On all-regime, high-order and well-balanced Lagrange-Projection type schemes for the shallow water equations

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MAISON DE LA SIMULATION

Contents

Introduction

- Construction of a Discontinuous Galerkin (DG) scheme for Shallow Water equations (SWE)
- Theory based on Finite Volume (FV) Lagrange-Projection (L-P) type schemes for Euler equations[1] and for SWE[2]
- Low Froude number : fast acoustic waves vs. slow material transport waves
- Acoustic - Transport operators decomposition (L-P like) :
  $\longrightarrow$ Impliciting fast phenomenons : less restrictive CFL condition
  $\longrightarrow$ Expliciting slow phenomenons : reasonable precision

[1] Christophe Chalons, Mathieu Girardin, and Samuel Kokh. "An all-regime Lagrange-Projection like scheme for the gas dynamics equations on unstructured meshes". In: *Communications in Computational Physics* 20.01 (2016), pp. 188–233.

[2] Christophe Chalons et al. "A large time-step and well-balanced Lagrange-Projection type scheme for the shallow-water equations". In: *Communic. Math. Sci.* 15.3 (2017), pp. 765–788.

Contents

Shallow Water Equations

Euler System in 1D

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x((\rho E + p)u) = 0. \end{cases}$$

Shallow Water Sytem in 1D

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\dfrac{h^2}{2}\right) = -gh\partial_x z. \end{cases}$$

$\longrightarrow$ Two similar systems

$\longrightarrow$ Non-conservative source term in SWE

Operators splitting

"Acoustic" / "Transport" decomposition

$$\begin{cases} \partial_t h + & h\,\partial_x u + & u\,\partial_x h = 0, \\ \partial_t (hu) + & hu\,\partial_x u + \partial_x \left(g\dfrac{h^2}{2} \right) + & u\,\partial_x (hu) = -gh\partial_x z. \end{cases}$$

Operators splitting

"Acoustic" / "Transport" decomposition

$$\underset{t^n \to t^{n+1-}}{\textit{Acoustic}} \begin{cases} \partial_t h + & h\,\partial_x u = 0, \\ \partial_t (hu) + & hu\,\partial_x u + \partial_x \left(g\dfrac{h^2}{2} \right) = -gh\partial_x z, \end{cases}$$

$$\underset{t^{n+1-} \to t^{n+1}}{\textit{Transport}} \begin{cases} \partial_t h + & u\,\partial_x h = 0, \\ \partial_t (hu) + & u\,\partial_x (hu) = 0. \end{cases}$$

Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^{1}/{}_{h}$
- Approximation of $\tau(\cdot,t)\,\partial_x X$ by $\tau(\cdot,t^n)\partial_x X = \partial_m X$
- Variable π : linearisation of the pressure $\frac{g}{2\tau^2}$

Acoustic System

$$\begin{cases} \partial_t h + & h\,\partial_x u = 0, \\ \partial_t(hu) + & hu\,\partial_x u + \partial_x\left(g\dfrac{h^2}{2}\right) = -gh\partial_x z, \end{cases}$$

Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^{1}/_{h}$
- Approximation of $\tau(\cdot,t)\,\partial_x X$ by $\tau(\cdot,t^n)\partial_x X = \partial_m X$
- Variable π : linearisation of the pressure $\frac{g}{2\tau^2}$

Acoustic System

$$\begin{cases} \partial_t \tau - \tau\,\partial_x u = 0, \\ \partial_t u + \tau\,\partial_x \left(\dfrac{g}{2\tau^2}\right) = -g\partial_x z, \\ \partial_t z = 0. \end{cases}$$

Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^1/_h$
- Approximation of $\tau(\cdot, t)\,\partial_x X$ by $\tau(\cdot, t^n)\partial_x X = \partial_m X$
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Acoustic System

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \left(\dfrac{g}{2\tau^2}\right) = -\dfrac{g}{\tau}\partial_m z, \\ \partial_t z = 0. \end{cases}$$

Relaxation Method

- Change of variable : $h \longrightarrow \tau = 1/h$
- Approximation of $\tau(\cdot,t)\,\partial_x X$ by $\tau(\cdot,t^n)\partial_x X = \partial_m X$
- Variable π : linearisation of the pressure $\frac{g}{2\tau^2}$

Relaxed Acoustic System

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi = -\dfrac{g}{\tau}\partial_m z, \\ \partial_t \pi + a^2 \partial_m u = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \\ \partial_t z = 0. \end{cases}$$

Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^1/_h$
- Approximation of $\tau(\cdot,t)\,\partial_x X$ by $\tau(\cdot,t^n)\partial_x X = \partial_m X$
- Variable π : linearisation of the pressure $\frac{g}{2\tau^2}$

Relaxed Acoustic System

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi = -\dfrac{g}{\tau}\partial_m z, \\ \partial_t \pi + a^2 \partial_m u = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \\ \partial_t z = 0. \end{cases}$$

Prop : Viscous approximation of the Acoustic system under the sub-characteristic condition : $a > \max(hc) = \max(\frac{1}{\tau}\sqrt{\frac{g}{\tau}})$.

Relaxation Method

Operators splitting :

- Instantaneous relaxation step
- Homogeneous relaxed Acoustic system

Relaxed Acoustic System

$$\begin{cases} \partial_t \tau = 0, \\ \partial_t u = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \\ \partial_t z = 0, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi + \dfrac{g}{\tau} \partial_m z = 0, \\ \partial_t \pi + a^2 \partial_m u = 0, \\ \partial_t z = 0. \end{cases}$$

Contents

FV Discretization

Acoustic step

$$
\begin{cases}
\tau_j^{n+1^-} = \tau_j^n + \dfrac{\Delta t}{\Delta x}\tau_j^n \left(u^{*,\alpha}_{j+1/2} - u^{*,\alpha}_{j-1/2}\right) = L_j^{\alpha}\tau_j^n, \\
u_j^{n+1^-} = u_j^n - \dfrac{\Delta t}{\Delta x}\tau_j^n \left(\pi^{*,\alpha}_{j+1/2} - \pi^{*,\alpha}_{j-1/2}\right) - \Delta t\, \tau_j^n \left\{gh\partial_x z\right\}_j^n, \\
\pi_j^{n+1^-} = \pi_j^n - a_j^2 \dfrac{\Delta t}{\Delta x}\tau_j^n \left(u^{*,\alpha}_{j+1/2} - u^{*,\alpha}_{j-1/2}\right).
\end{cases}
$$

Transport step

$$
\begin{cases}
h_j^{n+1} = L_j^{\alpha} h_j^{n+1^-} - \dfrac{\Delta t}{\Delta x}\left(h^{*,n+1^-}_{j+1/2}\, u^{*,\alpha}_{j+1/2} - h^{*,n+1^-}_{j-1/2}\, u^{*,\alpha}_{j-1/2}\right), \\
(hu)_j^{n+1} = L_j^{\alpha}(hu)_j^{n+1^-} - \dfrac{\Delta t}{\Delta x}\left((hu)^{*,n+1^-}_{j+1/2}\, u^{*,\alpha}_{j+1/2} - (hu)^{*,n+1^-}_{j-1/2}\, u^{*,\alpha}_{j-1/2}\right).
\end{cases}
$$

$\alpha = n$ (full explicit scheme) or $n + 1^-$ (implicit-explicit scheme)

