circ\ldots\circ\varphi_d^{-1}\,.$$

Let x be an element of D_R ; recall that $R>0$ is the limit of the decreasing sequences $(r_d)_d$ and $(\rho_d)_d$. Then x belongs to the ball $D_{r_{d+1}}$ for any d and if $d > d_0$, we get by (2.58), $\varphi_d^{-1}(x) \in D_{\rho_d} \subset D_{r_d}$. In the same way, we get $\varphi_{d-1}^{-1}\big(\varphi_d^{-1}(x)\big) \in D_{\rho_{d-1}} \subset D_{r_{d-1}}$ and iterating this process, we obtain

$$\varphi_{d_0}^{-1}\big(\varphi_{d_0+1}^{-1}\circ\ldots\circ\varphi_d^{-1}(x)\big) \in D_{r_{d_0}}\,. \tag{2.59}$$

If we put $M = \sup_{z\in D_{d_0}}\big|\varphi_0^{-1}\circ\ldots\circ\varphi_{d_0-1}^{-1}(z)\big|$, we then obtain, for all x in D_R and all $d > d_0$,

$$\big|\Psi_d(x)\big| \leq M\,. \tag{2.60}$$

The theorem follows. □

3. Hamiltonian vector fields on Poisson manifolds

In this section, we study normal forms of formal or analytic Hamiltonian vector fields in the neighborhood of the origin on the Poisson manifold $(\mathbb{K}^{2l+m},\Pi)$, where

$$\Pi = \Pi_{symp} + \Pi_{trans} = \sum_{i=1}^{l}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial z_j}. \tag{3.1}$$

Here $\Pi_{symp} = \sum_{i=1}^{l}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_i}$ is the standard symplectic Poisson structure on $\mathbb{K}^{2n}$, and $\Pi_{trans} = \Pi_{\mathfrak{g}} = \frac{1}{2}\sum_{i,j,k}c_{ij}^k z_k\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial z_j}$ is the associated linear Poisson structure on the dual of a given semisimple Lie algebra $\mathfrak{g}$ of dimension m over $\mathbb{K}$.

Let $H : (\mathbb{K}^{2l+m}, 0) \to (\mathbb{K}, 0)$ be a formal or local analytic function with $H(0) = 0$, and consider the Hamiltonian vector field X_H of H with respect to the above Poisson structure $\Pi = \Pi_{symp} + \Pi_{\mathfrak{g}}$. If $X_H(0) \neq 0$, then it is well-known that it can be rectified, i.e. there is a local canonical coordinate system $(x_1, y_1, \ldots, x_l, y_l, z_1, \ldots, z_m)$ in which $H = x_1$ and $X_H = \frac{\partial}{\partial y_1}$. Here we will assume that $X_H(0) = 0$

3.1. **Formal Poincaré-Birkhoff normalization.** In this subsection, we will show that the vector field X_H can be put formally into Poincaré-Birkhoff normal form. More precisely, we have:

Theorem 3.1. *With the above notations, for any formal or local analytic function $H : (\mathbb{K}^{2l+m}, 0) \to (\mathbb{K}, 0)$, there is a formal canonical coordinate system $(\hat{x}_i, \hat{y}_i, \hat{z}_j)$, in which the Poisson structure Π has the form*

$$\Pi = \sum_{i=1}^{l} \frac{\partial}{\partial \hat{x}_i} \wedge \frac{\partial}{\partial \hat{y}_i} + \frac{1}{2} \sum_{i,j,k} c_{ij}^k \hat{z}_k \frac{\partial}{\partial \hat{z}_i} \wedge \frac{\partial}{\partial \hat{z}_j}, \tag{3.2}$$

and in which we have

$$H = H_{ss} + \tilde{H}, \tag{3.3}$$

where H_{ss} is a function such that $X_{H_{ss}}$ is the semisimple part of the linear part of X_H, and

$$\{H, H_{ss}\} = 0. \tag{3.4}$$

Proof. For any function f on $\mathbb{K}^{2l+m}$, we can write $X_f = X_f^{symp} + X_f^{\mathfrak{g}}$ where X_f^{symp} (resp. $X_f^{\mathfrak{g}}$) is the Hamiltonian vector field of f with respect to Π_{symp} (resp. $\Pi_{\mathfrak{g}}$). We can write $H = \sum_{p,q} H^{p,q}$ where $H^{p,q}$ is a polynomial of degree p in x, y and of degree q in z.

A difficulty of our situation comes from the fact that Π is not homogeneous. If $p > 0$ then $X_{H^{p,q}}$ is not a homogeneous vector field but the sum of a homogeneous vector field of degree $p + q$ (given by $X_{H^{p,q}}^{\mathfrak{g}}$) and a homogeneous vector field of degree $p + q - 1$ (given by $X_{H^{p,q}}^{symp}$). Note that $X_{H^{0,q}}^{\mathfrak{g}}$ is homogeneous of degree q and of course $X_{H^{0,q}}^{symp} = 0$.

Denoting by $X^{(1)}$ the linear part of X_H, we have

$$X^{(1)} = X_{H^{0,1}} + X_{H^{2,0}} + X_{H^{1,1}}^{symp}. \tag{3.5}$$

This linear vector field $X^{(1)}$ is *not* a Hamiltonian vector field in general, but we will show that its semisimple part is Hamiltonian.

