Staggered pressure corrections scheme

for compressible flows

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CALIF^{3}S: `https://gforge.irsn.fr/gf/project/isis`

Outline

1. **Schemes for Euler equations**
   - Generic ingredients
   - A pressure correction scheme
   - Numerical tests
   - An explicit variant
2. **Entropy estimates**

Euler equations. . . and "derived" forms

- Euler (Navier-Stokes) equations:

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \\
&\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{\tau} + \boldsymbol{\nabla} p = 0, \\
&\partial_t(\varrho E) + \operatorname{div}\big[(\varrho E + p)\boldsymbol{u}\big] = \operatorname{div}(\boldsymbol{\tau} \boldsymbol{u}), \\
&p = (\gamma - 1)\ \varrho e, \quad E = \frac{1}{2}\,|\boldsymbol{u}|^2 + e.
\end{aligned}$$

- For regular functions, taking the scalar product of the momentum balance equation by $\boldsymbol{u}$ and using the mass balance equation yields the kinetic energy balance equation:

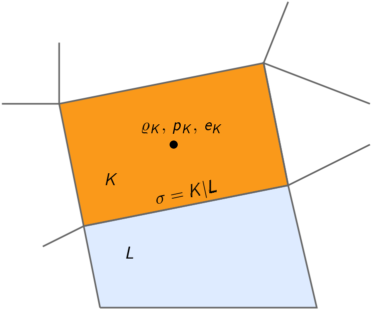
$$\partial_t(\varrho E_c) + \operatorname{div}(\varrho E_c \boldsymbol{u}) + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = \operatorname{div}(\boldsymbol{\tau}) \cdot \boldsymbol{u}, \qquad E_c = \frac{1}{2}\,|\boldsymbol{u}|^2.$$

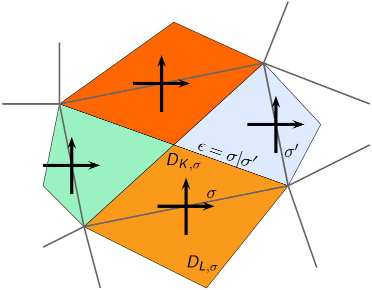
  Subtracting to the total energy balance yields the internal energy balance:

$$\partial_t(\varrho e) + \operatorname{div}(\varrho e \boldsymbol{u}) + p \operatorname{div} \boldsymbol{u} = \boldsymbol{\tau} : \boldsymbol{\nabla} \boldsymbol{u},$$

  and, from this equation, we get $e \geq 0$.

Objectives





Objective – derive a scheme for Euler (or Navier-Stokes) equations which is a natural extension of an existing scheme for low Mach number flows:

- staggered discretization,
- upwinding with respect to the material velocity,
- solution of the internal energy balance,
- pressure correction scheme.

Staggered schemes for compressible flows: Harlow & Amsden, Wesseling and co-workers, Goudon and co-workers. . .

Weak solutions and conservative schemes

- Weak solution:

$$\partial_t(s) + \mathrm{div}(F(s)) = 0$$

$$\hookrightarrow \quad \int_{\Omega\times(0,T)} -s\ \partial_t\varphi - F(s)\cdot\boldsymbol{\nabla}\varphi = 0, \quad \forall\varphi\in \mathrm{C}_c^\infty(\Omega\times(0,T)).$$

- Rankine-Hugoniot conditions.

  Let us suppose, in 1D, that s is discontinuous along a "line" in the $\Omega\times(0,T)$ "plane", and let w be the slope of this line (the propagation speed of the shock). Then, if s is a weak solution:

$$w = \frac{[F(s)]}{[s]}.$$

- Consistency: let $(s_h) \to \bar s$ (in strong enough norms); then $\bar s$ is a weak solution.

  Ex. : passage to the limit for a conservative finite volume approximation of the flux term:

$$\sum_K \varphi_K \sum_\sigma |\sigma|\ F(s)_\sigma\cdot\boldsymbol{n}_\sigma = \sum_\sigma |D_\sigma|\ F(s)_\sigma\cdot\frac{|\sigma|}{|D_\sigma|}\ (\varphi_L-\varphi_K)\,\boldsymbol{n}_\sigma \simeq \int_\Omega F(s)\cdot\widetilde{\boldsymbol{\nabla}}\varphi\,\mathrm{d}\boldsymbol{x}.$$

Strategy

How to obtain the correct weak solutions of Euler equations while solving the internal energy balance ?