FV Discretization

Acoustic step

$$\begin{cases} \tau_j^{n+1^-} = \tau_j^n + \dfrac{\Delta t}{\Delta x}\tau_j^n \left(u_{j+1/2}^{*,\alpha} - u_{j-1/2}^{*,\alpha}\right) = L_j^{\alpha}\tau_j^n, \\ u_j^{n+1^-} = u_j^n - \dfrac{\Delta t}{\Delta x}\tau_j^n \left(\pi_{j+1/2}^{*,\alpha} - \pi_{j-1/2}^{*,\alpha}\right) - \Delta t\, \tau_j^n \left\{gh\partial_x z\right\}_j^n, \\ \pi_j^{n+1^-} = \pi_j^n - a_j^2 \dfrac{\Delta t}{\Delta x}\tau_j^n \left(u_{j+1/2}^{*,\alpha} - u_{j-1/2}^{*,\alpha}\right). \end{cases}$$

Transport step

$$\begin{cases} h_j^{n+1} = L_j^{\alpha} h_j^{n+1^-} - \dfrac{\Delta t}{\Delta x}\left(h_{j+1/2}^{*,n+1^-} u_{j+1/2}^{*,\alpha} - h_{j-1/2}^{*,n+1^-} u_{j-1/2}^{*,\alpha}\right), \\ (hu)_j^{n+1} = L_j^{\alpha}(hu)_j^{n+1^-} - \dfrac{\Delta t}{\Delta x}\left((hu)_{j+1/2}^{*,n+1^-} u_{j+1/2}^{*,\alpha} - (hu)_{j-1/2}^{*,n+1^-} u_{j-1/2}^{*,\alpha}\right). \end{cases}$$

$\alpha = n$ (full explicit scheme) or $n+1^-$ (implicit-explicit scheme)

IMEX properties

Hypothesis :

- Subcharacteristic condition : $a > \max_j (h_j c_j)$
- CFL condition : $\frac{\Delta t}{\Delta x} \max_j \left| u^*_{j+1/2} \right| \leq \frac{1}{2}$

Properties :

- Conservative for h (and for hu if $z = \text{cst}$)
- Degeneration to classical L-P scheme if $z = \text{cst}$ ($\{gh\partial_x z\} = 0$)
- $h_j^n > 0, \forall j, n$, provided that $h_j^0 > 0, \forall j$.
- Well-balanced : preservation of the "lake at rest" conditions ($u = 0$ and $h + z = \text{cst}$)
- It satisfies a discrete entropy inequality of the form :

$$\mathcal{U}_j^{n+1} - \mathcal{U}_j^n + \frac{\Delta t}{\Delta x_j} \left(\mathcal{F}_{j+1/2}^{n+1-} - \mathcal{F}_{j-1/2}^{n+1-} \right) \leq -\Delta t \, \{ghu\partial_x z\}_j$$

IMEX properties

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- Subcharacteristic condition : $a > \max_j (h_j c_j)$
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Contents

Notations

- Based on work from Florent Renac[3] at ONERA
- Wrote for SWE without topography[4]
- Lagrange polynomials on Gauss-Lobatto quadrature:

$$\rho(x) = \sum_{k=0}^{p} \rho_{k,j}\phi_{k,j}(x), \qquad \forall x \in \left[x_{j-1/2}, x_{j+1/2}\right]$$

with $\phi_{k,j}(x) = \ell_k\left(\frac{2}{\Delta x}(x - x_j)\right)$, $\ell_k(s_i) = \delta_{k,i}$ and s_i are the Gauss-Lobatto quadrature points on $[-1, 1]$

- Numerical integration on the same Gauss-Lobatto quadrature points:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} f(x)\,\mathrm{d}x \simeq \frac{\Delta x}{2}\sum_{k=0}^{p}\omega_k f(x_{k,j}) = \frac{\Delta x}{2}\sum_{k=0}^{p}\omega_k f\left(x_j + \frac{\Delta x}{2}s_k\right)$$

[3] Florent Renac. "A robust high-order Lagrange-projection like scheme with large time steps for the isentropic Euler equations". In: *Numerische Mathematik* (2016), pp. 1–27.

[4] Christophe Chalons and Maxime Stauffert. "A High-Order Discontinuous Galerkin Lagrange Projection Scheme for the Barotropic Euler Equations". In: *FVCA 8, Lille, France, June 2017*.

Acoustic step

- Multiplication by a Lagrange polynomial and integration over a cell
- Development of the derivatives in time
- Numerical integration of the Lagrange polynomials
- Discretization of the derivatives in time

Time discretization

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi = -\dfrac{g}{\tau}\partial_m z, \\ \partial_t \pi + a^2 \partial_m u = 0. \end{cases}$$

Acoustic step

- Multiplication by a Lagrange polynomial and integration over a cell
- Development of the derivatives in time
- Numerical integration of the Lagrange polynomials
- Discretization of the derivatives in time

Time discretization

$$\begin{cases} \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_t \tau \, \mathrm{d}x - \int_{\kappa_j} \phi_{i,j}\, \partial_m u \, \mathrm{d}x = 0, \\ \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_t u \, \mathrm{d}x + \int_{\kappa_j} \phi_{i,j}\, \partial_m \pi \, \mathrm{d}x = - \int_{\kappa_j} \phi_{i,j}\, \frac{g}{\tau} \partial_m z, \\ \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_t \pi \, \mathrm{d}x + a^2 \int_{\kappa_j} \phi_{i,j}\, \partial_m u \, \mathrm{d}x = 0. \end{cases}$$

Acoustic step

- Multiplication by a Lagrange polynomial and integration over a cell
- Development of the derivatives in time
- Numerical integration of the Lagrange polynomials
- Discretization of the derivatives in time

Time discretization

$$\left\{\begin{aligned}
&\sum_{k=0}^{p}\left(\int_{\kappa_j}\phi_{i,j}\phi_{k,j}\,\mathrm{d}x\right)\partial_t\tau_{k,j}-\int_{\kappa_j}\phi_{i,j}\,\partial_m u\,\mathrm{d}x=0,\\
&\sum_{k=0}^{p}\left(\int_{\kappa_j}\phi_{i,j}\phi_{k,j}\,\mathrm{d}x\right)\partial_t u_{k,j}+\int_{\kappa_j}\phi_{i,j}\,\partial_m\pi\,\mathrm{d}x=-\int_{\kappa_j}\phi_{i,j}\,\frac{g}{\tau}\partial_m z,\\
&\sum_{k=0}^{p}\left(\int_{\kappa_j}\phi_{i,j}\phi_{k,j}\,\mathrm{d}x\right)\partial_t\pi_{k,j}+a^2\int_{\kappa_j}\phi_{i,j}\,\partial_m u\,\mathrm{d}x=0.
\end{aligned}\right.$$

Acoustic step

- Multiplication by a Lagrange polynomial and integration over a cell
- Development of the derivatives in time
- Numerical integration of the Lagrange polynomials
- Discretization of the derivatives in time

Time discretization

$$\begin{cases} \dfrac{\Delta x}{2}\omega_i\,\partial_t\tau_{i,j} - \displaystyle\int_{\kappa_j}\phi_{i,j}\,\partial_m u\,\mathrm{d}x = 0, \\ \dfrac{\Delta x}{2}\omega_i\,\partial_t u_{i,j} + \displaystyle\int_{\kappa_j}\phi_{i,j}\,\partial_m \pi\,\mathrm{d}x = -\displaystyle\int_{\kappa_j}\phi_{i,j}\,\dfrac{g}{\tau}\partial_m z, \\ \dfrac{\Delta x}{2}\omega_i\,\partial_t\pi_{i,j} + a^2\displaystyle\int_{\kappa_j}\phi_{i,j}\,\partial_m u\,\mathrm{d}x = 0. \end{cases}$$

Acoustic step

- Multiplication by a Lagrange polynomial and integration over a cell
- Development of the derivatives in time
- Numerical integration of the Lagrange polynomials
- Discretization of the derivatives in time

