The RS-IMEX scheme for the low-Froude shallow water equations¶

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joint work with: *Sebastian Noelle*

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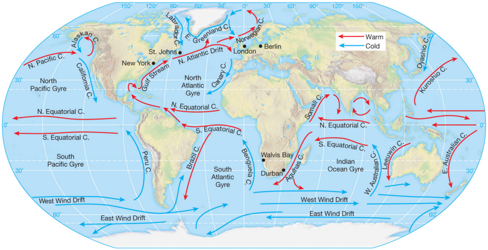
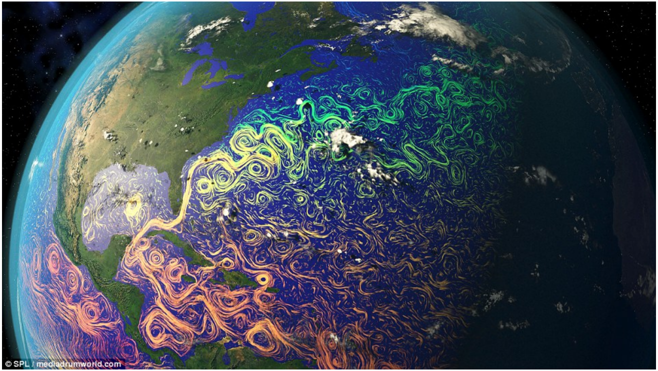
Outline

- Introduction
- RS-IMEX scheme
- 1d SWE
- 2d SWE
  - Numerical experiments
- 2d RSWE
  - Numerical experiments
- Recent progress

Outline


Warm
Cold
Alaskan C.
N. Pacific C.
California C.
North Pacific Gyre
N. Equatorial C.
S. Equatorial C.
South Pacific Gyre
West Wind Drift
East Wind Drift
Labrador C.
E. Greenland C.
St. Johns
New York
Gulf Stream
N. Atlantic Drift
North Atlantic Gyre
Canary C.
N. Equatorial C.
S. Equatorial C.
Peru C.
Brazil C.
South Atlantic Gyre
Benguela C.
Norwegian C.
London
Berlin
Somali C.
N. Equatorial C.
S. Equatorial C.
Walvis Bay
Durban
Agulhas C.
Indian Ocean Gyre
West Wind Drift
East Wind Drift
W. Australian C.
Leeuwin C.
E. Australian C.
Oyashio C.
Kuroshio C.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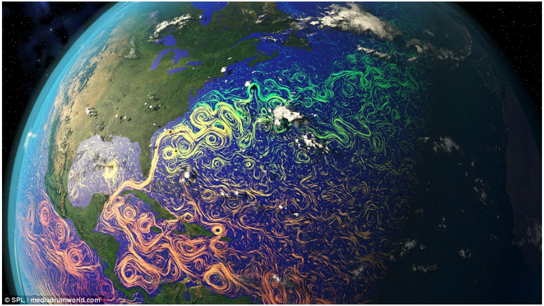
- g : gravity acceleration
- f : Coriolis parameter
- p : pressure
- $\boldsymbol{x} := (x_1, x_2, x_3)^T$
- $\boldsymbol{u} := (u_1, u_2, u_3)^T$



Compressible Euler Equations

$$\begin{cases} \partial_t \varrho + \mathrm{div}_{\boldsymbol{x}}(\varrho \boldsymbol{u}) = 0 \\ \partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_{\boldsymbol{x}}\left(\varrho \boldsymbol{u} \otimes \boldsymbol{u} + p\mathbb{I}_3\right) = \overbrace{-\varrho g \hat{\boldsymbol{k}}}^{\text{Gravitation}} \; \overbrace{-\varrho f \hat{\boldsymbol{k}} \times \boldsymbol{u}}^{\text{Coriolis}} \end{cases}$$

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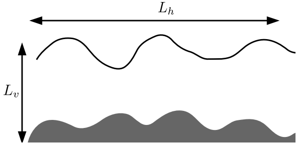
- shallowness $\implies \boxed{\partial_{x_3} p = -\varrho g} \implies u_3 \sim \mathcal{O}(\delta)$

$L_h \sim 10^2$–10^3 km
$L_v \sim 1$–5 km
$\delta := \frac{L_h}{L_v} \sim 10^{-3} - 10^{-2}$



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- incompressibility $\Longrightarrow \operatorname{div}_{\boldsymbol{x}} \boldsymbol{u} = 0$
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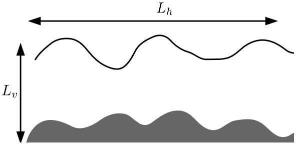

- boundary conditions:
  - no normal flow at bottom
  - free surface at top

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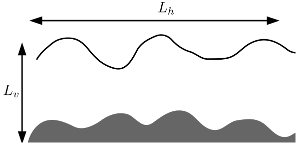

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Rotating Shallow Water Equations in horizontal plane (x_1, x_2)

$$\begin{cases} \partial_t h + \operatorname{div}_{\boldsymbol{x}}(h\boldsymbol{u}) = 0 \\ \partial_t(h\boldsymbol{u}) + \operatorname{div}_{\boldsymbol{x}}\left(h\boldsymbol{u} \otimes \boldsymbol{u} + \frac{gh^2}{2}\mathbb{I}_2\right) = -gh\nabla_{\boldsymbol{x}}\eta^b - fh\boldsymbol{u}^\perp \end{cases}$$

η^b is the bottom function, $\boldsymbol{u}^\perp := (-u_2, u_1)$.

Non-dimensionalisation

$$\hat{\boldsymbol{x}} := \frac{\boldsymbol{x}}{L_\circ}, \quad \hat{t} := \frac{t}{t_\circ}, \quad \hat{\boldsymbol{u}} := \frac{\boldsymbol{u}}{u_\circ}, \quad \hat{h} := \frac{h}{H_\circ}, \quad \hat{\eta}^b := \frac{\eta^b}{H_\circ}, \quad t_\circ = \frac{L_\circ}{u_\circ}$$

Non-dimensionalised RSWE

$$\begin{cases} St\,\partial_{\hat{t}}\hat{h} + \mathrm{div}_{\hat{\boldsymbol{x}}}(\hat{h}\hat{\boldsymbol{u}}) = 0 \\ St\,\partial_{\hat{t}}(\hat{h}\hat{\boldsymbol{u}}) + \mathrm{div}_{\hat{\boldsymbol{x}}}\left(\hat{h}\hat{\boldsymbol{u}} \otimes \hat{\boldsymbol{u}} + \frac{\hat{h}^2}{2Fr^2}\mathbb{I}_2\right) = -\frac{\hat{h}}{Fr^2}\nabla_{\hat{\boldsymbol{x}}}\hat{\eta}^b - \frac{\hat{h}}{Ro}\hat{\boldsymbol{u}}^\perp \end{cases}$$

