Low Mach number models: advantages in applications to thermodynamics

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Numerical schemes for low Mach number flows

Toulouse – November 20th. 2017

CDMATH origins

Once upon a time. . .

BASMAC project
at CEMRACS'11

CDMATH origins

Once upon a time. . .

Launching of the
CDMATH group

BASMAC project
at CEMRACS'11

CDMATH origins

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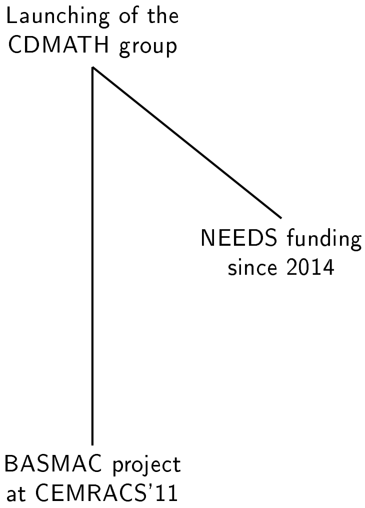

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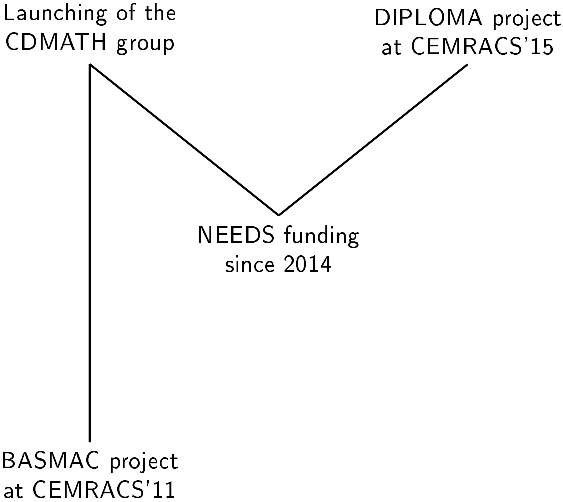

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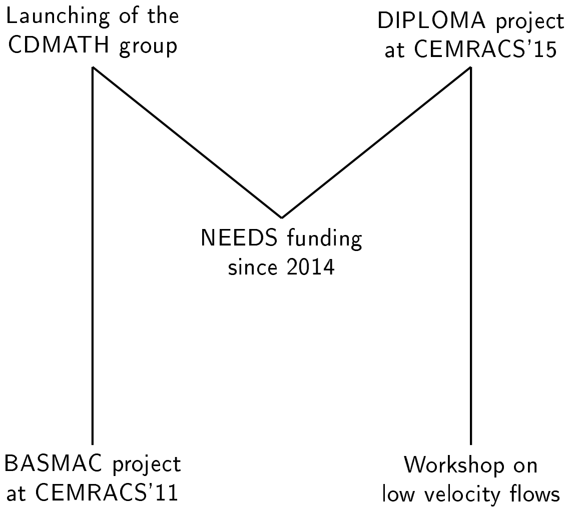
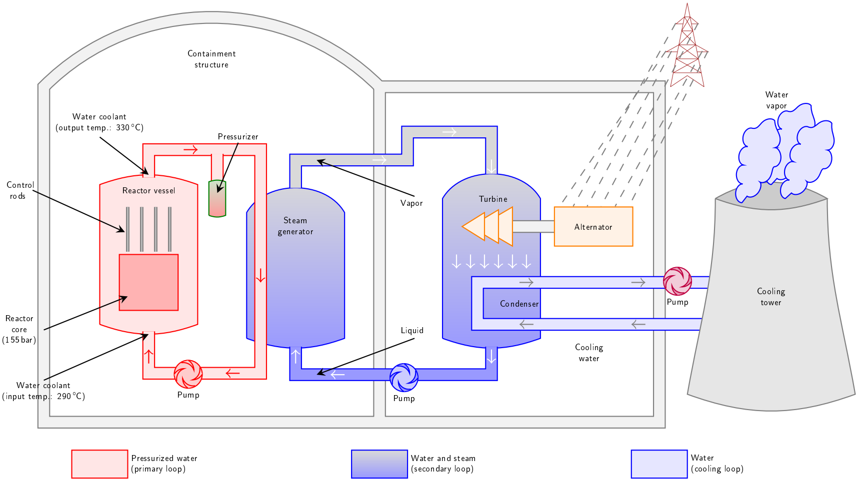


Pressurized water reactor



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Coupling

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Modelling

Dimensionless parameter: Mach number $\mathcal{M}$ accounting for the compressibility of the flow

Issues when $\mathcal{M} \ll 1$: $\mathcal{P}_{\mathcal{M}} \quad \stackrel{?}{\longleftrightarrow} \quad \mathcal{P}_0$

Applications

- Combustion: Majda, Klein, Najm, ...
- Astrophysics: Colella, Almgren, ...
- Nuclear: Bell, Paillère, Dellacherie, ...

Theoretical analysis

- How good is the approximation of a compressible model by its incompressible counterpart?
- What is the convergence rate?

Numerical analysis

- Balance between efficiency and accuracy?
- Theoretical vs. numerical convergence?

Some references

- Klainerman & Majda ('81, '82): barotropic models in infinite/periodic domains with well-prepared initial conditions
- Schochet ('94): analysis of convergence for the barotropic Euler equations for any kind of initial conditions based on the decomposition between fast waves and slow waves
- Desjardins, Grenier, Lions & Masmoudi ('99): barotropic case for bounded domains
- Danchin ('01, '05): extension to Besov spaces with critical indices
- Alazard ('05): extension to any kind of equations of state

Numerical aspects

Notices

- Explicit schemes: prohibitive stability conditions
- 2D cartesian grids: the numerical solution does not converge to the incompressible one when the Mach number decreases

Some references

- Klein ('95): operator splitting
- Guillard *et al.* ('99, '04, '08): asymptotic expansion wrt the Mach number in the schemes
- Dellacherie *et al.* ('10, '13): analysis of schemes thanks to the Schochet decomposition, stability of discrete kernels

Construction of intermediate models

Low Mach number: $\mathcal{M} \ll 1$ High heat transfers: $\operatorname{div} \mathbf{u} \neq 0$