Answer: Make the scheme "consistent" with total energy equation...

More precisely:

1- Build a (discrete) kinetic energy balance.

2- Suppose bounds and convergence for a sequence of discrete solutions, compatible with the regularity of the sought continuous solutions:

- control in BV and L^∞,
- convergence in L^p, for $p \geq 1$.

3- Let φ a regular function,

- interpolate,
- test the kinetic energy balance,
- test the internal energy balance,
- and pass to the limit in the scheme.

...and, on the basis of this computation, build corrective terms in the internal energy balance in such a way to recover, at the limit, the weak form of the total energy equation.

General form of the scheme

- Scheme (time semi-discrete setting):

Prediction step:
$$\frac{1}{\delta t}(\varrho^* \tilde{\boldsymbol u} - \varrho^{**} \boldsymbol u^*) + \mathrm{div}(\varrho^* \tilde{\boldsymbol u} \otimes \boldsymbol u^*) - \mathrm{div}\,\boldsymbol\tau(\tilde{\boldsymbol u}) + \xi \boldsymbol\nabla p^* = 0.$$

Correction step:
$$\left|\begin{array}{l} \dfrac{\varrho^*}{\delta t}(\boldsymbol u - \tilde{\boldsymbol u}) + \boldsymbol\nabla p - \xi \boldsymbol\nabla p^* = 0, \\ \dfrac{1}{\delta t}(\varrho - \varrho^*) + \mathrm{div}(\varrho \boldsymbol u) = 0, \\ \dfrac{1}{\delta t}(\varrho e - \varrho^* e^*) + \mathrm{div}(\varrho e \boldsymbol u) + p\,\mathrm{div}\,\boldsymbol u = \mathcal S, \\ p = \wp(\varrho, e) = (\gamma - 1)\ \varrho e. \end{array}\right.$$

- Time shift of the density discretization.
- Coupling of the mass and energy balance.
- Modification of the pressure gradient, corrective term in the internal energy balance.

Kinetic energy balance (1/2)

- Lemma – Since:

$$\frac{1}{\delta t}(\varrho^* - \varrho^{**}) + \mathrm{div}(\varrho^* \boldsymbol{u}^*) = 0,$$

  Then:

$$\tilde{\boldsymbol{u}} \cdot \Big[\frac{1}{\delta t}(\varrho^* \tilde{\boldsymbol{u}} - \varrho^{**} \boldsymbol{u}^*) + \mathrm{div}(\varrho^* \tilde{\boldsymbol{u}} \otimes \boldsymbol{u}^*)\Big] = \frac{1}{\delta t}(\varrho^* \tilde{E}_k - \varrho^{**} E_k^*) + \mathrm{div}(\varrho^* \tilde{E}_k \boldsymbol{u}^*) + \mathcal{R}_1,$$

  with $\tilde{E}_k = \dfrac{1}{2}|\tilde{\boldsymbol{u}}|^2$, $E_k^* = \dfrac{1}{2}|\boldsymbol{u}^*|^2$ and $\mathcal{R}_1 = \dfrac{1}{2\delta t}\varrho^{**}|\tilde{\boldsymbol{u}} - \boldsymbol{u}^*|^2$.

