Large density flows and congestion phenomena

Charlotte Perrin
CNRS, Institut de Mathématiques de Marseille

Schémas numériques pour les écoulements
à faible nombre de Mach

Toulouse, November 20-22, 2017

Applications

partially free surface flows

mixtures

collective motion

Fluid equations under maximal density constraint

- two-phase model (free / congested) $\rightarrow$ "hard" congestion model

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \operatorname{div} \mathbb{S} = 0 \\ 0 \leq \rho \leq \rho^*, \ (\rho^* - \rho) p = 0, \ p \geq 0 \end{cases}$$

in the following: $\rho^* = 1$

  - pressureles equations in the free domain $\rho < \rho^*$
  - $\operatorname{div} u \geq 0$ in the congested domain $\rho = \rho^*$
  - activation of the pressure $p \geq 0$ in the congested domain

    $\rightarrow$ Lagrange multiplier associated to the constraint on u

Literature

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \operatorname{div} \mathbb{S} = 0 \\ 0 \le \rho \le \rho^*, \ (\rho^* - \rho) p = 0, \ p \ge 0 \end{cases}$$

- **theoretical studies**
  - Lions, Masmoudi (1998), Perrin, Zatorska (2015) : $\mathbb{S} = 2\mu \mathrm{D}(u) + \lambda \operatorname{div} u \, \mathrm{Id}$
  - Berthelin (2002, 2016), Perrin, Westdickenberg (2017) : $\mathbb{S} = 0$
  - Perrin (2016) : $\mathbb{S} = \mathbb{S}(p)$ + effet de non-localité
- **numerical studies**
  - Degond, Hua, Navoret (2011), Bresch, Renardy (2016)
  - Maury, Preux (2016)
- **other references (similar systems) :** Bourdarias, Ersoy, Gerbi (2012), Godlewski, Parisot, Sainte-Marie, Wahl (2016), Lannes (2016)

Approximation by a "soft" congestion model

soft model

compressible singular pressure NS

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) \\ \quad - \nabla(\lambda \operatorname{div} u_\varepsilon) - 2\operatorname{div}(\mu \mathrm{D}(u_\varepsilon)) = 0 \\ 0 \le \rho < 1 \end{cases}$$

$$\xrightarrow[\varepsilon \to 0]{}$$

hard model

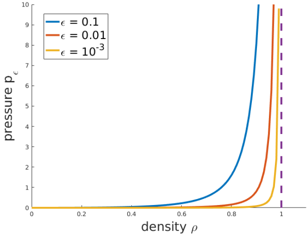
compressible/incompressible NS

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p \\ \quad - \nabla(\lambda \operatorname{div} u) - 2\operatorname{div}(\mu \mathrm{D}(u)) = 0 \\ 0 \le \rho \le 1, \; (1-\rho)p = 0, \; p \ge 0 \\ \operatorname{div} u = 0 \quad \text{on} \quad \{\rho = 1\} \end{cases}$$

$$p_\varepsilon(\rho) = \varepsilon \left(\frac{\rho}{1-\rho} \right)^\gamma$$

$\rightarrow$ singular repulsive forces

$$\lambda + \frac{2}{3}\mu \ge 0$$



ref : Perrin, Zatorska (2015)

Some elements of the theoretical proof

Ω a bounded domain of $\mathbb{R}^3$, Dirichlet boundary conditions on $\partial\Omega$

- **energy estimate + control of the density**

$$\sup_{t\in(0,T)} \int_\Omega \left[\frac{\rho_\varepsilon |u_\varepsilon|^2}{2} + \phi(\rho_\varepsilon)\frac{\varepsilon}{(1-\rho_\varepsilon)^{\gamma-1}}\right] + \int_0^T \int_\Omega \Big(2\mu |\,\mathrm{D}(u_\varepsilon)|^2 + \lambda\,(\,\mathrm{div}\, u_\varepsilon)^2\Big) \le E_\varepsilon^0$$

$$\mathrm{mes}\ \{\rho_\varepsilon \ge 1\} = 0$$

- **control of the pressure** the mom eq. is tested by $\mathcal{B}\big(\rho_\varepsilon - M^0\big)$, $\mathcal{B} \sim \mathrm{div}^{-1}$

$$\big(p_\varepsilon(\rho_\varepsilon)\big)_\varepsilon,\ \big(\rho_\varepsilon p_\varepsilon(\rho_\varepsilon)\big)_\varepsilon \ \text{ are bounded in } \ L^1\big((0,T)\times\Omega\big)$$