$\Downarrow$

Compressible Navier-Stokes system

$\rightarrow$ model with acoustics and with heat transfers

Asymptotic low Mach model

(obtained formally by filtering out the acoustics waves)

$\rightarrow$ model without acoustics but with heat transfers

Incompressible Navier-Stokes system

$\rightarrow$ model with no acoustics and no heat transfers

Construction of intermediate models

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Construction of a low Mach number model

Boundary conditions

$$\begin{cases} \rho(t,0) = \varrho_e(t) > 0, \\ (\rho u)(t,0) = D_e(t) \geqslant 0, \\ p(t,L) = p_s(t) > 0. \end{cases}$$

Euler–Φ Equations for a perfect gas

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(\rho E u + p u) = \Phi, \\ E = \dfrac{p}{(\gamma - 1)\rho} + \dfrac{1}{2}\rho |u|^2. \end{cases}$$

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Dimensionless equations

$$\begin{cases} \partial_{t'}\tilde{\rho} + \partial_{x'}(\tilde{\rho}\tilde{u}) = 0, \\ \partial_{t'}(\tilde{\rho}\tilde{u}) + \partial_{x'}(\tilde{\rho}\tilde{u}^2) + \dfrac{1}{\mathcal{M}^2}\partial_{x'}\tilde{p} = 0, \\ \partial_{t'}(\tilde{\rho}\tilde{E}) + \partial_{x'}(\tilde{\rho}\tilde{E}\tilde{u} + \tilde{p}\tilde{u}) = \tilde{\Phi}, \\ \qquad \tilde{E} = \dfrac{\tilde{p}}{(\gamma-1)\tilde{\rho}} + \dfrac{\mathcal{M}^2}{2}\tilde{\rho}|\tilde{u}|^2. \end{cases}$$

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Dimensionless equations

$$\tilde{*} = *^{(0)} + \mathcal{M} *^{(1)} + \mathcal{M}^2 *^{(2)} + \mathcal{O}(\mathcal{M}^3)$$

$$\begin{cases} \partial_{t'}\tilde{\rho} + \partial_{x'}(\tilde{\rho}\tilde{u}) = 0, \\ \partial_{t'}(\tilde{\rho}\tilde{u}) + \partial_{x'}(\tilde{\rho}\tilde{u}^2) + \dfrac{1}{\mathcal{M}^2}\partial_{x'}\tilde{p} = 0, \\ \partial_{t'}(\tilde{\rho}\tilde{E}) + \partial_{x'}(\tilde{\rho}\tilde{E}\tilde{u} + \tilde{p}\tilde{u}) = \tilde{\Phi}, \\ \qquad \tilde{E} = \dfrac{\tilde{p}}{(\gamma-1)\tilde{\rho}} + \dfrac{\mathcal{M}^2}{2}\tilde{\rho}|\tilde{u}|^2. \end{cases}$$

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Euler–LM equations: 0th-order of the asymptotic expansion

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \bar{p} = 0, \\ \partial_x u = \dfrac{(\gamma - 1)\Phi}{\gamma \mathcal{P}(t)} - \dfrac{\mathcal{P}'(t)}{\gamma \mathcal{P}(t)}. \end{cases}$$

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$p(t,x) = \mathcal{P}(t) + \bar{p}(t,x) + \mathcal{O}(\mathcal{M}^3)$

Construction of a low Mach number model

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$$\begin{cases} \rho(t,0) = \varrho_e(t) > 0, \\ (\rho u)(t,0) = D_e(t) \geqslant 0, \\ \cancel{p(t,L) = p_s(t) > 0.} \end{cases}$$

Euler–LM equations: 0th-order of the asymptotic expansion

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \bar{p} = 0, \\ \partial_x u = \dfrac{(\gamma-1)\Phi}{\gamma \mathcal{P}(t)} - \dfrac{\mathcal{P}'(t)}{\gamma \mathcal{P}(t)}. \end{cases}$$

$p(t,x) = \mathcal{P}(t) + \bar{p}(t,x) + \mathcal{O}(\mathcal{M}^3) \implies \mathcal{P}(t) = p_s(t),\ \bar{p}(t,L) = 0$

Models and numerical aspects

LMNC model (version 2011)

$$\begin{cases} \operatorname{div} \mathbf{u} = \dfrac{\beta(h, p_*)}{p_*}\Phi, & \text{(1a)} \\ \rho(h, p_*) \times \left[\partial_t h + \mathbf{u} \cdot \nabla h\right] = \Phi, & \text{(1b)} \\ \rho(h, p_*) \times \left[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] - \operatorname{div} \sigma(\mathbf{u}) + \nabla \bar{p} = \rho(h, p_*)\mathbf{g}. & \text{(1c)} \end{cases}$$

Equation of state (EoS): stiffened gas law for a monophasic fluid

$$\rho(h, p_*) = \frac{\gamma}{\gamma - 1} \frac{p_* + \pi}{h - q}$$

Dimension: 1

Numerical scheme: MOC (Matlab)

Models and numerical aspects

LMNC model (version 2012)

$$\begin{cases} \operatorname{div} \mathbf{u} = \dfrac{\beta(h, p_*)}{p_*}\Phi, & (1a) \\ \rho(h, p_*) \times \left[\partial_t h + \mathbf{u} \cdot \nabla h\right] = \Phi, & (1b) \\ \rho(h, p_*) \times \left[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] - \operatorname{div} \sigma(\mathbf{u}) + \nabla \bar{p} = \rho(h, p_*)\mathbf{g}. & (1c) \end{cases}$$

Equation of state (EoS): stiffened gas law with phase change

$$\rho(h, p_*) = \frac{\gamma(h)}{\gamma(h) - 1} \frac{p_* + \pi(h)}{h - q(h)}$$

Dimension: 1

Numerical scheme: INTMOC (Fortran)

Models and numerical aspects

LMNC model (version 2013)

$$
\begin{cases}
\operatorname{div} \mathbf{u} = \dfrac{\beta(h, p_*)}{p_*}\Phi, & \text{(1a)} \\
\rho(h, p_*) \times \left[\partial_t h + \mathbf{u} \cdot \nabla h\right] = \Phi, & \text{(1b)} \\
\rho(h, p_*) \times \left[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] - \operatorname{div} \sigma(\mathbf{u}) + \nabla \bar{p} = \rho(h, p_*)\mathbf{g}. & \text{(1c)}
\end{cases}
$$