By complexifying the system if necessary, we will assume that $\mathbb{K} = \mathbb{C}$. By a linear canonical change of coordinates, we can suppose that the semisimple part of $X_{H^{2,0}}$ is X_{h_2} where $h_2(x, y) = \sum_{j=1}^{l} \gamma_i x_j y_j$ ($\gamma_j \in \mathbb{C}$) and that the semisimple part of $X_{H^{0,1}}$ is X_{h_1} where h_1 belongs to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We write :

$$X_{h_2} = -\sum_{j=1}^{l} \gamma_j x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{l} \gamma_j y_j \frac{\partial}{\partial y_j} \quad \text{and} \quad X_{h_1} = \sum_{j=1}^{m} \alpha_j z_j \frac{\partial}{\partial z_j}. \tag{3.6}$$

Remark that we can assume that $\alpha_{s+1} = \ldots = \alpha_m = 0$ where $m-s$ is the dimension of the Cartan subalgebra $\mathfrak{h}$. Denote $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\gamma = (\gamma_1, \ldots, \gamma_l)$. If

$\lambda, \mu \in \mathbb{Z}_+^l$ and $\nu \in \mathbb{Z}_+^m$ then

$$\{h_1 + h_2, x^\lambda y^\mu z^\nu\} = (\langle \gamma, \mu - \lambda \rangle + \langle \alpha, \nu \rangle)\, x^\lambda y^\mu z^\nu \ , \tag{3.7}$$

where, for example, $\langle \alpha, \nu \rangle = \sum \alpha_j \nu_j$ denotes the standard scalar product of α and ν. In particular, $\{h_1 + h_2, .\}$ acts in a "diagonal" way on monomials.

We can arrange so that, written as a matrix, the terms coming from $X_{H^{0,1}-h_1}$, $X_{H^{2,0}-h_2}$ and $X^{symp}_{H^{1,1}}$ in the expression of $X^{(1)}$ are off-diagonal upper-triangular (and the terms coming from $X_{h_1+h_2}$ are on the diagonal).

If $\{h_1 + h_2, H^{1,1}\} = 0$, then $[X_{h_1+h_2}, X_{H^{1,1}}] = 0$, and $[X_{h_1+h_2}, X^{symp}_{H^{1,1}}] = 0$ because $X_{h_1+h_2}$ is linear and $X^{symp}_{H^{1,1}} = 0$ is the linear part of $X_{H^{1,1}}$, and, as a consequence, $X_{h_1+h_2}$ is the semisimple part of $X^{(1)}$.

If $\{h_1 + h_2, H^{1,1}\} \neq 0$ then we can apply some canonical changes of coordinates to make (the new) $H^{1,1}$ commute with $h_1 + h_2$ as follows. According to (3.7), there exist two polynomials $G^{1,1}_{(1)}$ and $\widetilde{G^{1,1}_{(1)}}$ of degree 1 in x, y and 1 in z such that

$$\{h_1 + h_2, G^{1,1}_{(1)}\} + \widetilde{G^{1,1}_{(1)}} = H^{1,1} \tag{3.8}$$

and

$$\{h_1 + h_2, \widetilde{G^{1,1}_{(1)}}\} = 0 \ . \tag{3.9}$$

Remark that, for any homogeneous polynomials $K^{0,1}, K^{2,0}, K^{1,1}$ of corresponding degrees in (x, y) and z, we have

$$[X^{symp}_{K^{1,1}}, [X^{symp}_{K^{1,1}}, X_{K^{0,1}}]] = [X^{symp}_{K^{1,1}}, [X^{symp}_{K^{1,1}}, X_{K^{2,0}}]] = 0 \ . \tag{3.10}$$

Denote by $F_1 = H^{0,1} + H^{2,0} - h_1 - h_2$ the "nilpotent part" of $H^{0,1} + H^{2,0}$.

Change the coordinate system by the push-forward of the time-1 flow $\varphi_{(1)} = \exp X_{G^{1,1}_{(1)}}$ of the Hamiltonian vector field $X_{G^{1,1}_{(1)}}$, i.e., $x_i^{new} = x_i \circ \varphi_{(1)}$ and so on. The new coordinate system is still a canonical coordinate system, because $\varphi_{(1)}$ preserves the Poisson structure Π. By this canonical change of coordinates, we can replace H by $H \circ \varphi_{(1)}$, and X_H by

$$X_H^{new} = \varphi_{(1)*} X_H = X_H + [X_{G^{1,1}_{(1)}}, X_H] + \frac{1}{2}[X_{G^{1,1}_{(1)}}, [X_{G^{1,1}_{(1)}}, X_H]] + \dots \tag{3.11}$$

It follows from (3.11) and (3.10) that the linear part of X_H^{new} is

$$X_{h_1+h_2} + X_{F_1} + X^{symp}_{\widetilde{G^{1,1}_{(1)}}} + X^{symp}_{\{G^{1,1}_{(1)}, F_1\}}. \tag{3.12}$$

In particular, by the above canonical change of coordinates, we have replaced $H^{1,1} = \{h_1 + h_2, G^{1,1}_{(1)}\} + \widetilde{G^{1,1}_{(1)}}$ by $\{G^{1,1}_{(1)}, F_1\} + \widetilde{G^{1,1}_{(1)}}$, while keeping h_1, h_2 and F_1 intact. (Note that $\{G^{1,1}_{(1)}, F_1\}$ is homogeneous of degree 1 in (x, y) and degree 1 in z).

By (3.7), (3.8), (3.9) we can write

$$G^{1,1}_{(1)} = \{h_1 + h_2, G^{1,1}_{(2)}\}, \tag{3.13}$$

which, together with $\{h_1 + h_2, F_1\} = 0$, gives

$$\{G^{1,1}_{(1)}, F_1\} = \{h_1 + h_2, \{G^{1,1}_{(2)}, F_1\}\}. \tag{3.14}$$

In other words, the new $H^{1,1}$ is $\{h_1 + h_2, \{G^{1,1}_{(2)}, F_1\}\} + \widetilde{G^{1,1}_{(1)}}$.

Repeating the above process, with the help of the time-1 flow $\varphi_{(2)} = \exp X_{\{G^{1,1}_{(2)},F_1\}}$ of the Hamiltonian vector field of $\{G^{1,1}_{(2)}, F_1\}$, we can replace $H^{1,1}$ by

$$\{h_1 + h_2, \{\{G^{1,1}_{(3)}, F_1\}, F_1\}\} + \widetilde{G^{1,1}_{(1)}}, \tag{3.15}$$

and so on.

