A self-adaptive IMEX splitting capturing the multi-scale waves of compressible low-velocity flows

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Plan :

1. Low-velocity flows endowed with a stiff equation of state
2. A dynamic Implicit-Explicit scheme
3. Numerical results

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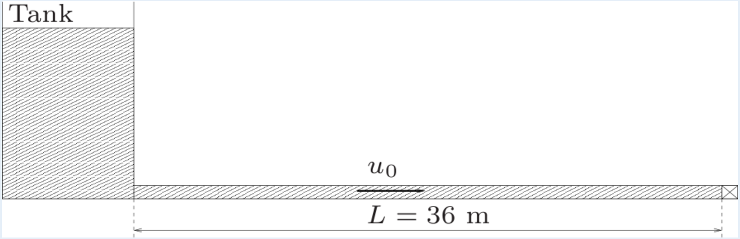
The Simpson's experiment : mechanical water hammer

Simpson's water hammer (Simpson, 1986, PhD)



Simpson's set up

- Liquid water : $p_0 = 3.281\,\text{bar}$, $T_0 = 23.9\,^\circ\text{C}$, $u_0 = 0.401\, m.s^{-1}$.
- At $t = 0$ valve closure.
- Strong shock/rarefaction waves propagating up and down.

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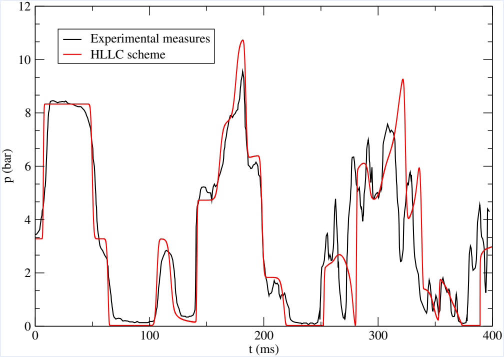

Time evolution of pressure : $p(x = 27\,m, t)$

- $(p_{\max} - p_0)\,/\,p_0 \approx 1.93.$

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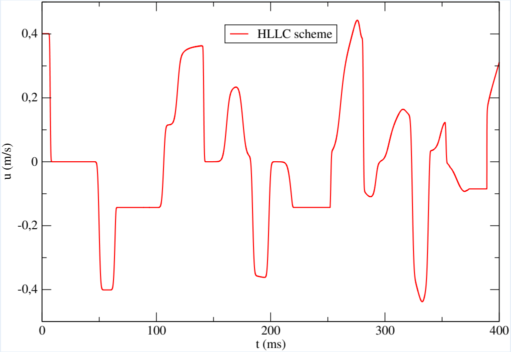

Time evolution of velocity : $u(x = 27\,m, t)$

- $(p_{max} - p_0)\,/\,p_0 \approx 1.93$.
- $|u_{max}| \approx 0.4\; m.s^{-1}$.

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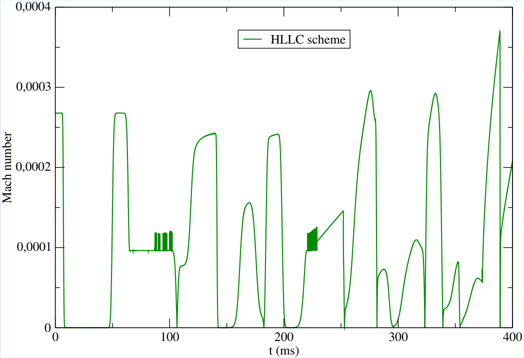

Time evolution of the Mach number : $M(x = 27\ m, t)$

- $\left(p_{\max} - p_0\right)/p_0 \approx 1.93$.
- $|u_{\max}| \approx 0.4\ m.s^{-1}$.
- $M_{\max} \approx 3.5 \times 10^{-4}$.

Analytical solution : symmetric double shock waves

Euler system with a passive tracer :

$$\begin{cases} \partial_t \rho & +\partial_x (\rho u) & = 0, \\ \partial_t (\rho Y) & +\partial_x (\rho Y u) & = 0, \\ \partial_t (\rho u) & +\partial_x (\rho u^2 + p) & = 0, \\ \partial_t (\rho e) & +\partial_x ((\rho e + p) u) & = 0. \end{cases} \qquad \begin{cases} Y : \text{passive tracer}, \\ e = \dfrac{|u|^2}{2} + \varepsilon, \\ \varepsilon = \varepsilon^{EOS}(\rho, p). \end{cases}$$

Stiffened gas equation of state :

$$\varepsilon^{EOS}(\rho, p) = \frac{p + \gamma P_\infty}{(\gamma - 1)\rho}$$

Analytical solution : symmetric double shock waves

	Left state	Right state
$\rho\ (kg.m^{-3})$	$\rho_{0,\,L} = \rho_0 = 10^3$	$\rho_{0,\,R} = \rho_0$
$u\ (m.s^{-1})$	$u_{0,\,L} = u_0 = 1$	$u_{0,\,R} = -u_0$
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$$\frac{P^* - P_0}{P_0} = M_0\,\gamma\left(\frac{\gamma+1}{4}\,M_0 + \sqrt{1 + \frac{(\gamma+1)^2}{16}\,M_0^2}\right).$$

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Analytical pressure jump :

$$\frac{P^* - P_0}{P_0} = M_0 \, \gamma \left(\frac{\gamma + 1}{4} \, M_0 + \sqrt{1 + \frac{(\gamma + 1)^2}{16} \, M_0^2} \right).$$

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Ideal gas EOS : $P_\infty = 0$

$p = P,\ p_0 = P_0.$

$$\frac{p^* - p_0}{p_0} = M_0 \times O(1) \text{ w.r.t } M_0.$$

$$\Rightarrow \lim_{M_0 \to 0} \frac{p^* - p_0}{p_0} = 0.$$

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Stiffened gas EOS : $P_\infty \gg 1$

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Numerical Application : $\gamma = 7.5$,
$P_\infty = 3 \times 10^8\ (Pa)$.
$\Rightarrow c_0 \approx 1500\ m.s^{-1},\ T_0 \approx 22^\circ C.$
$\Rightarrow M_0 \approx 7 \times 10^{-4},\ \alpha = 10^3.$
$\Rightarrow (p^* - p_0)/p_0 \approx 5.26.$

Allievi's model & Joukowski's jump conditions

Derivation of Allievi's model (Allievi, 1902)

Hypothesis :

- Euler system with constant temperature T_0 : $p = p^{\text{EOS}}(\rho,\ T_0) = p_0^{\text{EOS}}(\rho)$, $\rho = \left(p_0^{\text{EOS}}\right)^{-1}(p) = \rho_0^{\text{EOS}}(p)$, $1/c = \sqrt{\left(\rho_0^{\text{EOS}}\right)'(p)}$.

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$$\partial_t \rho + \partial_x (\rho u) = 0,$$
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- Reference scales : l_0, $\rho_0 \approx 10^3\ kg.m^{-3}$, $u_0 \approx 1\ m.s^{-1}$, $c_0 \approx 1.5 \times 10^3\ m.s^{-1}$, $p_0 = \rho_0\, u_0\, c_0 \approx 15\,\text{bar}$, ($\rho_0\, c_0^2 \approx 22500\,\text{bar}$, unphysical !), $t_0 = l_0/c_0$, $M_0 = u_0/c_0$.

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- Unsteady compressible low Mach number flows : $M_0 \ll 1 \Rightarrow \rho = \rho_0 + O(M_0),\ \ c = c_0 + O(M_0)$.

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$$\text{Allievi's model :} \begin{cases} \dfrac{1}{c_0^2}\partial_t\, p + \rho_0\, \partial_x\, u = 0, \\ \rho_0\, \partial_t\, u + \partial_x\ p = 0. \end{cases}$$

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- Unsteady compressible low Mach number flows : $M_0 \ll 1 \Rightarrow \rho = \rho_0 + O(M_0),\ \ c = c_0 + O(M_0)$.
- Eigenvalues and jump relations : $\lambda_0^{\pm} = \pm\, c_0$.

$$[p] = \pm\, \rho_0\, c_0\, [u] \quad \text{(Joukowski, 1898).}$$

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(*III*) $M_0 \ll 1$ and $\rho_0 \approx 10^3 \; kg.m^{-3}$ (water, $P_\infty \gg 1$), $c_0 \approx 1.5 \times 10^3 \; m.s^{-1}$, $p_0 = \rho_0 \, u_0 \, c_0$, $t_0 = l_0 / c_0$:

low velocity compressible flows asymptotically consistent with the Allievi's model.

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Question :

How to derive a numerical scheme able to be accurate on the different multi-scale waves when the flow goes through the regimes (*I*) and (*III*) ?

Plan

A time dynamic splitting

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&\partial_t \rho && +\partial_x (\rho\, u) && = 0,\\
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- Resolution based on the $\mathcal{C}/\mathcal{A}$ operator splitting :
  1. $\partial_t\ \mathbf{U} + \mathcal{C} = \mathbf{0} \quad (t^n \to t^{n+})$.
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  2. $\partial_t\ \mathbf{U} + \mathcal{A} = \mathbf{0} \quad \left(t^{n+} \rightarrow t^{n+1}\right)$.
- Time-dynamic evolution :
  - $\mathscr{E}_0^2(t) \rightarrow 1$.

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&\partial_t \rho && +\partial_x (\rho\, u) && && = 0\\
&\partial_t (\rho Y) && +\partial_x (\rho\, Y\, u) && && = 0\\
&\partial_t (\rho u) && +\partial_x (\rho\, u^2 + p) && + \; 0 && = 0\\
&\partial_t (\rho e) && +\underbrace{\partial_x \left((\rho\, e + p)\, u\right)}_{\mathcal{C}=\text{Euler}} && + \underbrace{0}_{\mathcal{A}} && = 0
\end{aligned}
$$

- Introduce : $\mathscr{E}_0^2(t) \in]0, 1]$, $\mathbf{U} = [\rho,\ \rho\, Y,\ \rho\, u, \rho\, e]^T$.
- Convective subsystem : $\mathcal{C}$. Acoustic subsystem $\mathcal{A}$.
- Resolution based on the $\mathcal{C}/\mathcal{A}$ operator splitting :
  1. $\partial_t\ \mathbf{U} + \mathcal{C} = \mathbf{0} \quad (t^n \to t^{n+})$.
  2. $\partial_t\ \mathbf{U} + \mathcal{A} = \mathbf{0} \quad \left(t^{n+} \to t^{n+1}\right)$.
- Time-dynamic evolution :
  - $\mathscr{E}_0^2(t) \to 1$.