Time discretization

$$\begin{cases} \tau_{i,j}^{n+1^-} = \tau_{i,j}^n + \dfrac{2\Delta t}{\omega_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_m u^\alpha \,\mathrm{d}x, \\ u_{i,j}^{n+1^-} = u_{i,j}^n - \dfrac{2\Delta t}{\omega_i \Delta x} \left(\displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_m \pi^\alpha \,\mathrm{d}x + \int_{\kappa_j} \phi_{i,j}\, \frac{g}{\tau^n} \partial_m z \,\mathrm{d}x \right), \\ \pi_{i,j}^{n+1^-} = \pi_{i,j}^n - a^2 \dfrac{2\Delta t}{\omega_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_m u^\alpha \,\mathrm{d}x. \end{cases}$$

Acoustic step

- How to write equation on τ as in finite volume ?
- Approximation of the integral of $\partial_m u$
- Integration by part (exact)
- Introduction of the numerical fluxes

Space discretization

$$\tau_{i,j}^{n+1^-} = \tau_{i,j}^n + \frac{2\Delta t}{\omega_i \Delta x} \int_{\kappa_j} \phi_{i,j}\, \partial_m u^\alpha \,\mathrm{d}x$$

$$\updownarrow$$

$$\tau_j^{n+1^-} = \tau_j^n + \frac{\Delta t}{\Delta x} \tau_j^n \left(u_{j+1/2}^{*,\alpha} - u_{j-1/2}^{*,\alpha} \right) = L_j^\alpha \tau_j^n$$

Acoustic step

- How to write equation on τ as in finite volume ?
- Approximation of the integral of $\partial_m u$
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- Introduction of the numerical fluxes

Space discretization

$$\int_{\kappa_j} \phi_{i,j}\, \partial_m u^\alpha \,\mathrm{d}x \simeq \frac{\Delta x}{2} \omega_i \tau_{i,j}^n \partial_x u_{i,j}^\alpha = \tau_{i,j}^n \int_{\kappa_j} \phi_{i,j}\, \partial_x u^\alpha \,\mathrm{d}x$$

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Space discretization

$$\int_{\kappa_j} \phi_{i,j}\, \partial_m u^\alpha \,\mathrm{d}x \simeq \frac{\Delta x}{2}\omega_i \tau^n_{i,j} \partial_x u^\alpha_{i,j} = \tau^n_{i,j} \int_{\kappa_j} \phi_{i,j}\, \partial_x u^\alpha \,\mathrm{d}x$$

$$\simeq \tau^n_{i,j} \left([\phi_{i,j} u^\alpha] - \int_{\kappa_j} u^\alpha \, \partial_x \phi_{i,j} \,\mathrm{d}x \right)$$

Acoustic step

- How to write equation on τ as in finite volume ?
- Approximation of the integral of $\partial_m u$
- Integration by part (exact)
- Introduction of the numerical fluxes

Space discretization

$$\int_{\kappa_j} \phi_{i,j}\, \partial_m u^\alpha \,\mathrm{d}x \simeq \frac{\Delta x}{2} \omega_i \tau_{i,j}^n \partial_x u_{i,j}^\alpha = \tau_{i,j}^n \int_{\kappa_j} \phi_{i,j}\, \partial_x u^\alpha \,\mathrm{d}x$$

$$\simeq \tau_{i,j}^n \left([\phi_{i,j} u^\alpha] - \int_{\kappa_j} u^\alpha\, \partial_x \phi_{i,j} \,\mathrm{d}x \right)$$

$$\simeq \tau_{i,j}^n \left(\delta_{i,p} u_{j+1/2}^{*,\alpha} - \delta_{i,0} u_{j-1/2}^{*,\alpha} - \sum_{k=0}^{p} \omega_k u_{k,j}^\alpha\, \partial_x \ell_i(s_k) \right)$$

Acoustic step

Source term treatment[a] with a naive discretization.

[a] Christophe Chalons and Maxime Stauffert. "A well-balanced Discontinuous-Galerkin Lagrange-Projection scheme for the Shallow Water Equations". Preprint. Oct. 2017.

Source term treatment

$$\int_{\kappa_j} \phi_{i,j} \, \frac{g}{\tau^n} \partial_m z \simeq \frac{\Delta x}{2} \omega_i \, \tau_{i,j}^n \frac{g}{\tau_{i,j}^n} \partial_x z \simeq \tau_{i,j}^n \int_{\kappa_j} \phi_{i,j} \, g h^n \partial_x z$$

$$\longrightarrow \tau_{i,j}^n \left(\delta_{i,p} \frac{\Delta x}{2} \, \{gh\partial_x z\}_{j+1/2}^n + \delta_{i,0} \frac{\Delta x}{2} \, \{gh\partial_x z\}_{j-1/2}^n + \frac{\Delta x}{2} \omega_i \, \{gh \, \partial_x z\}_{i,j} \right)$$

$$\text{with } \{gh \, \partial_x z\}_{i,j} = g h_{i,j}^n \, \partial_x z|_{i,j}$$

Acoustic step

Source term treatment[a] with a proposed discretization by Manuel Castro.

[a] Christophe Chalons and Maxime Stauffert. “A well-balanced Discontinuous-Galerkin Lagrange-Projection scheme for the Shallow Water Equations”. Preprint. Oct. 2017.

Source term treatment

$$\int_{\kappa_j} \phi_{i,j} \frac{g}{\tau^n} \partial_m z \simeq \frac{\Delta x}{2} \omega_i \tau_{i,j}^n \frac{g}{\tau_{i,j}^n} \partial_x z \simeq \tau_{i,j}^n \int_{\kappa_j} \phi_{i,j}\, g h^n \partial_x z$$
$$\longrightarrow \tau_{i,j}^n \Bigg(\delta_{i,p} \frac{\Delta x}{2} \left\{ gh\partial_x z \right\}_{j+1/2}^n + \delta_{i,0} \frac{\Delta x}{2} \left\{ gh\partial_x z \right\}_{j-1/2}^n + \frac{\Delta x}{2} \omega_i \left\{ gh\, \partial_x z \right\}_{i,j} \Bigg)$$

with $\left\{ gh\, \partial_x z \right\}_{i,j} = gh_{i,j}^n\, \partial_x (h^n + z)|_{i,j} - \partial_x \pi^n|_{i,j}$

Acoustic step

Global Acoustic step

$$\begin{cases} \tau_{i,j}^{n+1^-} = \tau_{i,j}^n + \dfrac{2\Delta t}{\omega_i \Delta x} \tau_{i,j}^n \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_x u^\alpha \,\mathrm{d}x = L_{i,j}^\alpha \tau_{i,j}^n, \\ u_{i,j}^{n+1^-} = u_{i,j}^n - \dfrac{2\Delta t}{\omega_i \Delta x} \tau_{i,j}^n \left(\displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_x \pi^\alpha \,\mathrm{d}x + \int_{\kappa_j} \phi_{i,j}\, g h^n \partial_x z \,\mathrm{d}x \right), \\ \pi_{i,j}^{n+1^-} = \pi_{i,j}^n - a^2 \dfrac{2\Delta t}{\omega_i \Delta x} \tau_{i,j}^n \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_x u^\alpha \,\mathrm{d}x. \end{cases}$$

Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X = \partial_x(Xu) - X\,\partial_x u$
- Approximation of the integral of $X\,\partial_x u^\alpha$ to bring out $L^\alpha_{i,j}$
- Integration by part (not exact)