- **weak cvg of the weak solutions** $\rightarrow$ compactness arguments of Lions
  - study the oscillation of ρ_ε $\rightarrow$ evolution of $\lim_{\varepsilon\to 0} \rho_\varepsilon \ln \rho_\varepsilon - \rho \ln \rho$
  - weak compactness property of the effective flux $(2\mu+\lambda)\,\mathrm{div}\, u_\varepsilon - p_\varepsilon(\rho_\varepsilon)$

Theorem (Perrin, Zatorska 2015)

Let $\gamma > 3$, *assume that initially* $0 \leq \rho_\varepsilon^0 < 1$ *a.e. and*

$$M_\varepsilon^0 = \int_\Omega \rho_\varepsilon^0 \,\mathrm{d}x \leq C, \qquad \int_\Omega E_\varepsilon(\rho_\varepsilon^0, m_\varepsilon^0) \leq C.$$

Then, for $\varepsilon \to 0$, *there exists a subsequence* $(\rho_\varepsilon, u_\varepsilon, p_\varepsilon)$ *converging to* (ρ, u, p) *a weak solution of the hard congestion system with*

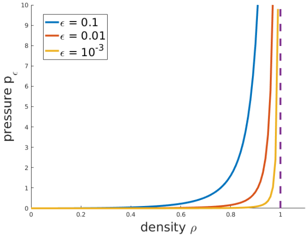
$$\begin{aligned} \rho_\varepsilon \to \rho \quad & \textit{strongly in} \quad L^q\big((0,T)\times\Omega\big) \\ u_\varepsilon \to u \quad & \textit{weakly in} \quad L^2\big(0,T;(W_0^{1,2}(\Omega))^d\big) \\ p_\varepsilon(\rho_\varepsilon) \to p \quad & \textit{weakly in} \quad \mathcal{M}^+\big((0,T)\times\Omega\big) \end{aligned}$$

- low regularity of the limit pressure
  - definition of the product ρp in the constraint $(1-\rho)p = 0$
- no result about the existence of local strong solutions
- no theoretical result about the singular limit in the inviscid case

Numerical approach

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) \ \big(- \operatorname{div} \mathbb{S}\big) = 0 \end{cases}$$

- singular pressure $p_\varepsilon(\rho) = \varepsilon \left(\dfrac{\rho}{1-\rho} \right)^\gamma$



- p_ε ensures the constraint $\rho_\varepsilon < 1$
- p_ε becomes stiffer and stiffer as $\varepsilon \to 0$
- explicit treatment $\Longrightarrow$ conditional stability

$$\Delta t \leq \frac{\sigma \Delta x}{\max\{|u_\varepsilon| + \sqrt{p'_\varepsilon(\rho_\varepsilon)}\}} \underset{\substack{\varepsilon \to 0 \\ \rho_\varepsilon \to 1}}{\longrightarrow} 0$$

$\rightarrow$ implicit treatment of some terms

Implicit treatment of the pressure

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \operatorname{div}(\rho u)^{n+1} = 0 \\ \dfrac{(\rho u)^{n+1}-(\rho u)^n}{\Delta t} + \operatorname{div}(\rho u \otimes u)^n + \nabla p_\varepsilon(\rho^{n+1}) = 0 \end{cases}$$

- reformulation: div (Mom eq) and insert the result into Mass eq.