A time dynamic splitting

$$
\begin{aligned}
&\partial_t \rho && +\partial_x (\rho\, u) && && = 0,\\
&\partial_t (\rho Y) && +\partial_x (\rho\, Y\, u) && && = 0,\\
&\partial_t (\rho u) && +\partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) && +((1-\mathscr{E}_0^2(t))\, \partial_x\, p && = 0,\\
&\partial_t (\rho e) && +\underbrace{\partial_x\, ((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u)}_{\mathcal{C}} && +\underbrace{((1-\mathscr{E}_0^2(t))\, \partial_x\, (p\, u)}_{\mathcal{A}} && = 0.
\end{aligned}
$$

- Introduce : $\mathscr{E}_0^2(t) \in]0,1]$, $\mathbf{U} = [\rho,\ \rho\, Y,\ \rho\, u, \rho\, e]^T$.
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  1. $\partial_t\ \mathbf{U} + \mathcal{C} = \mathbf{0} \quad (t^n \rightarrow t^{n+})$.
  2. $\partial_t\ \mathbf{U} + \mathcal{A} = \mathbf{0} \quad \left(t^{n+} \rightarrow t^{n+1}\right)$.
- Time-dynamic evolution :
  - $\mathscr{E}_0^2(t) \rightarrow 1$.

A time dynamic splitting

$$
\begin{aligned}
&\partial_t \rho &&+\partial_x (\rho\, u) && &&= 0,\\
&\partial_t (\rho Y) &&+\partial_x (\rho\, Y\, u) && &&= 0,\\
&\partial_t (\rho u) &&+\partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) &&+((1-\mathscr{E}_0^2(t))\,\partial_x\, p &&= 0,\\
&\partial_t (\rho e) &&+\underbrace{\partial_x\, ((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u)}_{\mathcal{C}} &&+\underbrace{((1-\mathscr{E}_0^2(t))\,\partial_x\,(p\, u)}_{\mathcal{A}} &&= 0.
\end{aligned}
$$

- Introduce : $\mathscr{E}_0^2(t) \in]0,1]$, $\mathbf{U} = [\rho,\, \rho\, Y,\, \rho\, u, \rho\, e]^T$.
- Convective subsystem : $\mathcal{C}$. Acoustic subsystem $\mathcal{A}$.
- Resolution based on the $\mathcal{C}/\mathcal{A}$ operator splitting :
  1. $\partial_t\ \mathbf{U} + \mathcal{C} = \mathbf{0} \quad (t^n \to t^{n+})$.
  2. $\partial_t\ \mathbf{U} + \mathcal{A} = \mathbf{0} \quad \left(t^{n+} \to t^{n+1}\right)$.
- Time-dynamic evolution :
  - $\mathscr{E}_0^2(t) \to 1$.
  - $\mathscr{E}_0^2(t) \to 0^+$.

A time dynamic splitting

$$
\begin{aligned}
&\partial_t \rho && +\partial_x (\rho u) && && = 0,\\
&\partial_t (\rho Y) && +\partial_x (\rho Y u) && && = 0,\\
&\partial_t (\rho u) && +\partial_x (\rho u^2) && +\partial_x p && = 0,\\
&\partial_t (\rho e) && +\underbrace{\partial_x (\rho e u)}_{\mathcal{C}} && +\underbrace{\partial_x (p u)}_{\mathcal{A}} && = 0.
\end{aligned}
$$

- Introduce : $\mathscr{E}_0^2(t) \in]0,1]$, $\mathbf{U} = [\rho, \rho Y, \rho u, \rho e]^T$.
- Convective subsystem : $\mathcal{C}$. Acoustic subsystem $\mathcal{A}$.
- Resolution based on the $\mathcal{C}/\mathcal{A}$ operator splitting :
  1. $\partial_t \mathbf{U} + \mathcal{C} = \mathbf{0} \quad (t^n \to t^{n+})$.
  2. $\partial_t \mathbf{U} + \mathcal{A} = \mathbf{0} \quad \left(t^{n+} \to t^{n+1}\right)$.
- Time-dynamic evolution :
  - $\mathscr{E}_0^2(t) \to 1$.
  - $\mathscr{E}_0^2(t) \to 0^+$.

Study of the Convective Subsystem $\mathcal{C}$

$$
\begin{aligned}
\partial_t \rho \quad & + \partial_x (\rho\, u) && = 0,\\
\partial_t (\rho Y) \quad & + \partial_x (\rho\, Y\, u) && = 0,\\
\partial_t (\rho u) \quad & + \partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) && = 0,\\
\partial_t (\rho e) \quad & + \underbrace{\partial_x \left((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u\right)}_{\mathcal{C}} && = 0.
\end{aligned}
$$

Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{C}}^2(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\mathscr{E}_0^2\, \frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ convective sound speed

Study of the Convective Subsystem $\mathcal{C}$

$$\begin{aligned}
\partial_t \rho \quad & + \partial_x (\rho\, u) && = 0, \\
\partial_t (\rho Y) \quad & + \partial_x (\rho\, Y\, u) && = 0, \\
\partial_t (\rho u) \quad & + \partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) && = 0, \\
\partial_t (\rho e) \quad & + \underbrace{\partial_x \left((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u\right)}_{\mathcal{C}} && = 0.
\end{aligned}$$

Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{C}}^2\, (\rho,\ p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\mathscr{E}_0^2\, \frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ convective sound speed
- $\rho\, c^2\, (\rho,\ p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ Euler sound speed

Study of the Convective Subsystem $\mathcal{C}$

$$
\begin{aligned}
&\partial_t \rho && + \partial_x (\rho\, u) && = 0,\\
&\partial_t (\rho Y) && + \partial_x (\rho\, Y\, u) && = 0,\\
&\partial_t (\rho u) && + \partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) && = 0,\\
&\partial_t (\rho e) && + \underbrace{\partial_x \left((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u\right)}_{\mathcal{C}} && = 0.
\end{aligned}
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Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{C}}^2\,(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\mathscr{E}_0^2\, \frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ convective sound speed
- $\rho\, c^2\,(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ Euler sound speed
- Stiffened gas thermodynamics : $c_{\mathcal{C}}^2 > 0$, and $\mathcal{C}$ is strictly hyperbolic.

Study of the Convective Subsystem $\mathcal{C}$

$$
\begin{aligned}
&\partial_t \rho && +\partial_x (\rho\, u) && = 0,\\
&\partial_t (\rho Y) && +\partial_x (\rho\, Y\, u) && = 0,\\
&\partial_t (\rho u) && +\partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) && = 0,\\
&\partial_t (\rho e) && +\underbrace{\partial_x \left((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u\right)}_{\mathcal{C}} && = 0.
\end{aligned}
$$

Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{C}}^2 (\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\mathscr{E}_0^2\, \frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ convective sound speed
- $\rho\, c^2 (\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ Euler sound speed
- Stiffened gas thermodynamics : $c_{\mathcal{C}}^2 > 0$, and $\mathcal{C}$ is strictly hyperbolic.
- Eigenvalues : $\lambda_1^{\mathcal{C}} = u - \mathscr{E}_0\, c_{\mathcal{C}} \leq \lambda_{2,\,3}^{\mathcal{C}} = u \leq \lambda_4^{\mathcal{C}} = u + \mathscr{E}_0\, c_{\mathcal{C}}$,

Study of the Convective Subsystem $\mathcal{C}$

$$
\begin{aligned}
&\partial_t \rho && +\partial_x (\rho\, u) && = 0,\\
&\partial_t (\rho Y) && +\partial_x (\rho\, Y\, u) && = 0,\\
&\partial_t (\rho u) && +\partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) && = 0,\\
&\partial_t (\rho e) && +\underbrace{\partial_x \left((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u\right)}_{\mathcal{C}} && = 0.
\end{aligned}
$$

Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{C}}^2 (\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\mathscr{E}_0^2\, \frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ convective sound speed
- $\rho\, c^2 (\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1} \left(\frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ Euler sound speed
- Stiffened gas thermodynamics : $c_{\mathcal{C}}^2 > 0$, and $\mathcal{C}$ is strictly hyperbolic.
- Eigenvalues : $\lambda_1^{\mathcal{C}} = u - \mathscr{E}_0\, c_{\mathcal{C}} \leq \lambda_{2,\,3}^{\mathcal{C}} = u \leq \lambda_4^{\mathcal{C}} = u + \mathscr{E}_0\, c_{\mathcal{C}}$,
- $\forall k \in \{1,\, 2,\, 3,\, 4\}$: $\lim\limits_{\mathscr{E}_0 \to 1} \lambda_k^{\mathcal{C}} = \lambda_k^{\text{Euler}}$, $\lim\limits_{\mathscr{E}_0 \to 0} \lambda_k^{\mathcal{C}} = \lambda_2^{\text{Euler}} = u$

Discretization of the convective subsystem $\mathcal{C}$

$$\begin{aligned}
&\partial_t \rho && +\partial_x (\rho\, u) && = 0,\\
&\partial_t (\rho Y) && +\partial_x (\rho\, Y\, u) && = 0,\\
&\partial_t (\rho u) && +\partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) && = 0,\\
&\partial_t (\rho e) && +\underbrace{\partial_x \left((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u\right)}_{\mathcal{C}} && = 0.
\end{aligned}$$

Discretization of the convective subsystem $\mathcal{C}$

$$
\begin{aligned}
&\partial_t \rho && +\partial_x (\rho\, u) && = 0,\\
&\partial_t (\rho Y) && +\partial_x (\rho\, Y\, u) && = 0,\\
&\partial_t (\rho u) && +\partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, p) && = 0,\\
&\partial_t (\rho e) && +\underbrace{\partial_x\, ((\rho\, e + \mathscr{E}_0^2(t)\, p)\, u)}_{\mathcal{C}} && = 0.
\end{aligned}
$$

- Suliciu-like (Suliciu, 1998, IJES) relaxation technique : Π, μ, $a_{\mathcal{C}}$.