Equation of state (EoS): tabulated law

$$\beta \in \mathbb{R}_p[h]$$

Dimension: 1

Numerical scheme: MOC (Fortran)

Models and numerical aspects

LMNC model (version 2014)

$$\begin{cases} \operatorname{div} \mathbf{u} = \dfrac{\beta(h, p_*)}{p_*}\Phi, & (1a) \\ \rho(h, p_*) \times \left[\partial_t h + \mathbf{u} \cdot \nabla h\right] = \Phi, & (1b) \\ \rho(h, p_*) \times \left[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] - \operatorname{div} \sigma(\mathbf{u}) + \nabla \bar{p} = \rho(h, p_*)\mathbf{g}. & (1c) \end{cases}$$

Equation of state (EoS): stiffened gas/tabulated law

$$\rho(h, p_*) = \frac{\gamma(h)}{\gamma(h) - 1} \frac{p_* + \pi(h)}{h - q(h)} \quad / \quad \beta \in \mathbb{R}_p[h]$$

Dimension: 2

Numerical scheme: FreeFem++ (convect)

Models and numerical aspects

LMNC model (version 2015)

$$\begin{cases} \operatorname{div} \mathbf{u} = \dfrac{\beta(h,p_*)}{p_*}\left[\Phi + \operatorname{div}\big(\lambda(h,p_*)\nabla T(h,p_*)\big)\right], & \text{(1a)} \\ \rho(h,p_*) \times \left[\partial_t h + \mathbf{u}\cdot\nabla h\right] = \Phi + \operatorname{div}\big(\lambda(h,p_*)\nabla T(h,p_*)\big), & \text{(1b)} \\ \rho(h,p_*) \times \left[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \operatorname{div}\sigma(\mathbf{u}) + \nabla\bar{p} = \rho(h,p_*)\mathbf{g}. & \text{(1c)} \end{cases}$$

Equation of state (EoS): stiffened gas/tabulated law

$$\rho(h,p_*) = \frac{\gamma(h)}{\gamma(h)-1}\,\frac{p_*+\pi(h)}{h-q(h)} \quad / \quad \beta \in \mathbb{R}_p[h]$$

Dimension: 1/2/3

Numerical scheme: MOC (Fortran)/FreeFem++ (convect)

Models and numerical aspects

LMNC model (version 2016)

$$
\begin{cases}
\operatorname{div} \mathbf{u} = -\dfrac{P_0'(t)}{\rho\big(h, P_0(t)\big) c^2\big(h, P_0(t)\big)} + \dfrac{\beta\big(h, P_0(t)\big)}{P_0(t)} \Phi, & (1a) \\
\rho\big(h, P_0(t)\big) \times \big[\partial_t h + \mathbf{u} \cdot \nabla h\big] = \Phi + P_0'(t), & (1b) \\
\rho\big(h, P_0(t)\big) \times \big[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\big] - \operatorname{div} \sigma(\mathbf{u}) + \nabla \bar{p} = \rho\big(h, P_0(t)\big)\mathbf{g}. & (1c)
\end{cases}
$$

Equation of state (EoS): stiffened gas/tabulated law

$$
\rho\big(h, P_0(t)\big) = \frac{\gamma\big(h, P_0(t)\big)}{\gamma\big(h, P_0(t)\big) - 1} \frac{P_0(t) + \pi\big(h, P_0(t)\big)}{h - q\big(h, P_0(t)\big)} \quad / \quad \beta \in \mathbb{R}_p[h, P_0(t)]
$$

Dimension: 1/2/3

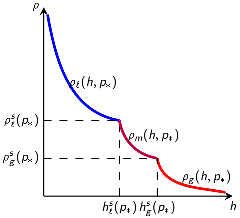
Numerical scheme: MOC (Fortran)/FreeFem++ (convect)

Diphasic EoS

- Liquid $\kappa = \ell$ and vapour $\kappa = g$ are characterized by their thermodynamic properties: $(h, p_*) \mapsto \rho_\kappa(h, p_*)$
- In the mixture, full equilibrium between liquid and vapour phases: $T = T^s(p_*)$ and we define values at saturation:

$$h^s_\kappa(p_*) := h_\kappa\big(p_*, T^s(p_*)\big), \qquad \rho^s_\kappa(p_*) := \rho_\kappa\big(p_*, T^s(p_*)\big) = \rho_\kappa\big(h^s_\kappa, p_*\big).$$

$$\boxed{\rho(h, p_*) = \begin{cases} \rho_\ell(h, p_*), & \text{if } h \le h^s_\ell(p_*), \\ \rho_m(h, p_*) & \text{if } h^s_\ell(p_*) < h < h^s_g(p_*), \\ \rho_g(h, p_*), & \text{if } h \ge h^s_g(p_*), \end{cases}}$$



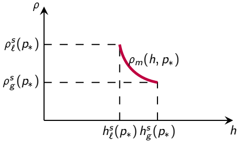
Mixture EoS

$$\begin{cases} \rho = \alpha \rho_g^s(p_*) + (1-\alpha)\rho_\ell^s(p_*) \\ \rho h = \alpha \rho_g^s(p_*) h_g^s(p_*) + (1-\alpha)\rho_\ell^s(p_*) h_\ell^s(p_*) \end{cases}$$

for $h \in [h_\ell^s(p_*); h_g^s(p_*)]$

$$\Downarrow$$

$$\boxed{\rho_m(h, p_*) = \frac{p_*/\beta_m(p_*)}{h - q_m(p_*)}}$$



where

$$\beta_m(p_*) := p_* \frac{\frac{1}{\rho_g^s} - \frac{1}{\rho_\ell^s}}{h_g^s - h_\ell^s} \qquad q_m(p_*) := \frac{\rho_g^s h_g^s - \rho_\ell^s h_\ell^s}{\rho_g^s - \rho_\ell^s}$$

Pure phase EoS: Noble Able Stiffened Gas law

$$\frac{1}{\rho_\kappa}(h, p_*) = \frac{\gamma_\kappa - 1}{\gamma_\kappa} \frac{h - q_\kappa}{p_* + \pi_\kappa} + b_\kappa$$