We consider two **singular limits**:

- **Non-rotating:** $f = 0$ and $Fr = \varepsilon \ll 1$
- **Rotating:** $Fr \sim Ro = \varepsilon \ll 1$

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$$\hat{\boldsymbol{x}} := \frac{\boldsymbol{x}}{L_\circ}, \quad \hat{t} := \frac{t}{t_\circ}, \quad \hat{\boldsymbol{u}} := \frac{\boldsymbol{u}}{u_\circ}, \quad \hat{h} := \frac{h}{H_\circ}, \quad \hat{\eta}^b := \frac{\eta^b}{H_\circ}, \quad t_\circ = \frac{L_\circ}{u_\circ}$$

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$$St := \frac{L_\circ/u_\circ}{t_\circ} = 1, \quad \boxed{Fr := \frac{u_\circ}{\sqrt{gH_\circ}}}, \quad \boxed{Ro := \frac{u_\circ}{L_\circ f}} = \frac{f^{-1}}{t_\circ}$$

We consider two **singular limits**:

- **Non-rotating:** $f = 0$ and $Fr = \varepsilon \ll 1$
- **Rotating:** $Fr \sim Ro = \varepsilon \ll 1$

Analytical/Numerical issues

What happens if $\varepsilon \to 0$?

- Continuous level: **Does the** $\lim_{\varepsilon \to 0}$ **system exist?**
  - comp. Euler $\to$ incomp. Euler [Klainerman and Majda, 1981]
  - RSWE $\xrightarrow[\text{distinguished limit}]{\text{quasi-geostrophic}}$ quasi-geostrophic equations [Majda, 2003]

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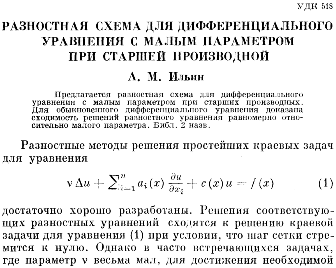
- Discrete/numerical level: **How does the scheme behave?**
  - Stiffness: wave speeds $\sim \mathcal{O}(\frac{1}{\varepsilon})$
    - Explicit: CFL condition $\Delta t \lesssim \varepsilon \, \Delta x$
    - Implicit: diffuses slow material waves
  - Inconsistency with the limit system

Asymptotic Preserving (AP) schemes

Introduced by [Jin, 1999]

- [Il'in, 1969]
- [Larsen et al., 1987]



УДК 518

РАЗНОСТНАЯ СХЕМА ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С МАЛЫМ ПАРАМЕТРОМ ПРИ СТАРШЕЙ ПРОИЗВОДНОЙ

А. М. Ильин

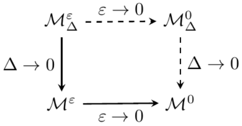
Предлагается разностная схема для дифференциального уравнения с малым параметром при старших производных. Для обыкновенного дифференциального уравнения доказана сходимость решений разностного уравнения равномерно относительно малого параметра. Библ. 2 назв.

Разностные методы решения простейших краевых задач для уравнения

$$\nu\,\Delta u + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + c(x)\,u = f(x) \qquad (1)$$

достаточно хорошо разработаны. Решения соответствующих разностных уравнений сходятся к решению краевой задачи для уравнения (1) при условии, что шаг сетки стремится к нулю. Однако в часто встречающихся задачах, где параметр ν весьма мал, для достижения необходимой

- **Asymptotic Efficiency (AEf):** uniform CFL, efficient implicit step
- **Asymptotic Consistency (AC):** consistent with the asymptotic system as $\varepsilon \to 0$



- **Asymptotic Stability (AS):** uniformly stable in ε

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Reference Solution IMEX scheme (I)

Hyperbolic system of balance laws for $\boldsymbol{U} \in \mathbb{R}^q$ in $\Omega \subset \mathbb{R}^d$

$$\partial_t \boldsymbol{U}(t, \boldsymbol{x}; \varepsilon) + \mathrm{div}_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{U}, t, \boldsymbol{x}; \varepsilon) = \boldsymbol{S}(\boldsymbol{U}, t, \boldsymbol{x}; \varepsilon),$$

- IMEX: stiff + non-stiff
  implicit + explicit
- **How to decompose?**
  - non-linear stiff part $\rightarrow$ non-linear iteration [Degond and Tang, 2011]
  - **linearly-implicit methods!**

$$\partial_t \boldsymbol{U} = \mathcal{N}(\boldsymbol{U}) \quad \Longrightarrow \boxed{\partial_t \boldsymbol{U} = \mathcal{L}(\boldsymbol{U}) + (\mathcal{N} - \mathcal{L})(\boldsymbol{U})}$$

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- for ODEs [Rosenbrock, 1963]:

$$x'(t) = f(x) \quad \Longrightarrow \boxed{x'(t) = f'(x)\,x(t) + \left[f(x(t)) - f'(x(t))\,x(t)\right]}$$

- for Euler with gravity [Restelli, 2007]
- *penalization method* [Filbet and Jin, 2010]: $\partial_t f + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f = \frac{1}{\varepsilon} Q(f)$

$$\boxed{Q(f) = P(f) + Q(f) - P(f), \qquad P(f) := Q'(\mathcal{M})(f - \mathcal{M})}$$

Reference Solution IMEX scheme (II)

$$\boldsymbol{U} = \underbrace{\overline{\boldsymbol{U}}}_{\textbf{Reference solution}} + D \underbrace{\boldsymbol{V}}_{\textbf{scaled perturbation}}$$

with $D = \mathrm{diag}(\varepsilon^{d_1}, \dots, \varepsilon^{d_q})$ for $\boldsymbol{U} = \boldsymbol{U}_{(0)} + \varepsilon \boldsymbol{U}_{(1)} + \varepsilon^2 \boldsymbol{U}_{(2)}$

$$\boldsymbol{F} = \boldsymbol{F}(\overline{\boldsymbol{U}}) + \overbrace{\boldsymbol{F}'(\overline{\boldsymbol{U}})\, D \boldsymbol{V}}^{\textit{Linear}} + \overbrace{\widehat{\boldsymbol{F}}(\overline{\boldsymbol{U}}, \boldsymbol{V})}^{\textit{Nonlinear}} = D\,\overbrace{(\overline{\boldsymbol{G}} + \widetilde{\boldsymbol{G}} + \widehat{\boldsymbol{G}})}^{\textit{RS+IM+EX}}$$

$$\boldsymbol{S} = \boldsymbol{S}(\overline{\boldsymbol{U}}) + \boldsymbol{S}'(\overline{\boldsymbol{U}})\, D \boldsymbol{V} + \widehat{\boldsymbol{S}}(\overline{\boldsymbol{U}}, \boldsymbol{V}) = D(\overline{\boldsymbol{Z}} + \widetilde{\boldsymbol{Z}} + \widehat{\boldsymbol{Z}})$$