Discretization of the convective subsystem $\mathcal{C}$

$$
\begin{aligned}
&\partial_t \rho && +\partial_x (\rho\, u) && = 0,\\
&\partial_t (\rho Y) && +\partial_x (\rho\, Y\, u) && = 0,\\
&\partial_t (\rho u) && +\partial_x (\rho\, u^2 + \mathscr{E}_0^2(t)\, \Pi) && = 0,\\
&\partial_t (\rho e) && +\partial_x \left((\rho\, e + \mathscr{E}_0^2(t)\, \Pi)\, u\right) && = 0,\\
&\partial_t (\rho \Pi) && +\underbrace{\partial_x \left((\rho\, \Pi + a_{\mathcal{C}}^2)\, u\right)}_{\mathcal{C}^\mu} && = \rho\, \left(p(\mathbf{U}) - \Pi\right)/\mu.
\end{aligned}
$$

- Suliciu-like (Suliciu, 1998, IJES) relaxation technique : Π, μ, $a_{\mathcal{C}}$.
- $\mathbf{U} = [\rho,\, \rho\, Y,\, \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\, \rho\, \Pi]$, $\mathbf{S} = [\mathbf{0},\, \rho\, \left(p(\mathbf{U}) - \Pi\right)/\mu]^T$.

Discretization of the convective subsystem $\mathcal{C}$

$$\begin{aligned} &\partial_t \mathbf{U} && + \partial_x \mathbf{F}^{\mathcal{C}}(\mathbf{W}) && = \mathbf{0}, \\ &\partial_t (\rho \Pi) && + \underbrace{\partial_x \left(\left(\rho \Pi + a_{\mathcal{C}}^2 \right) u \right)}_{\mathcal{C}^\mu} && = \rho \left(p(\mathbf{U}) - \Pi \right) / \mu . \end{aligned}$$

- Suliciu-like (Suliciu, 1998, IJES) relaxation technique : Π, μ, $a_{\mathcal{C}}$.
- $\mathbf{U} = [\rho, \rho Y, \rho u, \rho e]^T$, $\mathbf{W} = [\mathbf{U}, \rho \Pi]$, $\mathbf{S} = [\mathbf{0}, \rho \left(p(\mathbf{U}) - \Pi \right) / \mu]^T$.
- A two-steps resolution :
    1. Exact solution of the Riemann problem :

$$\partial_t \mathbf{W} + \mathcal{C}^\mu = \mathbf{0}, \quad \mathbf{W}(t = 0, .) = \begin{cases} \mathbf{W}_L & \text{if } x < 0 \\ \mathbf{W}_R & \text{if } x > 0 \end{cases}, \quad \mathbf{W}^{\text{God}}(x/t, \mathbf{W}_R, \mathbf{W}_L) .$$

Discretization of the convective subsystem $\mathcal{C}$

$$\begin{aligned} \partial_t \mathbf{U} \quad & + \partial_x \mathbf{F}^{\mathcal{C}}(\mathbf{W}) && = \mathbf{0}, \\ \partial_t (\rho \Pi) \quad & \underbrace{+ \partial_x \left((\rho \Pi + a_{\mathcal{C}}^2)\, u \right)}_{\mathcal{C}^\mu} && = \rho\, (p(\mathbf{U}) - \Pi)\,/\mu. \end{aligned}$$

- Suliciu-like (Suliciu, 1998, IJES) relaxation technique : Π, μ, $a_{\mathcal{C}}$.
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Hyperbolic relaxation system, LD fields, simple Riemann invariants.

Discretization of the convective subsystem $\mathcal{C}$

$$
\begin{aligned}
&\partial_t \mathbf{U} && + \partial_x \mathbf{F}^{\mathcal{C}}(\mathbf{W}) && = \mathbf{0}, \\
&\partial_t (\rho \Pi) && + \underbrace{\partial_x \left(\left(\rho \Pi + a_{\mathcal{C}}^2 \right) u \right)}_{\mathcal{C}^{\mu}} && = \rho \left(p(\mathbf{U}) - \Pi \right) / \mu .
\end{aligned}
$$

- Suliciu-like (Suliciu, 1998, IJES) relaxation technique : Π, μ, $a_{\mathcal{C}}$.
- $\mathbf{U} = [\rho,\ \rho Y,\ \rho u, \rho e]^T$, $\mathbf{W} = [\mathbf{U},\ \rho \Pi]$, $\mathbf{S} = [\mathbf{0},\ \rho \left(p(\mathbf{U}) - \Pi \right) / \mu]^T$.
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  1. Exact solution of the Riemann problem :

$$
\partial_t \mathbf{W} + \mathcal{C}^{\mu} = \mathbf{0}, \quad \mathbf{W}(t=0,.) = \begin{cases} \mathbf{W}_L & \text{if } x < 0 \\ \mathbf{W}_R & \text{if } x > 0 \end{cases}, \qquad \mathbf{W}^{\text{God}}(x/t,\ \mathbf{W}_R,\ \mathbf{W}_L).
$$

  Hyperbolic relaxation system, LD fields, simple Riemann invariants.

  2. Instantaneous projection on the equilibrium manifold :

$$
\mathscr{H}^{\text{eq}} = \{\mathbf{W},\ \text{s.t. } \Pi = p(\mathbf{U})\} \qquad \mathscr{P} : \mathbf{U} \to [\mathbf{U},\ \rho\, p(\mathbf{U})]^T .
$$

Discretization of the convective subsystem $\mathcal{C}$

$$\begin{aligned} &\partial_t \mathbf{U} && +\partial_x \mathbf{F}^{\mathcal{C}}(\mathbf{W}) && = \mathbf{0}, \\ &\partial_t (\rho\Pi) && \underbrace{+\partial_x \left(\left(\rho\,\Pi + a_{\mathcal{C}}^2\right) u\right)}_{\mathcal{C}^{\mu}} && = \rho\,\left(p(\mathbf{U}) - \Pi\right)/\mu. \end{aligned}$$

- Suliciu-like (Suliciu, 1998, IJES) relaxation technique : Π, μ, $a_{\mathcal{C}}$.
- $\mathbf{U} = [\rho,\, \rho\, Y,\, \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\, \rho\,\Pi]$, $\mathbf{S} = [\mathbf{0},\, \rho\,\left(p(\mathbf{U}) - \Pi\right)/\mu]^T$.
- A two-steps resolution :
  1. Exact solution of the Riemann problem :

  $$\partial_t\, \mathbf{W} + \mathcal{C}^{\mu} = \mathbf{0}, \quad \mathbf{W}\,(t=0,\,.) = \begin{cases} \mathbf{W}_L & \textit{if } x < 0 \\ \mathbf{W}_R & \textit{if } x > 0 \end{cases}, \qquad \mathbf{W}^{\text{God}}\,(x/t,\, \mathbf{W}_R,\, \mathbf{W}_L)\,.$$

     Hyperbolic relaxation system, LD fields, simple Riemann invariants.
  2. Instantaneous projection on the equilibrium manifold :

  $$\mathscr{H}^{\text{eq}} = \{\mathbf{W},\ \text{s.t. } \Pi = p\,(\mathbf{U})\} \qquad \mathscr{P} : \mathbf{U} \rightarrow [\mathbf{U},\, \rho\, p\,(\mathbf{U})]^T\,.$$

- Numerical flux : $\mathbf{H}_{\mathcal{C}}\,(\mathbf{U}^n_{i+1},\, \mathbf{U}^n_i) = \mathbf{F}^{\mathcal{C}}\left(\mathbf{W}^{\text{God}}\left(\mathscr{P}(\mathbf{U}^n_{i+1}),\, \mathscr{P}(\mathbf{U}^n_i)\right)\right)$.

Discretization of the convective subsystem $\mathcal{C}$

$$\begin{aligned}
\partial_t \mathbf{U} \quad & + \partial_x \mathbf{F}^{\mathcal{C}}(\mathbf{W}) && = \mathbf{0}, \\
\partial_t (\rho\Pi) \quad & \underbrace{+ \partial_x \left((\rho\,\Pi + a_{\mathcal{C}}^2)\, u \right)}_{\mathcal{C}^{\mu}} && = \rho\, (p(\mathbf{U}) - \Pi) / \mu.
\end{aligned}$$

- Suliciu-like (Suliciu, 1998, IJES) relaxation technique : Π, μ, $a_{\mathcal{C}}$.
- $\mathbf{U} = [\rho,\ \rho\, Y,\ \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\ \rho\,\Pi]$, $\mathbf{S} = [\mathbf{0},\ \rho\, (p(\mathbf{U}) - \Pi) / \mu]^T$.
- A two-steps resolution :
  1. Exact solution of the Riemann problem :

     $$\partial_t\ \mathbf{W} + \mathcal{C}^{\mu} = \mathbf{0}, \quad \mathbf{W}(t = 0, .) = \begin{cases} \mathbf{W}_L & \textit{if } x < 0 \\ \mathbf{W}_R & \textit{if } x > 0 \end{cases}, \qquad \mathbf{W}^{\text{God}}(x/t,\ \mathbf{W}_R,\ \mathbf{W}_L).$$

     Hyperbolic relaxation system, LD fields, simple Riemann invariants.
  2. Instantaneous projection on the equilibrium manifold :

     $$\mathscr{H}^{\text{eq}} = \{\mathbf{W},\ \text{s.t. } \Pi = p(\mathbf{U})\} \qquad \mathscr{P} : \mathbf{U} \to [\mathbf{U},\ \rho\, p(\mathbf{U})]^T.$$

- Numerical flux : $\mathbf{H}_{\mathcal{C}}(\mathbf{U}_{i+1}^n,\ \mathbf{U}_i^n) = \mathbf{F}^{\mathcal{C}}\left(\mathbf{W}^{\text{God}}(\mathscr{P}(\mathbf{U}_{i+1}^n),\ \mathscr{P}(\mathbf{U}_i^n))\right)$.
- Whitam's subcharacteristic condition (Whitham, 1974) : $a_{\mathcal{C}} > \rho\, c_{\mathcal{C}}$.