Since F_1 is "nilpotent", by iterating the above process a finite number of times, we can replace $H^{1,1}$ by $\widetilde{G^{1,1}_{(1)}}$, i.e. make it commute with $h_1 + h_2$. So we can assume that $\{h_1 + h_2, H^{1,1}\} = 0$. Then

$$H_{ss} = h_1 + h_2 \tag{3.16}$$

is a function such that $X_{H_{ss}}$ is the semisimple part of the linear part of X_H.

Now let us deal with higher degree terms. Write

$$X_H = X_{H_1} + X_{H_2} + X_{H_3} + \dots, \tag{3.17}$$

where each H_k is of the type

$$H_k = H^{0,k} + \sum_{p\geq 1} H^{p,k+1-p}. \tag{3.18}$$

(For example, $H_1 = H^{0,1} + H^{2,0} + H^{1,1} = H_{ss} + F_1 + H^{1,1}$).

By recurrence, assume that, for some $r \geq 2$, we have $\{H_{ss}, H_k\} = 0$ for all $k \leq r - 1$. We will change H_r by a canonical coordinate transformation to get the same equality for $k = r$.

In order to put H_r in normal form, we use the same method that we used to normalize $H^{1,1}$. Similarly to (3.8), we can write

$$H_r = \{H_{ss}, K_r\} + \tilde{K}_r, \tag{3.19}$$

where K_r and $\tilde{K}_r$ are of the same type as H_r (i.e., they are sums of monomials of bidegrees $(0, r)$ and $(p, r + 1 - p)$ with $p > 0$), $\{H_{ss}, \tilde{K}_r\} = 0$. Note K_r can be written as $K_r = \{H_{ss}, K_{(2)r}\}$ for some $K_{(2)r}$.

The canonical coordinate transformation given by the time-1 flow $\exp X_{K_r}$ of X_{K_r} leaves $H_1, \dots, H_{r-1}$ intact, and changes $H_r = \tilde{K}_r + \{H_{ss}, K_r\}$ to the sum of $\tilde{K}_r$ with the terms of appropriate bidegrees in $\{K_r, F_1 + H^{1,1}\}$. We will write it as

$$\tilde{K}_r + \{K_r, F_1 + H^{1,1}\} \; mod \; (terms \; of \; higher \; bidegrees). \tag{3.20}$$

It can also be written as

$$\tilde{K}_r + \{H_{ss}, \{K_{(2)r}, F_1 + H^{1,1}\}\} \; mod \; (terms \; of \; higher \; bidegrees). \tag{3.21}$$

Now apply the canonical coordinate transformation given by $\exp X_{\{K_{(2)r},F_1+H^{1,1}\}}$, and so on. Since $F_1 + H^{1,1}$ is "nilpotent", after a finite number of coordinate transformations like that, we can change H_r to $\tilde{K}_r$, which commutes with H_{ss}. Denote the composition of these coordinate changes (for a given r) as ϕ_r. Note that ϕ_r is of the type

$$\phi_r = Id + \; terms \, of \; degree \; \geq r \tag{3.22}$$

Thus, the sequence of local or formal Poisson-structure-preserving diffeomorphisms $(\Phi_r)_{r\geq 2}$, where $\Phi_r = \phi_r \circ \ldots \circ \phi_2$, converges formally and gives a formal normalization of H.

Finally, notice that, in the real case ($\mathbb{K} = \mathbb{R}$), by an argument similar to the one given in the previous section, all canonical coordinate transformations constructed above can be chosen real.

Theorem 3.1 is proved. □

Remark 3.2. In Theorem 3.1, if we forget the Lie algebra $\mathfrak{g}$ and just keep the symplectic structure, then we recover the classical Birkhoff normalization for Hamiltonian vector fields on symplectic manifolds (see, e.g., [1, 3, 11, 16]). On the other hand, if we forget the symplectic part and just deal with $\mathfrak{g}^*$ then we get the following result as a particular case:

Corollary 3.3. *Let h be a local analytic or formal function , with $h(0) = 0$ and $dh(0) \neq 0$, on the dual $\mathfrak{g}^*$ of a semisimple Lie algebra with the associated Lie-Poisson structure. Then the Hamiltonian vector field X_h admits a formal Poincaré-Birkhoff normalization, i.e., there exists a formal coordinate system in which the Poisson structure is linear and in which we have*

$$\{h, h_{ss}\} = 0\,,$$

where h_{ss} is the semisimple part of $dh(0)$ in $\mathfrak{g}$.

Example 3.4. The monomials $x^\lambda y^\mu z^\nu$ such that $\langle \gamma, \mu - \lambda\rangle + \langle \alpha, \nu\rangle = 0$ in (3.7) may be called *resonant* terms. In the two following examples we give the set of all resonant terms in the case of a trivial symplectic part.

a) $\mathfrak{g} = sl(2)$. In this case, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is of dimension 1 and there are only two roots $\{\alpha, -\alpha\}$. Denote by z_1, z_2, z_3 a basis of $\mathfrak{g}$ (or a coordinate system on $\mathfrak{g}^*$) such that z_1 (resp. z_2) spans the root space associated to α (resp. $-\alpha$) and z_3 spans the Cartan subalgebra. We suppose that in the decomposition (3.16) we have $h_1 = z_3$. Then the resonant terms are formal power expansion in the variables $\omega = z_1 z_2$ and z_3.

b) $\mathfrak{g} = sl(3)$. Here a Cartan subalgebra $\mathfrak{h}$ is of dimension 2 (see for instance [8]). There are 6 roots $\{\alpha_1, \alpha_2, \alpha_3, -\alpha_1, -\alpha_2, -\alpha_3\}$ and the relations between these roots are of type

$$\sum_i a_i \alpha_i - \sum_i b_i \alpha_i = 0\,, \tag{3.23}$$

with

$$a_1 - b_1 = a_2 - b_2 = a_3 - b_3\,. \tag{3.24}$$

If $\{\xi_1, \xi_2, \xi_3, \zeta_1, \zeta_2, \zeta_3, z_1, z_2\}$ is a basis of $\mathfrak{g}$ such that ξ_j (resp. ζ_j) spans the root space associated to α_j (resp. $-\alpha_j$) and $\{z_1, z_2\}$ spans $\mathfrak{h}$, then supposing that in the decomposition (3.16) h_1 is a linear combination of z_1 and z_2 we may write the resonant terms as formal power expansion formed by monomials of type

$$\xi_1^{a_1} \zeta_1^{b_1} \xi_2^{a_2} \zeta_2^{b_2} \xi_3^{a_3} \zeta_3^{b_3} z_1^{u_1} z_2^{u_2} \tag{3.25}$$

with $a_1 - b_1 = a_2 - b_2 = a_3 - b_3$.