Transport system

$$X^{n+1}_{i,j} = X^{n+1^-}_{i,j} - \frac{2\Delta t}{\omega_i \Delta x} \int_{\kappa_j} u^\alpha\, \phi_{i,j}\, \partial_x X^{n+1^-}\, \mathrm{d}x$$

$$\updownarrow$$

$$X^{n+1}_j = L^\alpha_j X^{n+1^-}_j - \frac{\Delta t}{\Delta x}\left(X^{*,n+1^-}_{j+1/2}\, u^{*,\alpha}_{j+1/2} - X^{*,n+1^-}_{j+1/2}\, u^{*,\alpha}_{j-1/2}\right)$$

$$\text{with } X^{*,n+1^-}_{j+1/2} = \begin{cases} X^{n+1^-}_j, & \text{if } u^{*,\alpha}_{j+1/2} \geq 0, \\ X^{n+1^-}_{j+1}, & \text{otherwise.} \end{cases}$$

Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X = \partial_x(Xu) - X\,\partial_x u$
- Approximation of the integral of $X\,\partial_x u^\alpha$ to bring out $L_{i,j}^\alpha$
- Integration by part (not exact)

Transport system

$$\int_{\kappa_j} u^\alpha\,\phi_{i,j}\,\partial_x X^{n+1^-} = \int_{\kappa_j} \phi_{i,j}\,\partial_x(X^{n+1^-}\,u^\alpha) - \int_{\kappa_j} X^{n+1^-}\,\phi_{i,j}\,\partial_x u^\alpha$$

Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
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- Approximation of the integral of $X\,\partial_x u^\alpha$ to bring out $L^\alpha_{i,j}$
- Integration by part (not exact)

Transport system

$$\begin{aligned}\int_{\kappa_j} u^\alpha\,\phi_{i,j}\,\partial_x X^{n+1^-} &= \int_{\kappa_j} \phi_{i,j}\,\partial_x(X^{n+1^-}\,u^\alpha) - \int_{\kappa_j} X^{n+1^-}\,\phi_{i,j}\,\partial_x u^\alpha \\ &\simeq \int_{\kappa_j} \phi_{i,j}\,\partial_x(X^{n+1^-}\,u^\alpha) - X^{n+1^-}_{i,j} \int_{\kappa_j} \phi_{i,j}\,\partial_x u^\alpha\end{aligned}$$

Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
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Transport system

$$\int_{\kappa_j} u^\alpha\,\phi_{i,j}\,\partial_x X^{n+1^-} = \int_{\kappa_j} \phi_{i,j}\,\partial_x(X^{n+1^-}\,u^\alpha) - \int_{\kappa_j} X^{n+1^-}\,\phi_{i,j}\,\partial_x u^\alpha$$

$$\simeq \int_{\kappa_j} \phi_{i,j}\,\partial_x(X^{n+1^-}\,u^\alpha) - X^{n+1^-}_{i,j}\int_{\kappa_j} \phi_{i,j}\,\partial_x u^\alpha$$

$$\simeq \left[\phi_{i,j}X^{n+1^-}\,u^\alpha\right] - \int_{\kappa_j} X^{n+1^-}\,u^\alpha\,\partial_x\phi_{i,j} - X^{n+1^-}_{i,j}\int_{\kappa_j}\phi_{i,j}\,\partial_x u^\alpha$$

$$\longrightarrow \left[\phi_{i,j}X^{n+1^-}\,u^\alpha\right] = \delta_{i,p}X^{*,n+1^-}_{j+1/2}\,u^{*,\alpha}_{j+1/2} - \delta_{i,0}X^{*,n+1^-}_{j+1/2}\,u^{*,\alpha}_{j-1/2}$$

Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X = \partial_x(Xu) - X\,\partial_x u$
- Approximation of the integral of $X\,\partial_x u^\alpha$ to bring out $L^\alpha_{i,j}$
- Integration by part (not exact)

Transport system

$$\begin{cases} h_{i,j}^{n+1} = L_{i,j}^{n+1-} h_{i,j}^{n+1-} - \dfrac{2\Delta t}{w_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j} \partial_x (h^{n+1^-} u^\alpha) \,\mathrm{d}x, \\ (hu)_{i,j}^{n+1} = L_{i,j}^{n+1-} (hu)_{i,j}^{n+1-} - \dfrac{2\Delta t}{w_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j} \partial_x ((hu)^{n+1^-} u^\alpha) \,\mathrm{d}x. \end{cases}$$

Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X = \partial_x(Xu) - X\,\partial_x u$
- Approximation of the integral of $X\,\partial_x u^\alpha$ to bring out $L^\alpha_{i,j}$
- Integration by part (not exact)

Global scheme

$$
\begin{cases}
h^{n+1}_{i,j} = h^n_{i,j} - \dfrac{2\Delta t}{w_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j} \partial_x h^{n+1^-} u^\alpha \,\mathrm{d}x, \\
(hu)^{n+1}_{i,j} = (hu)^n_{i,j} - \dfrac{2\Delta t}{w_i \Delta x} \left(\displaystyle\int_{\kappa_j} \phi_{i,j} \partial_x ((hu)^{n+1^-} u^\alpha + \pi^\alpha) \,\mathrm{d}x \right. \\
\qquad\qquad\qquad \left. + \displaystyle\int_{\kappa_j} \phi_{i,j}\, g h^n \partial_x z \,\mathrm{d}x \right).
\end{cases}
$$

Contents

IMEX DG scheme

Hypothesis :

- $a > \max_j max_i h_{i,j} \sqrt{g h_{i,j}}$
- $\frac{\Delta t}{\Delta x} \max_j \max_i c_{i,j} \leq 1$

with $c_{i,j} = \frac{2}{\omega_i} \left(\int_{\kappa_j} u_j^{n+1-} \partial_x \phi_{i,j} - \delta_{i,p} u^*_{j+1/2,-} + \delta_{i,0} u^*_{j-1/2,+} \right)$

Properties

- If $p = 0$: $c_j = u^*_{j-1/2,+} - u^*_{j+1/2,-} \longrightarrow$ same CFL as in FV
- Convex combination :

$$\overline{X}_j^{n+1} = \sum_{i=0}^{p} \frac{\omega_i}{2} \left(1 - \frac{\Delta t}{\Delta x} c_{i,j}\right) X_{i,j}^{n+1-} + \frac{\Delta t}{\Delta x} (-u^*_{j+1/2,-}) X_{0,j+1}^{n+1-} + \frac{\Delta t}{\Delta x} u^*_{j-1/2,+} X_{p,j-1}^{n+1-}$$

- $h_{i,j}^{n+1^-} > 0$ and thus $\overline{h}_j^{n+1} > 0$, provided that $h^n_{i,j} > 0$, $\forall i, j$

IMEX DG scheme

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with $c_{i,j} = \frac{2}{\omega_i}\left(\int_{\kappa_j} u_j^{n+1-}\partial_x \phi_{i,j} - \delta_{i,p} u^*_{j+1/2,-} + \delta_{i,0} u^*_{j-1/2,+}\right)$

Properties

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- $h_{i,j}^{n+1^-} > 0$ and thus $\overline{h}_j^{n+1} > 0$, provided that $h^n_{i,j} > 0$, $\forall i, j$

IMEX DG scheme

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- $a > \max_j max_i h_{i,j}\sqrt{gh_{i,j}}$
- $\frac{\Delta t}{\Delta x} \max_j \max_i c_{i,j} \leq 1$

with $c_{i,j} = \frac{2}{\omega_i}\left(\int_{\kappa_j} u_j^{n+1-}\partial_x \phi_{i,j} - \delta_{i,p} u^*_{j+1/2,-} + \delta_{i,0} u^*_{j-1/2,+}\right)$