- $\gamma_\kappa > 1$ adiabatic coefficient
- π_κ reference pressure
- q_κ binding energy
- b_κ covolume

$$\Downarrow$$

$$\beta_\kappa(p_*) = -\frac{p_*}{\rho_\kappa^2(h, p_*)} \left.\frac{\partial \rho}{\partial h}\right|_{p_*} = \frac{\gamma_\kappa - 1}{\gamma_\kappa} \frac{p_*}{p_* + \pi_\kappa}$$

independent from h

$$\Downarrow$$

$$\boxed{\rho_\kappa(h, p_*) = \frac{p_*/\beta_\kappa(p_*)}{h - \hat{q}_\kappa(p_*)}, \qquad \hat{q}_\kappa(p_*) := q_\kappa - \frac{p_*}{\beta_\kappa(p_*)} b_\kappa}$$

Pure phase EoS: Tabulated law

κ	h_κ [kJ · kg^{-1}]	ϱ_κ [kg · m^{-3}]	T_κ [K]	c^*_κ [m · s^{-1}]	$c_{p\kappa}$ [J · kg^{-1} · K^{-1}]	$c_{v\kappa}$ [J · kg^{-1} · K^{-1}]	λ_κ [W · m^{-1} · K^{-1}]	μ_κ [Pa · s]
ℓ	978.702	842.783	500.000	1293.67	4561.5	3218.0	0.657	1.209×10^{-4}
ℓ	980.223	842.359	500.336	1292.50	4563.5	3217.0	0.657	1.207×10^{-4}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ℓ	1627.450	595.733	617.667	624.66	8871.0	3098.1	0.459	6.850×10^{-5}
ℓ	h^s_ℓ	594.379	T^s	621.43	8950.0	3101.0	0.458	6.833×10^{-5}
g	h^s_g	101.930	T^s	433.40	14 000.6	3633.1	0.121	2.311×10^{-5}
g	2596.965	101.816	618.00	433.69	13 931.7	3627.6	0.121	2.310×10^{-5}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
g	3066.962	60.540	699.667	573.70	3670.7	2173.2	0.0803	2.615×10^{-5}
g	3068.184	60.473	700.000	574.02	3664.7	2171.5	0.0803	2.616×10^{-5}

Pure phase EoS: Tabulated law

Attempt 1: fitting of ρ from the tabulated values

$$\rho_\kappa(h, p_*) \approx \tilde{\rho}_\kappa(h) := \sum_{j=0}^{d_{\rho,\kappa}} r_{\kappa,j} \left(\frac{h}{10^6} \right)^j$$

$$\implies \beta_\kappa(h, p_*) = -\frac{p_*}{\rho_\kappa^2(h, p_*)} \frac{\partial \rho_\kappa}{\partial h}(h, p_*) \approx \tilde{\beta}_\kappa(h) = ?$$

Drawbacks

- Potential discontinuities of $h \mapsto \tilde{\rho}(h)$
- No monotonicity/positivity property for $\tilde{\rho}/\tilde{\beta}$
- No matching of the values at saturation
- Potential instabilities in the computation of $\tilde{\beta}$

Pure phase EoS: Tabulated law

Attempt 2 Deriving a new expression for β: $\beta = \dfrac{p}{\rho c^* \sqrt{T}} \sqrt{\dfrac{1}{c_v} - \dfrac{1}{c_p}}$.

Fitting of β from derived values

$$\beta_\kappa(h, p_*) \approx \tilde{\beta}_\kappa(h) := \sum_{j=0}^{d_{\beta,\kappa}} b_{\kappa,j} \left(\frac{h}{10^6}\right)^j.$$

Construction of ρ

$$\frac{1}{\rho}(h, p_*) \approx \frac{1}{\tilde{\rho}(h)} := \begin{cases} \dfrac{1}{\tilde{\rho}_\ell}(h) := \dfrac{1}{\rho_\ell^s(p_*)} + \displaystyle\int_{h_\ell^s}^{h} \frac{\tilde{\beta}_\ell(h)}{p_*} \mathrm{d}h, & \text{if } h \leq h_\ell^s, \\ \dfrac{1}{\tilde{\rho}_m}(h), & \text{if } h_\ell^s < h < h_g^s, \\ \dfrac{1}{\tilde{\rho}_g}(h) := \dfrac{1}{\rho_g^s(p_*)} + \displaystyle\int_{h_g^s}^{h} \frac{\tilde{\beta}_g(h)}{p_*} \mathrm{d}h, & \text{if } h \geq h_g^s. \end{cases}$$

Pure phase EoS: Tabulated law

Attempt 2 Deriving a new expression for β: $\beta = \frac{p}{\rho c^* \sqrt{T}} \sqrt{\frac{1}{c_v} - \frac{1}{c_p}}$.

Fitting of β from derived values

$$\beta_\kappa(h, p_*) \approx \tilde{\beta}_\kappa(h) := \sum_{j=0}^{d_{\beta,\kappa}} b_{\kappa,j} \left(\frac{h}{10^6} \right)^j.$$

Let us assume that $h \mapsto \tilde{\beta}(h)$ is defined over (h_{min}, h_{max}) with

$$h_{min} > h_\ell^s - \frac{p_*}{\rho_\ell^s \max_{[h_{min}, h_\ell^s]} \tilde{\beta}}.$$

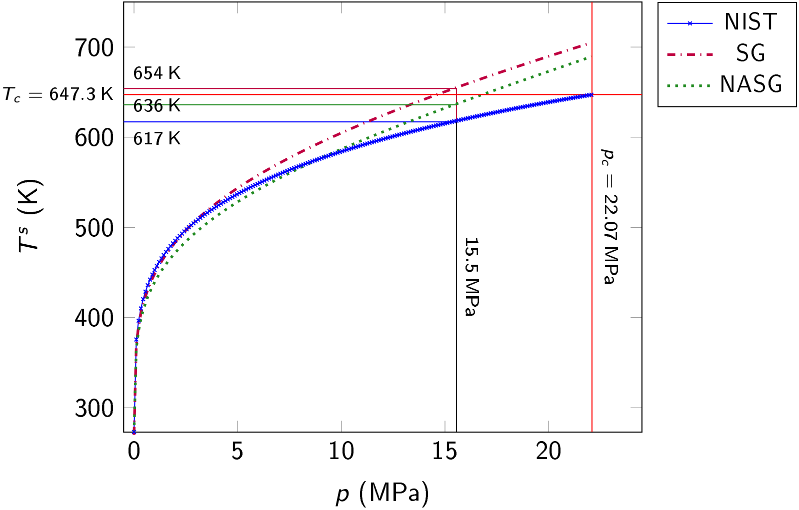
Then:

1. $h \mapsto \tilde{\rho}(h)$ is continuous and positive over (h_{min}, h_{max});
2. $h \mapsto \tilde{\rho}(h)$ is decreasing over (h_{min}, h_{max});
3. Relation $\tilde{\beta} = -\frac{p_*}{\tilde{\rho}^2} \frac{\partial \tilde{\rho}}{\partial h}$ exactly holds.

Pure phase EoS: comparison

	SG	NASG	NIST
h_ℓ^s	$1.627 \times 10^6\,\mathrm{J \cdot K^{-1}}$	$1.596 \times 10^6\,\mathrm{J \cdot K^{-1}}$	$1.629 \times 10^6\,\mathrm{J \cdot K^{-1}}$
h_g^s	$3.004 \times 10^6\,\mathrm{J \cdot K^{-1}}$	$2.861 \times 10^6\,\mathrm{J \cdot K^{-1}}$	$2.596 \times 10^6\,\mathrm{J \cdot K^{-1}}$
ρ_ℓ^s	$632.663\,\mathrm{kg \cdot m^{-3}}$	$737.539\,\mathrm{kg \cdot m^{-3}}$	$594.38\,\mathrm{kg \cdot m^{-3}}$
ρ_g^s	$52.937\,\mathrm{kg \cdot m^{-3}}$	$55.486\,\mathrm{kg \cdot m^{-3}}$	$101.93\,\mathrm{kg \cdot m^{-3}}$
T^s	$654.65\,\mathrm{K}$	$636.47\,\mathrm{K}$	$617.939\,\mathrm{K}$

Pure phase EoS: comparison



Heat conduction

Fitting of $1/c_p$ from tabulated values

$$\frac{1}{c_{p_\kappa}(h,p_*)} \approx \frac{1}{\tilde{c}_{p_\kappa}(h)} := \sum_{j=0}^{d_{c_p,\kappa}} c_{\kappa,j}\left(\frac{h}{10^6}\right)^j.$$

Construction of T

$$T(h,p_*) \approx \tilde{T}(h) := \begin{cases} \tilde{T}_\ell(h) := T^s + \displaystyle\int_{h_\ell^s}^{h} \frac{1}{\tilde{c}_{p_\ell}(h)}\mathrm{d}h, & \text{if } h \leq h_\ell^s,\\ T^s, & \text{if } h_\ell^s < h < h_g^s,\\ \tilde{T}_g(h) := T^s + \displaystyle\int_{h_g^s}^{h} \frac{1}{\tilde{c}_{p_g}(h)}\mathrm{d}h, & \text{if } h \geq h_g^s. \end{cases}$$

Steady state solution

$$(\rho_e^\infty, D_e^\infty > 0, \Phi^\infty(y)) := \lim_{t\to+\infty}(\rho_e(t), D_e(t), \Phi(t,y))$$

1. **Enthalpy**

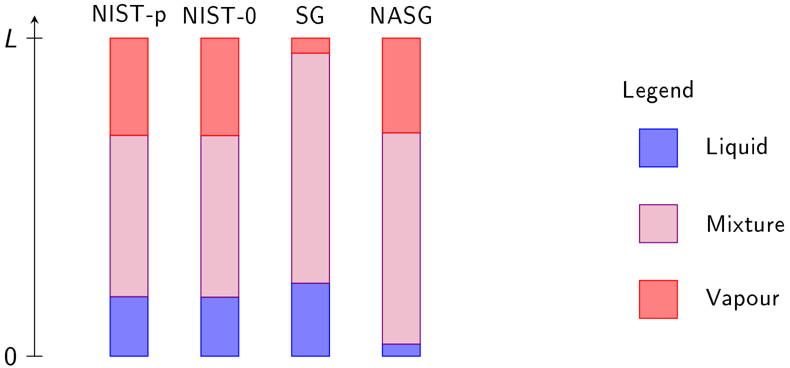
$$\boxed{h^\infty(y) = \rho^{-1}(\rho_e^\infty, p_*) + \frac{\Psi(y)}{D_e^\infty}, \qquad \Psi(y) := \int_0^y \Phi^\infty(z)\mathrm{d}z}$$

2. **Velocity**

$$\boxed{v^\infty(y) = \frac{D_e^\infty}{\rho(h^\infty(y), p_*)}}$$

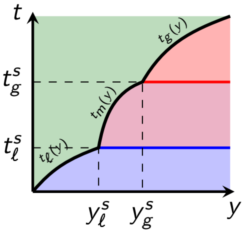
3. **Dynamic pressure**
   Direct integration of $\partial_y \bar{p} = \partial_y(\mu\partial_y v) - \partial_y(\rho v^2) - \rho g$.

EoS: comparisons



NASG two phases with phase transition

Φ, v_e, h_e, h_0: constant; IC and BC: liquid phase.



$$y_\ell^s = \frac{D_e}{\Phi}(h_\ell^s - h_e)$$
$$y_g^s = \frac{D_e}{\Phi}(h_g^s - h_e)$$
$$t_\ell^s = \frac{1}{\hat{\Phi}_\ell} \ln\left(\frac{h_\ell^s - \hat{q}_\ell}{h_0 - \hat{q}_\ell}\right)$$
$$t_g^s = t_\ell^s + \frac{1}{\hat{\Phi}_m} \ln\left(\frac{h_g^s - \hat{q}_m}{h_\ell^s - \hat{q}_m}\right)$$

Enthalpy: method of characteristics applied to $\partial_t h + v\partial_y h = \frac{\beta(h)\Phi}{p_*}(h - \hat{q}(h))$.

