Low Mach flows: non-stationary and high order aspects

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- Joint work with
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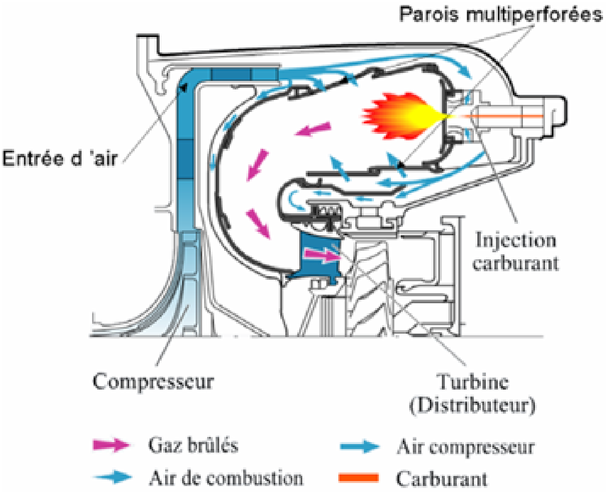
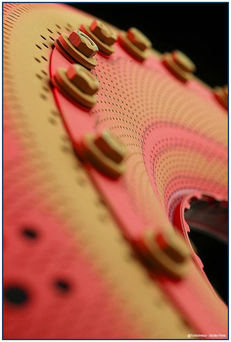
  - Jonathan Jung
  - Pascal Bruel

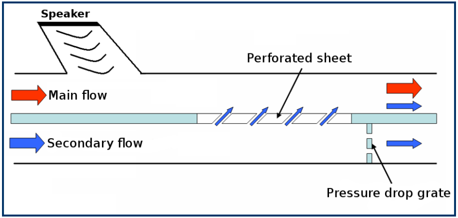
Context


Parois multiperforées
Entrée d 'air
Injection
carburant
Compresseur
Turbine
(Distributeur)
Gaz brûlés
Air compresseur
Air de combustion
Carburant

Context

Context





- Low Mach flows (but $Ma \neq 0$).
  - Compressible flows
  - Low velocity, high temperature
- With acoustic waves

Outline

Asymptotic models

Review of the steady (and one scale) case

Non-stationary (and two scales) case

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Asymptotic models

Review of the steady (and one scale) case

Non-stationary (and two scales) case

Euler equations

Euler equations

$$\partial_t \rho + \operatorname{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0$$
$$\partial_t(\rho \mathbf{u}) + \operatorname{div}_{\mathbf{x}}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0$$
$$\partial_t(\rho E) + \operatorname{div}_{\mathbf{x}}((\rho E + P)\mathbf{u}) = 0$$
$$p = (\gamma - 1)(\rho e - \frac{1}{2}\rho \mathbf{u}^2)$$

Adimensioning

- t_0, ℓ_0, ρ_0, P_0.

$$\hat{\rho} = \frac{\rho}{\rho_0} \qquad \hat{P} = \frac{P}{P_0} \qquad \hat{\mathbf{u}} = \frac{\mathbf{u} t_0}{\ell_0}$$
$$\widehat{\rho E} = \frac{\rho E}{\rho_0 P_0} \qquad \hat{\mathbf{x}} = \frac{\mathbf{x}}{\ell_0} \qquad \hat{t} = \frac{t}{t_0}$$

- Resulting system

$$\begin{cases} \partial_{\hat{t}}\hat{\rho} + \mathrm{div}_{\hat{\mathbf{x}}}(\hat{\rho}\hat{\mathbf{u}}) = 0 \\ \partial_{\hat{t}}(\hat{\rho}\hat{\mathbf{u}}) + \mathrm{div}_{\hat{\mathbf{x}}}(\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) + \dfrac{1}{\hat{M}^2}\nabla\hat{P} = 0 \\ \partial_{\hat{t}}(\widehat{\rho E}) + \mathrm{div}_{\hat{\mathbf{x}}}((\widehat{\rho E} + \hat{P})\hat{\mathbf{u}}) = 0 \end{cases}$$

  - with $\hat{M}^2 = \frac{\rho_0 \ell_0^2}{P_0 t_0^2}$

- Equation of state
  - $\hat{P} = (\gamma - 1)\left(\widehat{\rho E} - \dfrac{\hat{M}^2}{2}\hat{\rho}\hat{\mathbf{u}}^2\right)$

One scale asymptotic expansion

- Develop each variable in power of $\hat{M}$

$$\psi(\mathbf{x}, t, \hat{M}) = \psi^{(0)}(\mathbf{x}, t) + \hat{M}\psi^{(1)}(\mathbf{x}, t) + \hat{M}^2\psi^{(2)}(\mathbf{x}, t) + \dots$$