- So, multiplying the prediction step by $\tilde{u}$ yields:

$$\frac{1}{\delta t}(\varrho^* \tilde{E}_k - \varrho^{**} E_k^*) + \mathrm{div}(\varrho^* \tilde{E}_k \boldsymbol{u}^*) + \xi\, \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla} p^* + \mathcal{R}_1 = 0.$$

Kinetic energy balance (2/2)

- Velocity correction equation (a computation from J.-L. Guermond):

$$\frac{\varrho^*}{\delta t}(\boldsymbol{u} - \tilde{\boldsymbol{u}}) + \boldsymbol{\nabla} p - \xi \boldsymbol{\nabla} p^* = 0,$$

i.e.

$$\Big(\frac{\varrho^*}{\delta t}\Big)^{\frac{1}{2}}\, \boldsymbol{u} + \Big(\frac{\delta t}{\varrho^*}\Big)^{\frac{1}{2}} \boldsymbol{\nabla} p = \Big(\frac{\varrho^*}{\delta t}\Big)^{\frac{1}{2}}\, \tilde{\boldsymbol{u}} + \xi \Big(\frac{\delta t}{\varrho^*}\Big)^{\frac{1}{2}} \boldsymbol{\nabla} p^*.$$

Square this relation:

$$\frac{\varrho^*}{\delta t} E_k + \boldsymbol{u} \cdot \boldsymbol{\nabla} p + \frac{\delta t}{2\varrho^*} |\boldsymbol{\nabla} p|^2 = \frac{\varrho^*}{\delta t} \tilde{E}_k + \xi \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla} p^* + \xi \frac{\delta t}{2\varrho^*} |\boldsymbol{\nabla} p^*|^2.$$

- Sum to obtain:

$$\frac{1}{\delta t}(\varrho^* E_k - \varrho^{**} E_k^*) + \mathrm{div}(\varrho^* \tilde{E}_k \boldsymbol{u}^*) + \boldsymbol{u} \cdot \boldsymbol{\nabla} p + \mathcal{R}_1 + \mathcal{R}_2 = 0,$$

with $\mathcal{R}_2 = \dfrac{\delta t}{2\varrho^*} |\boldsymbol{\nabla} p|^2 - \xi \dfrac{\delta t}{2\varrho^*} |\boldsymbol{\nabla} p^*|^2$.

- So

$$\xi = \Big(\frac{\varrho^*}{\varrho^{**}}\Big)^{\frac{1}{2}} \quad \hookrightarrow \quad \mathcal{R}_2 = \frac{\delta t}{2\varrho^*} |\boldsymbol{\nabla} p|^2 - \frac{\delta t}{2\varrho^{**}} |\boldsymbol{\nabla} p^*|^2$$

Scheme, time semi-discrete setting

- Scheme:

Prediction step:

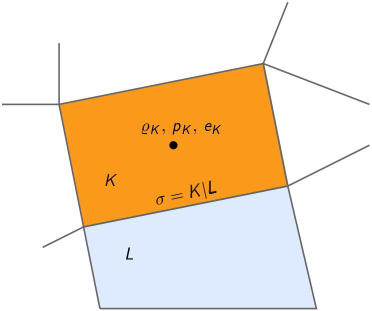
$$\frac{1}{\delta t}(\varrho^* \tilde{\boldsymbol{u}} - \varrho^{**}\boldsymbol{u}^*) + \mathrm{div}(\varrho^* \tilde{\boldsymbol{u}} \otimes \boldsymbol{u}^*) - \mathrm{div}\boldsymbol{\tau}(\tilde{\boldsymbol{u}}) + \Big[\frac{\varrho^*}{\varrho^{**}}\Big]^{1/2} \boldsymbol{\nabla} p^* = 0.$$

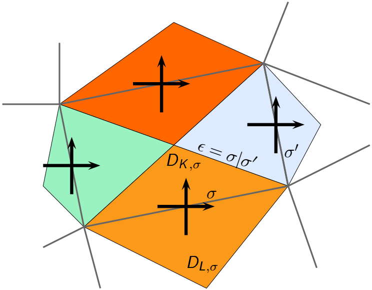
Correction step:

$$\left| \begin{array}{l} \dfrac{\varrho^*}{\delta t}(\boldsymbol{u} - \tilde{\boldsymbol{u}}) + \boldsymbol{\nabla} p - \Big[\dfrac{\varrho^*}{\varrho^{**}}\Big]^{1/2} \boldsymbol{\nabla} p^* = 0, \\ \dfrac{1}{\delta t}(\varrho - \varrho^*) + \mathrm{div}(\varrho \boldsymbol{u}) = 0, \\ \dfrac{1}{\delta t}(\varrho e - \varrho^* e^*) + \mathrm{div}(\varrho e \boldsymbol{u}) + p\, \mathrm{div} \boldsymbol{u} = \boldsymbol{\tau}(\tilde{\boldsymbol{u}}) : \boldsymbol{\nabla} \tilde{\boldsymbol{u}} + \mathcal{S}, \\ p = \wp(\varrho, e) = (\gamma - 1)\ \varrho e, \end{array} \right.$$

with $\mathcal{S} = \mathcal{R}_1$.