3.2. **Analytic normalization for integrable Hamiltonian systems.** Here, we assume that we work in the *complex analytic setup.*

Recall that we wrote in (3.6),

$$X_{h_2} = -\sum_{j=1}^{l} \gamma_j x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{l} \gamma_j y_j \frac{\partial}{\partial y_j} \quad \text{and} \quad X_{h_1} = \sum_{j=1}^{m} \alpha_j z_j \frac{\partial}{\partial z_j}\,, \tag{3.26}$$

and we had put $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{K}^m$ and $\gamma = (\gamma_1, \dots, \gamma_l) \in \mathbb{K}^l$.

Let $\mathcal{R} \subset \mathbb{Z}^{2l+m}$ be the sublattice of $\mathbb{Z}^{2l+m}$ formed by the vector $u \in \mathbb{Z}^{2l+m}$ written as $u = (\lambda, \mu, \nu)$, with λ and μ in $\mathbb{Z}^l$ and ν in $\mathbb{Z}^m$, and such that

$$\langle (-\gamma, \gamma, \alpha)\,,\, (\lambda, \mu, \nu)\rangle = -\sum \gamma_j \lambda_j + \sum \gamma_j \mu_j + \sum \alpha_j \nu_j = 0\,. \tag{3.27}$$

Of course, the elements (λ, μ, ν) of $\mathcal{R}$ correspond to the resonant monomials i.e. terms of type $x^\lambda y^\mu z^\nu$ such that $\{H_{ss}, x^\lambda y^\mu z^\nu\} = 0$. The dimension of $\mathcal{R}$ may be called the *degree of resonance* of H.

Now, we consider the sublattice $\mathcal{Q}$ of $\mathbb{Z}^{2l+m}$ formed by vectors $a \in \mathbb{Z}^{2l+m}$ such that $\langle a \,|\, u\rangle = 0$ for all u in $\mathcal{R}$. Let $\{\rho^{(1)}, \dots, \rho^{(r)}\}$ be a basis of $\mathcal{Q}$. The dimension r of $\mathcal{Q}$ is called the *toric degree* of X_H at 0. We then put for all $k = 1, \dots, r$

$$Z_k = \sum_{j=1}^{l} \rho_j^{(k)} x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{l} \rho_{l+j}^{(k)} y_j \frac{\partial}{\partial y_j} + \sum_{j=1}^{m} \rho_{2l+j}^{(k)} z_j \frac{\partial}{\partial z_j}\,. \tag{3.28}$$

The vector fields $iZ_1, \dots, iZ_r$ ($i = \sqrt{-1}$) are periodic with a real period in the sense that the real part of these vector fields is a periodic real vector field in $\mathbb{C}^{2l+m} = \mathbb{R}^{2(2l+m)}$; they commute pairwise and are linearly independent almost everywhere. Moreover, the vector field $X_{H_{ss}}$ is a linear combination (with coefficients in $\mathbb{C}$ a priori) of the iZ_k. We also have the trivial following property

Lemma 3.5. *If Λ is a p-vector ($p \geq 0$) then we have the equivalence*

$$[X_{H_{ss}}, \Lambda] = 0 \Leftrightarrow [Z_k, \Lambda] = 0 \;\forall k = 1, \dots, r$$

Proof : We just give here the idea of the proof of this lemma supposing that Λ is a 2-vector for instance ; but it works exactly in the same way for other multivectors. If Y is a vector field of type $\sum_{j=1}^{l} a_j x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{l} a_{l+j} y_j \frac{\partial}{\partial y_j} + \sum_{j=1}^{m} a_{2l+j} z_j \frac{\partial}{\partial z_j}$ and Λ of type $\Lambda = x^\lambda y^\mu z^\nu \frac{\partial}{\partial x_u} \wedge \frac{\partial}{\partial x_v}$, then

$$[Y, \Lambda] = \langle a\,,\, (\lambda, \mu, \nu) - (1_u, 1_v, 0)\rangle \Lambda \tag{3.29}$$

where $1_u = (0, \dots, 1, \dots, 0)$ is the vector of $\mathbb{Z}^l$ whose unique nonzero component is the u-component. Of course we get the same type of relation with 2-vectors in $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z}$, $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}$, etc.... Using this remark and the definition of the vectors $\rho^{(1)}, \dots, \rho^{(r)}$, the equivalence of the lemma is direct. □

According to this Lemma, since $X_{H_{ss}}$ preserves the Poisson structure, the vector fields $Z_1, \dots, Z_r$ will be Poisson vector fields for $(\mathbb{C}^{2l} \times \mathbb{C}^m, \{\,,\,\}_{symp} + \{\,,\,\}_{\mathfrak{g}^*})$. But according to Proposition 4.1 (see the Appendix), the Poisson cohomology space