Study of the acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \rho && && = 0,\\
&\partial_t (\rho Y) && && = 0,\\
&\partial_t (\rho u) && + \left(\left(1 - \mathscr{E}_0^2(t)\right) \partial_x\, p\right. && = 0,\\
&\partial_t (\rho e) && + \underbrace{\left(\left(1 - \mathscr{E}_0^2(t)\right) \partial_x\, (p\, u)\right)}_{\mathcal{A}} && = 0.
\end{aligned}
$$

Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{A}}^2\,(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1}\, \frac{p}{\rho}$ acoustic sound speed.

Study of the acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \rho && && = 0,\\
&\partial_t (\rho Y) && && = 0,\\
&\partial_t (\rho u) && +((1 - \mathscr{E}_0^2(t))\, \partial_x\, p && = 0,\\
&\partial_t (\rho e) && +\underbrace{((1 - \mathscr{E}_0^2(t))\, \partial_x\, (p\, u)}_{\mathcal{A}} && = 0.
\end{aligned}
$$

Hyperbolicity & Eigenvalues :

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- $\rho\, c_{\mathcal{C}}^2\,(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1}\, \left(\mathscr{E}_0^2\, \frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ convective sound speed.

Study of the acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \rho && = 0,\\
&\partial_t (\rho Y) && = 0,\\
&\partial_t (\rho u) \quad + \underbrace{((1-\mathscr{E}_0^2(t))\,\partial_x \, p}_{} && = 0,\\
&\partial_t (\rho e) \quad + \underbrace{((1-\mathscr{E}_0^2(t))\,\partial_x \,(p\, u)}_{\mathcal{A}} && = 0.
\end{aligned}
$$

Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{A}}^2\,(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1}\, \frac{p}{\rho}$ acoustic sound speed.
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- $c_{\mathcal{C}}^2 + \left(1-\mathscr{E}_0^2\right)\, c_{\mathcal{A}}^2 = c^2.$

Study of the acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \rho && = 0,\\
&\partial_t (\rho Y) && = 0,\\
&\partial_t (\rho u) \quad +\big((1-\mathscr{E}_0^2(t)\big)\,\partial_x\, p && = 0,\\
&\partial_t (\rho e) \quad +\underbrace{\big((1-\mathscr{E}_0^2(t)\big)\,\partial_x\,(p\,u)}_{\mathcal{A}} && = 0.
\end{aligned}
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Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{A}}^2\,(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1}\, \frac{p}{\rho}$ acoustic sound speed.
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- $c_{\mathcal{C}}^2 + \left(1-\mathscr{E}_0^2\right)\, c_{\mathcal{A}}^2 = c^2.$
- Stiffened gas thermodynamics : $\forall (x,t),\ p(x,t) > 0 \Rightarrow c_{\mathcal{A}}^2 > 0$, and $\mathcal{A}$ is strictly hyperbolic.

Study of the acoustic Subsystem $\mathcal{A}$

$$\begin{aligned}
&\partial_t \rho && && = 0,\\
&\partial_t (\rho Y) && && = 0,\\
&\partial_t (\rho u) && + \left(\left(1 - \mathscr{E}_0^2(t)\right) \partial_x\, p\right. && = 0,\\
&\partial_t (\rho e) && + \underbrace{\left(\left(1 - \mathscr{E}_0^2(t)\right) \partial_x\, (p\, u)\right)}_{\mathcal{A}} && = 0.
\end{aligned}$$

Hyperbolicity & Eigenvalues :

- $\rho\, c_{\mathcal{A}}^2\,(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1}\, \frac{p}{\rho}$ acoustic sound speed.
- $\rho\, c_{\mathcal{C}}^2\,(\rho,\, p) = \left(\partial_p\, \varepsilon_{|\rho}\right)^{-1}\, \left(\mathscr{E}_0^2\, \frac{p}{\rho} - \rho\, \partial_\rho\, \varepsilon_{|p}\right)$ convective sound speed.
- $c_{\mathcal{C}}^2 + \left(1 - \mathscr{E}_0^2\right)\, c_{\mathcal{A}}^2 = c^2$.
- Stiffened gas thermodynamics : $\forall (x,t),\; p(x,t) > 0 \Rightarrow c_{\mathcal{A}}^2 > 0$, and $\mathcal{A}$ is strictly hyperbolic.
- Eigenvalues : $\lambda_1^{\mathcal{A}} = -\left(1 - \mathscr{E}_0^2\right)\, c_{\mathcal{A}} < \lambda_{2,3}^{\mathcal{A}} = 0 < \lambda_4^{\mathcal{A}} = \left(1 - \mathscr{E}_0^2\right)\, c_{\mathcal{A}}$.

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \rho && && = 0, \\
&\partial_t (\rho Y) && && = 0, \\
&\partial_t (\rho u) && + ((1 - \mathcal{E}_0^2(t))\, \partial_x \, p && = 0, \\
&\partial_t (\rho e) && + \underbrace{((1 - \mathcal{E}_0^2(t))\, \partial_x \, (p\, u)}_{\mathcal{A}} && = 0.
\end{aligned}
$$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
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&\partial_t (\rho u) && + ((1 - \mathscr{E}_0^2(t))\, \partial_x\, p && = 0,\\
&\partial_t (\rho e) && + \underbrace{((1 - \mathscr{E}_0^2(t))\, \partial_x\, (p\, u)}_{\mathcal{A}} && = 0.
\end{aligned}
$$

- Suliciu-like relaxation technique : Π, μ, $a_{\mathcal{A}}$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \rho && && = 0\\
&\partial_t (\rho Y) && && = 0\\
&\partial_t (\rho u) && + ((1-\mathscr{E}_0^2(t))\,\partial_x \Pi && = 0\\
&\partial_t (\rho e) && + ((1-\mathscr{E}_0^2(t))\,\partial_x (\Pi\, u) && = 0\\
&\partial_t (\rho \Pi) && + \underbrace{((1-\mathscr{E}_0^2(t))\,\partial_x \left(a_{\mathcal{A}}^2\, u\right)}_{\mathcal{A}^{\mu}} && = \rho\,\left(p(\mathbf{U}) - \Pi\right)/\mu
\end{aligned}
$$

- Suliciu-like relaxation technique : Π, μ, $a_{\mathcal{A}}$
- $\mathbf{U} = [\rho,\, \rho\, Y,\, \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\, \rho\,\Pi]$, $\mathbf{S} = [\mathbf{0},\, \rho\,\left(p(\mathbf{U}) - \Pi\right)/\mu]^T$, $\tau = 1/\rho$, Whitam's condition : $a_{\mathcal{A}} > \rho c_{\mathcal{A}}$.

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \rho && = 0\\
&\partial_t (\rho Y) && = 0\\
&\partial_t (\rho u) + ((1-\mathscr{E}_0^2(t))\,\partial_x\, \Pi && = 0\\
&\partial_t (\rho e) + ((1-\mathscr{E}_0^2(t))\,\partial_x\, (\Pi\, u) && = 0\\
&\partial_t (\rho \Pi) + \underbrace{((1-\mathscr{E}_0^2(t))\,\partial_x\, (a_{\mathcal{A}}^2\, u)}_{\mathcal{A}^{\mu}} && = \rho\,(p(\mathbf{U})-\Pi)\,/\mu
\end{aligned}
$$

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- $\mathbf{U} = [\rho,\, \rho\, Y,\, \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\, \rho\,\Pi]$, $\mathbf{S} = [\mathbf{0},\, \rho\,(p(\mathbf{U})-\Pi)\,/\mu]^T$, $\tau = 1/\rho$, Whitam's condition : $a_{\mathcal{A}} > \rho c_{\mathcal{A}}$.

1. Resolution of the homogeneous system :

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \rho && = 0\\
&\partial_t (\rho Y) && = 0\\
&\partial_t (\rho u) \quad + ((1 - \mathscr{E}_0^2(t))\,\partial_x\, \Pi && = 0\\
&\partial_t (\rho e) \quad + ((1 - \mathscr{E}_0^2(t))\,\partial_x\, (\Pi\, u) && = 0\\
&\partial_t (\rho \Pi) \quad + \underbrace{((1 - \mathscr{E}_0^2(t))\,\partial_x\, (a_{\mathcal{A}}^2\, u)}_{\mathcal{A}^{\mu}} && = 0
\end{aligned}
$$

- Suliciu-like relaxation technique : Π, μ, $a_{\mathcal{A}}$
- $\mathbf{U} = [\rho,\ \rho\, Y,\ \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\ \rho\,\Pi]$, $\mathbf{S} = [\mathbf{0},\ \rho\ (p(\mathbf{U}) - \Pi)\,/\mu]^T$, $\tau = 1/\rho$, Whitam's condition : $a_{\mathcal{A}} > \rho c_{\mathcal{A}}$.

1. Resolution of the homogeneous system :
   - Eigenvalues :
     $\lambda^1_{\mathcal{A}} = -((1 - \mathscr{E}_0^2(t))a_{\mathcal{A}}\tau < \lambda^{2,3}_{\mathcal{A}} = 0 < \lambda^4_{\mathcal{A}} = +((1 - \mathscr{E}_0^2(t))a_{\mathcal{A}}\tau$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$\begin{aligned}
&\partial_t \rho && &&=0\\
&\partial_t (\rho Y) && &&=0\\
&\partial_t (\rho u) &&+((1-\mathscr{E}_0^2(t))\,\partial_x\, \Pi &&=0\\
&\partial_t (\rho e) &&+((1-\mathscr{E}_0^2(t))\,\partial_x\, (\Pi\, u) &&=0\\
&\partial_t (\rho \Pi) &&+\underbrace{((1-\mathscr{E}_0^2(t))\,\partial_x\, (a_{\mathcal{A}}^2\, u)}_{\mathcal{A}^{\mu}} &&=0
\end{aligned}$$

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- $\mathbf{U} = [\rho,\, \rho\, Y,\, \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\, \rho\, \Pi]$, $\mathbf{S} = [\mathbf{0},\, \rho\, (p(\mathbf{U}) - \Pi)\, / \mu]^T$, $\tau = 1/\rho$, Whitam's condition : $a_{\mathcal{A}} > \rho c_{\mathcal{A}}$.