Space discretization





The scheme (1/2)

- The scheme:

$$\forall \sigma \in \mathcal{E} \qquad \frac{|D_\sigma|}{\delta t}\,(\varrho^*_{D_\sigma}\tilde{\boldsymbol{u}}_\sigma - \varrho^{**}_{D_\sigma}\boldsymbol{u}^*_\sigma) + \sum_{\epsilon\in\mathcal{E}(D_\sigma)} F^*_{\sigma,\epsilon}\;\tilde{\boldsymbol{u}}_\epsilon + T_s + \Big[\frac{\varrho^*_{D_\sigma}}{\varrho^{**}_{D_\sigma}}\Big]^{1/2}(\boldsymbol{\nabla} p^*)_{\int D_\sigma} = 0.$$

$$\left|\begin{array}{ll} \forall \sigma \in \mathcal{E} & \dfrac{|D_\sigma|}{\delta t}\,\varrho^*_{D_\sigma}(\boldsymbol{u}_\sigma - \tilde{\boldsymbol{u}}_\sigma) + (\boldsymbol{\nabla} p)_{\int D_\sigma} - \Big[\dfrac{\varrho^*_{D_\sigma}}{\varrho^{**}_{D_\sigma}}\Big]^{1/2}(\boldsymbol{\nabla} p^*)_{\int D_\sigma} = 0, \\ \forall K \in \mathcal{M} & \dfrac{|K|}{\delta t}(\varrho_K - \varrho^*_K) + \displaystyle\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma} = 0 \\ \forall K \in \mathcal{M} & \dfrac{|K|}{\delta t}(\varrho_K\, e_K - \varrho^*_K\, e^*_K) + \displaystyle\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma} e_\sigma + p_K \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\;\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma} = S_K, \\ \forall K \in \mathcal{M} & p_K = (\gamma-1)\,\varrho_K\, e_K, \end{array}\right.$$

with:

$$\text{for } \sigma = K|L, \quad (\boldsymbol{\nabla} p)_{\int D_\sigma} = |\sigma|\;(p_L - p_K)\,\boldsymbol{n}_{K,\sigma},$$

$$F_{K,\sigma} = |\sigma|\;\varrho_\sigma\;\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma},\; \varrho_\sigma \text{ upwind}, \qquad e_\sigma \text{ upwind},$$

$$R_\sigma = \frac{|D_\sigma|}{\delta t}\;\varrho^{**}_{D_\sigma}\;\big|\tilde{\boldsymbol{u}}_\sigma - \boldsymbol{u}^*_\sigma\big|^2,$$

$$\tilde{\boldsymbol{u}}_\epsilon \text{ centered}, \quad T_s = \zeta \sum_{\epsilon\in\mathcal{E}(D_\sigma)} h_\epsilon^{d-1}[\tilde{\boldsymbol{u}}]_\epsilon, \quad R_\sigma {+}{=}\, \zeta\,\tilde{\boldsymbol{u}}_\sigma\cdot\sum_{\epsilon\in\mathcal{E}(D_\sigma)} h_\epsilon^{d-1}[\tilde{\boldsymbol{u}}]_\epsilon,$$

$$R_\sigma \hookrightarrow S_K.$$

The scheme (2/2)

- By construction, the mass balance is satisfied over the dual cells

$$\frac{|D_\sigma|}{\delta t}\,(\varrho^*_{D_\sigma} - \varrho^{**}_{D_\sigma}) + \sum_{\epsilon\in\mathcal{E}(D_\sigma)} F^*_{\sigma,\epsilon} = 0,$$

  but it is not so easy to obtain...