$H^1(\mathbb{C}^{2l} \times \mathbb{C}^m, \{\ ,\ \}_{symp} + \{\ ,\ \}_{\mathfrak{g}})$ is trivial therefore, these vector fields are actually Hamiltonian :

$$Z_k = X_{F_k} \quad \forall k = 1, \ldots, r\,. \tag{3.30}$$

Finally, we have r periodic Hamiltonian linear vector fields iZ_k which commute pairwise, are linearly independent almost everywhere. The real parts of these vector fields generate a Hamiltonian action of the real torus $\mathbb{T}^r$ on $(\mathbb{C}^{2n} \times \mathbb{C}^m, \{\ ,\ \}_{symp} + \{\ ,\ \}_{\mathfrak{g}})$. With all these notations, we can state the following proposition :

Proposition 3.6. *With the above notation, the following conditions are equivalent :*

a) There exists a holomorphic Poincaré-Birkhoff normalization of X_H in a neighborhood of 0 in $\mathbb{C}^{2l+m}$.

b) There exists an analytic Hamiltonian action of the real torus $\mathbb{T}^r$ in a neighborhood of 0 in $\mathbb{C}^{2l+m}$, which preserves X_H and whose linear part is generated by the (Hamiltonian) vector fields iZ_k, $k = 1, \ldots, r$.

Proof : Suppose that H is in holomorphic Poincaré-Birkhoff normal form. By Lemma 3.5, since $\{H, H_{ss}\} = 0$, the vector fields iZ_k preserve X_H.

Conversely, if the point *b)* is satisfied, then according to the holomorphic version of the Splitting Theorem (see [10]) we can consider that the action of the torus is "diagonal", i.e. the product of an action on $(\mathbb{C}^{2l}, \{\,,\}_{symp})$ by an action on $(\mathbb{C}^m, \{\ ,\ \}_{\mathfrak{g}})$ and moreover that the action on the symplectic part is linear. According to Proposition 4.2 (in Appendix), we can linearize the second part of the action by a Poisson diffeomorphism. We then can consider that the action of $\mathbb{T}^r$ is generated by the vector fields iZ_k, $k = 1, \ldots, r$. This action preserves X_H then we have $[iZ_k, X_H] = 0$ for all k. To conclude, just recall that $X_{H_{ss}}$ is a linear combination of the Z_k. □

Now, we are going to use Proposition 3.6 to clarify a link between the integrability of a Hamiltonian vector field X_H on an analytic Poisson manifold $(\mathbb{K}^n, \{\ ,\ \})$ and the existence of a convergent Poincaré-Birkhoff normalization. Recall first the definition (see for instance [15]) of the word *integrability* used here :

Definition 3.7. A Hamiltonian vector field X_H on a Poisson manifold (M, Π) (of dimension n) is called *integrable* (in the generalized Liouville sense) if there exist p $(1 \leq p \leq n)$ Hamiltonian vector fields $X_1 = X_H, X_2, \ldots, X_p$ and $n-p$ functions $f_1, \ldots, f_{n-p}$ such that

a) The vector fields commute pairwise, i.e.

$$[X_i, X_j] = 0 \;\forall i, j = 1, \ldots, p\,, \tag{3.31}$$

and they are linearly independent almost everywhere, i.e.

$$X_1 \wedge \ldots \wedge X_p \neq 0\,. \tag{3.32}$$

b) The functions are common first integrals for $X_1, \ldots, X_p$:

$$X_i(f_j) = 0 \;\forall i, j\,, \tag{3.33}$$

and they are functionally independent almost everywhere :

$$df_1 \wedge \ldots \wedge df_{n-p} \neq 0\,. \tag{3.34}$$

Of course this definition has a sense in the smooth category as well as in the analytic category. We can speak about *smooth* or *analytic* integrability.

Theorem 3.8. *Any analytically integrable Hamiltonian vector field in a neighborhood of a singularity on an analytic Poisson manifold admits a convergent Poincaré-Birkhoff normalization*

Proof : We can assume (see the beginning of the section) that we work in the neighborhood of 0 in

$$(\mathbb{C}^{2l+m}, \{\ ,\ \}) = (\mathbb{C}^{2l}, \{\ ,\ \}_{symp}) \times (\mathfrak{g}^*, \{\ ,\ \}_{\mathfrak{g}})$$

where $\{\ ,\ \}_{symp}$ is a symplectic Poisson structure and $\mathfrak{g}$ is a semisimple Lie algebra and $\{\ ,\ \}_{\mathfrak{g}}$ the standard Lie-Poisson structure on $\mathfrak{g}^*$. If X_H is integrable then, forgetting one moment the Hamiltonian feature, Theorem 1.1 and Proposition 2.1 in [14] give the existence of an action of a real torus $\mathbb{T}^r$ on $(\mathbb{K}^{2l+m}, 0)$ generated by vector fields $Y_1, \ldots, Y_r$ (r is the toric degree of X_H) where the linear parts of these vector fields are the iZ_k (see 3.28), and which preserves X_H. Moreover, the semisimple part X_H^{ss} of X_H is a linear combination of the Y_j : $X_H^{ss} = \sum_j \beta_j Y_j$ without any resonance relation between the β_j. Now, let us recall that we work in a Poisson manifold with a Hamiltonian vector field. Since the vector field X_H preserves the Poisson structure, its semisimple part also does and then we will have $[Y_j, \Pi] = 0$ for all $j = 1, \ldots, r$. Therefore, the action of the torus also preserves the Poisson structure. Proposition 3.6 allows to conclude. $\square$

Remark 3.9. If we suppose that H and the Poisson structure are real then it is natural to ask if all that we made is still valid. Note that in this case, we can consider H (and the Poisson structure) as complex analytic, with real coefficients.

Actually, in the same way as in [14, 16], we conjecture that we have the equivalence:

A real analytic Hamiltonian vector field X_H with respect to a real analytic Poisson structure admits a local real analytic Poincaré-Birkhoff normalization iff it admits a local holomorphic Poincaré-Birkhoff normalization.