- Gather by power of $\hat{M}$
- Resulting system
  - Scale $1/\hat{M}^2$

$$\nabla P^{(0)} = 0$$

  - Scale $1/\hat{M}$

$$\nabla P^{(1)} = 0$$

  - Scale $\hat{M}^0$

$$\left\{ \begin{array}{l} \partial_t \rho^{(0)} + \mathrm{div}_{\mathbf{x}}(\rho^{(0)}\mathbf{u}^{(0)}) = 0 \\ \partial_t(\rho^{(0)}\mathbf{u}^{(0)}) + \mathrm{div}_{\mathbf{x}}(\rho^{(0)}\mathbf{u}^{(0)} \otimes \mathbf{u}^{(0)} + P^{(2)}) = 0 \\ \partial_t(\rho^{(0)}E^{(0)}) + \mathrm{div}_{\mathbf{x}}((\rho^{(0)}E^{(0)} + P^{(0)})\mathbf{u}^{(0)}) = 0 \end{array} \right.$$
$$P^{(0)} = (\gamma - 1)\rho^{(0)}E^{(0)}$$

  - $P = P^{(0)}(t) + \hat{M}^2 P^{(2)}(\mathbf{x}, t) + \dots$

Two scales asymptotic expansion

- Two temporal scales: t and $\tau/\hat{M}$.

$$\psi(\mathbf{x}, t, \tau, \hat{M}) = \psi^{(0)}(\mathbf{x}, t, \tau) + \hat{M}\psi^{(1)}(\mathbf{x}, t, \tau) + \hat{M}^2\psi^{(2)}(\mathbf{x}, t, \tau) + \dots$$

  - Temporal derivative $\dfrac{\partial}{\partial \hat{t}} = \dfrac{1}{\hat{M}}\dfrac{\partial}{\partial \tau} + \dfrac{\partial}{\partial t}$
- Resulting system

Two scales asymptotic expansion

- Scale $1/\hat{M}^2$

$$\nabla P^{(0)} = 0$$

- Scale $1/\hat{M}$

$$\begin{cases} \partial_\tau \rho^{(0)} = 0 \\ \partial_\tau(\rho^{(0)}\mathbf{u}^{(0)}) = -\nabla P^{(1)} \\ \partial_\tau(\rho^{(0)}E^{(0)}) = 0 \end{cases}$$

- Scale $1/\hat{M}^2$

$$\begin{cases} \partial_\tau \rho^{(1)} + \partial_t \rho^{(0)} + \mathrm{div}_{\mathbf{x}}(\rho^{(0)}\mathbf{u}^{(0)}) = 0 \\ \partial_\tau(\rho^{(1)}\mathbf{u}^{(0)} + \rho^{(0)}\mathbf{u}^{(1)}) + \partial_t(\rho^{(0)}\mathbf{u}^{(0)}) + \mathrm{div}_{\mathbf{x}}(\rho^{(0)}\mathbf{u}^{(0)} \otimes \mathbf{u}^{(0)}) + \nabla P^{(2)} = 0 \\ \partial_\tau(\rho^{(1)}E^{(0)} + \rho^{(0)}E^{(1)}) + \partial_t(\rho^{(0)}E^{(0)}) + \mathrm{div}_{\mathbf{x}}\left(\rho^{(0)}E^{(0)}\mathbf{u}^{(0)} + P^{(0)}\mathbf{u}^{(0)}\right) = 0 \end{cases}$$

- Equation of state at scale $\hat{M}^0$ and $\hat{M}$

$$P^{(0)} = (\gamma-1)\rho^{(0)}E^{(0)} \qquad P^{(1)} = (\gamma-1)(\rho^{(1)}E^{(0)} + \rho^{(0)}E^{(1)})$$

Two scales asymptotic expansion

- Two temporal scales: t and $\tau/\hat{M}$.

$$\psi(\mathbf{x}, t, \tau, \hat{M}) = \psi^{(0)}(\mathbf{x}, t, \tau) + \hat{M}\psi^{(1)}(\mathbf{x}, t, \tau) + \hat{M}^2\psi^{(2)}(\mathbf{x}, t, \tau) + \dots$$

  - Temporal derivative $\dfrac{\partial}{\partial \hat{t}} = \dfrac{1}{\hat{M}} \dfrac{\partial}{\partial \tau} + \dfrac{\partial}{\partial t}$