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \operatorname{div}(\rho u)^{n} - \Delta t \Delta p_\varepsilon(\rho^{n+1}) - \Delta t \nabla^2 : (\rho u \otimes u)^n = 0 \\ \dfrac{(\rho u)^{n+1}-(\rho u)^n}{\Delta t} + \operatorname{div}(\rho u \otimes u)^n + \nabla p_\varepsilon(\rho^{n+1}) = 0 \end{cases}$$

$\rightarrow$ uniform stability condition

- $\rho^{n+1} = \rho(p_\varepsilon^{n+1}) \rightarrow$ nonlinear elliptic equation on p_ε^{n+1}

1) compute p_ε^{n+1}
2) deduce the new density $\rho^{n+1} \rightarrow \rho^{n+1}$ satisfies automatically the constraint
3) compute the new momentum $(\rho u)^{n+1}$

Degond, Hua & Navoret (2011): Implicit/explicit splitting of the pressure

Computation of the momentum, Gauge Method

- Gauge Decomposition $\quad \rho u = a - \nabla\varphi, \quad \mathrm{div}\, a = 0$
- time discretization

$$\Delta\varphi^{n+1} = \frac{1}{\Delta t}(\rho^{n+1} - \rho^n) \qquad \varphi^{n+1}{}_{|\partial\Omega} = 0$$

$$\Delta P^{n+1} = -\nabla^2 : (\rho u \otimes u)^n \qquad \rightsquigarrow \qquad P^{n+1} = p_\varepsilon(\rho^{n+1}) - \frac{\varphi^{n+1} - \varphi^n}{\Delta t}$$

$$\frac{a^{n+1} - a^n}{\Delta t} + \mathrm{div}\,(\rho u \otimes u)^n + \nabla P^{n+1} = 0$$

$$(\rho u)^{n+1} = a^{n+1} - \nabla\varphi^{n+1}$$

ref: Degond, Jin, Liu (2007), Degond, Hua, Navoret (2011)

Time-Space discretization in 1D

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{1}{\Delta x}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right] = 0$$

$$\frac{(\rho u)_j^{n+1} - (\rho u)_j^n}{\Delta t} + \frac{1}{\Delta x}\left[F_{j+1/2}^n - F_{j-1/2}^n\right] + \frac{1}{2\Delta x}\left[p_\varepsilon(\rho_{j+1}^{n+1}) - p_\varepsilon(\rho_{j-1}^{n+1})\right] = 0$$

with

$$Q_{j+1/2}^{n+1/2} = \frac{1}{2}\left[(\rho u)_j^{n+1} + (\rho u)_{j+1}^{n+1}\right] - \frac{D_{j+1/2}^n}{2}\left(\rho_{j+1}^n - \rho_j^n\right)$$

$$F_{j+1/2}^n = \frac{1}{2}\left[\frac{\left((\rho u)_j^n\right)^2}{\rho_j^n} + \frac{\left((\rho u)_{j+1}^n\right)^2}{\rho_{j+1}^n}\right] - \frac{D_{j+1/2}^n}{2}\left((\rho u)_{j+1}^n - (\rho u)_j^n\right)$$

$$D_{j+1/2}^n = \max\left\{|u_j^n|, |u_{j+1}^n|\right\}$$

$$\rho\big((p_\varepsilon)_j^{n+1}\big) - \frac{\Delta t^2}{4\Delta x^2}\left[p_\varepsilon(\rho_{j+2}^{n+1}) - 2p_\varepsilon(\rho_j^{n+1}) + p_\varepsilon(\rho_{j-2}^{n+1})\right]$$
$$= \rho_j^n - \frac{\Delta t}{2\Delta x}\big((\rho u)_{j+1}^n - (\rho u)_{j-1}^n\big) + \frac{\Delta t^2}{2\Delta x^2}\left[F_{j+3/2}^n - F_{j+1/2}^n - F_{j-1/2}^n + F_{j-3/2}^n\right]$$
$$+ \frac{\Delta t}{2\Delta x}\left[D_{j+1/2}^n(\rho_{j+1}^n - \rho_j^n) - D_{j-1/2}^n(\rho_j^n - \rho_{j-1}^n)\right]$$