4. Appendix

In this appendix, we give a proof of auxiliary results used in the previous sections. We first compute the first Poisson cohomology space of the Poisson manifold we consider in Section 3. Suppose that Π_S is a symplectic (i.e. nondegenerate) Poisson structure on $\mathbb{K}^{2l}$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). If $(x_1, \ldots, x_l, y_1, \ldots, y_l)$ are coordinates on $\mathbb{K}^{2l}$, we can write

$$\Pi_S = \sum_{i=1}^{l} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}\,.$$

Let $\mathfrak{g}$ be a m-dimensional (real or complex) semisimple Lie algebra and consider $\Pi_{\mathfrak{g}}$ the corresponding linear Poisson structure on $\mathbb{K}^m$. Suppose that $(z_1, \ldots, z_m)$ are coordinates on $\mathbb{K}^m$. We then show the following :

Proposition 4.1. *Under the hypotheses above, if* $\mathrm{H}^1(\mathbb{K}^{2l} \times \mathbb{K}^m, \Pi_S + \Pi_{\mathfrak{g}})$ *denotes the first (formal or analytic) Poisson cohomology space of the product of* $(\mathbb{K}^{2l}, \Pi_S)$

by $(\mathbb{K}^m, \Pi_{\mathfrak{g}})$ *then*

$$\mathrm{H}^1(\mathbb{K}^{2l} \times \mathbb{K}^m, \Pi_S + \Pi_{\mathfrak{g}}) = \{0\}\,.$$

Proof : If X is a (formal or analytic) vector field on $\mathbb{K}^{2l} \times \mathbb{K}^m$ we write $X = X^S + X^{\mathfrak{g}}$ where X^S is a vector field which only has components in the $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ and, in the same way, $X^{\mathfrak{g}}$ only has components in the $\frac{\partial}{\partial z_i}$. Before computing the Poisson cohomology space, let us make the following two remarks :

If $[X^S, \Pi_S] = 0$ then $X^S = [f, \Pi_S]$ where f is a (formal or analytic) function on $\mathbb{K}^{2l+m}$. Indeed, recalling that (because Π_S is symplectic) the Poisson cohomology of $(\mathbb{K}^{2l}, \Pi_S)$ is isomorphic to the de Rham cohomology of $\mathbb{K}^{2l}$ (see for instance [12]), the relation $[X^S, \Pi_S] = 0$ may be translated as $d\alpha = 0$ where α is a 1-form on $\mathbb{K}^{2l}$ depending (formally or analytically) on parameters $z_1, \dots, z_m$. Then we can write $\alpha = df$ where f is a function on $\mathbb{K}^{2l}$ depending (formally or analytically) on parameters $z_1, \dots, z_m$.

In the same way, if $[X^{\mathfrak{g}}, \Pi_S] = 0$ then, writing $X^{\mathfrak{g}} = \sum_i X_i^{\mathfrak{g}}(x, y, z)\frac{\partial}{\partial z_i}$, we get $[X_i^{\mathfrak{g}}, \Pi_S] = 0$ for all i. Thus, each $X_i^{\mathfrak{g}}$ depends only on z. Indeed, here $X_i^{\mathfrak{g}}$ may be seen as a function on $\mathbb{K}^{2l}$ depending (formally or analytically) on parameters $z_1, \dots, z_m$ such that $dX_i^{\mathfrak{g}} = 0$.

Now if $X = X^S + X^{\mathfrak{g}}$ is a vector field on $\mathbb{K}^{2l} \times \mathbb{K}^m$, it is easy to see that $[X, \Pi_S + \Pi_{\mathfrak{g}}] = 0$ is equivalent to the three equations

$$0 = [X^S, \Pi_S] \tag{4.1}$$
$$0 = \left[X^S, \Pi_{\mathfrak{g}}\right] + [X^{\mathfrak{g}}, \Pi_S] \tag{4.2}$$
$$0 = [X^{\mathfrak{g}}, \Pi_{\mathfrak{g}}] \tag{4.3}$$

According to the first remark we made above, equation (4.1) gives $X^S = [f, \Pi_S]$ where f is a (formal or analytic) function on $\mathbb{K}^{2l+m}$. Now, replacing X^S by $[f, \Pi_S]$ in (4.2) and using the graded Jacobi identity of the Schouten bracket, we get

$$\left[X^{\mathfrak{g}} - [f, \Pi_{\mathfrak{g}}], \Pi_S\right] = 0\,. \tag{4.4}$$

Since $X^{\mathfrak{g}} - [f, \Pi_{\mathfrak{g}}]$ is a vector field which only has component in $\frac{\partial}{\partial z}$, the second remark we made above gives

$$X^{\mathfrak{g}} = [f, \Pi_{\mathfrak{g}}] + Y \tag{4.5}$$

where Y is a vector field on $\mathbb{K}^m$ (i.e. only has components in $\frac{\partial}{\partial z}$ and whose coefficients are functions of z). Finally, (4.3) gives $[Y, \Pi_{\mathfrak{g}}] = 0$ i.e. Y is a 1-cocycle for the Poisson cohomology of $(\mathbb{K}^m, \Pi_{\mathfrak{g}})$. Since the Lie algebra $\mathfrak{g}$ is semisimple, the Poisson cohomology space $\mathrm{H}^1(\mathbb{K}^m, \Pi_{\mathfrak{g}})$ is trivial (see for instance [4]). We then obtain $Y = [h, \Pi_{\mathfrak{g}}]$ where h is a function on $\mathbb{K}^m$.