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$$\partial_t \boldsymbol{V} = -\overline{\boldsymbol{T}} + \overbrace{\left(-\mathrm{div}_{\boldsymbol{x}} \widetilde{\boldsymbol{G}} + \widetilde{\boldsymbol{Z}}\right)}^{=:\widetilde{\boldsymbol{R}}} + \overbrace{\left(-\mathrm{div}_{\boldsymbol{x}} \widehat{\boldsymbol{G}} + \widehat{\boldsymbol{Z}}\right)}^{=:\widehat{\boldsymbol{R}}}$$

with scaled residual of the reference solution

$$\boxed{\overline{\boldsymbol{T}} := D^{-1} \partial_t \overline{\boldsymbol{U}} + \mathrm{div}_{\boldsymbol{x}} \overline{\boldsymbol{G}} - \overline{\boldsymbol{Z}}}$$

Reference Solution IMEX scheme (III)

RS-IMEX scheme

$$D_t \boldsymbol{V}_\Delta^n = -\overline{\boldsymbol{T}}_\Delta^{n+1} + \widetilde{\boldsymbol{R}}_\Delta^{n+1} + \widehat{\boldsymbol{R}}_\Delta^n$$

- Time integration $\quad D_t\phi(t,\boldsymbol{x}) := \frac{\phi(t+\Delta t,\boldsymbol{x})-\phi(t,\boldsymbol{x})}{\Delta t}$
- Rusanov-type flux $\quad f_{i+1/2} := \frac{f(u_i)+f(u_{i+1})}{2} - \frac{\alpha_{i+1/2}}{2}(u_{i+1}-u_i)$
- Central discretization of the source term

1: get $\overline{\boldsymbol{U}}_\Delta^n$ and $\boldsymbol{V}_\Delta^n$

2: find $\overline{\boldsymbol{U}}_\Delta^{n+1} \rightarrow$ compute $\overline{\boldsymbol{T}}_\Delta^{n+1}$

3: Explicit step $D_t \boldsymbol{V}_\Delta^n = \widehat{\boldsymbol{R}}_\Delta^n \quad \rightarrow \boldsymbol{V}_\Delta^{n+1/2}$

4: Implicit step $D_t \boldsymbol{V}_\Delta^{n+1/2} = -\overline{\boldsymbol{T}}_\Delta^{n+1} + \widetilde{\boldsymbol{R}}_\Delta^{n+1} \quad \rightarrow \boldsymbol{V}_\Delta^{n+1}$

5: $\boldsymbol{U}_\Delta^{n+1} = \overline{\boldsymbol{U}}_\Delta^{n+1} + D\boldsymbol{V}_\Delta^{n+1}$

a bit of literature review

	System	Reference solution $\overline{U}$
[Bispen et al., 2014]	SWE	LaR
[Schütz and Kaiser, 2016]	Van der Pol	∞ damping
[Zakerzadeh, 2016a]	1dSWE	LaR and lake eq.
[Kaiser et al., 2016]	isent. Euler	incompressible eq.
[Zakerzadeh, 2016b]	2dSWE	lake eq.
[Bispen et al., 2017]	Euler + gravity	hydrostatic equilib.
[Zakerzadeh, 2017b]	2dSWE + Coriolis	barotropic vorticity eq.
[Kaiser and Schütz, 2017]	extends [Kaiser et al., 2016] to high order dG	

- **Non-rotating:**
  [Degond and Tang, 2011; Haack et al., 2012; Noelle et al., 2014; Bispen et al., 2014; Dimarco et al., 2016; Zakerzadeh, 2017a], . . .

- **Rotating:**
  [Bouchut et al., 2004; Audusse et al., 2009, 2011, 2015, 2017; Lukáčová-Medvid'ová et al., 2007; Hundertmark-Zaušková et al., 2011]

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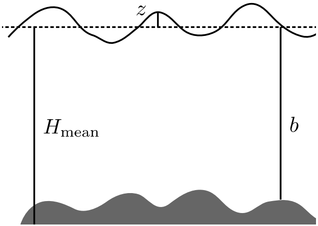
1d Shallow water equations with topography

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \dfrac{h^2}{2\varepsilon^2} \right) = -\dfrac{h}{\varepsilon^2} \partial_x \eta^b \end{cases}$$

Define: [Bispen et al., 2014]

- $H_{\text{mean}} - \eta^b =: -b > 0$
- $h = z - b$
- $m := hu$



$$\boldsymbol{U} = \begin{bmatrix} z \\ m \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} m \\ \dfrac{m^2}{z-b} + \dfrac{z^2 - 2zb}{2\varepsilon^2} \end{bmatrix}, \qquad \boldsymbol{S} = \begin{bmatrix} 0 \\ -\dfrac{z}{\varepsilon^2} \partial_x b \end{bmatrix}.$$

