# Convergence of the MAC scheme for incompressible flows


T. Gallouët[⋆], R. Herbin[⋆], J.-C. Latché[⋆⋆], K. Mallem[⋆⋆⋆]

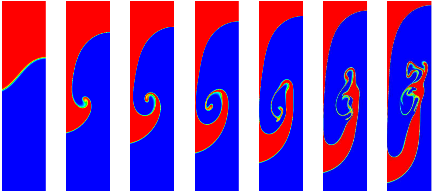
⋆ Aix-Marseille Université
⋆⋆ I.R.S.N. Cadarache
⋆ ⋆ ⋆ Université d'Annaba


Calanque de Cortiou
Marseille main sewer

# Incompressible Variable density flows

- Bounded domain $\Omega \times (0, T)$, $\Omega \subset \mathbb{R}^d$, $T > 0$.
- Non homogeneous fluid : density $\rho$ is not constant
- Incompressible fluid : $\mathrm{div}\, \bar{\boldsymbol{u}} = 0$,
- Incompressible variable density Navier-Stokes equations:

$$\left.\begin{array}{l} \partial_t \bar\rho + \mathrm{div}(\bar\rho \bar{\boldsymbol{u}}) = 0, \\ \\ \partial_t \bar\rho\, \bar{\boldsymbol{u}} + \mathrm{div}(\bar\rho\, \bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}) - \Delta \bar{\boldsymbol{u}} + \boldsymbol{\nabla} \bar{p} = 0, \\ \\ \mathrm{div}\, \bar{\boldsymbol{u}} = 0, \\ \\ \boldsymbol{u}|_{\partial\Omega} = 0, \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \rho|_{t=0} = \rho_0 \geq \rho_{\min} > 0, \end{array}\right| \qquad \text{(VDNS)}$$



# Variable density NS equations: *a priori* estimates

- maximum principle for $\rho$

$$\mathrm{div}\bar{u} = 0 \text{ and } \partial_t \bar{\rho} + \mathrm{div}(\bar{\rho}\bar{\boldsymbol{u}}) = 0 \ \rightsquigarrow \partial_t \bar{\rho} + \bar{\boldsymbol{u}} \cdot \nabla \bar{\rho} = 0 \text{ (transport equation)}$$

$$0 < \min_{\Omega} \rho_0 \leq \bar{\rho} \leq \max_{\Omega} \rho_0.$$

- kinetic energy balance:

Multiply momentum balance eq by $\bar{\boldsymbol{u}}$ and use mass balance (twice) $\rightsquigarrow$

$$\partial_t(\frac{1}{2}\bar{\rho}|\bar{\boldsymbol{u}}|^2) + \mathrm{div}(\frac{1}{2}\bar{\rho}|\bar{\boldsymbol{u}}|^2\boldsymbol{u}) - \Delta\bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{u}} + \boldsymbol{\nabla}\bar{p} \cdot \bar{\boldsymbol{u}} = 0.$$

- $\mathrm{L}^\infty((0,T);\mathrm{L}^2(\Omega)^d)$ and $\mathrm{L}^2((0,T);\mathrm{H}^1_0(\Omega)^d)$ bound:

$$\frac{1}{2}\int_\Omega \bar{\rho}(.,t)\,|\bar{\boldsymbol{u}}(.,t)|^2\,\mathrm{d}\boldsymbol{x} + \int_0^t \int_\Omega |\boldsymbol{\nabla}\bar{\boldsymbol{u}}|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \frac{1}{2}\int_\Omega \bar{\rho}_0\,|\bar{\boldsymbol{u}}_0|^2\,\mathrm{d}\boldsymbol{x}, \quad \forall t \in (0,T).$$

# Variable density NS equations: weak solutions

- **Weak Solutions:**

  $\rho_0 \in \mathrm{L}^\infty(\Omega)$ such that $\rho_0 > 0$ and $\boldsymbol{u}_0 \in \mathrm{L}^2(\Omega)^d$. A weak solution is a pair $(\rho, \boldsymbol{u})$ such that:

  - $\bar\rho \in \mathrm{L}^\infty((0,T)\times\Omega)$ and $\bar\rho > 0$ *a.e.* in $\Omega \times (0,T)$.
  - $\bar{\boldsymbol{u}} \in \mathrm{L}^\infty((0,T);\mathrm{L}^2(\Omega)^d) \cap \mathrm{L}^2((0,T);\mathrm{H}^1_0(\Omega)^d)$ et $\mathrm{div}\bar{\boldsymbol{u}} = 0$ *a.e.* in $\Omega \times (0,T)$.
  - For any $\phi$ in $\mathrm{C}^\infty_c(\Omega \times [0,T))$,

    $$-\int_0^T \int_\Omega (\bar\rho\partial_t\phi + \bar\rho\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\phi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \bar\rho_0\phi(.,0)\,\mathrm{d}\boldsymbol{x}.$$

  - for any $\boldsymbol{v}$ in $\mathrm{C}^\infty_c(\Omega\times[0,T))^d$ such that $\mathrm{div}\boldsymbol{v} = 0$,

    $$\int_0^T \int_\Omega \Big(-\bar\rho\bar{\boldsymbol{u}}\cdot\partial_t\boldsymbol{v} - (\bar\rho\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}}) : \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\nabla}\boldsymbol{v}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \bar\rho_0\bar{\boldsymbol{u}}_0\cdot\boldsymbol{v}(.,0)\,\mathrm{d}\boldsymbol{x}.$$

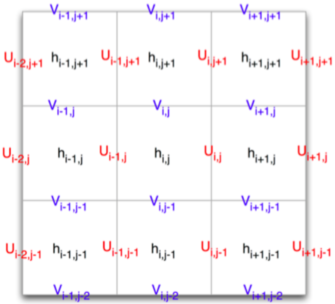
- Existence of a weak solution Simon '90.

- Discretization
-   - Liu Walkington *Discontinuous Galerkin* 2007
    - Latché Saleh *Rannacher Turek* 2016

# The standard Marker-And-Cell scheme

(Harlow Welsh 65)

- structured grids,
- pressure at the cell-center and normal velocity components at mid-edges,
- staggered meshes.




**Advantages and drawback**

+ minimal number of unknowns,
+ no pressure stabilization needed,
+ simplicity and robustness of the scheme,
– needs local refinement for complex geometries.

**Convergence analysis:**
Shin Strickwerda 96, 97, stability, inf-sup condidion
Nicolaides Wu '96, $(\omega, \psi)$ formulation
Blanc '05 irregular rectangles, finite volume approach

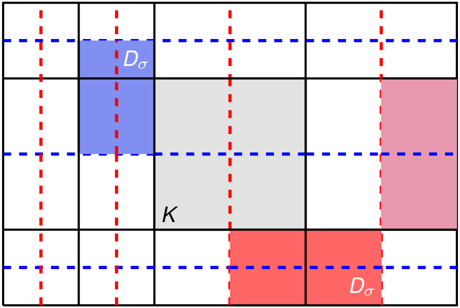
# The MAC grid (or Arakawa C)

Primal mesh (pressure) $\mathcal{T}$ :

- $K$ : Primal cell.
- $\mathcal{E}$ set of faces of $\mathcal{T}$.
- $\mathcal{E}^{(i)}$ ; $i = 1, \ldots, d$ : set of faces orthogonal to the $i$-th component of $\boldsymbol{e}_i$.

Dual mesh (velocity) :

- $D_\sigma$ dual cell associate to the face $\sigma \in \mathcal{E}^{(i)}$.




$$\rho = \sum_{K \in \mathcal{T}} \rho_K \chi_K, \qquad u^{(i)} = \sum_{\sigma \in \mathcal{E}^{(i)}} u_\sigma \chi_{D_\sigma}$$