- Resulting system
  - $P^{(0)}(\mathbf{x}, t, \tau) = P^{(0)}(t)$
  - Coupling of $P^{(1)}, \rho^{(0)}\mathbf{u}^{(0)}$

$$\begin{cases} \partial_\tau P^{(1)} + \dfrac{1}{\gamma} \mathrm{div}_{\mathbf{x}} \left({c^{(0)}}^2 \rho^{(0)} \mathbf{u}^{(0)} \right) = -\dfrac{\mathrm{d}P^{(0)}}{\mathrm{d}t} \\ \partial_\tau (\rho^{(0)} \mathbf{u}^{(0)}) + \nabla P^{(1)} = 0 \end{cases}$$

Summary for Euler system

One scale asymptotic expansion

$$\begin{cases} \partial_t \rho^{(0)} + \mathrm{div}_{\mathbf{x}}(\rho^{(0)}\mathbf{u}^{(0)}) = 0 \\ \partial_t(\rho^{(0)}\mathbf{u}^{(0)}) + \mathrm{div}_{\mathbf{x}}(\rho^{(0)}\mathbf{u}^{(0)} \otimes \mathbf{u}^{(0)} + P^{(2)}) = 0 \\ \partial_t(\rho^{(0)}E^{(0)}) + \mathrm{div}_{\mathbf{x}}((\rho^{(0)}E^{(0)} + P^{(0)})\mathbf{u}^{(0)}) = 0 \end{cases}$$
$$P^{(0)} = (\gamma - 1)\rho^{(0)}E^{(0)}$$

$$P = P^{(0)}(t) + \hat{M}^2 P^{(2)}(\mathbf{x}, t) + \dots$$

Two scales asymptotic expansion

- $P^{(0)}(\mathbf{x}, t, \tau) = P^{(0)}(t)$
- Coupling of $(P^{(1)}, \rho^{(0)}\mathbf{u}^{(0)})$

$$\begin{cases} \partial_\tau P^{(1)} + \dfrac{1}{\gamma}\mathrm{div}_{\mathbf{x}}\left({c^{(0)}}^2 \rho^{(0)}\mathbf{u}^{(0)}\right) = -\dfrac{\mathrm{d}P^{(0)}}{\mathrm{d}t} \\ \partial_\tau(\rho^{(0)}\mathbf{u}^{(0)}) + \nabla P^{(1)} = 0 \end{cases}$$

Barotropic Euler equations

Original system

$$\begin{cases} \partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = 0 \\ P = \kappa \rho^\gamma \end{cases}$$

Scaling variables

- t_0, ℓ_0, ρ_0.
- $\hat{P} = P/P(\rho_0)$, $\hat{\mathbf{u}} = (\mathbf{u} t_0)/\ell_0$ $\hat{M} = u_0/\sqrt{P'(\rho_0)}$

Adimensioned system

$$\begin{cases} \partial_{\hat{t}} \hat{\rho} + \mathrm{div}_{\mathbf{x}} \cdot (\hat{\rho} \hat{\mathbf{u}}) = 0 \\ \partial_{\hat{t}}(\hat{\rho} \hat{\mathbf{u}}) + \mathrm{div}_{\mathbf{x}}(\hat{\rho} \hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) + \dfrac{1}{\hat{M}^2} \nabla_{\hat{x}} \hat{P} = 0 \end{cases}$$

Asymptotic of barotropic Euler equations

One scale asymptotic expansion

$$\begin{cases} \mathrm{div}_{\mathbf{x}} \mathbf{u}^{(0)} = 0 \\ \partial_t(\rho^{(0)}\mathbf{u}^{(0)}) + \mathrm{div}_{\mathbf{x}}(\rho^{(0)}\mathbf{u}^{(0)} \otimes \mathbf{u}^{(0)}) + \nabla_{\mathbf{x}} P^{(2)} = 0 \end{cases}$$

$$P = P^{(0)}(t) + \hat{M}^2 P^{(2)}(\mathbf{x}, t) + \dots$$

Two scales asymptotic expansion

- $\rho^{(0)}(\mathbf{x}, t, \tau) = \rho^{(0)}(t)$
- Coupling of $(P^{(1)}, (\rho\mathbf{u})^{(0)})$

$$\begin{cases} \partial_\tau P^{(1)} + c_0^2 \nabla_{\mathbf{x}} \cdot (\rho\mathbf{u})^{(0)} = -\dfrac{\mathrm{d}P^{(0)}}{\mathrm{d}t} \\ \partial_\tau (\rho\mathbf{u})^{(0)} + \nabla_{\mathbf{x}} P^{(1)} = 0. \end{cases}$$