To resume, we get

$$X = X^S + X^{\mathfrak{g}} = [f + h, \Pi_S + \Pi_{\mathfrak{g}}]\,, \tag{4.6}$$

which means that X is a 1-cobord for the Poisson cohomology of $(\mathbb{K}^{2l} \times \mathbb{K}^m, \Pi_S + \Pi_{\mathfrak{g}})$. □

The second result is an analytic version of a smooth linearization theorem due to V. Ginzburg. In the Appendix of [6], he states that the G-action of a compact Lie group on a Poisson manifold (P, Π) (everything is smooth here), fixing a point x of P and such that the Poisson structure is linearizable at x, can be linearized by a diffeomorphism which preserves the Poisson structure. Here, we state the following :

Proposition 4.2. *Consider an analytic action of a compact (analytic) Lie group on $(\mathbb{K}^n, \Pi)$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$), where Π is an analytic Poisson structure on $\mathbb{K}^n$. Suppose that the action fix the origin 0 and that the Poisson structure is linearizable at 0. Then, the action can be linearized by a Poisson diffeomorphim.*

Proof : The proof is the same as in the smooth case : we use Moser's path method. If g is an element of G, we put φ^g the corresponding diffeomorphism of $\mathbb{K}^n$ and φ^g_{lin} its linear part at 0. We construct a path of analytic actions of G on $(\mathbb{K}^n, \Pi)$ given by the following diffeomorphisms :

$$\varphi^g_t(x) = \begin{cases} \varphi^g(tx)/t & \text{if} \quad 0 < t \leq 1 \\ \varphi^g_{lin}(x) & \text{if} \quad t = 0 \end{cases}$$

for any g in G and x in $\mathbb{K}^n$. These actions preserve Π and fix 0. We want now to show that there exists a path of diffeomorphisms ψ_t, with $\psi_0 = Id$, preserving the Poisson structure Π and such that

$$\psi_t \circ \varphi^g_t \circ \psi_t^{-1} = \varphi^g_0 = \varphi^g_{lin} \,, \tag{4.7}$$

for all $t \in [0, 1]$ and all g in G.

Let $C_t(g)$ be the time-depending vector field associated to φ^g_t :

$$C_t(g)(\varphi^g_t(x)) = \frac{\partial \varphi^g_t}{\partial t}(x) \,. \tag{4.8}$$

Derivating the condition (4.7), we are led to look for a time-depending vector field X_t (corresponding to ψ_t) verifying

$$C_t(g) = {\varphi^g_t}_* X_t - X_t \,, \tag{4.9}$$

for all $t \in [0, 1]$ and all g in G.

We put

$$X_t = -\int_G {\varphi^h_t}_* C_t(h) dh \,, \tag{4.10}$$

dh is a bi-invariant Haar measure on G such that the volume of G is 1. This vector field is analytic and depends smoothly on t. Moreover, since each $C_t(h)$ preserves the Poisson structure Π, so does X_t. Finally, one can check that X_t satisfies the condition (4.9). □

Proof of Proposition 2.6. We denote by α the linear application from the Cartan subalgebra $\mathfrak{h}$ to $\mathbb{K}^n$ defined by $\alpha(h) = (\alpha_1(h), \ldots, \alpha_n(h))$ for any h in $\mathfrak{h}$ and by W its image. We show that the subset of W formed by the elements γ such that the $\omega_d(\gamma)$ (defined as in (2.20) replacing $\langle \alpha_i \,, \, h_1 \rangle$ by γ_i) do not satisfy the ω-condition is of measure 0 (in W). Since α is a linear surjection from $\mathfrak{h}$ to W, it will show Proposition 2.6.

Note that if $\gamma \in \mathbb{K}^n$ satisfies the condition (which is a condition of type "Siegel")

$$(\exists c > 0)\ (\forall \lambda \in \mathbb{Z}_+^n),\ s.t.\ \big||\lambda| - 1 + \langle \gamma\,,\, \lambda\rangle\big| \geq \frac{c}{|\lambda|^s}\,, \tag{4.11}$$

where $s > n$, then $\omega_d(\gamma)$ satisfies the ω-condition (2.21). We then show that the set of the γ in W which do not satisfy the condition (4.11) is of measure 0 in W.

For any positive integer k and any positive real number c, if $\|\ \|$ denotes the norm associated to $\langle\,,\,\rangle$, we put

$$\begin{array}{rcl} W_k & = & \big\{\gamma \in W \;\big|\; \|\gamma\| \leq k\big\} \\ V_c & = & \big\{\gamma \in \mathbb{K}^n \;\big|\; (\exists \lambda \in \mathbb{Z}_+^n)\ s.t.\ \big||\lambda| - 1 + \langle \gamma\,,\, \lambda\rangle\big| \,\leq \dfrac{c}{|\lambda|^s}\ \big\} \\ V & = & \cap_{c>0} V_c \end{array}$$

Actually, we show here that $W_1 \cap V$ is of measure 0 but the same technic works to prove that $W_k \cap V$ is also of measure 0 for each k. Therefore $\cup_k(W_k \cap V)$ is of measure 0 too, which proves the proposition.

Now, for any λ in $\mathbb{Z}_+^n$ we consider the affine subspace $\mathcal{V}_\lambda$ of $\mathbb{K}^n$ formed by the vectors γ such that $\langle \gamma\,,\, \lambda\rangle = 1 - |\lambda|$ and we put for $c > 0$,

$$\mathcal{V}_{\lambda,c} = \big\{\gamma \in \mathbb{K}^n\,;\ \big||\lambda| - 1 + \langle \gamma\,,\, \lambda\rangle\big| \,\leq \frac{c}{|\lambda|^s}\ \big\}\,. \tag{4.12}$$

This last set is like a tubular neighborhood of $\mathcal{V}_\lambda$ of thickness $\frac{2c}{|\lambda|^s}$. We look now at $K_{\lambda,c} = \mathcal{V}_{\lambda,c} \cap W_1$. If it is not empty, it is a kind of "band" in W_1 of thickness smaller than $S\frac{2c}{|\lambda|^s}$ where S is a positive constant which only depends on the dimension of W (and on the metric). Therefore, we get

$$Vol(W_1 \cap V_c) \leq \sum_{\lambda \in \mathbb{Z}_+^n} Vol(K_{\lambda,c}) \leq cS \sum_{\lambda \in \mathbb{Z}_+^n} \frac{1}{|\lambda|^s}\,. \tag{4.13}$$

This latest sum converges (because $s > n$) and we then get $Vol(W_1 \cap V) = Vol(\cap_{c>0} W_1 \cap V_c) = 0$. □

Proof of Lemma 2.8. a) The first inequality of (2.24) is obvious. To prove the second one, we use the Cauchy inequality

$$|a_\lambda| \leq \frac{\sup_{z \in D_\rho} |f(z)|}{\rho^{|\lambda|}}$$

for all λ, which induces $|a_\lambda|{\rho'}^{|\lambda|} \leq \|f\|_\rho \big(\frac{\rho'}{\rho}\big)^{|\lambda|}$. The inequality follows.