# From the continuous to the discrete equations: time-implicit discretization

- Continuous equations

$$\bar\rho \in \mathrm{L}^\infty((0,T)\times\Omega),\ \bar{\boldsymbol{u}} \in \mathrm{L}^\infty((0,T);\mathrm{L}^2(\Omega)^d)\cap \mathrm{L}^2((0,T);\mathrm{H}^1_0(\Omega)^d),$$

$$\operatorname{div} \bar{\boldsymbol{u}} = 0,$$

$$\partial_t \bar\rho + \operatorname{div}(\bar\rho \bar{\boldsymbol{u}}) = 0,$$

$$\partial_t(\bar\rho)\ \bar{\boldsymbol{u}} + \operatorname{div}(\bar\rho\ \bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}) - \Delta \bar{\boldsymbol{u}} + \boldsymbol{\nabla} \bar p = 0,$$

- Discrete equations

$$\rho^{(n+1)}, p^{(n+1)} \in L_{\mathcal T},\ \boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal E},$$
$$\operatorname{div}_{\mathcal T} \boldsymbol{u}^{(n+1)} = 0,$$
$$\eth_t \rho_{\mathcal T}^{(n+1)} + \operatorname{div}_{\mathcal T}((\rho\boldsymbol{u})^{(n+1)}) = 0$$
$$\eth_t(\rho\boldsymbol{u})^{(n+1)} + \boldsymbol{c}_{\mathcal E}((\rho\boldsymbol{u})^{(n+1)})\boldsymbol{u}^{(n+1)} - \Delta_{\mathcal E}\boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla}_{\mathcal E}\ p^{(n+1)} = 0,$$

- Discrete time derivative:

$$\eth_t(\phi)^{(n+1)} = \frac{1}{\delta t}(\phi(\cdot,t_{n+1}) - \phi(\cdot,t_n)),\ \phi \in H_{\mathcal T} \text{ or } \phi \in \mathbf{H}_{\mathcal E}.$$

- Discrete operators:

$$\operatorname{div}_{\mathcal T}\boldsymbol{u},\quad \operatorname{div}_{\mathcal T}\rho\boldsymbol{u},\quad \boldsymbol{c}_{\mathcal E}((\rho\boldsymbol{u}))\boldsymbol{u},\quad -\Delta_{\mathcal E}\boldsymbol{u},\quad \boldsymbol{\nabla}_{\mathcal E}\ p\ldots$$

# The time-implicit MAC scheme

- Discrete equations

  $\rho$, $p$ piecewise constant on the primal mesh,

  $\boldsymbol{u}$ piecewise constant on the dual mesh

$$
\begin{aligned}
&\rho^{(n+1)}, p^{(n+1)} \in L_{\mathcal{T}}, \boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E}},\\
&\mathrm{div}_{\mathcal{T}} \boldsymbol{u}^{(n+1)} = 0,\\
&\eth_t \rho_{\mathcal{T}}^{(n+1)} + \mathrm{div}_{\mathcal{T}}((\rho \boldsymbol{u})^{(n+1)}) = 0\\
&\eth_t (\rho \boldsymbol{u})^{(n+1)} + \boldsymbol{c}_{\mathcal{E}}((\rho \boldsymbol{u})^{(n+1)}) \boldsymbol{u}^{(n+1)} - \Delta_{\mathcal{E}} \boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla}_{\mathcal{E}}\, p^{(n+1)} = 0,
\end{aligned}
$$

- Discrete operators $\mathrm{div}_{\mathcal{T}} \boldsymbol{u}$, $\mathrm{div}_{\mathcal{T}}(\rho \boldsymbol{u})$ :

  For $K \in \mathcal{T}$ :

$$
\mathrm{div}_K \boldsymbol{u}^{(n+1)} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma|\, u_{K,\sigma} = 0, \text{ with } u_{K,\sigma} = \pm u_\sigma,
$$

$$
\frac{1}{\delta t}(\rho_K^{(n+1)} - \rho_K^{(n)}) + \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{(n+1)} = 0,
$$

  with $F_{K,\sigma} = |\sigma|\, \rho_\sigma^{\mathrm{up}} u_{K,\sigma}$, $\rho_\sigma^{\mathrm{up}}$ : upwind approximation of $\rho$ on $\sigma \Longrightarrow \rho_K^{(n+1)} > 0$.