$$h(t,y) = \begin{cases} q_\ell + (h_0 - \hat{q}_\ell)e^{\hat{\Phi}_\ell t} & \text{if } (t,y) \in \mathcal{L} \text{ and } t < t_\ell(y), \\ q_m + (h_\ell^s - \hat{q}_m)e^{\hat{\Phi}_m (t - t_\ell^s)} & \text{if } (t,y) \in \mathcal{M} \text{ and } t < t_m(y), \\ q_g + (h_g^s - \hat{q}_g)e^{\hat{\Phi}_g (t - t_g^s)} & \text{if } (t,y) \in \mathcal{G} \text{ and } t < t_g(y), \\ h_e + \frac{\Phi}{D_e} y & \text{otherwise.} \end{cases}$$

Heat conduction

$$\Lambda \text{ discontinuous} \implies \text{weak solutions} \implies \text{jump conditions}$$

Let us define

$$\Gamma_\ell(t) = \{\mathbf{x} \in \Omega^d \ : \ h(t,\mathbf{x}) = h_\ell^s\} \quad \text{and} \quad \Gamma_g(t) = \{\mathbf{x} \in \Omega^d \ : \ h(t,\mathbf{x}) = h_g^s\}.$$

Then we assume

- In each phase domain, $(h, \mathbf{u}, \bar{p})$ is a strong solution;
- Where phase transition occurs, $[\![\Lambda(h, p_*)\nabla h]\!] \cdot \boldsymbol{n} = 0$ over $\Gamma_\ell(t)$ and $\Gamma_g(t)$;
- $[\![h]\!] = 0$ only over $\{\mathbf{x} \in \Gamma_\kappa(t) \ : \mathbf{u}(t,\mathbf{x}) \cdot \boldsymbol{n}_m(t,\mathbf{x}) > 0\}$.

Steady solutions

Let us set

$$\mathcal{H}(z) = h_\ell^s - h_e - \frac{\Phi_0}{D_e} z + \frac{\Phi_0 \kappa_\ell}{D_e}\left(1 - e^{-z/\kappa_\ell}\right), \qquad \text{where } \kappa_\ell := \frac{\Lambda_\ell}{D_e}.$$

Then

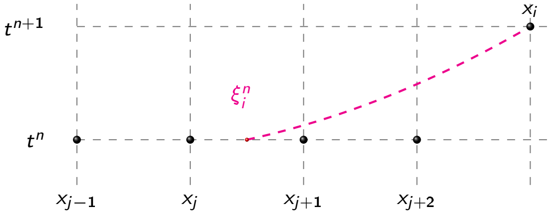
1. If $\mathcal{H}(L_y) > 0$, there exists a unique steady (liquid) solution

$$h^\infty(y) = h_e + \frac{\Phi_0}{D_e} y - \frac{\Phi_0 \kappa_\ell}{D_e} e^{-L_y/\kappa_\ell}\left(e^{y/\kappa_\ell} - 1\right).$$

2. Let us assume that $\mathcal{H}(L_y) \leq 0$ and that $\mathcal{H}(L_y) > \frac{\Phi_0 \kappa_\ell}{D_e}\left(e^{-y_\ell^s/\kappa_\ell} - e^{-L_y/\kappa_\ell}\right) - (h_g^s - h_\ell^s)$ where $\mathcal{H}(y_\ell^s) = 0$. Then there exists a unique steady (liq-mix) solution

$$h^\infty(y) = \begin{cases} h_e + \dfrac{\Phi_0}{D_e} y - \dfrac{\Phi_0 \kappa_\ell}{D_e} e^{-y_\ell^s/\kappa_\ell}\left(e^{y/\kappa_\ell} - 1\right), & \text{if } y \in (0, y_\ell^s), \\[2ex] h_\ell^s + \dfrac{\Phi_0}{D_e}(y - y_\ell^s), & \text{if } y \in (y_\ell^s, L_y). \end{cases}$$

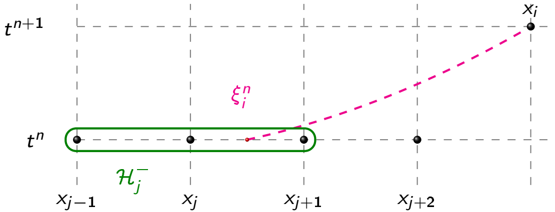
1D numerical scheme ($\Lambda = 0$)



$$\frac{h_i^{n+1} - \hat{h}_i^n}{\Delta t} = \frac{\Phi(t^n, \xi_i^n)}{\rho(\hat{h}_i^n, p_*)} \qquad \hat{h}_i^n := \alpha_{ij}^n h_{ij}^- + (1 - \alpha_{ij}^n) h_{ij}^+ \qquad \theta_{ij}^n = \frac{x_{j+1} - \xi_i^n}{\Delta x}$$

	$\mathcal{P}_j^+(\theta_{ij}^n) \geqslant 0$	$\mathcal{P}_j^+(\theta_{ij}^n) < 0$
$\mathcal{P}_j^-(\theta_{ij}^n) \geqslant 0$	$\alpha_{ij}^n = \frac{1+\theta_{ij}^n}{3}$, $h_{ij}^\pm = \mathcal{H}_j^\pm$	$\alpha_{ij}^n = 1$, $h_{ij}^\pm = \mathcal{H}_j^\pm$
$\mathcal{P}_j^-(\theta_{ij}^n) < 0$	$\alpha_{ij}^n = 0$, $h_{ij}^\pm = \mathcal{H}_j^\pm$	$\alpha_{ij}^n = \theta_{ij}^n$, $h_{ij}^- = h_j^n$, $h_{ij}^+ = h_{j+1}^n$

1D numerical scheme ($\Lambda = 0$)



$$\frac{h_i^{n+1} - \hat{h}_i^n}{\Delta t} = \frac{\Phi(t^n, \xi_i^n)}{\rho(\hat{h}_i^n, p_*)} \qquad \hat{h}_i^n := \alpha_{ij}^n h_{ij}^- + (1 - \alpha_{ij}^n) h_{ij}^+ \qquad \theta_{ij}^n = \frac{x_{j+1} - \xi_i^n}{\Delta x}$$