- $\overline{\boldsymbol{U}} := (\overline{z}, 0)^T$ lake at rest $\Longrightarrow \overline{\boldsymbol{T}} = \mathbf{0}$
- $D := \mathrm{diag}(\varepsilon^2, 1) \quad \Longleftarrow \eta_{(0)} = const.$

$$\widehat{\boldsymbol{G}} = \begin{bmatrix} 0 \\ \dfrac{v_2^2}{\overline{z} + \varepsilon^2 v_1 - b} + \dfrac{\varepsilon^2}{2} v_1^2 \end{bmatrix} \qquad \widetilde{\boldsymbol{G}} = \begin{bmatrix} v_2/\varepsilon^2 \\ (\overline{z} - b) v_1 \end{bmatrix}$$

$$\widehat{\boldsymbol{Z}} = \mathbf{0} \qquad \widetilde{\boldsymbol{Z}} = \begin{bmatrix} 0 \\ -\partial_x b \, v_1 \end{bmatrix}$$

$$\widehat{\lambda} = 0, 2\, u_{pert} \qquad \widetilde{\lambda} = \pm \frac{\sqrt{\overline{z} - b}}{\varepsilon}$$

$$\boldsymbol{V}_i^{n+1/2} = \boldsymbol{V}_i^n - \frac{\Delta t}{\Delta x} \left(\widehat{\boldsymbol{G}}_{i+1/2}^n - \widehat{\boldsymbol{G}}_{i-1/2}^n \right) \qquad \text{(Explicit step)}$$

$$\boldsymbol{V}_i^{n+1} = \boldsymbol{V}_i^{n+1/2} - \frac{\Delta t}{\Delta x} \left(\widetilde{\boldsymbol{G}}_{i+1/2}^{n+1} - \widetilde{\boldsymbol{G}}_{i-1/2}^{n+1} \right) + \Delta t \, \widetilde{\boldsymbol{Z}}_i^{n+1} \qquad \text{(Implicit step)}$$

Asymptotic analysis in 1d

Theorem [Zakerzadeh, 2016a]

For 1d SWE with topography in $\Omega = \mathbb{T}$ and under an ε-uniform CFL condition, with well-prepared initial data and LaR reference solution, the RS-IMEX scheme is

(i) solvable: a unique solution for all $\varepsilon > 0$

(ii) ε-stable: $\lim_{\varepsilon \to 0} \| \boldsymbol{V}_\Delta^{n+1} \| = \mathcal{O}(1)$

(iii) Rigorously AC: for the fully-discrete settings

(iv) AS: there exists a constant C_{N,T_f} such that

$$\| \boldsymbol{V}_\Delta^{n} \|_{\ell_2} \leq C_{N,T_f} \| \boldsymbol{V}_\Delta^{0} \|_{\ell_2}$$

(v) well-balanced: preserves the lake at rest equilibrium state

(vi) possible $\mathcal{O}(\varepsilon^2)$ checker-board oscillations for z

Sketch of the proof: asymptotic consistency

- recast the linear implicit step as $J_\varepsilon \boldsymbol{V}_\Delta^{n+1} = \underbrace{\boldsymbol{V}_\Delta^{n+1/2}}_{\mathcal{O}(1)}$

$$J_\varepsilon := \begin{bmatrix} \mathbb{I}_N & \frac{\beta}{\varepsilon^2} Q \\ \beta R_b & \mathbb{I}_N \end{bmatrix}, \qquad \beta := \frac{\Delta t}{2\Delta x}, \quad \widetilde{\alpha} = 0$$

$$Q := \textbf{\textit{Circ}}\,(0,1,0,\ldots,0,-1)$$

$$(R_b)_i = (b_{i+1} - b_{i-1}, \overline{h}_{i+1}, 0, \ldots, 0, -\overline{h}_{i-1})$$

- show that $\lim_{\varepsilon\to 0} \|J_\varepsilon^{-1}\| < \infty$

  How? singular values of J_ε do not approach zero

  $\Longrightarrow$ study the numerical range of $J_\varepsilon^* J_\varepsilon$

$$\boxed{W(J_\varepsilon^* J_\varepsilon) = \|\beta R_b \boldsymbol{w}_1 + \boldsymbol{w}_2\|_{\ell_2}^2 + \left\|\frac{\beta}{\varepsilon^2} Q \boldsymbol{w}_2 + \boldsymbol{w}_1\right\|_{\ell_2}^2 \to 0? \qquad \|\boldsymbol{w}\|_2 = 1}$$

- take the usual formal approach

Sketch of the proof: asymptotic stability

$$\boldsymbol{V}_\Delta^n = \mathcal{E}_{imp}\,\mathcal{E}_{exp}\,\boldsymbol{V}_\Delta^{n-1}$$

- Implicit: $\|\mathcal{E}_{imp}\|_{\ell_2} \leq 1 + C_{imp}\Delta t$
- Explicit: $\|\mathcal{E}_{exp}\boldsymbol{V}_\Delta^{n-1}\|_{\ell_2} \leq \|\boldsymbol{V}_\Delta^{n-1}\|_{\ell_2} + C_{exp}\Delta t \|\boldsymbol{V}_\Delta^{n-1}\|_{\ell_2}^2$

$$\|\boldsymbol{V}_\Delta^n\|_{\ell_2} \leq (1 + C_{imp}\Delta t)\left(1 + C_{exp}\Delta t\,\|\boldsymbol{V}_\Delta^{n-1}\|_{\ell_2}\right)\|\boldsymbol{V}_\Delta^{n-1}\|_{\ell_2}$$

discrete Gronwall's inequality [Willett and Wong, 1965]
$\Longrightarrow$ requires smallness of the initial datum

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$$\boldsymbol{F} = \begin{bmatrix} m_1 & m_2 \\ \dfrac{m_1^2}{z-b} + \dfrac{z^2-2zb}{2\varepsilon^2} & \dfrac{m_1 m_2}{z-b} \\ \dfrac{m_1 m_2}{z-b} & \dfrac{m_2^2}{z-b} + \dfrac{z^2-2zb}{2\varepsilon^2} \end{bmatrix}, \qquad \boldsymbol{S} = \begin{bmatrix} 0 \\ -z\partial_x b/\varepsilon^2 \\ -z\partial_y b/\varepsilon^2 \end{bmatrix}.$$

$$\overline{\boldsymbol{G}} := \boldsymbol{G}(\overline{\boldsymbol{U}}), \qquad \widetilde{\boldsymbol{G}} := \boldsymbol{G}'(\overline{\boldsymbol{U}})\boldsymbol{V}, \qquad \widehat{\boldsymbol{G}} := \boldsymbol{G} - \overline{\boldsymbol{G}} - \widetilde{\boldsymbol{G}}$$
$$\overline{\boldsymbol{Z}} := \boldsymbol{Z}(\overline{\boldsymbol{U}}), \qquad \widetilde{\boldsymbol{Z}} := \boldsymbol{Z}'(\overline{\boldsymbol{U}})\boldsymbol{V}, \qquad \widehat{\boldsymbol{Z}} := \boldsymbol{Z} - \overline{\boldsymbol{Z}} - \widetilde{\boldsymbol{Z}}$$