# The time-implicit MAC scheme

Discrete equations, $\rho$, $p$ piecewise constant on the primal mesh, and $\boldsymbol{u}$ piecewise constant on the dual mesh

$$
\begin{aligned}
&\rho^{(n+1)}, p^{(n+1)} \in L_{\mathcal{T}}, \boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E}},\\
&\mathrm{div}_{\mathcal{T}} \boldsymbol{u}^{(n+1)} = 0,\\
&\eth_t \rho_{\mathcal{T}}^{(n+1)} + \mathrm{div}_{\mathcal{T}}((\rho \boldsymbol{u})^{(n+1)}) = 0\\
&\eth_t (\rho \boldsymbol{u})^{(n+1)} + \boldsymbol{c}_{\mathcal{E}}((\rho \boldsymbol{u})^{(n+1)}) \boldsymbol{u}^{(n+1)} - \Delta_{\mathcal{E}} \boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla}_{\mathcal{E}}\, p^{(n+1)} = 0,
\end{aligned}
$$

For $\sigma \in \mathcal{E}$ :

$$
\frac{1}{\delta t}(\rho_\sigma^{(n+1)} \boldsymbol{u}_\sigma - \rho_\sigma^{(n)} \boldsymbol{u}_\sigma^{(n)}) + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^{(n+1)} \boldsymbol{u}_\epsilon^{(n+1)} - (\Delta \boldsymbol{u})_\sigma^{(n+1)} + (\boldsymbol{\nabla} p)_\sigma^{(n+1)} = 0, \sigma \in \mathcal{E}_{\mathrm{int}}.
$$

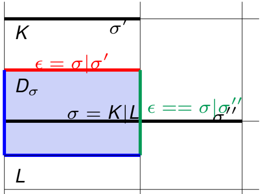
with:

$\rho_\sigma$ and $F_{\sigma,\epsilon}$ chosen later.

centered approximation for $\boldsymbol{u}_\epsilon$,

$-(\Delta \boldsymbol{u})_\sigma =$ two point flux FV approximation on the velocity meshes,

$$
\boldsymbol{\nabla}_{\mathcal{E}}\, p^{(n+1)} = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} (\boldsymbol{\nabla} p)_\sigma, \qquad (\boldsymbol{\nabla} p)_\sigma = \frac{|\sigma|}{|D_\sigma|}(p_L - p_K)\boldsymbol{n}_{K,\sigma},
$$



# Discrete duality, coercivity, inf-sup property

- *Discrete divergence-gradient duality*

$$\int_\Omega q \,\mathrm{div}_\mathcal{T} \boldsymbol{v} + \int_\Omega \nabla_\mathcal{E} q \cdot \boldsymbol{v} = 0.$$

- *Coercivity*
  - $\|\boldsymbol{u}\|_{1,\mathcal{E}} = \sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}^{(i)}} |D_\sigma| \Delta_{\mathcal{E}^{(i)}} u^{(i)} u_\sigma = \|\nabla_\mathcal{E} \boldsymbol{u}\|_{L^2(\Omega)^d} \geq C \|\boldsymbol{u}\|_{L^2(\Omega)^d}$

- *Inf-sup property*
  - $\forall q \in L_\mathcal{T}, \exists \boldsymbol{v} \in H_\mathcal{E}; q = \mathrm{div}_\mathcal{T} \boldsymbol{v}$ and $\|\boldsymbol{v}\|_\mathcal{E} \leq C \|p\|_{L^2(\Omega)^d}$

(discrete version of Necas' Lemma, Strickwerda '90, Blanc '05)

# Discrete kinetic energy inequality, choice of $\rho_\sigma$ and

- Discrete equations
  If a mass balance holds on the dual cells:

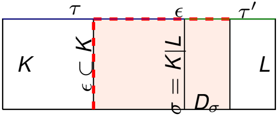
$$\frac{|D_\sigma|}{2\delta t}(\rho_\sigma - \rho_\sigma^*) + \sum_{\sigma \in \mathcal{E}} F_{\sigma,\epsilon} = 0,$$

  and $\rho_\sigma > 0$, then

$$\frac{1}{2\delta t}\left(\rho_\sigma u_\sigma^2 - \rho_\sigma^*(u_\sigma^*)^2\right) + \frac{1}{2|D_\sigma|}\sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)\\ \epsilon=\sigma|\sigma'}} F_{\sigma,\epsilon} u_\sigma u_{\sigma'} - (\Delta u)_\sigma u_\sigma + (\eth p)_\sigma u_\sigma \leq 0$$