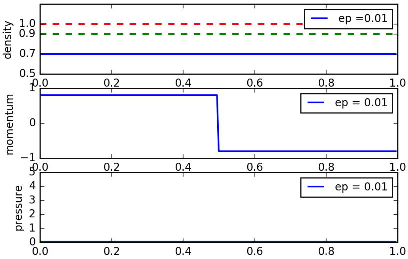
$$\frac{1}{\Delta x^2}\left[\varphi_{j+1}^{n+1} - 2\varphi_j^{n+1} + \varphi_{j-1}^{n+1}\right]$$
$$= \frac{1}{\Delta t}(\rho_j^{n+1} - \rho_j^n) - \frac{1}{2\Delta x}\left[D_{j+1/2}^n(\rho_{j+1}^n - \rho_j^n) - D_{j-1/2}^n(\rho_j^n - \rho_{j-1}^n)\right]$$

$$a^{n+1} = a^n - \Delta t\Big((\rho u \otimes u)^n + p_\varepsilon(\rho^{n+1})\Big)\Big|_0^1$$
$$+ \frac{\Delta t}{2}\sum_{j=1}^{Nx}\left[D_{j+1/2}^n(\rho_{j+1}^n - \rho_j^n) - D_{j-1/2}^n(\rho_j^n - \rho_{j-1}^n)\right]$$

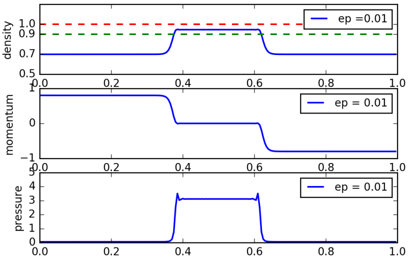
$$(\rho u)_j^{n+1} = a^{n+1} - \frac{1}{2\Delta x}\left[\varphi_{j+1}^{n+1} - \varphi_{j-1}^{n+1}\right]$$

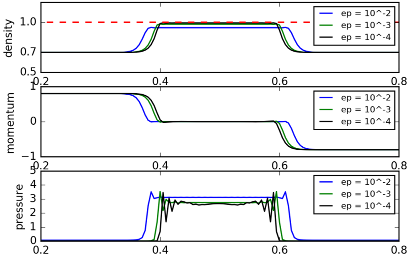
Numerical simulations: $\gamma = 2$, $\Delta t = 5.10^{-4}$, $\Delta x = 5.10^{-3}$

$t = 0$



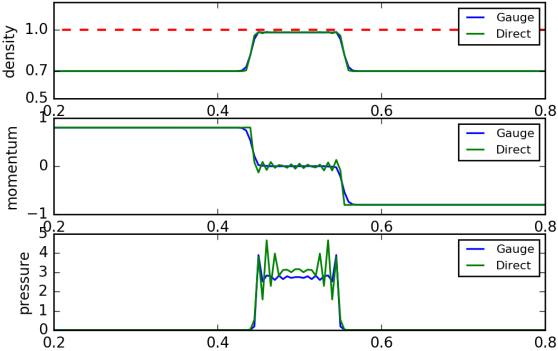
$t = 0.04$





Gauge Method vs Direct Method

$\varepsilon = 10^{-3}$, $\gamma = 2$, $\Delta t = 5.10^{-4}$, $\Delta x = 5.10^{-3}$



Lagrangian approach in 1d, $\mathbb{S} = 0$

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0 \\ 0 \leq \rho \leq 1, \quad (1-\rho)p = 0, \quad p \geq 0 \end{cases}$$

- we introduce the monotone rearrangement $X_t : (0,1) \to \mathbb{R}$ such that

$$\rho(t, X_t(w)) = \frac{1}{\partial_w X_t(w)} \quad \text{for} \quad w \in (0,1)$$

$$\rho_t \leq 1, \ \int_{\mathbb{R}} |x|^2 \rho(t,x)\, \mathrm{d}x < +\infty \quad \Leftrightarrow \quad X_t = \mathrm{Id} + S_t \in L^2(0,1) \ \text{ avec } \ \partial_w S_t \geq 0$$