1. Resolution of the homogeneous system :
    - Eigenvalues :
      $\lambda_{\mathcal{A}}^1 = -((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau < \lambda_{\mathcal{A}}^{2,\,3} = 0 < \lambda_{\mathcal{A}}^4 = +((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau$
    - Strong Riemann invariants :
      $W \equiv u - \frac{\Pi}{a_{\mathcal{A}}},\;\; R \equiv u + \frac{\Pi}{a_{\mathcal{A}}}$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \tau && = 0\\
&\partial_t Y && = 0\\
&\partial_t W \quad -((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau\,\partial_x W && = 0\\
&\partial_t R \quad +((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau\,\partial_x R && = 0\\
&\partial_t (\rho e) \quad +((1-\mathscr{E}_0^2(t))\,\partial_x (\Pi\, u) && = 0
\end{aligned}
$$

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- $\mathbf{U} = [\rho,\ \rho\, Y,\ \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\ \rho\,\Pi]$, $\mathbf{S} = [\mathbf{0},\ \rho\ (p(\mathbf{U}) - \Pi)\,/\mu]^T$, $\tau = 1/\rho$, Whitam's condition : $a_{\mathcal{A}} > \rho c_{\mathcal{A}}$.

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   - Eigenvalues :
     $\lambda^1_{\mathcal{A}} = -((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau < \lambda^{2,3}_{\mathcal{A}} = 0 < \lambda^4_{\mathcal{A}} = +((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau$
   - Strong Riemann invariants :
     $W \equiv u - \frac{\Pi}{a_{\mathcal{A}}},\ \ R \equiv u + \frac{\Pi}{a_{\mathcal{A}}}$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \tau && &&= 0\\
&\partial_t Y && &&= 0\\
&\partial_t W && +\lambda^1_{\mathcal{A}}\,\partial_x W &&= 0\\
&\partial_t R && +\lambda^4_{\mathcal{A}}\,\partial_x R &&= 0\\
&\partial_t (\rho e) && +((1-\mathscr{E}_0^2(t))\,\partial_x(\Pi\, u) &&= 0
\end{aligned}
$$

- Suliciu-like relaxation technique : Π, μ, $a_{\mathcal{A}}$
- $\mathbf{U} = [\rho,\ \rho\, Y,\ \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\ \rho\,\Pi]$, $\mathbf{S} = [\mathbf{0},\ \rho\,(p(\mathbf{U}) - \Pi)\,/\mu]^T$, $\tau = 1/\rho$, Whitam's condition : $a_{\mathcal{A}} > \rho c_{\mathcal{A}}$.

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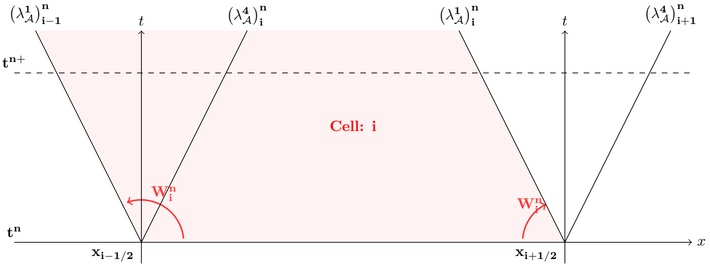
   - Eigenvalues :
     $\lambda^1_{\mathcal{A}} = -((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau < \lambda^{2,3}_{\mathcal{A}} = 0 < \lambda^4_{\mathcal{A}} = +((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau$
   - Strong Riemann invariants :
     $W \equiv u - \frac{\Pi}{a_{\mathcal{A}}},\ \ R \equiv u + \frac{\Pi}{a_{\mathcal{A}}}$

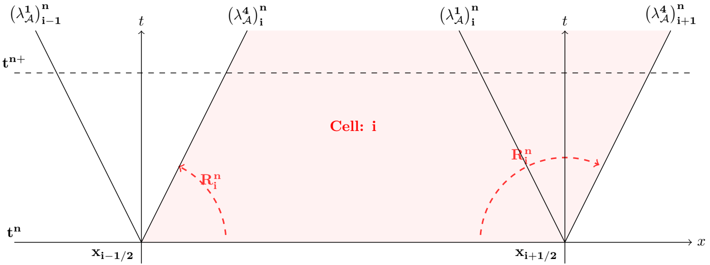
Discretization of the Acoustic Subsystem $\mathcal{A}$



- Suliciu-like relaxation technique : Π, μ, $a_{\mathcal{A}}$
- $\mathbf{U} = [\rho,\ \rho\, Y,\ \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\ \rho\,\Pi]$, $\mathbf{S} = [\mathbf{0},\ \rho\ (p(\mathbf{U}) - \Pi)\ /\mu]^T$, $\tau = 1/\rho$, Whitam's condition : $a_{\mathcal{A}} > \rho c_{\mathcal{A}}$.

1. Resolution of the homogeneous system :
   - Eigenvalues :
     $\lambda^1_{\mathcal{A}} = -((1 - \mathscr{E}^2_0(t))a_{\mathcal{A}}\tau < \lambda^{2,3}_{\mathcal{A}} = 0 < \lambda^4_{\mathcal{A}} = +((1 - \mathscr{E}^2_0(t))a_{\mathcal{A}}\tau$
   - Strong Riemann invariants :
     $W \equiv u - \frac{\Pi}{a_{\mathcal{A}}},\ \ R \equiv u + \frac{\Pi}{a_{\mathcal{A}}}$

Discretization of the Acoustic Subsystem $\mathcal{A}$



- Suliciu-like relaxation technique : Π, μ, $a_{\mathcal{A}}$
- $\mathbf{U} = [\rho,\ \rho\, Y,\ \rho\, u, \rho\, e]^T$, $\mathbf{W} = [\mathbf{U},\ \rho\, \Pi]$, $\mathbf{S} = [\mathbf{0},\ \rho\, (p(\mathbf{U}) - \Pi)\, /\mu]^T$, $\tau = 1/\rho$, Whitam's condition : $a_{\mathcal{A}} > \rho c_{\mathcal{A}}$.

1. Resolution of the homogeneous system :
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    - Strong Riemann invariants :
      $W \equiv u - \frac{\Pi}{a_{\mathcal{A}}},\ R \equiv u + \frac{\Pi}{a_{\mathcal{A}}}$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\partial_t \, \tau && &&= 0\\
&\partial_t \, Y && &&= 0\\
&\partial_t \, W && -((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau \, \partial_x \, W &&= 0\\
&\partial_t \, R && +((1-\mathscr{E}_0^2(t))a_{\mathcal{A}}\tau \, \partial_x \, R &&= 0\\
&\partial_t \, (\rho e) && +((1-\mathscr{E}_0^2(t)) \, \partial_x \, (\Pi \, u) &&= 0
\end{aligned}
$$

- Time-implicit scheme based on a strong Riemann invariants formulation (Coquel et al., 2010, Math. Comp.)

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\tau_i^{n+} = \tau_i^n,\\
&Y_i^{n+} = Y_i^n,\\
&\frac{W_i^{n+} - W_i^n}{\Delta t} - \frac{\left(1 - (\mathscr{E}_0^n)^2\right)\,(a_{\mathcal{A}})^n\,\tau_i^n}{\Delta x}\left(W_{i+1/2}^{n+} - W_{i-1/2}^{n+}\right) = 0,\\
&\frac{R_i^{n+} - R_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right)\,(a_{\mathcal{A}})^n\,\tau_i^n}{\Delta x}\left(R_{i+1/2}^{n+} - R_{i-1/2}^{n+}\right) = 0,\\
&\frac{(\rho e)_i^{n+} - (\rho e)_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right)}{\Delta x}\left((\Pi u)_{i+1/2}^{n+} - (\Pi u)_{i-1/2}^{n+}\right) = 0.
\end{aligned}
$$

- Time-implicit scheme based on a strong Riemann invariants formulation (Coquel et al., 2010, Math. Comp.)

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$\tau_i^{n+} = \tau_i^n,$$

$$Y_i^{n+} = Y_i^n,$$

$$\frac{W_i^{n+} - W_i^n}{\Delta t} - \frac{\left(1 - (\mathscr{E}_0^n)^2\right) \, (a_{\mathcal{A}})^n \, \tau_i^n}{\Delta x} \left(W_{i+1/2}^{n+} - W_{i-1/2}^{n+}\right) = 0,$$

$$\frac{R_i^{n+} - R_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right) \, (a_{\mathcal{A}})^n \, \tau_i^n}{\Delta x} \left(R_{i+1/2}^{n+} - R_{i-1/2}^{n+}\right) = 0,$$

$$\frac{(\rho e)_i^{n+} - (\rho e)_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right)}{\Delta x} \left((\Pi u)_{i+1/2}^{n+} - (\Pi u)_{i-1/2}^{n+}\right) = 0.$$

- Time-implicit scheme based on a strong Riemann invariants formulation (Coquel et al., 2010, Math. Comp.)
- Face values :

$$W_{i+1/2}^{n+} \equiv W_{i+1}^{n+}, \; R_{i+1/2}^{n+} \equiv R_i^{n+}$$

$$u_{i+1/2}^{n+} = \frac{R_i^{n+} + W_{i+1}^{n+}}{2}, \; \Pi_{i+1/2}^{n+} = \frac{a_{\mathcal{A}}^n \left(R_i^{n+} - W_{i+1}^{n+}\right)}{2}$$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\tau_i^{n+} = \tau_i^n, \\
&Y_i^{n+} = Y_i^n, \\
&\frac{W_i^{n+} - W_i^n}{\Delta t} - \frac{\left(1 - (\mathscr{E}_0^n)^2\right)\ (a_{\mathcal{A}})^n\ \tau_i^n}{\Delta x} \left(W_{i+1}^{n+} - W_i^{n+}\right) = 0, \\
&\frac{R_i^{n+} - R_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right)\ (a_{\mathcal{A}})^n\ \tau_i^n}{\Delta x} \left(R_i^{n+} - R_{i-1}^{n+}\right) = 0 \\
&\frac{(\rho e)_i^{n+} - (\rho e)_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right)}{\Delta x} \left((\Pi u)_{i+1/2}^{n+} - (\Pi u)_{i-1/2}^{n+}\right) = 0.
\end{aligned}
$$