- Proof
- Choice of $\rho_\sigma$ and $F_{\sigma,\epsilon}$

Herbin, Kheriji, Latché 2013




$$|D_\sigma|\rho_\sigma = |D_{K,\sigma}|\rho_K + |D_{L,\sigma}|\rho_L,$$

$$F_{\sigma,\epsilon} = \left| \begin{array}{l} \frac{1}{2}(F_{K,\tau} + F_{K,\tau'}) \text{ if } \epsilon \perp \sigma, \epsilon \subset \tau \cup \tau' \\ \\ \frac{1}{2}(-F_{K,\sigma} + F_{K,\sigma'}) \text{ if } \epsilon \subset K = [\sigma, \sigma'] \end{array} \right.$$

# Convergence analysis

**Theorem :** $(\mathcal{T}_m, \mathcal{E}_m)$, $\delta t_m$ sequence of meshes and time steps $h_m \to 0$, $\delta t_m \to 0$ + suitable assumptions on the mesh,

There exists $\rho_m$, $\boldsymbol{u}_m$ solution to the scheme, and, up to a subsequence

$\rho_m \to \bar{\rho}$ in $L^p(\Omega \times (0,T))$, $p \in [1, +\infty[$,

$\boldsymbol{u}_m \to \bar{\boldsymbol{u}}$ in $L^2(\Omega \times (0,T))$

$(\bar{\rho}, \bar{\boldsymbol{u}})$ is a weak solution to (VDNS)

**Sketch of proof**

- *A priori estimates* on the approximate solutions $\rho_m, \boldsymbol{u}_m \rightsquigarrow$ existence of a solution to the scheme.
- *Compactness* thanks to discrete functional analysis tools:
  Up to a subsequence, $\rho_m \to \bar{\rho}$, $\boldsymbol{u}_m \to \bar{\boldsymbol{u}}$ for some norms.
- *Passage to the limit* in the weak form of the scheme, $\rightsquigarrow$ $(\bar{\rho}, \bar{\boldsymbol{u}})$ is a weak solution to (VDNS).

# Convergence analysis: estimates and existence

- 1 - *A priori* estimates.
  - Maximum principle (upwind discretization of the mass balance):
    $$\min_{\boldsymbol{x}\in\Omega} \rho_0(\boldsymbol{x}) \leq \rho \leq \max_{\boldsymbol{x}\in\Omega} \rho_0(\boldsymbol{x}).$$
  - Kinetic energy identity + grad-div duality:
    $$\|\boldsymbol{u}\|_{\mathrm{L}^\infty((0,T);\mathrm{L}^2(\Omega)^d)} + \|\boldsymbol{u}\|_{\mathrm{L}^2((0,T);H^1_{\mathcal{E},0})} \leq C$$
  - Estimate of the momentum balance convection form:
    $$\begin{aligned} |\boldsymbol{C}_{\mathcal{E}}(\rho\boldsymbol{u})\boldsymbol{v}\cdot\boldsymbol{w}| &\leq C_{\eta_{\mathcal{T}}}\, \|\rho\|_{L^\infty(\Omega)}\, \|\boldsymbol{u}\|_{L^4(\Omega)^d}\, \|\boldsymbol{v}\|_{L^4(\Omega)^d}\, \|\boldsymbol{w}\|_{1,\mathcal{E},0} \\ &\leq C_{\eta_{\mathcal{T}}}\, \|\rho\|_{L^\infty(\Omega)}\, \|\boldsymbol{u}\|_{1,\mathcal{E},0}\, \|\boldsymbol{v}\|_{1,\mathcal{E},0}\, \|\boldsymbol{w}\|_{1,\mathcal{E},0}. \end{aligned}$$
- 2 - Existence of a solution thanks to a topological degree argument.