- Using the mass balance at the previous time step requires to use a constant time step...

- There is no "local" total energy equation: the kinetic energy balance is associated to the dual cells, while the internal energy one is associated to primal ones.

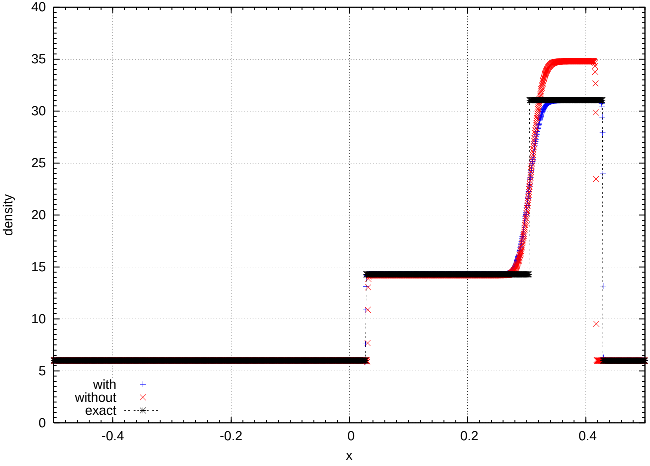
$$R_\sigma = \frac{|D_\sigma|}{\delta t}\,\varrho^{**}_{D_\sigma}\,\left|\tilde{\boldsymbol{u}}_\sigma - \boldsymbol{u}^*_\sigma\right|^2 \hookrightarrow S_K = \sum_{\sigma\in\mathcal{E}(K)} \frac{|D_{K,\sigma}|}{\delta t}\,\varrho^{**}_K\,\left|\tilde{\boldsymbol{u}}_\sigma - \boldsymbol{u}^*_\sigma\right|^2,$$

  with $|D_\sigma|\;\varrho_\sigma = |D_{K,\sigma}|\;\varrho_K + |D_{L,\sigma}|\;\varrho_L$.
  When performing the consistency study of the scheme, we use:

$$\sum_{(0,T)} \delta t\Big[\sum_{\sigma\in\mathcal{E}} \mathcal{R}_\sigma\varphi_\sigma - \sum_{K\in\mathcal{M}} \mathcal{S}_K\varphi_K\Big] \longrightarrow 0.$$

- The rest $\mathcal{R}$ and corrective $\mathcal{S}$ terms compensate exactly when integrated over the domain, so the "total energy" is conserved.
  This total energy includes an L^2 norm of the pressure gradient (in other words, the scheme ensures a control of the pressure in $L^\infty(0,T;H^1(\Omega))$, with a $\delta t/\rho^{1/2}$ weight).

A 1D Riemann problem



Riemann problem – Results obtained with and without corrective terms in the internal energy balance.

Explicit variant

- Scheme (time semi-discrete setting):

$$\begin{aligned}
&\frac{1}{\delta t}(\varrho - \varrho^*) + \operatorname{div}(\varrho^* \boldsymbol{u}^*) = 0, && \rightsquigarrow \ \varrho \\
&\frac{1}{\delta t}(\varrho e - \varrho^* e^*) + \operatorname{div}(\varrho^* e^* \boldsymbol{u}^*) + p^* \operatorname{div} \boldsymbol{u}^* = \mathcal{S}^*, && \rightsquigarrow \ e \\
&p = \wp(\varrho, e) = (\gamma - 1)\ \varrho e, && \\
&\frac{1}{\delta t}(\varrho \boldsymbol{u} - \varrho^* \boldsymbol{u}^*) + \operatorname{div}(\varrho^* \boldsymbol{u}^* \otimes \boldsymbol{u}^*) + \boldsymbol{\nabla} p = 0. && \rightsquigarrow \ \boldsymbol{u}
\end{aligned}$$

- The kinetic energy balance is (still) derived by taking the inner product of the momentum energy balance by $\boldsymbol{u} \rightsquigarrow p^* \operatorname{div} \boldsymbol{u}^*$ in the internal energy balance.

- Upwinding of the convection term in the momentum balance (also).