Properties

- It satisfies a discrete entropy inequality of the form :

$$
\begin{aligned}
(hE)(\overline{\boldsymbol{U}}_j^{n+1}) - \overline{(hE)}_j^n \\
&+ \frac{\Delta t}{\Delta x}\Big[((hE)^*_{j+1/2} + \pi^*_{j+1/2})u^*_{j+1/2} \\
&\qquad - ((hE)^*_{j-1/2} + \pi^*_{j-1/2})u^*_{j-1/2}\Big] \\
&\leq -\Delta t \left\{ghu\partial_x z\right\}_j .
\end{aligned}
$$

WB properties

With naive discretization

Mean values

Hypothesis :

$$h^0 + z^0 = K \text{ and } u^0 = 0 \text{ with } h^0 \text{ and } z^0 \text{ polynomials of order } \leq p$$

Result :

WB for the mean values and only for the EXEX scheme

Nodal values

Hypothesis :

$$h^0 + z^0 = K \text{ and } u^0 = 0 \text{ with } h^0 \text{ and } z^0 \text{ polynomials of order } \leq p/2$$

Result :

WB for the nodal values for both the EXEX and the IMEX schemes

WB properties

With naive discretization

Mean values

Hypothesis :

$h^0 + z^0 = K$ and $u^0 = 0$ with h^0 and z^0 polynomials of order $\leq p$

Result :

WB for the mean values and only for the EXEX scheme

Nodal values

Hypothesis :

$h^0 + z^0 = K$ and $u^0 = 0$ with h^0 and z^0 polynomials of order $\leq p/2$

Result :

WB for the nodal values for both the EXEX and the IMEX schemes

WB properties

With discretization proposed by Manuel Castro

Unconditionnal WB property

Hypothesis :

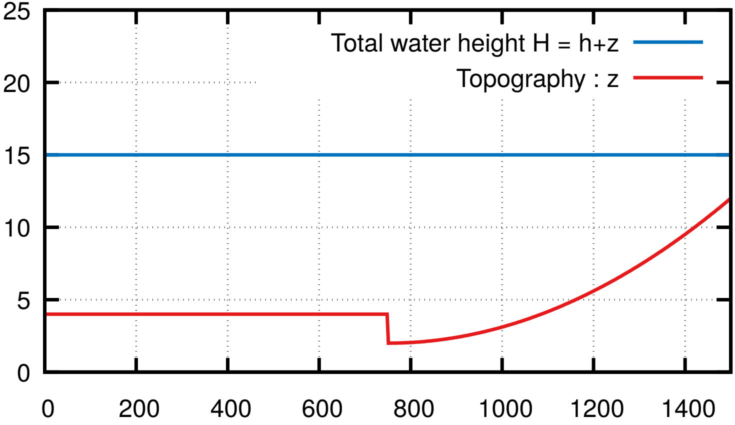
$$h^0 + z^0 = K \text{ and } u^0 = 0 \text{ with any } h^0 \text{ and } z^0$$

Result :

WB for the nodal values for both the EXEX and the IMEX schemes

Contents

WB property



Total water height H = h+z
Topography : z
0
5
10
15
20
25
0
200
400
600
800
1000
1200
1400

WB property

With naive discretization

$r=2$ 500-cell grid		$T=\Delta t$, mean values		$T=\Delta t$, nodal values	
		$\left\lVert\overline{h+z}-15\right\rVert_\infty/15$	$\left\lVert\overline{q}/\overline{h}\right\rVert_\infty$	$\left\lVert h+z-15\right\rVert_\infty/15$	$\left\lVert q/h\right\rVert_\infty$
EXEX	$p=0$	9.87 E−17	0.00 E−17	9.87 E−17	0.00 E−17
	$p=1$	9.87 E−17	4.47 E−16	9.87 E−17	**3.88 E−5**
	$p=2$	**1.97 E−16**	**3.05 E−16**	9.87 E−17	**4.67 E−8**
	$p=3$	**1.97 E−16**	**2.63 E−16**	9.87 E−17	**1.34 E−11**
	$p=4$	1.98 E−16	0.00 E−17	9.87 E−17	0.00 E−17
IMEX	$p=0$	9.87 E−17	0.00 E−17	9.87 E−17	0.00 E−17
	$p=1$	**1.97 E−6**	**3.89 E−5**	2.67 E−6	**1.93 E−4**
	$p=2$	**4.70 E−10**	**9.74 E−9**	6.83 E−9	**1.72 E−7**
	$p=3$	**3.45 E−14**	**6.86 E−13**	1.78 E−12	**3.97 E−11**
	$p=4$	1.98 E−16	0.00 E−17	9.87 E−17	0.00 E−17

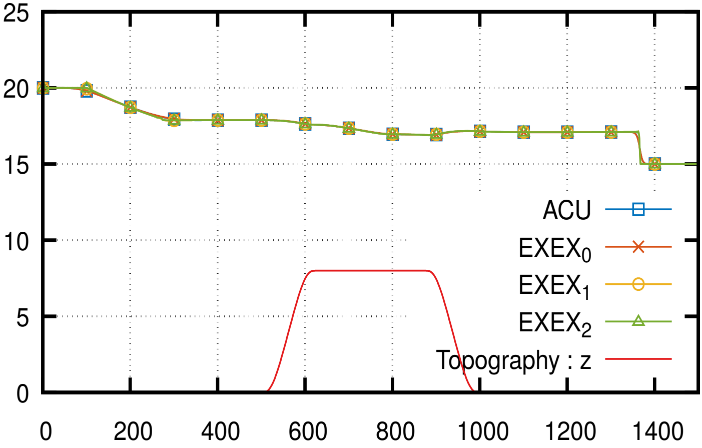
WB property

With discretization proposed by Manuel Castro

$r=2$ 500-cell grid		$T=\Delta t$, mean values		$T=\Delta t$, nodal values	
		$\left\lVert \overline{h+z}-15 \right\rVert_\infty / 15$	$\left\lVert \overline{q}/\overline{h} \right\rVert_\infty$	$\left\lVert h+z-15 \right\rVert_\infty / 15$	$\left\lVert q/h \right\rVert_\infty$
EXEX	$p=0$	9.87 E−17	0.00 E−17	9.87 E−17	0.00 E−17
	$p=1$	9.87 E−17	0.00 E−17	9.87 E−17	**0.00 E−17**
	$p=2$	**1.97 E−16**	**0.00 E−17**	9.87 E−17	**0.00 E−17**
	$p=3$	**1.97 E−16**	**0.00 E−17**	9.87 E−17	**0.00 E−17**
	$p=4$	1.97 E−16	0.00 E−17	9.87 E−17	0.00 E−17
IMEX	$p=0$	9.87 E−17	0.00 E−17	9.87 E−17	0.00 E−17
	$p=1$	**9.87 E−17**	**0.00 E−17**	9.87 E−17	**0.00 E−17**
	$p=2$	**1.97 E−16**	**0.00 E−17**	9.87 E−17	**0.00 E−17**
	$p=3$	**1.97 E−16**	**0.00 E−17**	9.87 E−17	**0.00 E−17**
	$p=4$	1.97 E−16	0.00 E−17	9.87 E−17	0.00 E−17