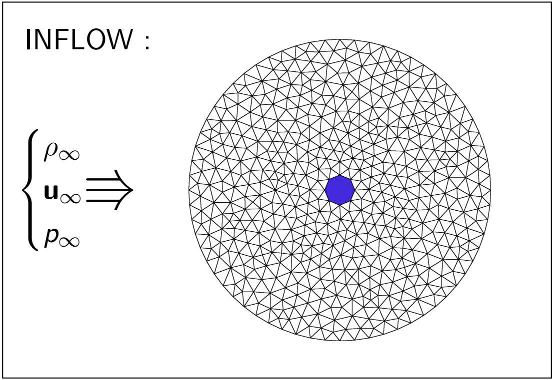
Outline

Asymptotic models

Review of the steady (and one scale) case

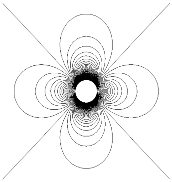
Non-stationary (and two scales) case

A steady test case

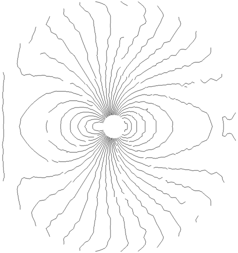


- Unstructured/Structured mesh.
- Roe Riemann Solver.

A steady test case



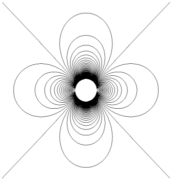
Incompressible exact



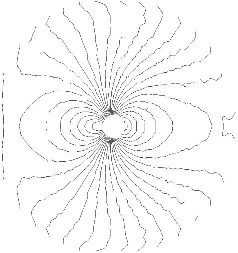
Roe, Quad

- The **compressible discrete** solution does not converge towards the **incompressible** one as the Mach number tends to zero on quadrangular meshes. (Here at $M = 10^{-3}$)

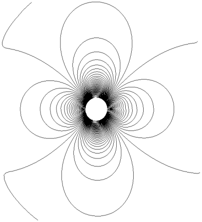
A steady test case



Incompressible exact



Roe, Quad



Roe, Tri

- The **compressible discrete** solution does not converge towards the **incompressible** one as the Mach number tends to zero on quadrangular meshes. (Here at $M = 10^{-3}$)
- H. Guillard, On the behavior of upwind schemes in the low Mach number limit. IV: *P*0 approximation on triangular and tetrahedral cells, Computers & Fluids, 2009, 38 (10).

Comparison between continuous and discrete cases

- Continuous :

Order $\mathbf{M^{-2}}$:

$$\nabla p^{(0)} = 0$$

Order $\mathbf{M^{-1}}$:

$$\nabla p^{(1)} = 0$$

Order $\mathbf{M^{0}}$:

$$\nabla.(\rho^{(0)}\mathbf{u}^{(0)}) = 0$$

$$\nabla.(\rho^{(0)}\mathbf{u}^{(0)} \otimes \mathbf{u}^{(0)}) + \nabla p^{(2)} = 0$$

- Discrete (Roe scheme) :

Order $\mathbf{M^{-2}}$:

$$\sum p^{(0)}\mathbf{n} = 0$$

Order $\mathbf{M^{-1}}$:

$$\frac{1}{2}\sum \frac{\Delta \bar{p}^{(0)}}{\bar{c}^{(0)}} = 0$$

$$\frac{1}{2}\sum \bar{p}^{(0)}\bar{c}^{(0)}\mathbf{n}\Delta \bar{U}^{(0)} + \sum p^{(1)}\mathbf{n} = 0$$

Comparison between continuous and discrete cases

- Continuous :

Order $\mathbf{M^{-2}}$:

$$\nabla p^{(0)} = 0$$

Order $\mathbf{M^{-1}}$:

$$\color{red}{\nabla p^{(1)} = 0}$$

Order $\mathbf{M^{0}}$:

$$\nabla.(\rho^{(0)}\mathbf{u}^{(0)}) = 0$$

$$\nabla.(\rho^{(0)}\mathbf{u}^{(0)} \otimes \mathbf{u}^{(0)}) + \color{blue}{\nabla p^{(2)}} = 0$$

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$$\frac{1}{2}\sum \bar{p}^{(0)}\bar{c}^{(0)}\mathbf{n}\Delta \bar{U}^{(0)} + \sum p^{(1)}\mathbf{n} = 0$$