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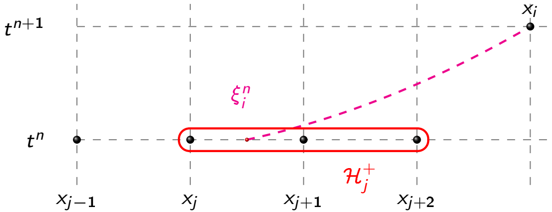
1D numerical scheme ($\Lambda = 0$)



$$\frac{h_i^{n+1} - \hat{h}_i^n}{\Delta t} = \frac{\Phi(t^n, \xi_i^n)}{\rho(\hat{h}_i^n, p_*)} \qquad \hat{h}_i^n := \alpha_{ij}^n h_{ij}^- + (1 - \alpha_{ij}^n) h_{ij}^+ \qquad \theta_{ij}^n = \frac{x_{j+1} - \xi_i^n}{\Delta x}$$

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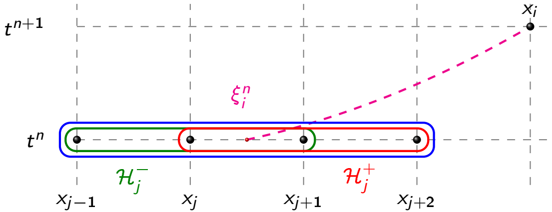
1D numerical scheme ($\Lambda = 0$)



$$\frac{h_i^{n+1} - \hat{h}_i^n}{\Delta t} = \frac{\Phi(t^n, \xi_i^n)}{\rho(\hat{h}_i^n, p_*)} \qquad \hat{h}_i^n := \alpha_{ij}^n h_{ij}^- + (1 - \alpha_{ij}^n) h_{ij}^+ \qquad \theta_{ij}^n = \frac{x_{j+1} - \xi_i^n}{\Delta x}$$

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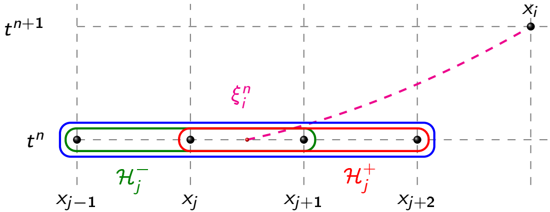
1D numerical scheme ($\Lambda = 0$)



Properties

- Unconditionally stable (L^∞, L^2) and 2nd-order in (almost) time and space.
- The 1st-order version for β-cst EoS admits a discrete steady state which satisfies $h^\infty_{d,i} \leq h^\infty(y_i)$ and which converges to the continuous steady state.
- The NASG version can be extended to a more accurate integrated formulation.

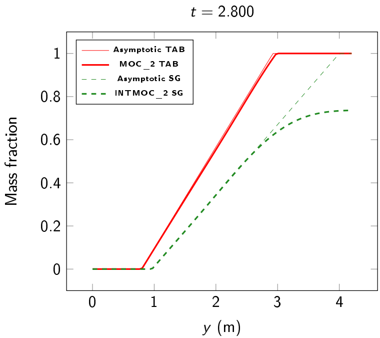
1D numerical scheme ($\Lambda \neq 0$)

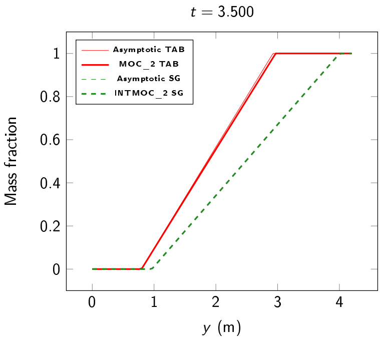
Diffusive version

$$\frac{h_i^{n+1} - \hat{h}_i^n}{\Delta t} = \frac{1}{\rho(\hat{h}_i^n, p_*)} \times \left[\Phi(t^n, \xi_i^n) + \frac{\Lambda_{i+1/2}^n \left(h_{i+1}^{n+1} - h_i^{n+1} \right) - \Lambda_{i-1/2}^n \left(h_i^{n+1} - h_{i-1}^{n+1} \right)}{\Delta y^2} \right].$$

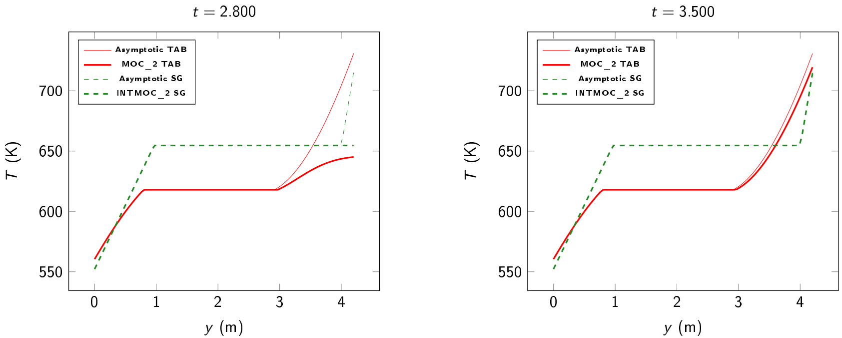
Satisfies the maximum principle.

SG (INTMOC 2) vs TAB (MOC 2)

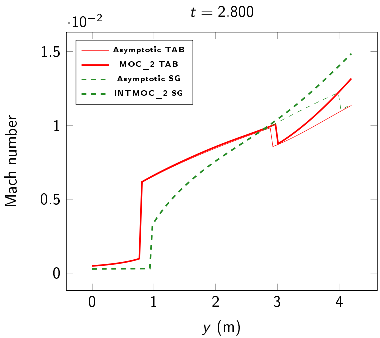


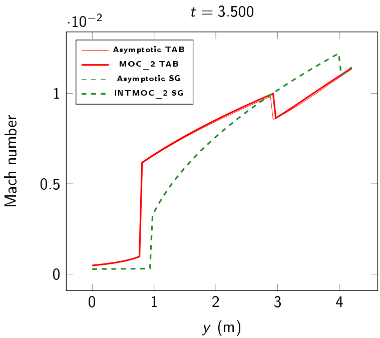


SG (INTMOC 2) vs TAB (MOC 2)



SG (INTMOC 2) vs TAB (MOC 2)





