Well-balanced numerical schemes for the wave equation with Coriolis source term

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Outline

Shallow water equations with Coriolis source term and their expansion at low Froude number

Modified equation analysis

(Only a few of) the proposed modified schemes

Conclusions and perspectives

Shallow water equations with Coriolis source term and their expansion at low Froude number

The Shallow water equations

Water height h and velocity $\mathbf{u}$:

$$\begin{aligned} \partial_t h + \nabla \cdot (h\mathbf{u}) &= 0, \\ \partial_t (h\mathbf{u}) + \nabla \cdot (h\mathbf{u} \otimes \mathbf{u}) + \nabla \left(g \frac{h^2}{2} \right) &= -g h \nabla b - h \Omega \mathbf{u}^{\perp}, \end{aligned}$$

Nondimensionalization: $\bar{\mathbf{x}} = \frac{\mathbf{x}}{L}$, $\quad \bar{t} = \frac{t}{T}$, $\quad \bar{h} = \frac{h}{H}$, $\quad$ and $\quad \bar{\mathbf{u}} = \frac{\mathbf{u}}{U}$.
One gets

$$\begin{aligned} \mathrm{St}\, \partial_t h + \nabla \cdot (h\mathbf{u}) &= 0, \\ \mathrm{St}\, \partial_t (h\mathbf{u}) + \nabla \cdot (h\mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Fr}^2} \nabla \left(\frac{h^2}{2} \right) &= -\frac{1}{\mathrm{Fr}^2} h \nabla b - \frac{1}{\mathrm{Ro}} h \mathbf{u}^{\perp}, \end{aligned}$$

where the Strouhal, Froude and Rossby numbers are defined by

$$\mathrm{St} = \frac{L}{UT}, \quad \mathrm{Fr} = \frac{U}{\sqrt{gH}}, \quad \mathrm{Ro} = \frac{U}{\Omega L}.$$

We shall focus here on cases where

$$\mathrm{Ro} = \mathcal{O}(M) \quad \text{and} \quad \mathrm{Fr} = \mathcal{O}(M)$$

with M a small parameter.

Low Froude asymptotics with $b \equiv$ cst

$$\begin{aligned} \mathrm{St}\,\partial_t h + \nabla\cdot(h\mathbf{u}) &= 0, \\ \mathrm{St}\,\partial_t(h\mathbf{u}) + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \frac{1}{M^2}\nabla\left(\frac{h^2}{2}\right) &= -\frac{1}{M}h\mathbf{u}^\perp. \end{aligned}$$

If we consider $\mathrm{St} = 1$, ($T =$ convection time scale), then an asymptotic expansion provides

$$\mathcal{O}(M^{-2}) \ : \ \nabla h_0 = 0, \tag{1}$$

$$\mathcal{O}(M^{-1}) \ : \ \nabla h_1 = -\boldsymbol{u}_0^\perp \ \text{(geostrophic equilibrium)} \tag{2}$$

h_0 is a constant in space. With periodic boundary conditions h_0 is also a constant in time. Therefore

$$h(t,x) = h_\star + Mh_1(x,t) + \cdots$$

If $\mathrm{St} = \mathcal{O}(M^{-1})$, then we obtain

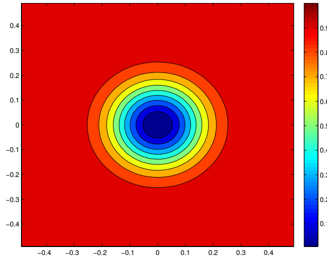
$$\begin{aligned} \mathcal{O}(1) \ &: \ \partial_t h_1 + h_\star\nabla\cdot\boldsymbol{u}_0 = 0, \\ \mathcal{O}(M^{-1}) \ &: \ \partial_t\boldsymbol{u}_0 + \nabla h_1 = -\boldsymbol{u}_0^\perp. \end{aligned}$$

Statement of our study

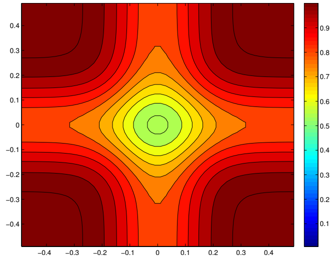
$$\partial_t r + a\nabla \cdot \boldsymbol{u} = 0,$$

$$\partial_t \boldsymbol{u} + a\nabla r = -\omega \boldsymbol{u}^{\perp}.$$

The Godunov scheme does not preserve the steady states of this system and destroy them quite fast. What can we do about that? How much does it differ from the case $\omega = 0$? What about the difference between cartesian and triangular grids?