Some fixes

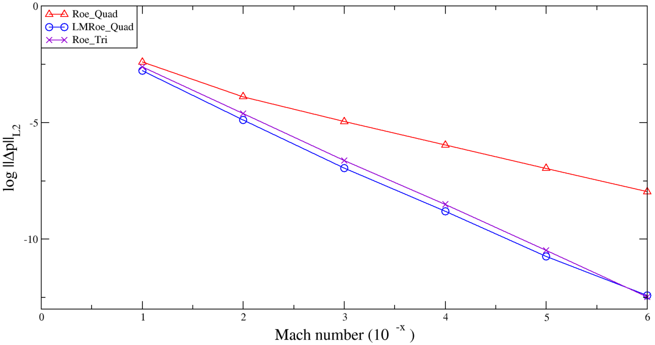
$$\frac{1}{2}\sum \bar{p}^{(0)}\bar{c}^{(0)}\mathbf{n}\Delta \bar{U}^{(0)} + \sum p^{(1)}\mathbf{n} = 0$$

- Guillard & Viozat (1999)

$$\Phi_{ij}^{\text{Roe-Turkel}} = \frac{\mathbf{f}(\mathcal{U}_i)+\mathbf{f}(\mathcal{U}_j)}{2}\cdot\mathbf{n}_{ij} - P(\mathcal{U}_{ij})^{-1}\frac{|P(\mathcal{U}_{ij})A_{\mathbf{n}_{ij}}(\mathcal{U}_{ij})|}{2}\cdot(\mathcal{U}_j-\mathcal{U}_i)$$

- Rieper & Bader (2009)
  - Replace all terms in $\Delta(\mathbf{u}\cdot\mathbf{n})$ by $M_{i,j}\Delta(\mathbf{u}\cdot\mathbf{n})$
- Dellacherie (2010)
  - Replace the terms in $\Delta(\mathbf{u}\cdot\mathbf{n})$ in the momentum equation by $M_{i,j}\Delta(\mathbf{u}\cdot\mathbf{n})$

Results



continuous $\quad P = P^{(0)} + \quad 0 \quad + P^{(2)} M^2$

Quad $\quad P = P^{(0)} + P^{(1)} M + P^{(2)} M^2$

What about higher order?

- Any of the higher order schemes work well with the cylinder test.
- Problems occur with e.g. flow around a NACA on quads
  - F. Bassi, C. De Bartolo, R. Hartmann and A. Nigro, A discontinuous Galerkin method for inviscid low Mach number flows, Journal of Computational Physics, 2009.
  - A. Nigro, S. Renda, C. De Bartolo, R. Hartmann and F. Bassi A high-order accurate discontinuous Galerkin finite element method for laminar low Mach number flows International Journal for Numerical Methods in Fluids, 2013.

Outline

Asymptotic models

Review of the steady (and one scale) case

Non-stationary (and two scales) case

Two scales asymptotic expansion

Two scales asymptotic expansion

- $\rho^{(0)}(\mathbf{x}, t, \tau) = \rho^{(0)}(t)$
- Coupling of $(P^{(1)}, (\rho\mathbf{u})^{(0)})$

$$\begin{cases} \partial_\tau P^{(1)} + c_0^2 \nabla_{\mathbf{x}} \cdot (\rho\mathbf{u})^{(0)} = -\dfrac{\mathrm{d}P^{(0)}}{\mathrm{d}t} \\ \partial_\tau (\rho\mathbf{u})^{(0)} + \nabla_{\mathbf{x}} P^{(1)} = 0. \end{cases}$$

Test case

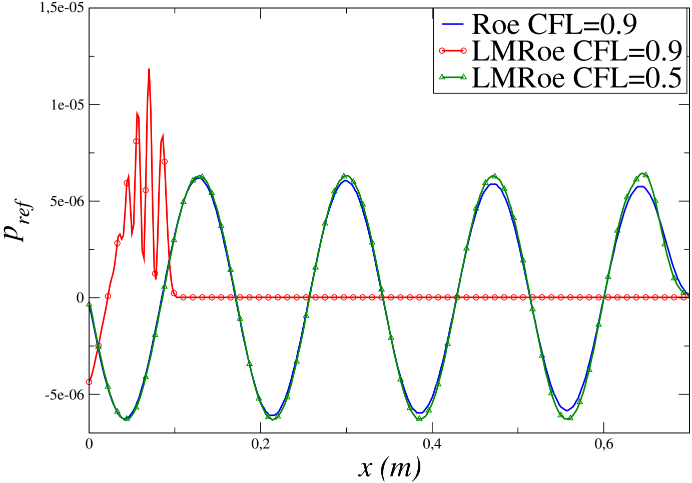
- Take a base flow at low Mach $(\rho^{(0)}, \mathbf{u}^{(0)})$
- Add an acoustic perturbation
  - $\mathrm{d}(\mathbf{u} \cdot \mathbf{n}) + \dfrac{c}{\rho}\mathrm{d}\rho = 0$
  - $\mathbf{u} = O(1)$
  - $\rho = O(\hat{M})$