Explicit variant: $\mathcal{S}$

- Kinetic energy identity (with an upwind discretization of the convection term):

$$\boldsymbol{u}_\sigma \cdot \Big[\frac{|D_\sigma|}{\delta t}(\varrho_{D_\sigma}\boldsymbol{u}_\sigma - \varrho^*_{D_\sigma}\boldsymbol{u}^*_\sigma) + \sum_{\epsilon=D_\sigma|D'_\sigma} F^*_{\sigma,\epsilon}\, \boldsymbol{u}^*_\epsilon\Big]$$
$$= \frac{1}{2}\frac{|D_\sigma|}{\delta t}(\varrho_{D_\sigma}|\boldsymbol{u}_\sigma|^2 - \varrho^*_{D_\sigma}|\boldsymbol{u}^*_\sigma|^2) + \frac{1}{2}\sum_{\epsilon=D_\sigma|D'_\sigma} F^*_{\sigma,\epsilon}\, \boldsymbol{u}^*_\sigma \cdot \boldsymbol{u}^*_{\sigma'} + R_\sigma$$

with

$$R_\sigma = \frac{1}{2}\frac{|D_\sigma|}{\delta t}\ \varrho_{D_\sigma}\ |\boldsymbol{u}_\sigma - \boldsymbol{u}^*_\sigma|^2 + \sum_{\epsilon=D_\sigma|D'_\sigma} \frac{|F^*_{\sigma,\epsilon}|}{2}\ (\boldsymbol{u}^*_\sigma - \boldsymbol{u}^*_{\sigma'})\cdot \boldsymbol{u}^*_\sigma$$
$$+ \sum_{\epsilon=D_\sigma|D'_\sigma} F^*_{\sigma,\epsilon}\ (\boldsymbol{u}_\sigma - \boldsymbol{u}^*_\sigma)\cdot(\boldsymbol{u}^*_\sigma - \boldsymbol{u}^*_{\sigma'}).$$

- Up to a term tending to zero (under L^∞ and BV estimates for $\boldsymbol{u}$),

$$R_\sigma \simeq \frac{1}{2}\frac{|D_\sigma|}{\delta t}\ \varrho_{D_\sigma}\ |\boldsymbol{u}_\sigma - \boldsymbol{u}^*_\sigma|^2 + \sum_{\epsilon=D_\sigma|D'_\sigma} \frac{|F^*_{\sigma,\epsilon}|}{2}\ |\boldsymbol{u}^*_\sigma - \boldsymbol{u}^*_{\sigma'}|^2$$
$$+ \sum_{\epsilon=D_\sigma|D'_\sigma} F^*_{\sigma,\epsilon}\ (\boldsymbol{u}_\sigma - \boldsymbol{u}^*_\sigma)\cdot(\boldsymbol{u}^*_\sigma - \boldsymbol{u}^*_{\sigma'}),$$

which is non-negative under a CFL condition.

- The corrective term $\mathcal{S}$ compensates this remainder term.

Entropy estimates: formal computations (1/3)

- Expected result: obtain an *entropy inequality* for one specific entropy:

$$\partial_t \eta(\rho, e) + \operatorname{div}\big[\eta(\rho, e)\,\boldsymbol{u}\big] \leq 0, \quad \eta(\rho, e) = \underbrace{\rho \log(\rho)}_{\varphi_\rho(\rho)} + \rho \underbrace{\Big(-\frac{1}{\gamma - 1}\log(e)\Big)}_{\varphi_e(e)}.$$

- Note:

$$\varphi'_\rho(\rho) = 1 + \log(\rho) \qquad \varphi''_\rho(\rho) = \frac{1}{\rho} \qquad (\varphi_\rho \text{ is convex})$$

$$\varphi'_e(e) = -\frac{1}{\gamma - 1}\frac{1}{e} \leq 0 \qquad \varphi''_e(e) = \frac{1}{\gamma - 1}\frac{1}{e^2} \qquad (\varphi_e \text{ is convex})$$

Entropy estimates: formal computations (2/3)