- Time-implicit scheme based on a strong Riemann invariants formulation (Coquel et al., 2010, Math. Comp.)
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$$
W_{i+1/2}^{n+} \equiv W_{i+1}^{n+}, \ \ R_{i+1/2}^{n+} \equiv R_i^{n+}
$$

$$
u_{i+1/2}^{n+} = \frac{R_i^{n+} + W_{i+1}^{n+}}{2}, \ \ \Pi_{i+1/2}^{n+} = \frac{a_{\mathcal{A}}^n \left(R_i^{n+} - W_{i+1}^{n+}\right)}{2}
$$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
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&Y_i^{n+} = Y_i^n, \\
&\frac{W_i^{n+} - W_i^n}{\Delta t} - \frac{\left(1 - (\mathscr{E}_0^n)^2\right) \, (a_{\mathcal{A}})^n \, \tau_i^n}{\Delta x} \left(W_{i+1}^{n+} - W_i^{n+}\right) = 0, \\
&\frac{R_i^{n+} - R_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right) \, (a_{\mathcal{A}})^n \, \tau_i^n}{\Delta x} \left(R_i^{n+} - R_{i-1}^{n+}\right) = 0 \\
&\frac{(\rho e)_i^{n+} - (\rho e)_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right)}{\Delta x} \left((\Pi u)_{i+1/2}^{n+} - (\Pi u)_{i-1/2}^{n+}\right) = 0.
\end{aligned}
$$

2. Instantaneous projection on the equilibrium manifold :

$$
\mathscr{H}^{\mathrm{eq}} = \{\mathbf{W}, \text{ s.t. } \Pi = p\,(\mathbf{U})\} \qquad \mathscr{P} : \mathbf{U} \to [\mathbf{U}, \, \rho \, p\,(\mathbf{U})]^T
$$

Discretization of the Acoustic Subsystem $\mathcal{A}$

$$
\begin{aligned}
&\tau_i^{n+} = \tau_i^n, \\
&Y_i^{n+} = Y_i^n, \\
&\frac{W_i^{n+} - W_i^n}{\Delta t} - \frac{\left(1 - (\mathscr{E}_0^n)^2\right)\,(a_{\mathcal{A}})^n\,\tau_i^n}{\Delta x}\left(W_{i+1}^{n+} - W_i^{n+}\right) = 0, \\
&\frac{R_i^{n+} - R_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right)\,(a_{\mathcal{A}})^n\,\tau_i^n}{\Delta x}\left(R_i^{n+} - R_{i-1}^{n+}\right) = 0 \\
&\frac{(\rho e)_i^{n+} - (\rho e)_i^n}{\Delta t} + \frac{\left(1 - (\mathscr{E}_0^n)^2\right)}{\Delta x}\left((\Pi u)_{i+1/2}^{n+} - (\Pi u)_{i-1/2}^{n+}\right) = 0.
\end{aligned}
$$

2. Instantaneous projection on the equilibrium manifold :

$$
\mathscr{H}^{\text{eq}} = \{\mathbf{W},\ \text{s.t. } \Pi = p\,(\mathbf{U})\} \qquad \mathscr{P} : \mathbf{U} \to [\mathbf{U},\ \rho\, p\,(\mathbf{U})]^T
$$

$$
\begin{aligned}
&\rho_i^{n+1} = \rho_i^{n+},\ (\rho u)_i^{n+1} = (\rho u)_i^{n+},\ (\rho e)_i^{n+1} = (\rho e)_i^{n+} \Rightarrow \mathbf{U}_i^{n+1} = \mathbf{U}_i^{n+} \\
&\Pi_i^{n+1} = p\left(\mathbf{U}_i^{n+1}\right)
\end{aligned}
$$

Definition of the Splitting Parameter $\mathscr{E}_0(t)$

$$
\begin{aligned}
&\partial_t \rho && + \partial_x (\rho u) && && = 0\\
&\partial_t (\rho Y) && + \partial_x (\rho Y u) && && = 0\\
&\partial_t (\rho u) && + \partial_x (\rho u^2 + \mathscr{E}_0^2(t)\, p) && + ((1 - \mathscr{E}_0^2(t))\, \partial_x p && = 0\\
&\partial_t (\rho e) && + \underbrace{\partial_x ((\rho e + \mathscr{E}_0^2(t)\, p)\, u)}_{\mathcal{C}} && + \underbrace{((1 - \mathscr{E}_0^2(t))\, \partial_x (p u)}_{\mathcal{A}} && = 0
\end{aligned}
$$

$$\mathscr{E}_0(t) = \min\left(M_{max}(t),\, 1\right)$$

$$M_{max}(t) = \sup_{x \in \Omega} \left(\left(|u| / c\right)(x,\, t)\right)$$

$M_{max}(t) \geq 1$:

$$\mathcal{C} + \mathscr{E}_0^2\, \mathcal{A} \approx \mathcal{C} + \mathcal{A}$$

$$\left(1 - \mathscr{E}_0^2\right)\, \mathcal{A} \approx 0$$

Time-explicit Riemann solver
Compressible waves captured

$M_{max}(t) \ll 1$:

$$\mathcal{C} + \mathscr{E}_0^2\, \mathcal{A} \approx \mathcal{C}$$

$$\left(1 - \mathscr{E}_0^2\right)\, \mathcal{A} \approx \mathcal{A}$$

Time-implicit scheme on $\mathcal{A}$
Slow material waves captured

Definition of the Splitting Parameter $\mathscr{E}_0(t)$

$$
\begin{aligned}
&\partial_t \rho && +\partial_x(\rho u) && && = 0\\
&\partial_t(\rho Y) && +\partial_x(\rho Y u) && && = 0\\
&\partial_t(\rho u) && +\partial_x(\rho u^2 + \mathscr{E}_0^2(t)\, p) && +((1-\mathscr{E}_0^2(t))\,\partial_x p && = 0\\
&\partial_t(\rho e) && +\underbrace{\partial_x\,((\rho e + \mathscr{E}_0^2(t)\, p)\, u)}_{\mathcal{C}} && +\underbrace{((1-\mathscr{E}_0^2(t))\,\partial_x(p u)}_{\mathcal{A}} && = 0
\end{aligned}
$$

$$\mathscr{E}_0(t) = \min\left(M^S_{max}(t),\, 1\right)$$

$$M^S_{max}(t) = \max\left(M_{max}(t),\, \mathcal{S}(t)\right)$$

$M_{max}(t) \ll 1$, $\mathcal{S}(t) \approx 1$:

$$\mathcal{C} + \mathscr{E}_0^2\,\mathcal{A} \approx \mathcal{C} + \mathcal{A}$$

$$\left(1 - \mathscr{E}_0^2\right)\,\mathcal{A} \approx 0$$

Time-explicit Riemann solver
Compressible waves captured

$M_{max}(t) \ll 1$, $\mathcal{S}(t) \ll 1$:

$$\mathcal{C} + \mathscr{E}_0^2\,\mathcal{A} \approx \mathcal{C}$$

$$\left(1 - \mathscr{E}_0^2\right)\,\mathcal{A} \approx \mathcal{A}$$

Time-implicit scheme on $\mathcal{A}$
Slow material waves captured

Definition of the shock detector $\mathcal{S}(t)$

Isothermal water hammer : Joukowski's jump relation

$$[p] = \pm\, \rho_0\, c_0\, [u] \quad \text{(Joukowski, 1898)}$$

Definition of the shock detector $\mathcal{S}(t)$

Isothermal water hammer : Joukowski's jump relation

$$[p] = \pm\, \rho_0\, c_0\, [u] \quad \text{(Joukowski, 1898)}$$

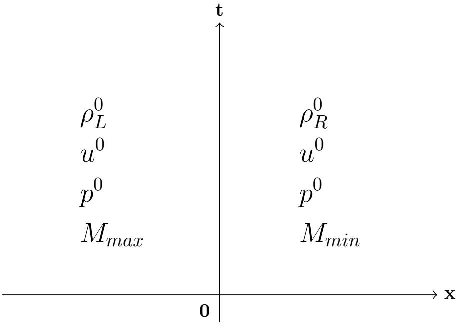
Discrete shock detector : $\mathcal{S}(t^n)$

$$\mathcal{S}(t^n) = \sup_{i+1/2} \left(\frac{\left|(\sigma_S)^n_{i+1/2}\right|}{\max\left(c^n_{i+1}, c^n_i\right)} \right),$$

$$(\sigma_S)^n_{i+1/2} = \begin{cases} \dfrac{p^n_{i+1} - p^n_i}{\frac{\rho^n_{i+1}+\rho^n_i}{2}\left(u^n_{i+1} - u^n_i\right)} & \text{if } |u^n_{i+1} - u^n_i| > \epsilon^{\text{thres}} \max\left(|u^n_{i+1}|, |u^n_i|\right) \\ 0 & \text{otherwise,} \end{cases}$$

Plan

A Pragmatic Stability Study

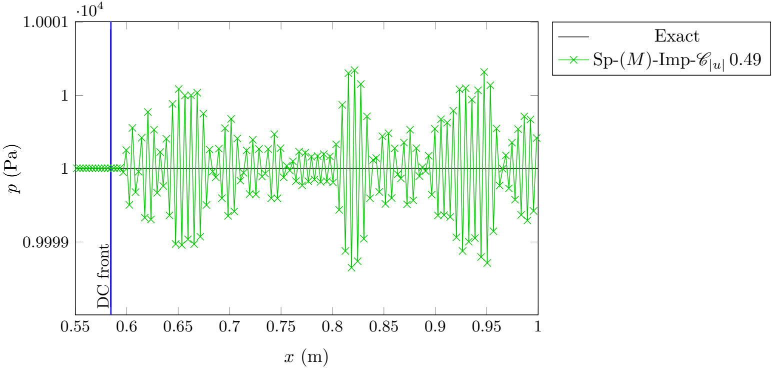


Initial conditions : single contact discontinuity

- $\rho_L^0 = 1\, kg.m^{-3}$,
  $\rho_R^0 = 0.125\, kg.m^{-3}$
- $p^0 = 0.1$ bar
- Ideal gas : $\gamma = 7/5$
- M_{min} input parameter,
  $u^0 = M_{min}\, c_R^0$,
  $c_R^0 = \sqrt{(\gamma p^0)/\rho_R^0}$,
  $M_{max} = u^0/c_L^0 =$
  $M_{min}\sqrt{\rho_L^0/\rho_R^0}$