Unsteady test case

- Guillard & Viozat (1999)

Unsteady test case

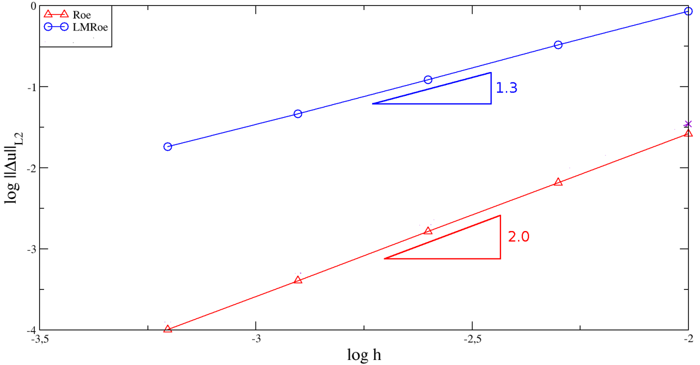
- Rieper & Bader (2009)



Unsteady test case

Convergence for the wave equation (DG^1)

- Rieper & Bader (2009)



Discrete two time scales asymptotic development

Roe first order discrete wave equation

$$\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + \frac{a}{2}\sum \Delta_{il}\mathbf{u} = 0$$

$$\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 = 0$$

Diffusion given by classical upwind flux for the wave system.
Diffusion is a SPD matrix.

- E. Burman, Alexandre Ern, Miguel Angel Fernandez. Explicit Runge–Kutta schemes and finite elements with symmetric stabilization for first-order linear PDE systems, SIAM Journal on Numerical Analysis, 2010, **48** (6),

Discrete two time scales asymptotic development

Roe first order discrete wave equation

$$\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + \frac{a}{2}\sum \Delta_{il}\mathbf{u} &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}$$

Modified Roe first order discrete wave equation

$$\begin{aligned}
\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + 0 &= 0\\
\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 &= 0
\end{aligned}$$

Modifying dissipation

Roe : not accurate in steady case

$$\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + \frac{a}{2}\sum \Delta_{il}\mathbf{u} = 0$$

$$\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 = 0$$

Modifying dissipation

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$$\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 = 0$$

Modified Roe : not stable (L^2 but not H^1 stable)

$$\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + 0 + 0 = 0$$
$$\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + \frac{a}{2}\sum \Delta_{il} p + 0 = 0$$

Modifying dissipation

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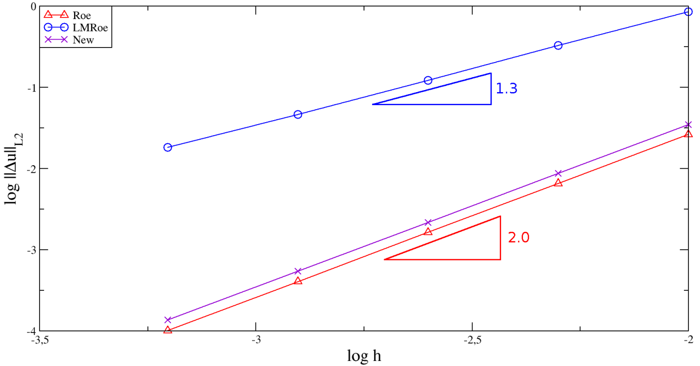
A new set of dissipative terms

$$\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + \frac{1}{2}\sum \Delta_{il} p + 0 = 0$$

$$\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + a\sum \Delta_{il} p + -\frac{a^2}{2}\sum \Delta_{il}\mathbf{u} = 0$$

Convergence for the wave equation (DG^1)



Unsteady test case

- New set of disspative terms for wave equations

Multidimensional scheme for Barotropic Euler equations (1/2)

New scheme in dimension 1

$$\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + \frac{1}{2}\sum \Delta_{il} p + 0 = 0$$
$$\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + a\sum \Delta_{il} p + -\frac{a^2}{2}\sum \Delta_{il}\mathbf{u} = 0$$