- $\overline{\boldsymbol{U}}$: zero-Froude limit (lake equations)

$$\begin{cases} \mathrm{div}_{\boldsymbol{x}}\boldsymbol{m} = 0 \\ \partial_t \boldsymbol{m} - \mathrm{div}_{\boldsymbol{x}}\left(\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{b}\right) - b\nabla_{\boldsymbol{x}}\pi = \boldsymbol{0} \end{cases}$$

- Chorin's projection method to update $\overline{\boldsymbol{U}}$
- $D = \mathrm{diag}(\varepsilon^2, 1, 1)$

$$
\begin{aligned}
\boldsymbol{V}_{ij}^{n+1/2} &= \boldsymbol{V}_{ij}^{n} \quad - \frac{\Delta t}{\Delta x}\left(\widehat{\boldsymbol{G}}^{n}_{1,i+1/2j} - \widehat{\boldsymbol{G}}^{n}_{1,i-1/2j}\right) - \frac{\Delta t}{\Delta y}\left(\widehat{\boldsymbol{G}}^{n}_{2,ij+1/2} - \widehat{\boldsymbol{G}}^{n}_{2,ij-1/2}\right)\\
\boldsymbol{V}_{ij}^{n+1} &= \boldsymbol{V}_{ij}^{n+1/2} - \frac{\Delta t}{\Delta x}\left(\widetilde{\boldsymbol{G}}^{n+1}_{1,i+1/2j} - \widetilde{\boldsymbol{G}}^{n+1}_{1,i-1/2j}\right)\\
&\quad - \frac{\Delta t}{\Delta y}\left(\widetilde{\boldsymbol{G}}^{n+1}_{2,ij+1/2} - \widetilde{\boldsymbol{G}}^{n+1}_{2,ij-1/2}\right) + \Delta t\, \widetilde{\boldsymbol{Z}}_{ij}^{n+1} - \Delta t\, \overline{\boldsymbol{T}}_{ij}^{n+1}
\end{aligned}
$$

$$
\begin{aligned}
\overline{\boldsymbol{T}}^{n+1}_{1,ij} &= \left(\nabla_{h,x}\overline{m_1}_{ij}^{n+1} + \nabla_{h,x}\overline{m_2}_{ij}^{n+1}\right)/\varepsilon^2\\
\overline{\boldsymbol{T}}^{n+1}_{2,ij} &= D_t\overline{m_1}_{ij}^{n} + \nabla_{h,x}\left(\frac{\overline{m_1}_{ij}^{n+1,2}}{\overline{z} - b_{ij}}\right) + \nabla_{h,y}\left(\frac{\overline{m_1}_{ij}^{n+1}\overline{m_2}_{ij}^{n+1}}{\overline{z} - b_{ij}}\right)\\
\overline{\boldsymbol{T}}^{n+1}_{3,ij} &= D_t\overline{m_2}_{ij}^{n} + \nabla_{h,x}\left(\frac{\overline{m_1}_{ij}^{n+1}\overline{m_2}_{ij}^{n}}{\overline{z} - b_{ij}}\right) + \nabla_{h,y}\left(\frac{\overline{m_1}_{ij}^{n+1,2}}{\overline{z} - b_{ij}}\right)
\end{aligned}
$$

Asymptotic analysis in 2d

Theorem [Zakerzadeh, 2016b]

For 2d SWE with topography in $\Omega = \mathbb{T}^2$ and under an ε-uniform CFL condition, with well-prepared initial data and zero-Froude limit reference solution, the RS-IMEX scheme is

(i) solvable: a unique solution for all $\varepsilon > 0$

(ii) "ε-stable": $\lim_{\varepsilon \to 0} \| \boldsymbol{V}_\Delta^{n+1} \| = \mathcal{O}(1)$ **(if $\overline{\boldsymbol{U}} = \mathbf{0}$ and $\nabla_{\boldsymbol{x}} \eta^b = 0$)**

(iii) Rigorously AC: for the fully-discrete settings

(iv) "AS": there exists a constant C_{N,T_f} such that

$$\| \boldsymbol{V}_\Delta^n \|_{\ell_2} \leq C_{N,T_f} \| \boldsymbol{V}_\Delta^0 \|_{\ell_2}$$

provided the reference solver is stable in some suitable sense

(v) "well-balanced": preserves the LaR state **(if $\overline{\boldsymbol{U}}_\Delta, \boldsymbol{V}_\Delta \in \mathcal{U}_\Delta^{LaR}$)**

(vi) possible $\mathcal{O}(\varepsilon^2)$ checker-board oscillations for z

Travelling vortex

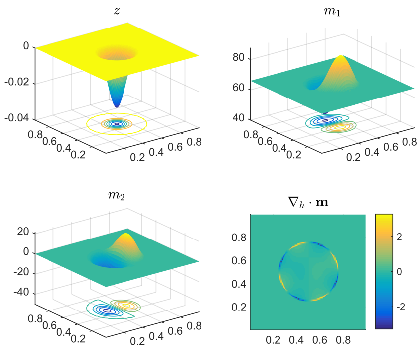
- Exact solution is available [Ricchiuto and Bollermann, 2009]
- Initial condition as [Bispen et al., 2014] with periodic domain $\Omega = [0,1)^2$:

$$\begin{cases} z(0,x,y) &= \mathbf{1}_{[r\leq \frac{\pi}{\omega}]} \left(\frac{\Gamma\varepsilon}{\omega}\right)^2 \left(g(\omega r) - g(\pi)\right), \\ u_1(0,x,y) &= u_0 + \mathbf{1}_{[r\leq \frac{\pi}{\omega}]}\Gamma\left(1+\cos(\omega r)\right)(y_c - y), \\ u_2(0,x,y) &= \mathbf{1}_{[r\leq \frac{\pi}{\omega}]}\Gamma\left(1+\cos(\omega r)\right)(x - x_c), \end{cases}$$

with $b(x,y) = -110$, $u_0 = 0.6$ and

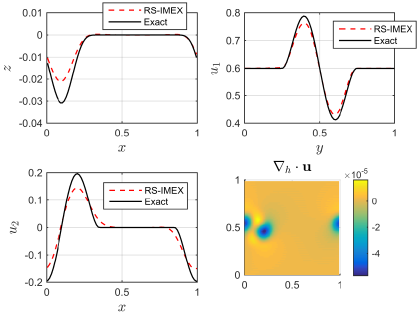
$$r := \mathrm{dist}(\boldsymbol{x}, \boldsymbol{x}_c), \quad \boldsymbol{x}_c = (0.5, 0.5)^T, \quad \Gamma = 1.4, \quad \omega = 4\pi,$$
$$g(r) := 2\cos r + 2r\sin r + \frac{1}{8}\cos 2r + \frac{r}{4}\sin 2r + \frac{3}{4}r^2.$$

Travelling vortex: initial condition

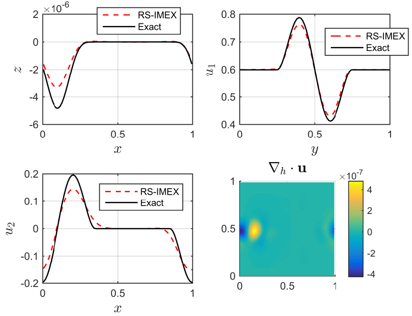


Initial condition for the travelling vortex example with $\varepsilon = 0.8$, computed on the 100×100 grid.