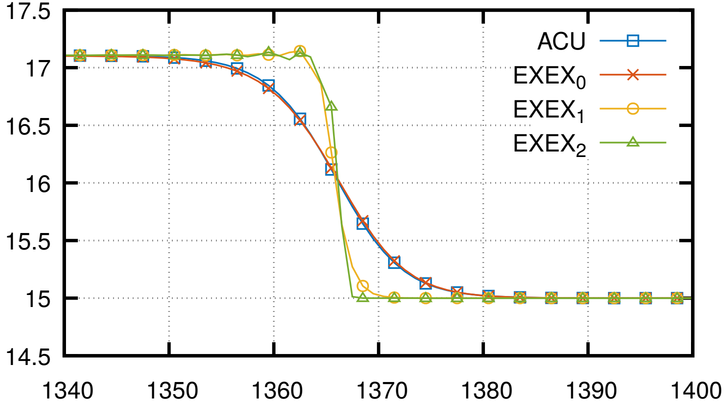
Dam Break

NbCell : 1500, Tf : 50, variable $H = h + z$



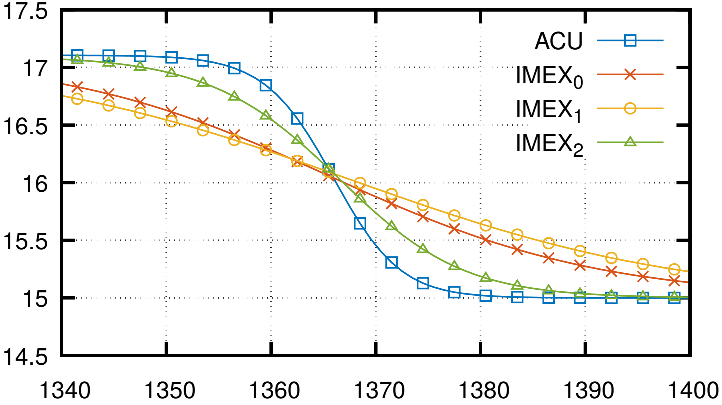
Dam Break

NbCell : 1500, Tf : 50, variable $H = h + z$



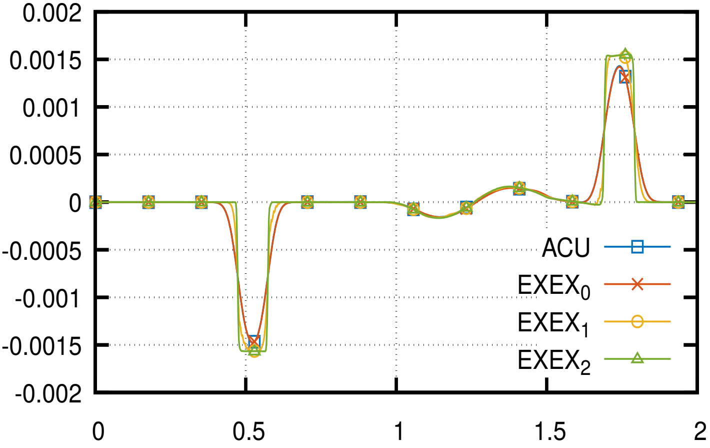
Dam Break

NbCell : 1500, Tf : 50, variable $H = h + z$



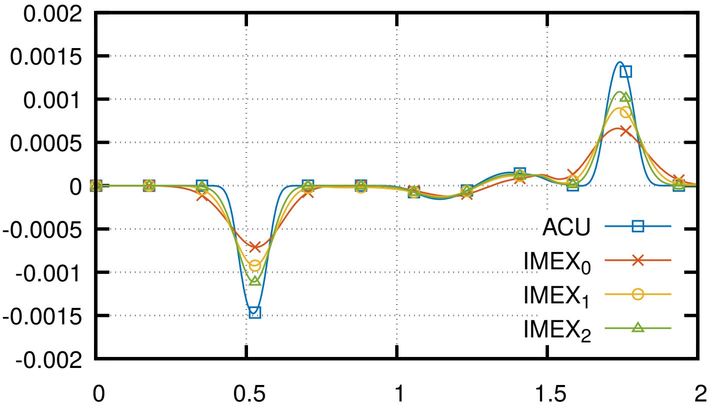
Propagation of perturbation

NbCell : 1000, Tf : 0.2, variable u

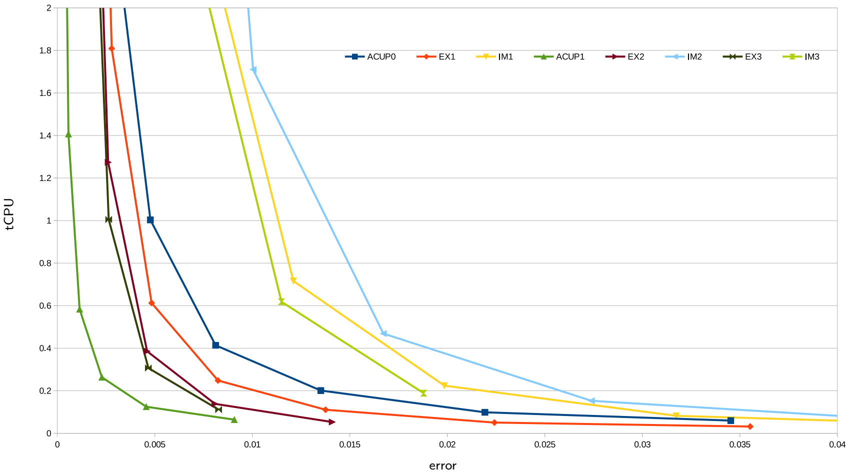


Propagation of perturbation

NbCell : 1000, Tf : 0.2, variable u

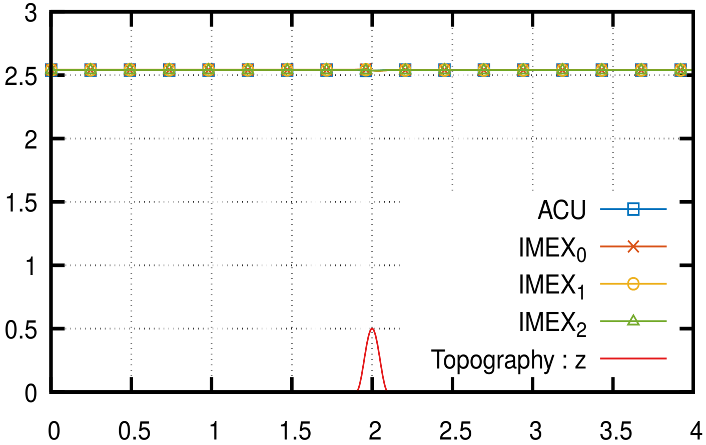


Efficiency curves



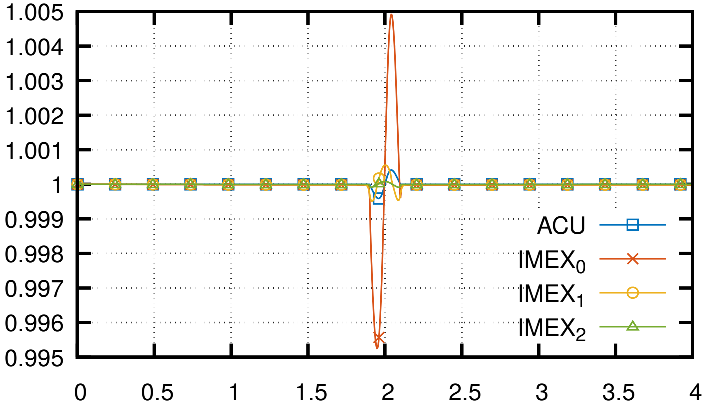
Fluvial regime

NbCell : 1600, Tf : 200, variable $H = h + z$



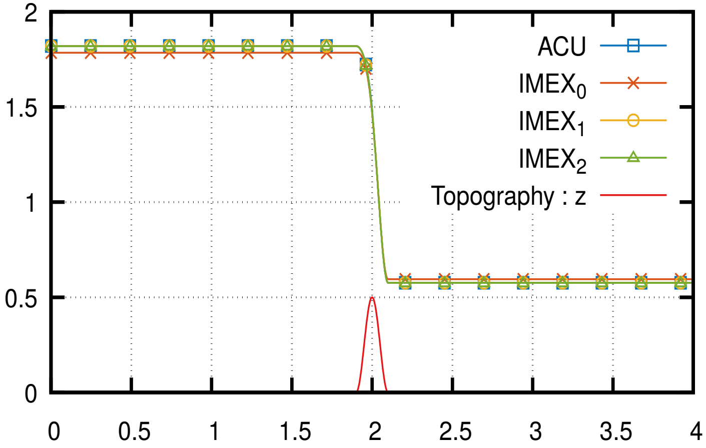
Fluvial regime

NbCell : 1600, Tf : 200, variable q



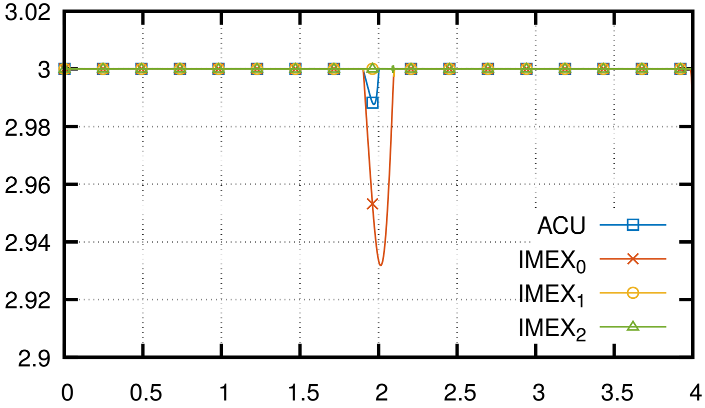
Transcritical regime without shock

NbCell : 1600, Tf : 200, variable $H = h + z$



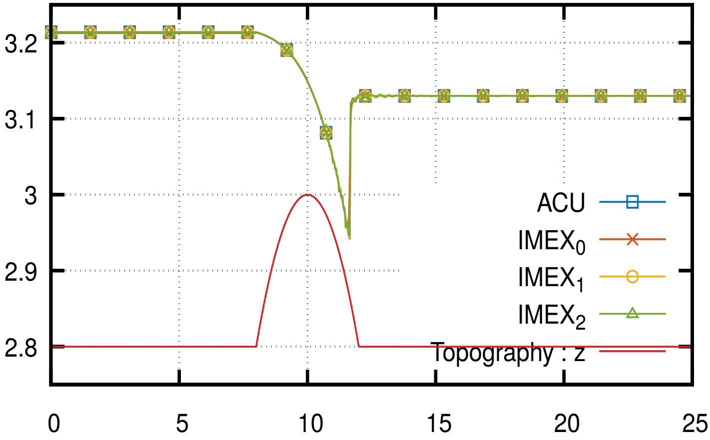
Transcritical regime without shock

NbCell : 1600, Tf : 200, variable q



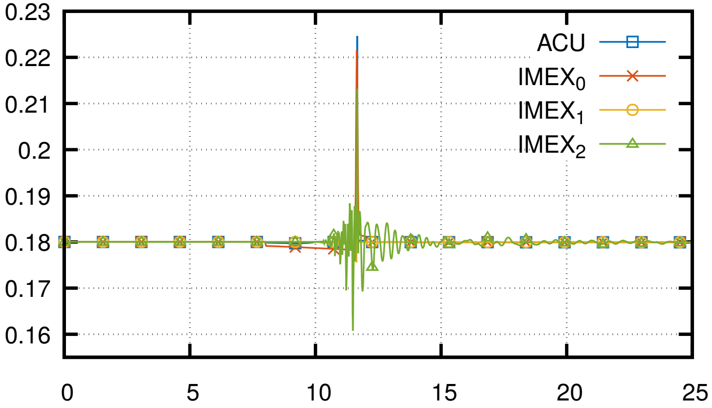
Transcritical regime with a shock

NbCell : 1600, Tf : 200, variable $H = h + z$