New scheme in dimension d

$$\partial_\tau \mathbf{u} + \frac{1}{2}\sum p_l \mathbf{n} + \frac{\mathbf{C}_{12}}{2\sqrt{d}}\sum \Delta_{il} p + 0 = 0$$
$$\partial_\tau p + \frac{a^2}{2}\sum \mathbf{u}_l.\mathbf{n} + a\sum \Delta_{il} p + -\frac{a^2\mathbf{C}_{21}}{2}\cdot\sum \Delta_{il}\mathbf{u} = 0$$

$\mathbf{C}_{12} = \mathbf{C}_{12}^T = \mathbf{1}_d$

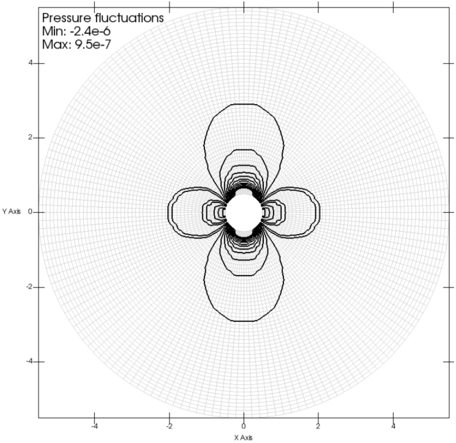
Multidimensional scheme for Barotropic Euler equations (2/2)

Modification of the Roe solver

$$\Phi_{ij}^{\mathrm{New}} = \Phi_{ij}^{\mathrm{Roe}} - (1-\theta_{ij})\frac{\rho_{ij} a_{ij}}{2}\begin{pmatrix} 0 \\ \left[(\mathbf{u}_i - \mathbf{u}_j)\cdot \mathbf{n}_{ij}\right]\mathbf{n}_{ij} \end{pmatrix}$$
$$+(1-\theta_{ij})\begin{pmatrix} \frac{1}{2} a_{ij}(\rho_i - \rho_j) \pm \frac{1}{2\sqrt{d}}\mathbf{1}_d \cdot (\rho_i \mathbf{u}_i - \rho_j \mathbf{u}_j) \\ \mp \frac{a_{ij}^2}{2\sqrt{d}}(\rho_i - \rho_j)\mathbf{1}_d \end{pmatrix}$$

Results on the steady cylinder test (1/3)

- Comparison of different fixes
- Exact



Results on the steady cylinder test (1/3)

- Comparison of different fixes
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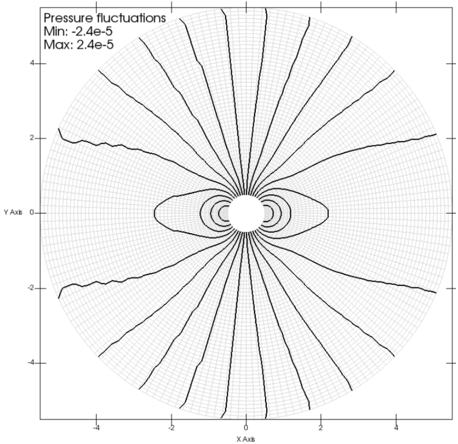
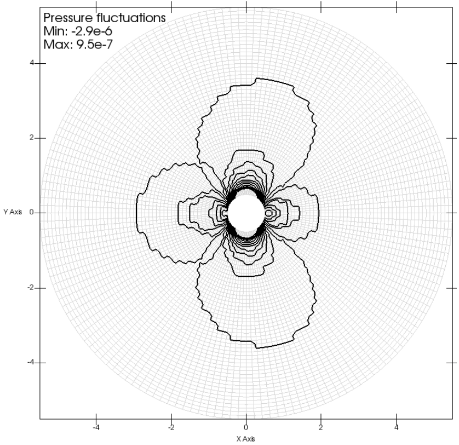


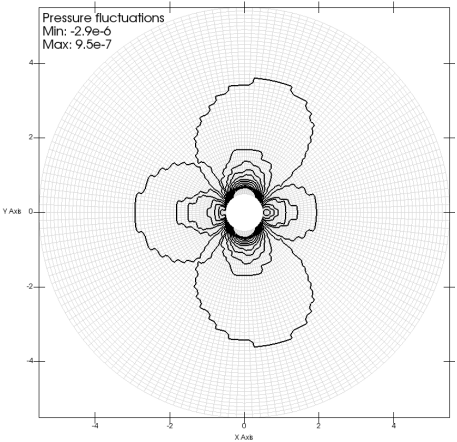
Results on the steady cylinder test (1/3)