Initial condition



Solution with the Godunov scheme

Modified equation analysis

Modified equation analysis

The kernel and orthogonal subspaces are

$$\mathcal{E} = \left\{ q = (r, \boldsymbol{u}),\ a\nabla r + \omega \boldsymbol{u}^\perp = 0 \right\}$$

$$\mathcal{E}^\perp = \{ q = (p, \boldsymbol{v}), \omega p - a\nabla \times \boldsymbol{v} = 0 \}.$$

The original system has $\mathcal{E}$ as steady-states and is "orthogonality preserving": if $q(t = 0)$ is in $\mathcal{E}^\perp$ then $q(t > 0)$ remains in $\mathcal{E}^\perp$.
The modified equation associated to the (semi-discrete) Godunov scheme is (with $\kappa_r = \kappa_u = \frac{1}{2}ah$)

$$\partial_t r + a\nabla \cdot \boldsymbol{u} - \kappa_r \Delta r = 0,$$

$$\partial_t \boldsymbol{u} + a\nabla r - \kappa_u \mathbf{D}\boldsymbol{u} + \omega \boldsymbol{u}^\perp = 0.$$

with $\mathbf{D}\boldsymbol{u} := (\partial_{xx} u, \partial_{yy} v)^T$. There holds:

$$\frac{1}{2}\frac{d}{dt}||q||^2 = -\kappa_r ||\nabla r||^2 - \kappa_u (||\partial_x u||^2 + ||\partial_y v||^2).$$

proves the stability of the scheme and allows one to find the kernel.

Modified equation analysis

$$a\nabla \cdot \boldsymbol{u} - \kappa_r \Delta r = 0,$$

$$a\nabla r - \kappa_u \mathbf{D}\boldsymbol{u} + \omega \boldsymbol{u}^\perp = 0.$$

with $\mathbf{D}\boldsymbol{u} := (\partial_{xx} u, \partial_{yy} v)^T$ implies that

$$\kappa_r ||\nabla r||^2 + \kappa_u (||\partial_x u||^2 + ||\partial_y v||^2) = 0.$$

Implies that the kernel verifies $\nabla r = 0$ and $(\partial_x u, \partial_y v) = (0, 0)$ which, in turns, implies that $\boldsymbol{u} = 0$.

The "low Mach" ($\kappa_u = 0$) strategy by Dellacherie et al. is not useful: the modified kernel still verifies $\nabla r = 0$ and thus $\boldsymbol{u} = 0$.

The "low Mach" strategy on κ_r doesn't work either since then $u(y)$ and $v(x)$ is not a good representation of the original kernel.

The "low Mach" strategy on both κ_r and κ_u leads to an unstable explicit scheme (or very low CFL if $\mathcal{O}(M)$) and the implicit scheme has oscillations

Modified equation analysis

The "Apparent topography" scheme by Bouchut et al. has the following modified equation

$$\partial_t r + a\nabla \cdot \boldsymbol{u} - \kappa_r \nabla \cdot (\nabla r + \frac{\omega}{a}\boldsymbol{u}^{\perp}) = 0,$$

$$\partial_t \boldsymbol{u} + a\nabla r - \kappa_u \mathbf{D}\boldsymbol{u} + \omega \boldsymbol{u}^{\perp} = 0.$$

The diffusion in the pressure equation does not affect the equilibrium. Works well in 1D because $\mathbf{D}\boldsymbol{u} = 0$ for $\boldsymbol{u} \in \mathcal{E}$ in 1D.

In 2D not sufficient because $\mathbf{D}\boldsymbol{u} \neq 0$ for $\boldsymbol{u} \in \mathcal{E}$.

No energy estimate but Fourier analysis shows that the modified equation is stable. No preservation of orthogonality.

Modified modified equation

We propose to construct schemes that have the following modified equation (Apparent topography + "Divergence Penalization" idea by Dellacherie in 2010)

$$\partial_t r + a\nabla\cdot\boldsymbol{u} - \kappa_r \nabla\cdot(\nabla r + \frac{\omega}{a}\boldsymbol{u}^\perp) = 0,$$

$$\partial_t \boldsymbol{u} + a\nabla r - \kappa_u \nabla(\nabla\cdot\boldsymbol{u}) + \omega\boldsymbol{u}^\perp = 0.$$

The kernel contains at least $\mathcal{E}$ and is exactly $\mathcal{E}$ when $\kappa_r = 0$

Energy damping when $\kappa_r = 0$ and stability through Fourier analysis when $\kappa_r > 0$

Orthogonality preservation only when $\kappa_r = 0$

Wait for the conclusive remarks for (better?) results when $\kappa_r > 0$.