Uniform accuracy (I)



$\varepsilon = 0.8$.



$\varepsilon = 0.01$.

Error of the RS-IMEX scheme, computed on the 80 $\times$ 80 grid with $\mathrm{CFL} = 0.45$ and $T_f = 1$

Uniform accuracy (II)

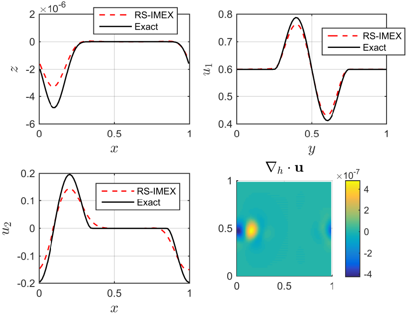
Experimental order of convergence for the travelling vortex example with $T_f = 1$.

N	$\varepsilon = 0.8$			
	e_{z,ℓ_∞}	EOC_{z,ℓ_∞}	e_{u_1,ℓ_∞}	EOC_{u_1,ℓ_∞}
20	2.61e-2	-	1.04e-1	-
40	2.00e-2	0.38	6.80e-2	0.61
80	1.23e-2	0.70	3.63e-2	0.91
160	6.20e-3	0.99	1.65e-3	1.14

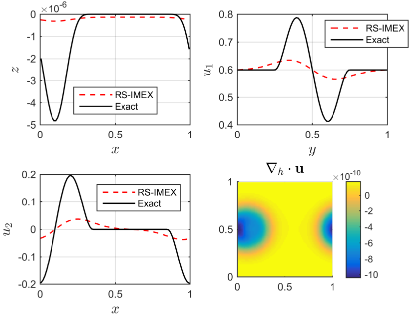
N	$\varepsilon = 10^{-6}$			
	e_{z,ℓ_∞}	EOC_{z,ℓ_∞}	e_{u_1,ℓ_∞}	EOC_{u_1,ℓ_∞}
20	4.08e-14	-	1.04e-1	-
40	3.13e-14	0.38	6.80e-2	0.61
80	1.92e-14	0.71	3.63e-2	0.91
160	9.69e-15	0.99	1.65e-3	1.14

Should we invest in $\overline{\boldsymbol{U}}$?

- solving for $\overline{\boldsymbol{U}}$ takes around 1% of the total time
- the quality matters!



$\varepsilon = 0.01$, $\overline{\boldsymbol{U}} = \boldsymbol{U}_{(0)}$.



$\varepsilon = 0.01$, $\overline{\boldsymbol{U}} = \mathbf{0}$.

Outline

- Introduction
- RS-IMEX scheme
- 1d SWE
- 2d SWE
  - Numerical experiments
- 2d RSWE
  - Numerical experiments
- Recent progress

2d Rotating shallow water equations with topography

$$\begin{cases} \partial_{\hat{t}}(\Theta \hat{z}) + \mathrm{div}_{\hat{\mathbf{x}}}(\hat{h}\hat{\mathbf{u}}) = 0, \\ \partial_{\hat{t}}(\hat{h}\hat{\mathbf{u}}) + \mathrm{div}_{\hat{\mathbf{x}}}\left(\hat{h}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} + \frac{\hat{h}^2}{2Fr^2}\mathbb{I}_2\right) = -\frac{\Theta}{Fr^2}\hat{h}\nabla_{\hat{\mathbf{x}}}\hat{\eta}^b - \frac{\hat{h}}{Ro}\mathbf{u}^\perp, \end{cases}$$

- two height scales: $H_\circ$ for H_{mean}, and $Z_\circ$ for z and η^b
- $\hat{h} = 1 + \Theta(\hat{z} - \hat{\eta}^b)$
- $\Theta := \frac{Z_\circ}{H_\circ}$
- $F^{1/2} := fL_\circ/\sqrt{gH_\circ} = \mathcal{O}(1)$

Quasi-geostrophic distinguished limit [Majda, 2003]

$$Ro = \varepsilon \ll 1, \quad Fr = F^{1/2}\varepsilon, \quad \Theta = F\varepsilon$$

$\Theta \sim \varepsilon \Longrightarrow$ the variation of η^b and z are mild:

$$\|z\|, \|\nabla_{\mathbf{x}}\eta^b\| = \mathcal{O}(\varepsilon)$$

$$\begin{cases}\partial_t z + \frac{1}{\Theta}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{m} = 0,\\ \partial_t \boldsymbol{m} + \mathrm{div}_{\boldsymbol{x}}\left(\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\Theta z - b} + \frac{\Theta z^2 - 2bz}{2\varepsilon}\mathbb{I}_2\right) = -\frac{1}{\varepsilon}z\nabla_{\boldsymbol{x}}b - \frac{1}{\varepsilon}\boldsymbol{m}^{\perp},\end{cases}$$

with $\boldsymbol{m} := (\Theta z - b)\boldsymbol{u}$ and $1 - \Theta\eta^b = -b$.

$$\boldsymbol{F} = \begin{bmatrix} m_1/\Theta & m_2/\Theta \\ \frac{m_1^2}{\Theta z - b} + \frac{\Theta z^2 - 2zb}{2\varepsilon} & \frac{m_1 m_2}{\Theta z - b} \\ \frac{m_1 m_2}{\Theta z - b} & \frac{m_2^2}{\Theta z - b} + \frac{\Theta z^2 - 2zb}{2\varepsilon} \end{bmatrix}$$

$$\boldsymbol{S}^B = \begin{bmatrix} 0 \\ -z\partial_x b/\varepsilon \\ -z\partial_y b/\varepsilon \end{bmatrix} \qquad \boldsymbol{S}^C = \begin{bmatrix} 0 \\ m_2/\varepsilon \\ -m_1/\varepsilon \end{bmatrix}$$