- existence of global weak solutions characterized by the formula

$$\boxed{X_t = \mathrm{P}_{\widetilde{K}}(X_0 + tU_0)} \qquad \text{where} \ \ \widetilde{K} = \{X \in L^2 \,|\, X = \mathrm{id} + S, \ \partial_w S \geq 0\}$$

$$\to \quad U_t = \frac{\mathrm{d}}{\mathrm{d}t} X_t \ \text{ a.e } t, \quad \partial_w P_t = -\frac{\mathrm{d}}{\mathrm{d}t} U_t \ \text{ in } \mathcal{D}'$$

ref : Perrin, Westdickenberg (2017)

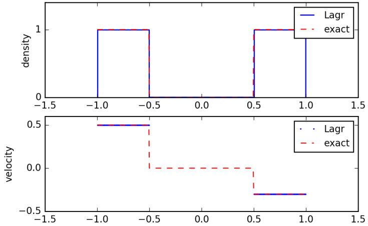
Associated numerical scheme

$$X_t = \mathrm{P}_{\widetilde{K}}(X_0 + tU_0)$$

- minimization at each time step of $\|X_0 + tU_0 - X\|_2^2$ under the constraint $X \in \widetilde{K}$
- comparison with exact solutions:
  $\rightarrow$ case of sticky blocks (ref: Berthelin, 2002)

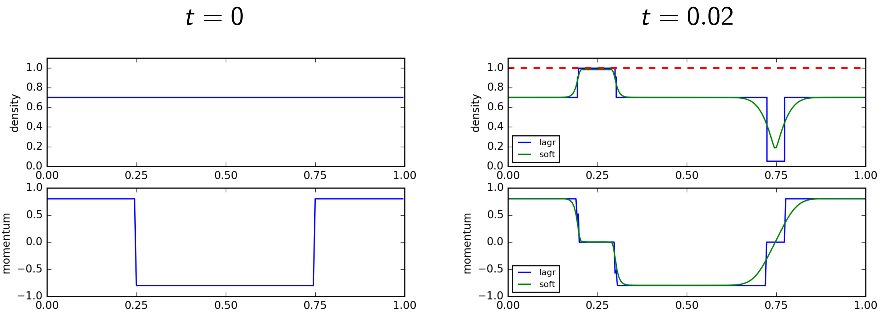
- dynamics of congested blocks
- free dynamics until a collision at $t = t^*$
- from time t^* the blocks form a bigger block

$$l^* = l_1 + l_2$$
$$u^* = \frac{l_1 u_1 + l_2 u_2}{l_1 + l_2}$$



Comparison with the soft approach

$\Delta x = 5.10^{-3}$, $\Delta t = 10^{-4}$, $\varepsilon = 10^{-4}$



Conclusion, ongoing works

- singular limit "soft" $\rightarrow$ "hard"
  - extension to inviscid fluids, complex fluids, etc.
  - qualitative properties of the limit solutions
  - other numerical schemes

- on the limit "hard" system
  - extension of the Lagrangian method to 2d/3d ?

Implicit / Explicit scheme (Degond, Hua, Navoret)

$$p_\varepsilon(\rho) = \varepsilon \left(\frac{\rho}{1-\rho} \right)^\gamma$$

splitting of the pressure $p_\varepsilon = p_\varepsilon^{\mathrm{ex}}(\rho^n) + p_\varepsilon^{\mathrm{im}}(\rho^{n+1})$

$$p_\varepsilon^{\mathrm{ex}}(\rho) = \begin{cases} p_\varepsilon(\rho)/2 & \text{if } \rho \leq 1-\delta \\ \text{poly of order 2} & \text{if } \rho > 1-\delta \end{cases}$$

$\rightarrow$ good choice of parameter: $\boxed{\delta = \varepsilon^{\frac{1}{\gamma+2}}}$

$$\Longrightarrow \sqrt{(p_\varepsilon^{\mathrm{ex}})'(1)},\ (p_\varepsilon^{\mathrm{ex}})''(1) \leq C$$