The point *b)* is obvious.

c) If $f = \sum_{|\lambda| \geq q} a_\lambda x^\lambda$ then

$$\begin{array}{rcl} \Big|\dfrac{\partial f}{\partial x_j}\Big|_{\rho'} & = & \displaystyle\sum_{|\lambda| \geq q} \lambda_j |a_\lambda| {\rho'}^{|\lambda|-1} \\ & = & \displaystyle\sum_{|\lambda| \geq q} |a_\lambda| \rho^{|\lambda|} \times \frac{\lambda_j}{\rho'} \Big(\frac{\rho'}{\rho}\Big)^{|\lambda|} \end{array}$$

When $\rho' = \left(\frac{1}{(2d)(2^d)}\right)^{1/(2^d+1)} \rho \geq R > 0$, $q = 2^d + 1$ and $d \geq 1$, each number $\frac{\lambda_j}{\rho'}\left(\frac{\rho'}{\rho}\right)^{|\lambda|}$ can be majored by

$$\frac{2^d+1}{R}\left(\left(\frac{1}{(2d)(2^d)}\right)^{1/(2^d+1)}\right)^{2^d+1} .$$

It is easy to see that these numbers are smaller than 1, provided that d is large enough. $\square$

Proof of Lemma 2.9. a) Since the sequence $(r_d)_d$ decreases and converges to a positive real number $R > 0$, we have $r_d > R$ for all d. We write $r_d - \rho_d = r_d \frac{1}{d^2} > \frac{R}{d^2}$, thus for d sufficiently large, we get $r_d - \rho_d > \frac{1}{2^d}$.

b) We have $\rho_d - r_{d+1} = \rho_d \left[1 - \left(\frac{\omega_{d+1}}{2^{d+1}}\right)^{\frac{1}{2^{d+1}+1}}\right]$. Since the sequence $(\rho_d)_d$ decreases and converges to $R > 0$, we have $\rho_d > R > 0$ for all d. We then show that if d is sufficiently large, then

$$R\left[1 - \left(\frac{\omega_{d+1}}{2^{d+1}}\right)^{\frac{1}{2^{d+1}+1}}\right] > \frac{1}{2^d} .$$

We have $\left(\frac{\omega_{d+1}}{2^{d+1}}\right)^{\frac{1}{2^{d+1}+1}} = e^{\gamma_d}$ where $\gamma_d = \frac{1}{2^{d+1}+1}\ln\left(\frac{\omega_{d+1}}{2^{d+1}}\right)$. By the ω-condition, the sequence $(\gamma_d)_d$ converges to 0 and is negative for all d sufficiently large. Then, if ε is a small positive real number (for instance $\varepsilon = 1/2$), we have for all d sufficiently large, $1 - e^{\gamma_d} > -(1-\varepsilon)\gamma_d$. We deduce that

$$R(1 - e^{\gamma_d}) > -R(1-\varepsilon)\frac{\ln\left(\frac{\omega_{d+1}}{2^{d+1}}\right)}{2^{d+2}} \tag{4.14}$$

which gives

$$R(1 - e^{\gamma_d}) > \frac{1}{2^d}\left[\frac{R(1-\varepsilon)}{4}\left(\ln(2^{d+1}) - \ln\omega_{d+1}\right)\right] . \tag{4.15}$$

Therefore, for d sufficiently large, $R(1 - e^{\gamma_d}) > \frac{1}{2^d}$. $\square$

References

[1] G.D. Birkhoff, *Dynamical Systems*, 2nd ed., AMS Colloq. Publ., No 9, Providence, 1927.

[2] A.D. Bruno, *Analytic form of differential equations*, Trans. Moscow Math. Soc. **25** (1971), 131–288.

[3] A.D. Bruno, *Local methods in nonlinear differential equations*, Springer-Verlag, Berlin, 1989.

[4] J. Conn, *Normal forms for analytic Poisson structures*, Ann. of Math. (2) **119** (1984), No. 3, 577–601.

[5] J.-P. Dufour, N. T. Zung, Poisson structures and their normal forms, Progress in Mathematics, Vol. 242, Birkhäuser, 2005.

[6] V. Ginzburg, *Momentum mappings and Poisson cohomology*, Internat. J. Math. **7** (1996), no. 3, 329–358.

[7] P. Dazord, A. Lichnerowicz and Ch.-M. Marle, *Structure locale des variétés de Jacobi*, J. Math. Pures Appl. **70** (1991), no. 1, 101–152.

[8] A.W. Knapp, Lie groups beyond an introduction, Progress in Mathematics, Vol. 140, Birkhäuser, 2002.

[9] J. Martinet, *Normalisation des champs de vecteurs holomorphes (d'après A.D. Brjuno)*, Séminaire Bourbaki **564** (1980).
[10] E. Miranda and N.T. Zung, *Equivariant normal forms of Poisson structures*, in preparation.
[11] C.L. Siegel and J. Moser, *Lectures on celestial mechanics*, Springer-Verlag, 187, 1971.
[12] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Math., vol. 118, Birkhauser Boston, 1994.
[13] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** (1983), no. 3, 523–557.
[14] N.T. Zung, *Convergence versus integrability in Poincaré-Dulac normal forms*, Math. Res. Lett. **9** (2002), 217-228.
[15] N.T. Zung, *Torus action and integrable systems*, preprint math.DS/0407455 (2004).
[16] N.T. Zung, *Convergence versus integrability in Birkhoff normal forms*, Ann. of Math. **161** (2005), No. 1, 139-154.

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