- From the mass balance equation:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0 \hookrightarrow$$

$$\partial_t\big(\varphi_\rho(\rho)\big) + \mathrm{div}\big(\varphi_\rho(\rho)\boldsymbol{u}\big) + \underbrace{\big(\rho\varphi'_\rho(\rho) - \varphi(\rho)\big)}_{=\rho}\,\mathrm{div}\boldsymbol{u} = 0.$$

- From the internal energy balance equation:

$$\partial_t(\rho e) + \mathrm{div}(\rho e \boldsymbol{u}) + p\,\mathrm{div}(u) = S \hookrightarrow$$

$$\partial_t\big(\rho\varphi_e(e)\big) + \mathrm{div}\big(\rho\varphi_e(e)\boldsymbol{u}\big) + \underbrace{\varphi'_e(e)\,p}_{=-\rho}\,\mathrm{div}\boldsymbol{u} = \underbrace{\varphi'_e(e)\,S}_{\leq 0}.$$

- Sum ...

Entropy estimates: formal computations (3/3)

At the discrete level:

- From the mass balance equation:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0 \hookrightarrow$$

$$\partial_t \big(\varphi_\rho(\rho)\big) + \mathrm{div}\big(\varphi_\rho(\rho)\boldsymbol{u}\big) + \underbrace{\big(\rho\varphi'_\rho(\rho) - \varphi(\rho)\big)}_{=\rho} \mathrm{div}\boldsymbol{u} + \mathcal{R}_\rho = 0.$$

- From the internal energy balance equation:

$$\partial_t(\rho e) + \mathrm{div}(\rho e \boldsymbol{u}) + p\mathrm{div}(u) = S \hookrightarrow$$

$$\partial_t \big(\rho\varphi_e(e)\big) + \mathrm{div}\big(\rho\varphi_e(e)\boldsymbol{u}\big) + \underbrace{\varphi'_e(e)\, p}_{=-\rho} \mathrm{div}\boldsymbol{u} + \mathcal{R}_e = \underbrace{\varphi'_e(e)\, S}_{\leq 0}.$$

Then:

$\mathcal{R}_\rho \geq 0,\ \mathcal{R}_e \geq 0$: entropy estimate,

$\mathcal{R}_\rho + \mathcal{R}_e \geq \mathcal{R}_\eta$ with $\mathcal{R}_\eta$ tending to zero: weak entropy estimate.

Entropy estimates: implicit upwind scheme (1/2)

For the pressure correction scheme, the mass and internal energy balance equations read:

$$\begin{aligned} & \frac{1}{\delta t}(\varrho - \varrho^*) + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \\ & \frac{1}{\delta t}(\varrho e - \varrho^* e^*) + \operatorname{div}(\varrho e \boldsymbol{u}) + p \operatorname{div} \boldsymbol{u} = +\mathcal{S}, \\ & p = (\gamma - 1)\ \varrho e, \end{aligned}$$

i.e. as a standard implicit scheme.

Entropy estimates: implicit upwind scheme (2/2)

- Lemma (m) – In time semi-discrete notations, with an upwind discretization of the convection term and if φ is convex:

$$\varphi'(\rho)\Big[\frac{1}{\delta t}(\varrho-\varrho^*)+\mathrm{div}(\varrho\boldsymbol{u})\Big] = \frac{1}{\delta t}\big(\varphi(\varrho)-\varphi(\varrho^*)\big)+\mathrm{div}\big(\varphi(\varrho)\boldsymbol{u}\big)+\big[\rho\varphi'(\rho)-\varphi(\rho)\big]\,\mathrm{div}(\boldsymbol{u})+\mathcal{R}_\rho,$$

with $\mathcal{R}_\rho \geq 0$.

- Lemma (e) – Let us suppose that:

$$\frac{1}{\delta t}(\varrho-\varrho^*)+\mathrm{div}(\varrho\boldsymbol{u})=0.$$

Then, with an upwind discretization of the convection term and if φ is convex:

$$\varphi'(e)\Big[\frac{1}{\delta t}(\varrho e-\varrho^* e^*)+\mathrm{div}(\varrho e\boldsymbol{u})\Big] = \frac{1}{\delta t}\big(\varrho\varphi(e)-\varrho^*\varphi(e^*)\big)+\mathrm{div}\big(\varrho\varphi(e)\boldsymbol{u}\big)+\mathcal{R}_e,$$

with $\mathcal{R}_e \geq 0$.