- $\overline{\boldsymbol{U}}$: quasi-geostrophic equations [Majda, 2003]

$$
\begin{aligned}
&\boldsymbol{u} = \nabla_{\boldsymbol{x}}^{\perp} z && \text{(geostrophic balance)}\\
&\Delta_{\boldsymbol{x}} z = \zeta, && \textit{with } \zeta := \|\nabla_{\boldsymbol{x}} \times \boldsymbol{u}\|\\
&(\partial_t + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}})\,(\zeta - Fz + F\eta^b) = 0 && \text{(potential vorticity eq.)},
\end{aligned}
$$

- Arakawa method to update $\overline{\boldsymbol{U}}$ [Arakawa, 1966]
- $D = \mathbb{I}_3$

$$
\begin{aligned}
\overline{\boldsymbol{G}} &:= \boldsymbol{G}(\overline{\boldsymbol{U}}), & \widetilde{\boldsymbol{G}} &:= \boldsymbol{G}'(\overline{\boldsymbol{U}})\boldsymbol{V}, & \widehat{\boldsymbol{G}} &:= \boldsymbol{G} - \overline{\boldsymbol{G}} - \widetilde{\boldsymbol{G}}\\
\overline{\boldsymbol{Z}} &:= \boldsymbol{Z}(\overline{\boldsymbol{U}}), & \widetilde{\boldsymbol{Z}} &:= \boldsymbol{Z}'(\overline{\boldsymbol{U}})\boldsymbol{V}, & \widehat{\boldsymbol{Z}} &:= \boldsymbol{Z} - \overline{\boldsymbol{Z}} - \widetilde{\boldsymbol{Z}}
\end{aligned}
$$

Asymptotic analysis in 2d rotating case

Theorem [Zakerzadeh, 2017b]

For 2d RSWE with topography in $\Omega = \mathbb{T}^2$ and under an ε-uniform CFL condition, with well-prepared initial data and the QGE reference solution, the RS-IMEX scheme is

(i) solvable: a unique solution for all $\varepsilon > 0$

(ii) "ε-stable": $\lim_{\varepsilon \to 0} \| \boldsymbol{V}_\Delta^{n+1} \| = \mathcal{O}(1)$

(iii) Rigorously AC: for the fully-discrete settings

(iv) "AS": there exists a constant C_{N,T_f} such that

$$\| \boldsymbol{V}_\Delta^{n} \|_{\ell_2} \leq C_{N,T_f} \| \boldsymbol{V}_\Delta^{0} \|_{\ell_2}$$

provided the reference solver is stable in some suitable sense

(v) "well-balanced": preserves the LaR state if $\overline{\boldsymbol{U}}_\Delta, \boldsymbol{V}_\Delta \in \mathcal{U}_\Delta^{LaR}$

(vi) possible $\mathcal{O}(\varepsilon)$ checker-board oscillations for the surface perturbation

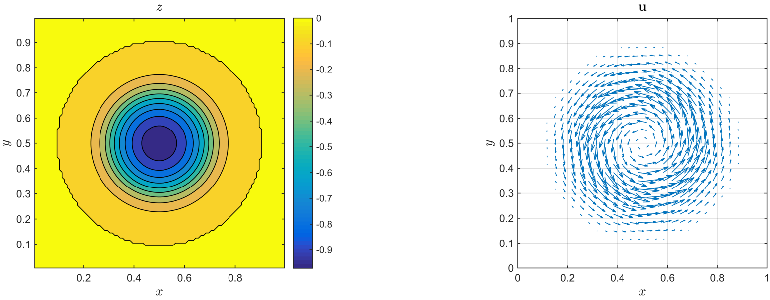
2d stationary vortex

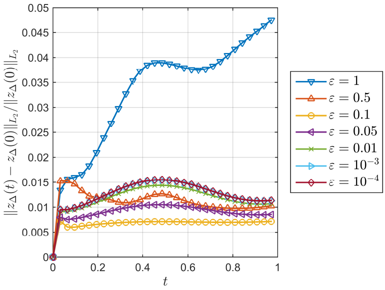
in periodic domain $[0,1)^2$, pressure gradient is balanced with the Coriolis force and the advective terms [Audusse et al., 2009]:

$$\boldsymbol{u}_0(r,\theta) = \vartheta_\theta(r)\hat{\theta}, \qquad \vartheta_\theta(r) := 5r\mathbf{1}_{[r<\frac{1}{5}]} + (2-5r)\mathbf{1}_{[\frac{1}{5}\leq r<\frac{2}{5}]},$$

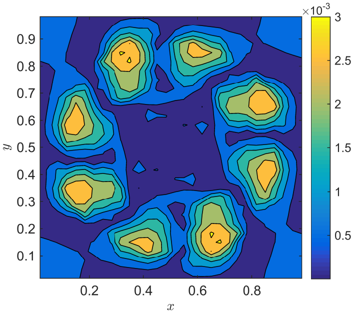
$$z_0'(r) = \vartheta_\theta + \varepsilon\frac{\vartheta_\theta^2}{r},$$

where r is the distance to the vortex center $(0.5, 0.5)^T$ and $H_{\mathrm{mean}} = 2$.



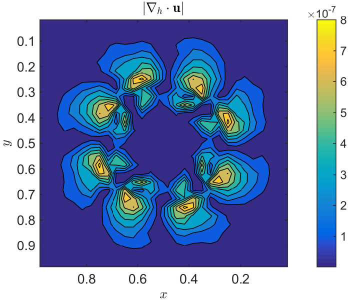


Relative error for z.

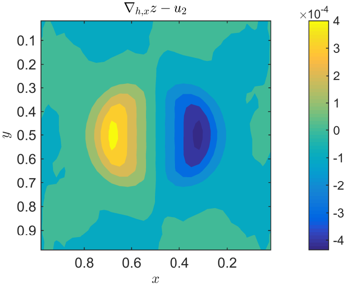


$|z_\Delta(t) - z_\Delta(0)|$ for $t = 10$ and $\varepsilon = 10^{-4}$.

$\implies$ the scheme is uniformly accurate and AS!



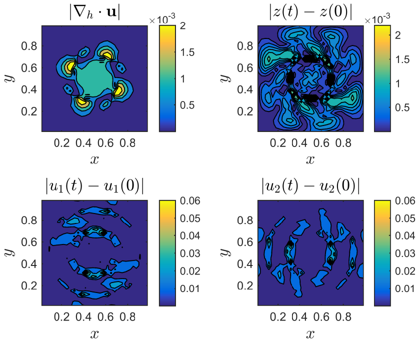
Absolute divergence of the velocity field for $\varepsilon = 10^{-4}$ at $t = 1$.