2D numerical scheme

"**At time** t^{n+1}, **find** $(\mathbf{u}^{n+1}, h^{n+1}, \bar{p}^{n+1}) \in (\mathbf{u}_e + \mathcal{U}) \times (h_e + \mathcal{H}) \times \mathrm{L}^2(\Omega_2)$ **such that**

$$\iint_{\Omega_2} p_t \,\mathrm{div}\,\mathbf{u}^{n+1}\,\mathrm{d}\mathbf{x} = \iint_{\Omega_2} \frac{\beta(h^n, p_*)\rho(h^n, p_*)}{p_*} \frac{h^{n+1} - h^n(\xi^n)}{\Delta t} p_t \,\mathrm{d}\mathbf{x}, \qquad \forall\, p_t \in \mathrm{L}^2,$$

$$\iint_{\Omega_2} \rho(h^n, p_*) \frac{h^{n+1} - h^n(\xi^n)}{\Delta t} h_t \,\mathrm{d}\mathbf{x} = \iint_{\Omega_2} \Phi(t^{n+1}, \cdot) h_t \,\mathrm{d}\mathbf{x} - \iint_{\Omega_2} \Lambda(h^n, p_*) \nabla h^{n+1} \cdot \nabla h_t \,\mathrm{d}\mathbf{x}, \qquad \forall\, h_t \in \mathcal{H},$$

$$\iint_{\Omega_2} \rho(h^n, p_*) \frac{\mathbf{u}^{n+1} - \mathbf{u}^n(\xi^n)}{\Delta t} \cdot \mathbf{u}_t \,\mathrm{d}\mathbf{x} + \iint_{\Omega_2} \frac{\mu(h^n, p_*)}{2} \left(\nabla \mathbf{u}^{n+1} + (\nabla \mathbf{u}^{n+1})^T\right) :: \left(\nabla \mathbf{u}_t + \nabla {\mathbf{u}_t}^T\right) \mathrm{d}\mathbf{x}$$

$$+ \iint_{\Omega_2} \eta(h^n, p_*)\,(\mathrm{div}\,\mathbf{u}^{n+1})(\mathrm{div}\,\mathbf{u}_t)\,\mathrm{d}\mathbf{x} - \iint_{\Omega_2} \bar{p}^{n+1}\,\mathrm{div}\,\mathbf{u}_t \,\mathrm{d}\mathbf{x} = \iint_{\Omega_2} \rho(h^n, p_*)\mathbf{g} \cdot \mathbf{u}_t \,\mathrm{d}\mathbf{x}, \qquad \forall\, \mathbf{u}_t \in \mathcal{U}\text{"}$$

where

$$\mathcal{H} = \left\{\theta \in \mathrm{H}^1(\Omega_2) \colon \theta(x, 0) = 0\right\},$$

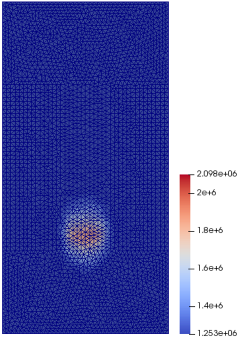
$$\mathcal{U} = \left\{\mathbf{v} \in \left(\mathrm{H}^1(\Omega_2)\right)^2 \colon \mathbf{v}(x, 0) = \mathbf{0},\ \mathbf{v} \cdot \boldsymbol{n}(0, y) = \mathbf{v} \cdot \boldsymbol{n}(L_x, y) = 0\right\}.$$

Jump and boundary conditions taken into account.

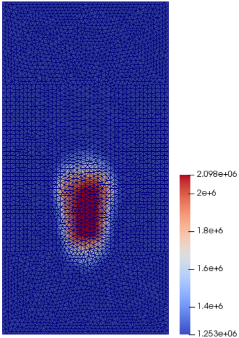
Trick

$$\mathrm{div}\,\mathbf{u} = \frac{\beta(h, p_*)\rho(h, p_*)}{p_*} \left[\partial_t h + \mathbf{u} \cdot \nabla h\right].$$

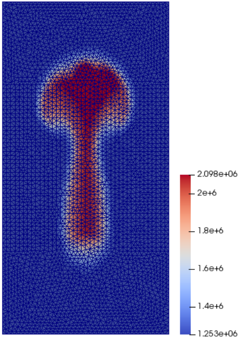
Numerical results



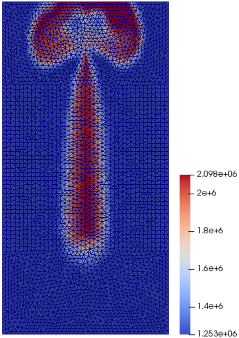
Numerical results



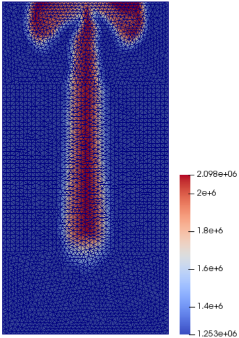
Numerical results



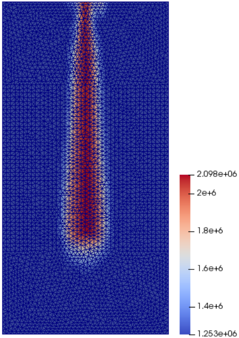
Numerical results



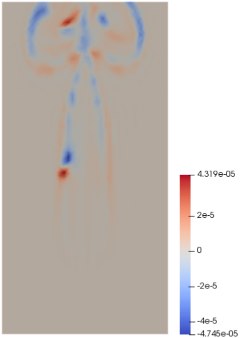
Numerical results



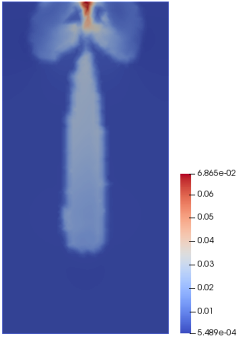
Numerical results



Numerical results



Numerical results



Conclusion

Summary

- Qualitative results to assess compressible codes
- Development of 1D/2D specific numerical methods
- Implementation of an innovative numerical method for the EoS

Perspectives

- Derivation of a new hierarchy of models
- Enrichment of the modelling by taking into account neutronics
- Development of our own 2D method
- Theoretical study in dimension 2
- Coupling of codes
- Extension to other reactors


THANK YOU FOR YOUR ATTENTION