- So exactly what is needed to obtain a discrete entropy inequality.

Entropy estimates: explicit upwind scheme (1/3)

For the pressure correction scheme, the mass and internal energy balance equations read:

$$\frac{1}{\delta t}(\varrho - \varrho^*) + \mathrm{div}(\varrho^* \boldsymbol{u}^*) = 0,$$
$$\frac{1}{\delta t}(\varrho e - \varrho^* e^*) + \mathrm{div}(\varrho^* e^* \boldsymbol{u}^*) + p^* \,\mathrm{div} \boldsymbol{u}^* = +\mathcal{S},$$
$$p^* = (\gamma - 1)\ \varrho^* e^*,$$

i.e. as a standard explicit scheme.

- Lemma (e) still hods, under a CFL condition.
- But Lemma (m) is no more valid (some kind of mixing of time levels. . .).
- Possibility to derive a weak entropy estimate ?

Entropy estimates: explicit upwind scheme (2/3)

- Some definitions:

$$\|z\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{n=0}^{N} \sum_{K\in\mathcal{M}} |K|\ |z_K^{n+1} - z_K^n|$$

$$\|z\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}} |\sigma|\ |z_L^n - z_K^n|$$

$$\|R\|_{-1,1,\star} = \sup_{\psi\in \mathrm{C}_c^\infty([0,T)\times\bar\Omega)} \frac{1}{\displaystyle\sup_{x\in\Omega,\ t\in(0,T)} \|\boldsymbol{\nabla}\psi(\boldsymbol{x},t)\|} \Big[\sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{M}} |K|\ R_K^n \psi_K^n\Big]$$

$$\|\boldsymbol{u}\|^q_{L^q(0,T;W^{1,q}_{\mathcal{M}})} = \sum_{i=1}^{d}\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{M}} \sum_{(\sigma,\sigma')\in\mathcal{E}(K)^2} |K| \left(\frac{u^n_{\sigma,i} - u^n_{\sigma',i}}{h_K}\right)^q.$$

$$\underline{h}_{\mathcal{M}} = \min_{K\in\mathcal{M}} \frac{|K|}{\displaystyle\sum_{\sigma\in\mathcal{E}(K)} |\sigma|}.$$

Entropy estimates: explicit upwind scheme (3/3)

- Estimate of the lim inf of the rest – Under a (mild) regularity assumption on the mesh, a CFL assumption:

$$\|\mathcal{R}_\eta\|_{L^1} \leq C \left(\|\rho\|_{\mathcal{T},t,\mathrm{BV}} + \|e\|_{\mathcal{T},t,\mathrm{BV}}\right) \frac{\delta t}{\underline{h}_{\mathcal{M}}}$$

and

$$\|\mathcal{R}_\eta\|_{L^1} \leq C \ \|\rho\|_{\mathcal{T},t,\mathrm{BV}}{}^{1/p} \ \|\boldsymbol{u}\|_{L^q(0,T;W^{1,q}_{\mathcal{M}})} \ \delta t^{1/p}$$

where $p \geq 1$, $q \geq 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and C is an increasing function of $\max(\|\rho\|_\infty, \|1/\rho\|_\infty, \|e\|_\infty, \|1/e\|_\infty)$.

- How to use it: nonlinear stabilization ?

Entropy estimates: under a "mild upwinding assumption"

- Under a "mild upwinding assumption":

- Strong (weak) entropy inequalities $\hookrightarrow$ weak entropy inequalities with a (an additional) remainder term $\mathcal{R}_{\eta,\mathrm{add}}$ satisfying:

$$\|\mathcal{R}_{\eta,\mathrm{add}}\|_{-\mathbf{1},\mathbf{1},\star} \leq C\Big(\|\rho\|_{\mathcal{T},x,\mathrm{BV}} + \|e\|_{\mathcal{T},x,\mathrm{BV}}\Big)\ h_{\mathcal{M}}$$

...unfortunately excepted the last one ...