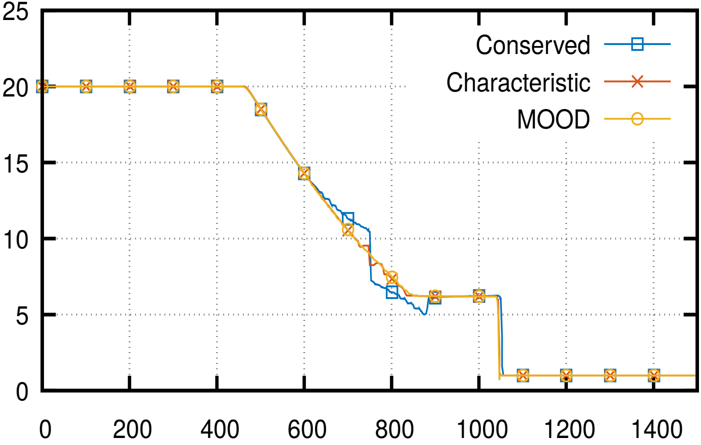
Transcritical regime with a shock

NbCell : 1600, Tf : 200, variable q



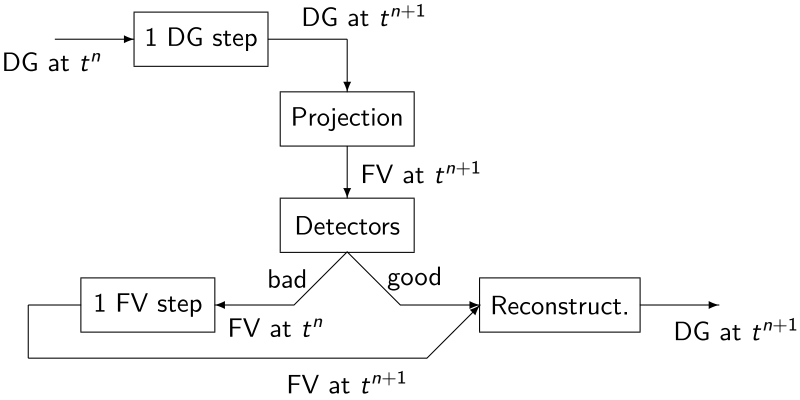
Limitors

NbCell = 500, Tf = 20, p = 2, variable h

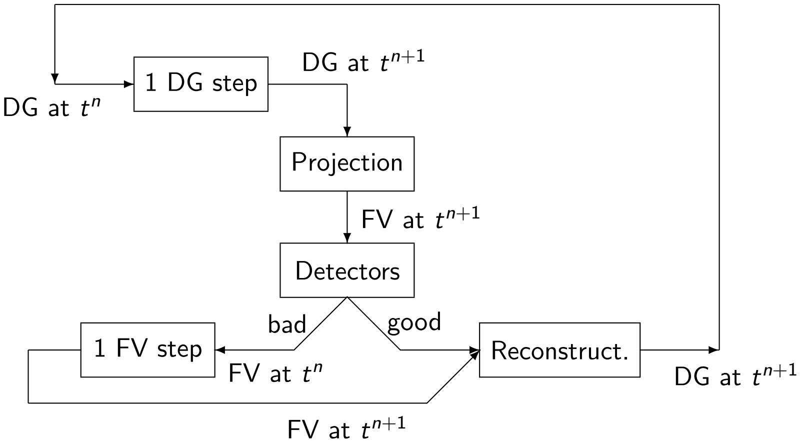


Contents

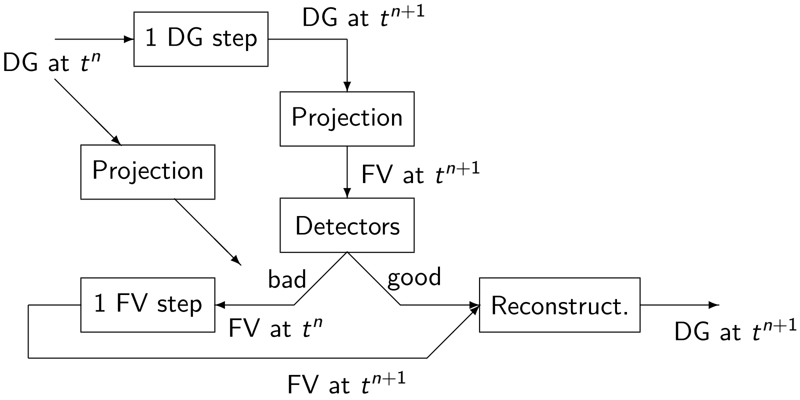
Naive approach



Naive approach

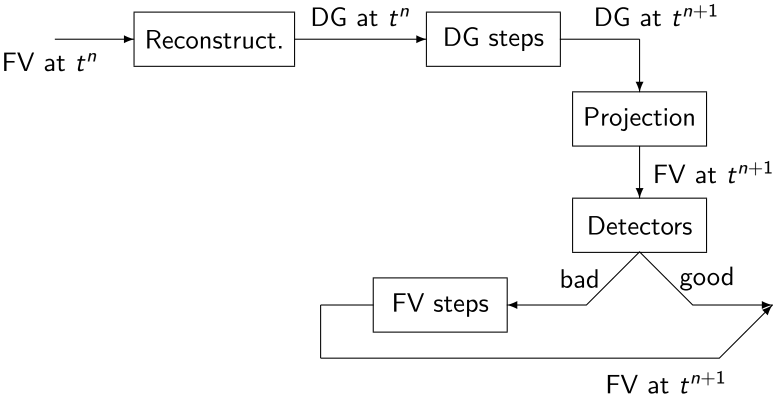


Naive approach

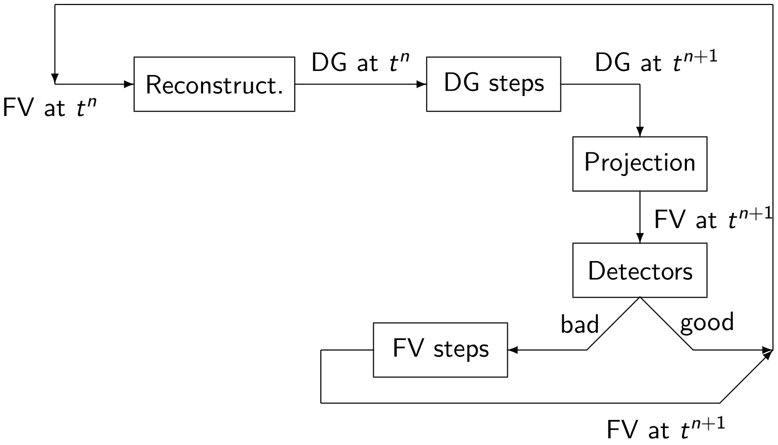


Problem : Reconstruction $\circ$ Projection $\neq$ Identity

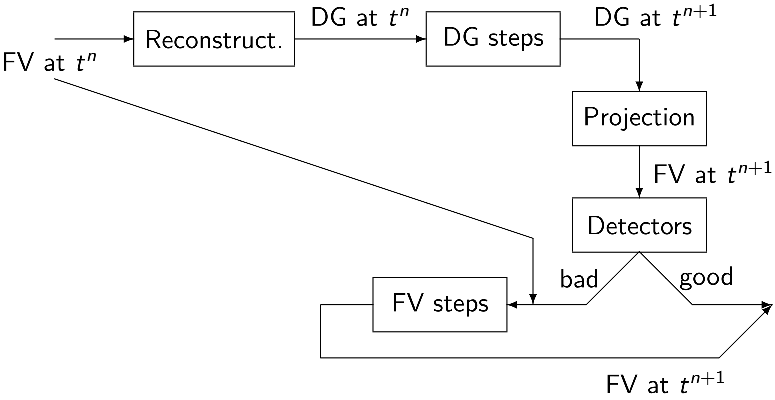
Robustness approach



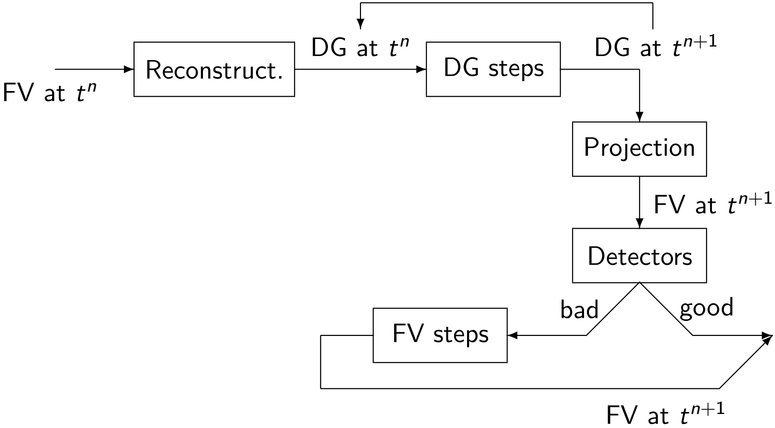
Robustness approach



Robustness approach



Robustness approach



Projection $\circ$ Reconstruction $=$ Identity

$\Rightarrow$ no recomputation of DG solution when detector $= 0$

Contents

2D scheme

Acoustic step

$$\begin{cases} \mathbf{u}_j^{n+1-} = \mathbf{u}_j^n - \tau_j^n \Delta t \sum_{k\in\mathcal{N}(j)} \sigma_{jk}\, \pi_{jk}^{*,\theta} \mathbf{n}_{jk}, \\ \pi_j^{n+1-} = \pi_j^n - \tau_j^n \Delta t \sum_{k\in\mathcal{N}(j)} \sigma_{jk}\, (a_{jk})^2 u_{jk}^*, \\ \tau_j^{n+1-} = \tau_j^n + \tau_j^n \Delta t \sum_{k\in\mathcal{N}(j)} \sigma_{jk}\, u_{jk}^*, \end{cases}$$