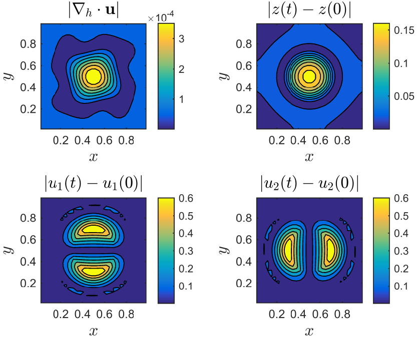
Geostrophic balance for $\varepsilon = 10^{-4}$ at $t = 1$.

$\Longrightarrow$ the scheme is AC!

Should we invest in $\overline{\boldsymbol{U}}$?



$\varepsilon = 0.1$, $\overline{\boldsymbol{U}} = \boldsymbol{U}_{(0)}$.



$\varepsilon = 0.1$, $\overline{\boldsymbol{U}} = \mathbf{0}$.

Outline

How to refine estimates?

- following [Giesselmann, 2015]: (semi-discrete)
  - for $E_{tot} := \frac{1}{2}\||h|\boldsymbol{u}|^2\|_{L_1(\Omega)} + \frac{1}{2\varepsilon^2}\|z\|^2_{L_2(\Omega)}$ and all $\varepsilon > 0$
  $$E^{n+1}_{tot} \leq E^{n}_{tot} + \mathcal{O}(\Delta t^2)$$
  - for $E_{kin,(0)} := \frac{1}{2}\||\boldsymbol{u}_{(0)}|^2\|_{L_1(\Omega)}$, when $\varepsilon \to 0$
  $$E^{n+1}_{kin,(0)} \leq E^{n}_{kin,(0)} + \mathcal{O}(\Delta t^2)$$
- following [Bispen et al., 2017]: (fully-discrete)
  - L_1 estimate for non-linear explicit step
  - L_2 estimate for linear implicit step
  - interpolation between the norms
- follwoing [Gallouët et al., 2017; Feireisl et al., 2016; Berthon et al., 2016; Fischer, 2015]?

New applications?

- Euler with congestion? (with Charlotte Perrin)

Conclusion

We have analyzed the RS-IMEX scheme for shallow water equations:

- 1d, 2d, 2d + Coriolis
- "rigorous" asymptotic analysis $\Longrightarrow$ AP!
- reasonable numerical results

- **H.Z.**, Asymptotic analysis of the RS-IMEX scheme for the shallow water equations in one space dimension, HAL: hal-01491450.
- **H.Z.**, Asymptotic consistency of the RS-IMEX scheme for the low-Froude shallow water equations: Analysis and numerics, *XVI International Conference on Hyperbolic Problems*.
- **H.Z.**, The RS-IMEX scheme for the rotating shallow water equations with the Coriolis force, In *International Conference on Finite Volumes for Complex Applications*, pp. 199–207. Springer, Cham (2017).

Merci de votre attention !

The basic idea: *stability of the modified equation*

Linear system $\partial_t \boldsymbol{U} + A\partial_x \boldsymbol{U} = \boldsymbol{0}$ [Schütz and Noelle, 2014]:

$$\partial_t \boldsymbol{U} + A\partial_x \boldsymbol{U} = D_\nu \partial_x^2 \boldsymbol{U}, \quad D_\nu := \frac{\Delta t}{2}\left(\frac{\alpha \Delta x}{\Delta t}\mathbb{I}_q - \widehat{A}^2 + \widetilde{A}^2 + [\widetilde{A}, \widehat{A}]\right)$$

is stable if $\mathcal{P}(\xi) := -iA\xi - \xi^2 D_\nu$ has only eva with negative real parts.

$$\left[\widehat{\boldsymbol{G}}', \widetilde{\boldsymbol{G}}'\right] = \left[\boldsymbol{G}'(\boldsymbol{U}), \boldsymbol{G}'(\overline{\boldsymbol{U}})\right] \quad \Longrightarrow \quad \text{smaller, for smaller } \|\boldsymbol{U} - \overline{\boldsymbol{U}}\|$$

modified version as in [Zakerzadeh and Noelle, 2016]

$$\widetilde{\mathcal{P}}(\xi) := -i\xi\Lambda - \xi^2 \frac{\Delta t}{2}\left[\frac{\alpha \Delta x}{\Delta t}\mathbb{I}_q - \Lambda^2 + 2Q_{R\to\widetilde{R}}\widetilde{\Lambda}Q^{-1}_{R\to\widetilde{R}}\Lambda\right].$$

$$\lim_{\varepsilon\to 0}\|\boldsymbol{U} - \overline{\boldsymbol{U}}\| = 0 \quad \Longrightarrow \quad R \text{ and } \widetilde{R} \text{ get closer} \quad \Longrightarrow \quad Q_{R\to\widetilde{R}} \to \mathbb{I}_q$$

$$\overline{\boldsymbol{T}}_{1,ij}^{n+1} = D_t \overline{z}_{ij}^n + \frac{1}{\boxed{\Theta}} \left(\nabla_{h,x} \overline{m_1}_{ij} + \nabla_{h,x} \overline{m_2}_{ij} \right)^{n+1}$$

$$\overline{\boldsymbol{T}}_{2,ij}^{n+1} = D_t \overline{m_1}_{ij}^n + \nabla_{h,x} \left(\frac{\overline{m_1}_{ij}^2}{\Theta \overline{z}_{ij} - b_{ij}} \right)^{n+1} + \nabla_{h,y} \left(\frac{\overline{m_1}_{ij} \overline{m_2}_{ij}}{\Theta \overline{z}_{ij} - b_{ij}} \right)^{n+1}$$
$$+ \frac{1}{2\boxed{\varepsilon}} \nabla_{h,x} \left(\Theta \overline{z}_{ij}^2 - 2 b_{ij} \overline{z}_{ij} \right)^{n+1} + \frac{1}{\boxed{\varepsilon}} \overline{z}_{ij}^{n+1} \nabla_{h,x} b_{ij} - \frac{1}{\boxed{\varepsilon}} \overline{m_2}_{ij}^{n+1}$$

$\rightarrow$ one can check that $\| \overline{\boldsymbol{T}}_{\Delta}^{n+1} \| = \mathcal{O}(1)$

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