Actual construction of the schemes will rely on two main ingredients: duality in discrete gradient and divergence, and location of the discrete version of the kernel definition (cell based, edge based or vertex based equality between $a\nabla r$ and $\omega\boldsymbol{u}^\perp$)

(Only a few of) the proposed modified schemes

Colocated schemes on 2D cartesian meshes

Cell based kernel definition : $a\nabla_{2h} r + \omega \boldsymbol{u}^{\perp} = 0$ with

$$(\nabla_{2h} r)_{i,j} := (\frac{(r_{i+1,j} - r_{i-1,j})}{2\Delta x}, \frac{(r_{i,j+1} - r_{i,j-1})}{2\Delta y})^T$$

This implies that $\nabla_{2h} \cdot \boldsymbol{u} = 0$ with

$$(\nabla_{2h} \cdot \boldsymbol{u})_{i,j} := \frac{(u_{i+1,j} - u_{i-1,j})}{2\Delta x} + \frac{(v_{i,j+1} - v_{i,j-1})}{2\Delta y}$$

and ∇_{2h} and $-\nabla_{2h}\cdot$ are discrete adjoint operators.

$$\partial_t r + a\nabla_{2h} \cdot \boldsymbol{u} - \kappa_r \nabla_{2h} \cdot (\nabla_{2h} r + \frac{\omega}{a} \boldsymbol{u}^{\perp}) = 0,$$

$$\partial_t \boldsymbol{u} + a\nabla_{2h} r - \kappa_u \nabla_{2h} (\nabla_{2h} \cdot \boldsymbol{u}) + \omega \boldsymbol{u}^{\perp} = 0.$$

Large stencil; odd/even decoupling

Colocated schemes on 2D cartesian meshes – continued

Edge based kernel definition

$$a \begin{pmatrix} \frac{\hat{r}_{i+1,j}-\hat{r}_{i,j}}{\Delta x} \\ \frac{\hat{r}_{i,j+1}-\hat{r}_{i,j}}{\Delta y} \end{pmatrix} + \omega \begin{pmatrix} -\frac{\hat{v}_{i+1,j}+\hat{v}_{i,j}}{2} \\ \frac{\hat{u}_{i,j+1}+\hat{u}_{i,j}}{2} \end{pmatrix} = 0$$

The scheme with the kernel at the interface is given by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} r_{i,j}(t) + a_\star f_h^{\mathrm{c}} \left[\nabla_h^{\mathrm{v}} \cdot \boldsymbol{u}_h\right]_{i,j} - \nu_r \nabla_h^{\mathrm{c}} \cdot \left[\nabla_h^{\mathrm{v}} r_h + \omega f_h^{\mathrm{v}}(\boldsymbol{u}_h^{\perp})\right]_{i,j} = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{u}_{i,j}(t) + a_\star f_h^{\mathrm{c}} \left[\nabla_h^{\mathrm{v}} r_h\right]_{i,j} - \nu_u \nabla_h^{\mathrm{c}} \left[\nabla_h^{\mathrm{v}} \cdot \boldsymbol{u}_h\right]_{i,j} = -\omega f_h^{\mathrm{c}} \left[f_h^{\mathrm{v}}(\boldsymbol{u}_h^{\perp})\right]_{i,j}. \end{cases}$$

with f_h^c and f_h^v are averaging operators from vertices to cells and from cells to vertices,

$\nabla_h^{\mathrm{v}}\cdot$ and ∇_h^{v} are divergence and gradient operators defined at the vertices from values at the cells

$\nabla_h^{\mathrm{c}}\cdot$ and ∇_h^{c} are divergence and gradient operators defined at the cells from values at the vertices

Staggered schemes on triangular meshes

The velocity equation of the standard ($\kappa_u = 1$) Godunov scheme can be written on cell K in the following way

$$\partial_t \boldsymbol{u}_K + a\nabla^{\mathrm{NC}}\{r\} - \kappa_u a\nabla^{\mathrm{NC}}[\boldsymbol{u}\cdot\boldsymbol{n}] + \omega \boldsymbol{u}_K^{\perp} = 0$$

where ∇^{NC} is the gradient of the non-conforming P1 function defined by its values at the edge midpoints, $\{r\}_{\sigma=K|L} = \frac{1}{2}(r_K + r_L)$ and the diffusive term is $[\boldsymbol{u}\cdot\boldsymbol{n}]_{\sigma=K|L} = \frac{1}{2}(\boldsymbol{u}_L - \boldsymbol{u}_K)\cdot\boldsymbol{n}_{KL}$.