Numerical fluxes

$$a_{jk} \geq \max[(\rho c)_j^n, (\rho c)_k^n],$$

$$u_{jk}^* = \frac{1}{2}\mathbf{n}_{jk}^T(\mathbf{u}_j^\alpha + \mathbf{u}_k^\alpha) - \frac{1}{2a_{jk}}(\pi_k^\alpha - \pi_j^\alpha) - \frac{g}{2a_{jk}}\frac{h_j^n + h_k^n}{2}(b_k - b_j),$$

$$\pi_{jk}^{*,\theta} = \frac{1}{2}(\pi_j^\alpha + \pi_k^\alpha) - \frac{a_{jk}\theta_{jk}}{2}\mathbf{n}_{jk}^T(\mathbf{u}_k^\alpha - \mathbf{u}_j^\alpha) + \frac{g}{2}\frac{h_j^n + h_k^n}{2}(b_k - b_j).$$

Contents

Conclusion

Achievements

- DG discretization for L-P schemes in framework of SWE
- Well-balanced properties
- Implementation of a compiled code compatible with MOOD
- Study of low Froude truncation errors

Perspectives

- Numerical results for MOOD
- Implementation of a 2D code with unstructured mesh
- Numerical results for low Froude regime flows
- Experimental order of accuracy for the DG schemes
- Study other systems that have some asymptotic regime (eg. MHD)
- Use those schemes with AMR techniques in CanoP

Bibliography

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Thank you for your attention