- Comparison of different fixes
- Roe, Rieper fix



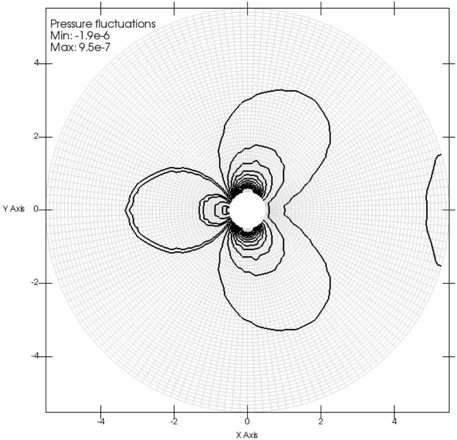
Results on the steady cylinder test (1/3)

- Comparison of different fixes
- Roe, Dellacherie fix



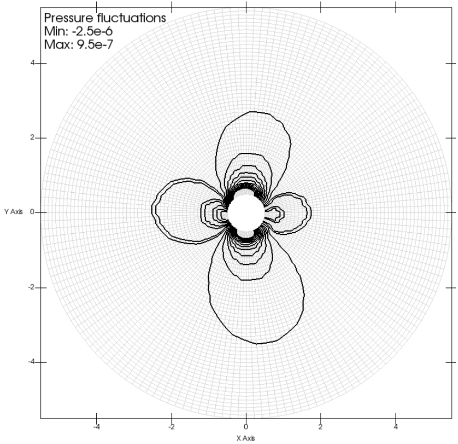
Results on the steady cylinder test (1/3)

- Comparison of different fixes
- Roe, Guillard fix



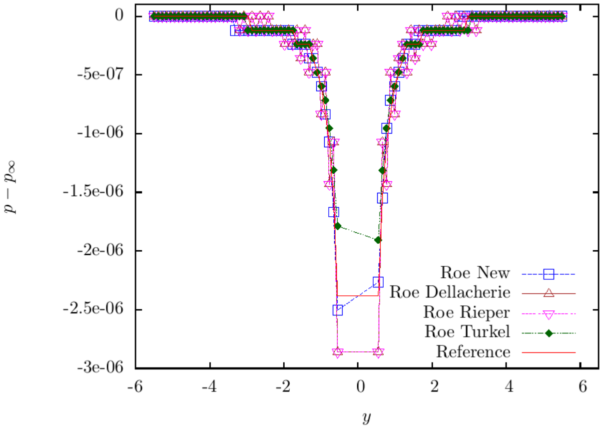
Results on the steady cylinder test (1/3)

- Comparison of different fixes
- Roe, New scheme



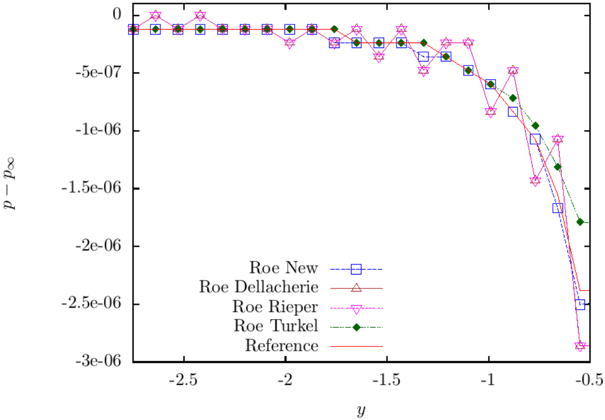
Results on the steady cylinder test (2/3)

- Checkerboard instability



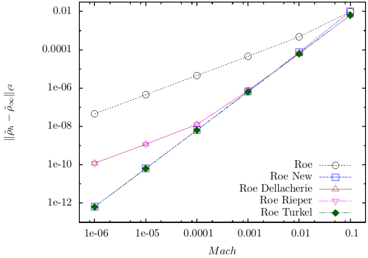
Results on the steady cylinder test (2/3)

- Checkerboard instability

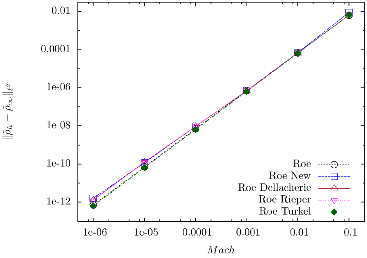


Results on the steady cylinder test (3/3)

- Convergence of $\rho_h - \rho_\infty$ wih respect to the Mach number



Quads



Tri

Conclusion

- Classical low Mach solver are not stable for high order acoustic calculation.
- A stable and accurate low Mach scheme for both steady and unsteady flow computation.
  - Stabilization of the incompressible system is not achieved by centering the pressure
  - Stabilization of the wave system is not symmetric .