So two options:

- either set $\kappa_u = 0$ (or $\mathcal{O}(M)$) and then the kernel is $\omega\boldsymbol{u}_K^{\perp} = -a\nabla^{\mathrm{NC}}\{r\}$ but is not consistent ($\frac{1}{2}(r_K + r_L) = r_\sigma + \mathcal{O}(h)$)

-either keep $\kappa_u = 1$ but then $[\boldsymbol{u}\cdot\boldsymbol{n}]_{K|L} = 0$ for all $K|L$ is equivalent to $\boldsymbol{u} = \nabla\times\phi^L$, where ϕ^L is a (conforming Lagrange) $P1$ function defined by its values at the nodes: this would be coherent with $\omega\boldsymbol{u}_K^{\perp} = -a\nabla r^L \implies$ **define r at the nodes**

Staggered schemes on triangular meshes

The velocity equation is now written in the following way

$$\partial_t \boldsymbol{u}_K + a\nabla^{\mathrm{L}} r - \kappa_u a \nabla^{\mathrm{NC}}[\boldsymbol{u}\cdot\boldsymbol{n}] + \omega \boldsymbol{u}_K^{\perp} = 0$$

where ∇^{L} is the gradient of the conforming P1 function defined by its values at the nodes

With r defined at the nodes, we now have to write a new equation for r on barycentric dual cells centered on the nodes

$$\partial_t r + a\nabla^{\mathrm{D}}\cdot\boldsymbol{u} - \kappa_r \nabla^{\mathrm{D}}\cdot(\nabla^{\mathrm{L}} r + \frac{\omega}{a}\boldsymbol{u}^{\perp}) = 0.$$

with ∇^{D} the standard node-centered divergence which is the adjoint of the discrete ∇^{L} gradient.

This scheme is energy damping and orthogonality preserving when $\kappa_r = 0$; it is cheked to be stable numericaly when $\kappa_r \neq 0$.

Conclusions and Perspectives

Conclusions and perspectives

- Conclusions:
  - Analysis of the limit regime of the shallow water equation with Coriolis source term
  - Need for the numerical schemes to perserve discrete stationary states $a\nabla r + \omega \boldsymbol{u}^\perp = 0$
  - Modified equation analysis of the standard Godunov scheme and suggestion of a new modified equation
  - Construction of the schemes in various configurations
  - Better numerical results are obtained
- Not discussed here but in Hieu's PhD thesis:
  - Fully discrete implementations were proposed and analyzed in terms of stability conditions and dispersion relations (staggered schemes seem to be better in that respect, $\kappa_r = 1$ seem to be better than $\kappa_r = 0$)
  - Non linear generalizations have been proposed, implemented and tested but still need comprehensive assessment

Conclusions and perspectives

- Perspectives

Modification of the Apparent Topography scheme to obtain stability through energy estimates and orthogonality preservation

$$\partial_t r + a\nabla \cdot \boldsymbol{u} - \kappa_r \nabla \cdot (\nabla r + \frac{\omega}{a}\boldsymbol{u}^\perp) = 0,$$

$$\partial_t \boldsymbol{u} + a\nabla r - \kappa_u \nabla(\nabla \cdot \boldsymbol{u}) + \omega \boldsymbol{u}^\perp - \kappa_r \frac{\omega}{a}(\nabla^\perp r - \frac{\omega}{a}\boldsymbol{u}) = 0.$$

Energy dissipation:

$$\frac{1}{2} d_t \left(\|r\|^2 + \|\mathbf{u}\|^2 \right) = -\kappa_r \|\nabla r + \frac{\omega}{a}\mathbf{u}^\perp\|^2 - \kappa_u \|\nabla \cdot \mathbf{u}\|^2,$$

Kernel: exactly $a\nabla r + \omega \mathbf{u}^\perp = 0$

Orthogonality preserving

Conclusions and perspectives

$$\partial_t r + a\nabla \cdot \boldsymbol{u} - \kappa_r \nabla \cdot (\nabla r + \frac{\omega}{a}\boldsymbol{u}^\perp) = 0,$$

$$\partial_t \boldsymbol{u} + a\nabla r - \kappa_u \nabla(\nabla \cdot \boldsymbol{u}) + \omega \boldsymbol{u}^\perp - \kappa_r \frac{\omega}{a}(\nabla^\perp r - \frac{\omega}{a}\boldsymbol{u}) = 0.$$

may be rewritten as

$$\partial_t r + a\nabla \cdot \boldsymbol{U} = 0,$$

$$\partial_t \boldsymbol{u} + a\nabla R + \omega \boldsymbol{U}^\perp = 0.$$

with some new variables

$$\boldsymbol{U} = \boldsymbol{u} - \frac{\kappa_r}{a}(\nabla r + \frac{\omega}{a}\boldsymbol{u}^\perp),$$

$$R = r - \frac{\kappa_u}{a}\nabla \cdot \boldsymbol{u}.$$

Link with the "regularized" variables of Parisot and Vila?