# “Easy case” : homogeneous fluid, constant density

- Estimate on $u_m$ in $L^2(0,T;H^1_{\mathcal{E},0})$
- Estimate on $\eth_t u_m$

$$\|\eth_t\, \boldsymbol{u}_m\|_{L^1(0,T;\boldsymbol{E}'_{\mathcal{E}})} = \sum_{n=0}^{N-1} \delta t \|\eth_t \boldsymbol{u}\|_{\boldsymbol{E}'_{\mathcal{E}}} \leq C \text{ where } \|\boldsymbol{v}\|_{\boldsymbol{E}'_{\mathcal{E}}} = \max_{\substack{\boldsymbol{\varphi}\in \boldsymbol{E}_{\mathcal{E}} \\ \|\boldsymbol{\varphi}\|_{1,\mathcal{E},0}\leq 1}} \left| \int_\Omega \boldsymbol{v}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} \right| ;$$

- Discrete Aubin-Simon theorem $\Longrightarrow$ compactness on $\boldsymbol{u}_m$ in $L^2((0,T),L^2(\Omega))$

**Discrete Aubin-Simon theorem** $1 \leq p < +\infty$, $B$ Banach space, $B_m \subset B$, dim $B_m < +\infty$, $\|\cdot\|_{X_m}$ and $\|\cdot\|_{Y_m}$ norms on $B_m$ s.t. : if $\|w_m\|_{X_m}$ is bounded then:

(i) $\exists w \in B$; $w_m \to w$ in $B$ up to a subsequence,
(ii) If $w_m \to w$ in $B$, and $\|w_m\|_{Y_m} \to 0$ then $w = 0$.

Let $T > 0$ and $(u_m)_{m\in\mathbb{N}}$ be a sequence of $L^p((0,T),B)$, piecewise constant on the time intervals, such that

1. $(\int_0^T \|u_m\|^p_{X_m} dt)_{m\in\mathbb{N}}$ is bounded,
2. $(\int_0^T \|\eth u_m\|_{Y_m} dt)_{m\in\mathbb{N}}$ is bounded.

Then there exists $u \in L^p((0,T),B)$ such that, up to a subsequence, $u_m \to u$ in $L^p((0,T),B)$.

- $u_m \to \bar{u} \in L^2((0,T),L^2(\Omega))$ + $(u_m)_{m\in\mathbb{N}}$ bounded in $L^2((0,T),H^1_{\mathcal{E},0})$
$\rightsquigarrow \bar{u} \in L^2((0,T),H^1_0(\Omega))$.

# Variable density, estimate on the translates

- No estimate on $\eth_t u_m$
- Estimate on the time translates (continuous case: Boyer Fabrie '13 chapter 2);

$$\int_0^{T-\tau}\int_\Omega |\boldsymbol{u}_m(\boldsymbol{x}, t+\tau) - \boldsymbol{u}_m(\boldsymbol{x}, t)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le C_{\eta_{\mathcal{T}},T}\frac{\rho_{\max}}{\rho_{\min}}(\|\boldsymbol{u}_m\|^3_{L^2(\boldsymbol{H}_{\mathcal{E},0})} + 1)\sqrt{\tau+\delta t}$$

  Kolmogorov compactness theorem $\rightsquigarrow$ compactness of $\boldsymbol{u}_m$ in $L^2((0,T), L^2(\Omega))$.

- $\sqrt{\tau+\delta t}$ : No estimate on time derivative.
- Will not generalize to the compressible case because of $\rho_{\min}$