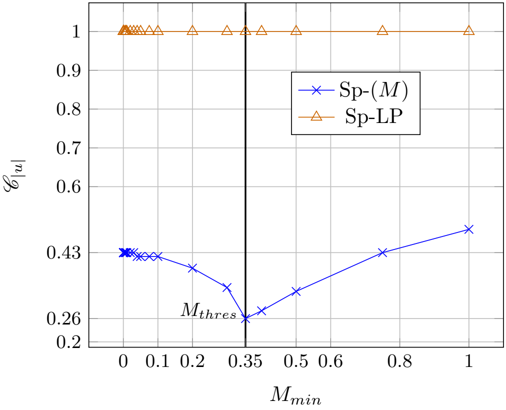
- $\mathcal{S}(t) = 0$ imposed
- $\Delta t^n = \mathscr{C}_{|u|} \frac{\Delta x}{\max\limits_i(|u_i^n|)}$
- Transmissive boundary conditions

A Pragmatic Stability Study



Pressure, $M_{min} = 10^{-2}$, with $\mathbf{N_{cells}} = \mathbf{10^3}$, $\mathscr{C}_{|u|} = 0.49$, iteration 270, $(t = 2.496 \times 10^{-2}\ s)$

A Pragmatic Stability Study



Numerically measured stable Courant numbers : $M_{min} \in \left[10^{-4},\, 1\right]$

A Pragmatic Stability Study

Tentative of explanation of the curve :

$$\mathscr{C}_{\mathcal{C}} = \frac{\left(\left|u^0\right| + \mathscr{E}_0^n \, c_{\mathcal{C}}^{0,\,R}\right) \Delta t}{\Delta x}, \mathscr{C}_{|u|} = \frac{\left|u^0\right| \Delta t}{\Delta x}$$

$$\text{with : } c_{\mathcal{C}}^{0,\,R} = c_{\mathcal{C}}\left(\rho_R^0,\, p^0\right).$$

$$\mathscr{C}_{|u|} = \left(1 + \mathscr{E}_0^n \, \frac{c_{\mathcal{C}}^{0,\,R}}{\left|u^0\right|}\right)^{-1} \mathscr{C}_{\mathcal{C}} = \left(1 + \frac{\mathscr{E}_0^n}{M_{min}} \, \frac{c_{\mathcal{C}}^{0,\,R}}{c^{0,\,R}}\right)^{-1} \mathscr{C}_{\mathcal{C}}$$

$$\text{and : } \frac{c_{\mathcal{C}}^{0,\,R}}{c^{0,\,R}} = \sqrt{(\mathscr{E}_0^n)^2 \frac{\gamma - 1}{\gamma} + \frac{1}{\gamma}} \in [1/\gamma,\, 1]$$

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$$\overbrace{\mathscr{C}_{\mathcal{C}} \approx 1}^{\text{Stability criterion}} \Rightarrow \mathscr{C}_{|u|} = \begin{cases} \left(1 + \dfrac{1}{M_{min}}\right)^{-1}, & \text{if : } M_{min} \geq M_{thres} = \sqrt{\rho_R^0/\rho_L^0} \approx 0.3535 \\ \left(1 + \dfrac{1}{M_{thres}}\sqrt{\left(\dfrac{M_{min}}{M_{thres}}\right)^2 \dfrac{\gamma-1}{\gamma} + \dfrac{1}{\gamma}}\right)^{-1}, & \text{otherwise} \end{cases}$$

A Pragmatic Stability Study

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$$\mathscr{C}_{\mathcal{C}} = \frac{\left(\left|u^0\right| + \mathscr{E}_0^n \, c_{\mathcal{C}}^{0,\,R}\right) \Delta t}{\Delta x}, \mathscr{C}_{|u|} = \frac{\left|u^0\right| \, \Delta t}{\Delta x}$$

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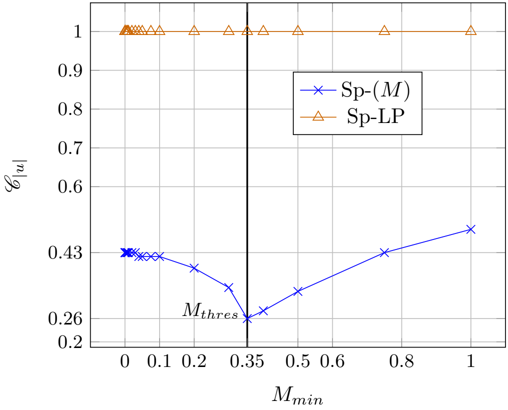
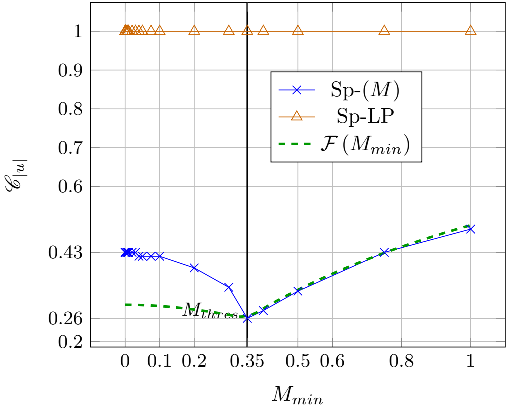
$$\overbrace{\mathscr{C}_{\mathcal{C}} \approx 1}^{\text{Stability criterion}} \quad \Rightarrow \mathscr{C}_{|u|} = \mathcal{F}\left(M_{min}\right)$$

A Pragmatic Stability Study



Numerically measured stable Courant numbers : $M_{min} \in [10^{-4}, 1]$

A Pragmatic Stability Study



Numerically measured stable Courant numbers : $M_{min} \in [10^{-4}, 1]$

A Double Riemann Problem with Stiff Thermodynamics

Shock tube initial conditions : $L = 2\,m$, $x_0 = 0.55\,m$, $x_1 = 1.23\,m$

	Left state ($x < x_0$)	Intermediate state ($x_0 < x < x_1$)	Right state ($x_1 < x$)
$\rho\,(kg.m^{-3})$	$\rho_L^0 = 10^3$	$\rho_{\text{interm}}^0 = 9.98 \times 10^2$	$\rho_R^0 = 9.97 \times 10^2$
$u\,(m.s^{-1})$	$u_L^0 = 1$	$u_{\text{interm}}^0 = 1$	$u_R^0 = 1$
$p\,(bar)$	$p_L^0 = 10^3$	$p_{\text{interm}}^0 = 10$	$p_R^0 = 1$
Y	$Y_L^0 = 0.7$	$Y_{\text{interm}}^0 = 0.2$	$Y_R^0 = 0.1$

- Thermodynamics : stiffened gas, $\rho\varepsilon = (p + \gamma\,P_\infty)\,/(\gamma - 1)$ with $\gamma = 7.5$ and $P_\infty = 3 \times 10^3$ bar.

Time-steps :

Convective time steps Sp-(M)-Imp :

$$\Delta t^n_{\mathcal{C}} = \mathscr{C}_{\mathcal{C}} \frac{\Delta x}{\max\limits_{i+1/2}\left(\max\left(\left| u^n_i - \mathscr{E}^n_0 \, (a_{\mathcal{C}})^n_{i+1/2} \, \tau^n_i \right|, \left| u^n_{i+1} + \mathscr{E}^n_0 \, (a_{\mathcal{C}})^n_{i+1/2} \, \tau^n_{i+1} \right| \right) \right)}, \ \mathscr{C}_{\mathcal{C}} = 0.9$$

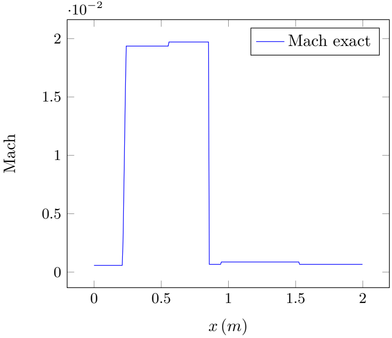
- $(a^n_{\mathcal{C}})_{i+1/2} = K \, max \left(\rho^n_i \, (c_{\mathcal{C}})^n_i, \, \rho^n_{i+1} \, (c_{\mathcal{C}})^n_{i+1} \right), \ K > 1$

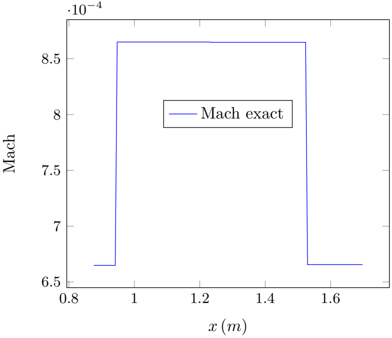
Convective time steps LP-Imp (Chalons et al., 2016, Com. in Comp. Phys.) :

$$\Delta t^n_{\mathcal{C}} = \mathscr{C}_{\mathcal{C}} \frac{\Delta x}{\max\limits_{i+1/2}\left(\left((u^*_{\mathcal{A}})^n_{i-1/2} \right)^+ - \left((u^*_{\mathcal{A}})^n_{i+1/2} \right)^- \right)} \ , \ \mathscr{C}_{\mathcal{C}} = 0.9$$

- $(u^*_{\mathcal{A}})^n_{i+1/2} = \frac{u^n_{i+1} + u^n_i}{2} - \frac{1}{2 \, a^n_{i+1/2}} \left(p^n_{i+1} - p^n_i \right)$
- $a^n_{i+1/2} = K \, max \left(\rho^n_i \, c^n_i, \, \rho^n_{i+1} \, c^n_{i+1} \right), \ K > 1$