# Variable density, convergence

- Convergence proof, sketch :
  - $\boldsymbol{u}_m \to \bar{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega)^d)$, $\rho_m \to \bar{\rho}$ weakly in $L^2(0,T;L^2(\Omega))$
    $\rightsquigarrow \displaystyle\int_0^T \int_\Omega \rho_m\, \boldsymbol{u}_m\, \boldsymbol{\psi} \to \int_0^T \int_\Omega \bar{\rho}\, \bar{\boldsymbol{u}}\, \boldsymbol{\psi}$ for any $\boldsymbol{\psi} \in L^\infty(\Omega \times (0,T))^d \qquad (*)$
  - In particular for $\boldsymbol{\psi} = \nabla_m \varphi$,
$$\int_0^T \int_\Omega \mathrm{div}_m(\rho_m\, \boldsymbol{u}_m)\, \varphi\, \mathrm{d}\boldsymbol{x} dt = \int_0^T \int_\Omega \rho_m\, \boldsymbol{u}_m\, \nabla_m \varphi\, \mathrm{d}\boldsymbol{x} dt \to \int_\Omega (\bar{\rho}\, \bar{\boldsymbol{u}})\, \nabla \varphi\, \mathrm{d}\boldsymbol{x} dt$$
    (thanks to weak BV estimate)
  - $\eth_t \rho_m \to \partial_t \rho$, $\eth_t(\rho_m\, \boldsymbol{u}_m) \to \partial_t(\rho \boldsymbol{u})$, $\mathrm{div}\boldsymbol{u}_m \to \mathrm{div}\bar{\boldsymbol{u}} \Rightarrow \mathrm{div}_m \bar{\boldsymbol{u}} = 0$ (linear operators)
  - $\rho_m \boldsymbol{u}_m$ bounded in $L^2(0,T;L^2(\Omega)^d) \Longrightarrow \rho_m \boldsymbol{u}_m$ converges weakly in $L^2(0,T;L^2(\Omega)^d)$. Hence by $(*)$, $\rho_m \boldsymbol{u}_m \to \bar{\rho}\bar{\boldsymbol{u}}$ weakly in $L^2(0,T;L^2(\Omega)^d)$ and since $\boldsymbol{u}_m \to \boldsymbol{u}$ in $L^2(0,T;L^2(\Omega)^d)$
$$\int_\Omega \rho_m \boldsymbol{u}_m \otimes \boldsymbol{u}_m \nabla \boldsymbol{\varphi} \to \int_\Omega \bar{\rho} \bar{\boldsymbol{u}} \otimes \boldsymbol{u} \nabla \boldsymbol{\varphi}$$

# Variable density case, convergence, alternate proof

- Compactness of $\rho_m$ in $L^2(H^{-1})$ thanks to Aubin-Simon with $B = H^{-1}(\Omega)$ $X = L^2(\Omega)$ $Y = W^{-1,1}(\Omega)$
  - $X = L^2(\Omega)$ compactly embedded in $B = H^{-1}(\Omega)$
  - $\partial_t \rho_m$ bounded in $L^1(0, T, W^{-1,1}(\Omega))$
  - $\rho_m$ bounded in $L^2(0, T, L^2(\Omega)) \rightsquigarrow \rho_m$ compact in $L^2(0, T, H^{-1}(\Omega))$
- $\boldsymbol{u}_m$ bounded in $L^2(0, T, H^1_0(\Omega))$ hence (weak strong product)
- $\rho_m \boldsymbol{u}_m \to \bar{\rho}\bar{u}$ weakly in $L^1$.
- Same ideas for the convergence of $\rho_m \boldsymbol{u}_m \otimes \boldsymbol{u}_m$ ...

  Thierry Gallouët, 2017

# Recent and on going work

- Locally refined grids : with Chénier, Eymard, Gallouët for modified MAC,
  Latché, Piar, Saleh for RT grids.
- Stability and weak consistency for the full NS equations (with D. Grapsas, W. Kheriji, J.-C. Latché)
- Convergence for steady-state barotropic NS ($\gamma \geq 3$) (with J.-C. Latché, T. Gallouët and D. Maltese)
- Error estimates for barotropic NS, adaptation of strong weak uniqueness (with D. Maltese, and A. Novotny).
- Stability and weak consistency for the Euler equations with higher order schemes and entropy consistency (with T. Gallouët, J.-C. Latché, N. Therme)
- Low Mach limit for barotropic NS (with J.-C. Latché, K. Saleh).
- Convergence of the fully discrete pressure correction scheme. TODO
- Convergence for the time-dependent barotropic compressible Navier-Stokes. TODO