A Double Riemann Problem with Stiff Thermodynamics





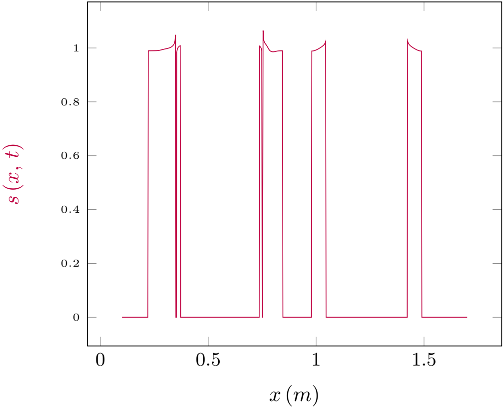
Mach number profile : overall area   Mach number profile : low velocity area

A Double Riemann Problem with Stiff Thermodynamics

Discrete shock detector : $\mathcal{S}(t^n)$

$$\mathcal{S}(t^n) = \sup_{i+1/2} \left(\frac{\left| (\sigma_S)^n_{i+1/2} \right|}{\max\left(c^n_{i+1}, c^n_i\right)} \right) = \sup_{i+1/2} s^n_{i+1/2},$$

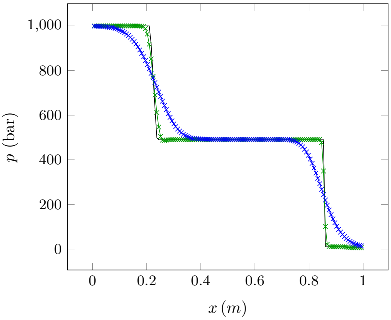
$$(\sigma_S)^n_{i+1/2} = \begin{cases} \dfrac{p^n_{i+1} - p^n_i}{\frac{\rho^n_{i+1}+\rho^n_i}{2}\left(u^n_{i+1} - u^n_i\right)} & \text{if } \left|u^n_{i+1} - u^n_i\right| > \epsilon^{\text{thres}} \max\left(\left|u^n_{i+1}\right|, \left|u^n_i\right|\right) \\ 0 & \text{otherwise,} \end{cases}$$



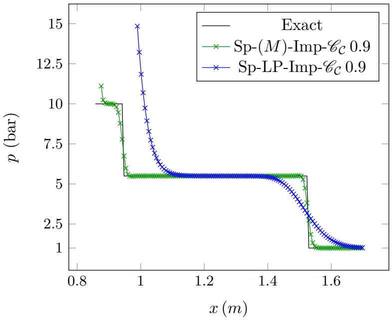
Local shock detector : overall area

Time : $t = 1.46 \times 10^{-4}\ s$

A Double Riemann Problem with Stiff Thermodynamics

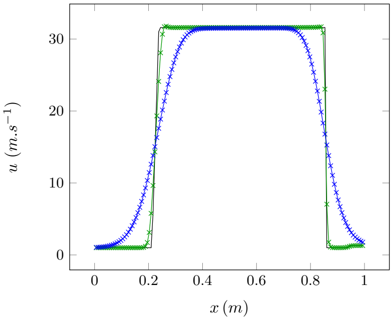


Pressure profile : high velocity area

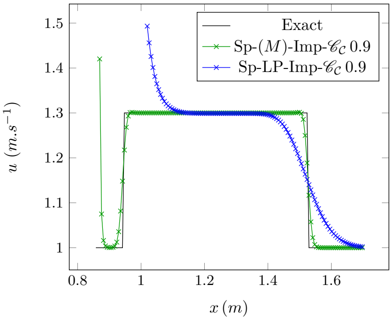


Pressure profile : low velocity area

A Double Riemann Problem with Stiff Thermodynamics

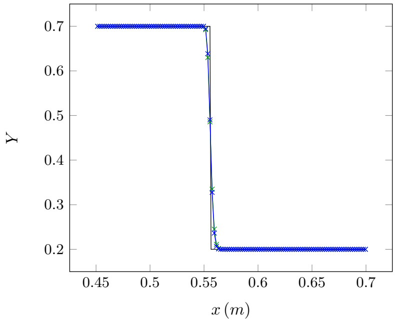


Velocity profile : high velocity area

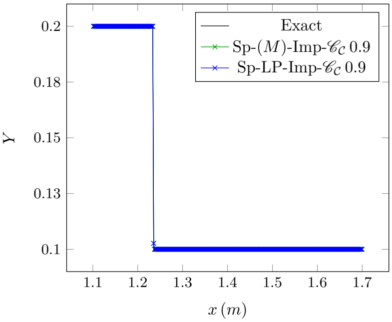


Velocity profile : low velocity area

A Double Riemann Problem with Stiff Thermodynamics



Y profile : high velocity area

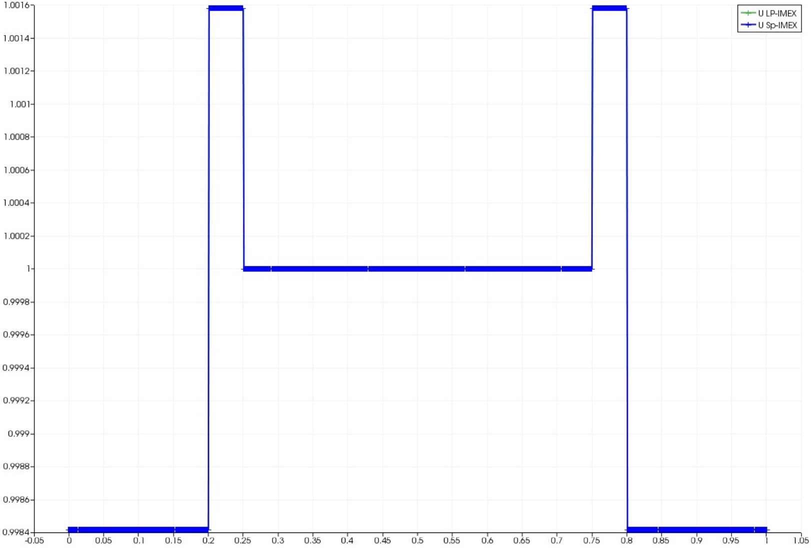


Y profile : low velocity area

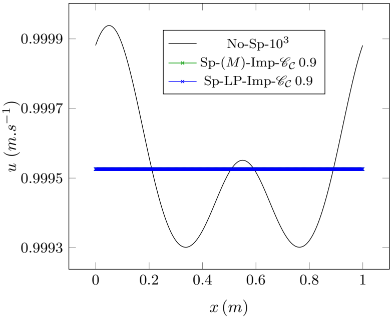
	(zone 1) : $x < 0.2$ or $x > 0.8$	(zone 2) : $x \in [0.2, 0.25]$ or $x \in [0.75, 0.8]$	(zone 3) : $x \in [0.25, 0.75]$
$\rho\,(kg.m^{-3})$	ρ^0	ρ^0	ρ^0
$u\,(m.s^{-1})$	$u_L^0 = u^0 \times \left(1 - M^0/2\right)$	$u_R^0 = u^0 \times \left(1 + M^0/2\right)$	$u_{\mathrm{m}}^0 = u^0$
$p\,(bar)$	p^0	p^0	p^0

Table: (Dimarco et al., 2017)'s Riemann Problem : initial conditions

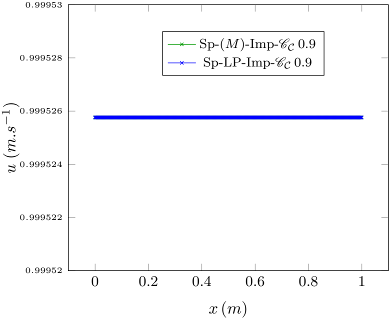
- $\rho^0 = 1\,kg.m^{-3}$, $p^0 = 1$ bar, $c^0 \equiv \sqrt{p^0/\rho^0}$
- $u^0 \equiv M^0 \times c^0$, M^0 input parameter : $M^0 = 3.2 \times 10^{-3} \Rightarrow u^0 \approx 1\,m.s^{-1}$
- Length of reference $L^0 = 1\,m$. Time of reference $t^0 = L^0/u^0$
- Physical time of simulation : $T_{\mathrm{end}} = 0.05 \times t^0$
- Ideal gas thermodynamics $\gamma = 7/5$, $\mathcal{S}(t) = 0$
- Periodic boundary conditions



Velocity, $M^0 = 3.2 \times 10^{-3}$



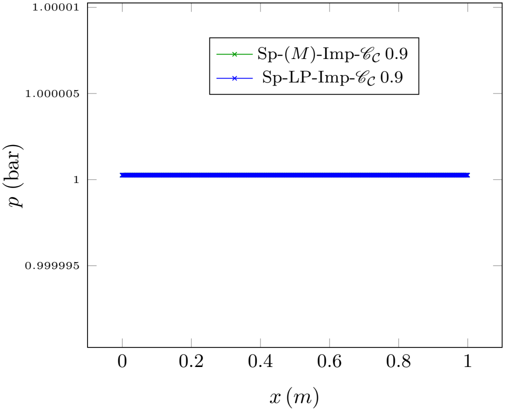
Velocity profile : $M^0 = 3.2 \times 10^{-3}$



Velocity profile (zoom) : $M^0 = 3.2 \times 10^{-3}$

- $u\,(T_{\text{end}})$ such that :

$$\int_\Omega \rho u(.,\, t = 0)\, d\Omega = \int_\Omega \rho u(.,\, t = T_{\text{end}})\, d\Omega$$



Pressure profile (zoom) : $M^0 = 3.2 \times 10^{-3}$

Conclusion and Perspectives

Main ideas :

- Construction of a self-adaptive IMEX scheme
- "self-adaptative" aspect due to $\mathscr{E}_0(t)$ $\Rightarrow$ automatically select the appropriate spatial flux discretization
- $\Delta t^n \leftrightarrow |u_i^n \pm \mathscr{E}_0^n\,(c_{\mathcal{C}})_i^n|$ $\Rightarrow$ automatically select the appropriate time-step

Perspectives :

- Amelioration of the shock detector $\mathcal{S}(t^n)$ in the ideal gas thermodynamics case
- Local formulation of the dynamic parameter : $\mathscr{E}_0(t) \rightarrow \mathscr{E}_0(x,\, t)$
- Perform a proper stability analysis (linearized case, periodic BC...)

Merci de votre attention !

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