Fast waves and incompressible models

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Fast waves in hyperbolic problem

$$\partial_t \boldsymbol{W}^\varepsilon + \sum_j A_j(\boldsymbol{W}^\varepsilon, \varepsilon)\partial_{x_j} \boldsymbol{W}^\varepsilon = 0$$

Smooth solutions are non-linear waves moving with velocity $\lambda_j(\boldsymbol{W})$.

Two time scales pb :

$$\exists \; 2 \text{ sets } F, S \quad s.t \quad \lambda_k >> \lambda_j, k \in F, j \in S$$

- Usually associated to the existence of a small parameter
- The "limit" system $\varepsilon \to 0$ is no more hyperbolic : singular limit
- Usually associated to an stationary incompressible constraint :

$$\exists \;\; \mathbb{L} \;\; s.t \;\; \mathbb{L}(\boldsymbol{W}^0) = 0$$

why "random" interaction fast waves $\Rightarrow$ incompressible constraint ?
why non-linear interactions of fast waves do not modify the "slow" dynamics ?

Singular limit of hyperbolic PDEs

Let $\boldsymbol{W} \in \boldsymbol{R}^N$ solution of the hyperbolic system with a large operator

$$\begin{cases} \partial_t \boldsymbol{W} + \sum_j [A_j(\boldsymbol{W}, \varepsilon) + \frac{1}{\varepsilon} C_j] \partial_{x_j} \boldsymbol{W} = 0 \\ \boldsymbol{W}(0, \boldsymbol{x}, \varepsilon) = \boldsymbol{W}_0(\boldsymbol{x}, \varepsilon) \end{cases}$$

What is the behavior of the solutions when $\varepsilon \to 0$?

Let $\boldsymbol{n}$ be a arbitrary direction, then some eigenvalues of $\sum_j n_j(A_j + \frac{1}{\varepsilon} C_j)$ are of the form $a_k + \frac{1}{\varepsilon} c_k \to \pm\infty$ while the others (kernel of $\sum_j n_j C_j$) are simply a_k

What is the behavior of the solutions when Slow and Fast waves co-exist ?

Singular limit of hyperbolic PDEs : Slow limit

$$\partial_t \boldsymbol{W} + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W} + \frac{1}{\varepsilon}\sum_j C_j \partial_{x_j} \boldsymbol{W} = 0$$

$\mathbb{L}\boldsymbol{W} = \sum_j C_j \partial_{x_j} \boldsymbol{W}$ has to be $\mathcal{O}(\varepsilon)$

Look for the solution as $\boldsymbol{W} = \boldsymbol{W}_0 + \varepsilon \boldsymbol{W}_1$ with $\mathbb{L}\boldsymbol{W}_0 = 0$, obtain :

$$\partial_t \boldsymbol{W}_0 + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W}_0 + \mathbb{L}\boldsymbol{W}_1 = \mathcal{O}(\varepsilon)$$

and the solutions converge to $\boldsymbol{W}_0$ defined by :

$$\begin{cases} \mathbb{L}\boldsymbol{W}_0 = 0 \\ \partial_t \boldsymbol{W}_0 + \mathbb{P}\sum_j A_j(\boldsymbol{W}_0, 0)\partial_{x_j} \boldsymbol{W} = 0 \end{cases}$$

$\mathbb{P}$ projection on the kernel of $\mathbb{L}$

But the system has also a fast limit

$$\partial_t \boldsymbol{W} + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W} + \frac{1}{\varepsilon}\sum_j C_j \partial_{x_j} \boldsymbol{W} = 0$$

Let us do the simple change of variable : $t = \varepsilon\tau$:

$$\frac{1}{\varepsilon}\partial_\tau \boldsymbol{W} + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W} + \frac{1}{\varepsilon}\sum_j C_j \partial_{x_j} \boldsymbol{W} = 0$$

and when $\varepsilon \to 0$ the limiting form becomes :

$$\partial_\tau \boldsymbol{W} + \sum_j C_j \partial_{x_j} \boldsymbol{W} = 0$$

Solution are fast waves moving at velocity $\dfrac{1}{\varepsilon}$

First example : Low Mach number flows

Superposition incompressible + acoustics

Compressible Euler equations :

$$\partial_t \rho + \text{ div } \rho \boldsymbol{u} + = 0 \qquad \rho = \boxed{\rho_*} \quad \rho$$

$$\partial_t \rho \boldsymbol{u} + \text{ div } \rho \boldsymbol{u} \otimes \boldsymbol{u} + \nabla p = 0 \qquad \boldsymbol{u} = \boxed{u_*} \quad \boldsymbol{u}$$

$$\partial_t p + \boldsymbol{u}.\nabla p + \rho a^2 \text{ div } \boldsymbol{u} + = 0 \qquad p = \boxed{\rho_* (a_*)^2} \quad p$$

$$x_i = \boxed{L_*} \; x_i; \quad t = \boxed{L_* / u_*} \; t \qquad \boxed{\varepsilon = u_* / a_*}$$

$$\boxed{\begin{aligned} &\partial_t \rho + \text{ div } \rho \boldsymbol{u} = 0 \\ &\partial_t \rho \boldsymbol{u} + \text{ div } \rho \boldsymbol{u} \otimes \boldsymbol{u} + \frac{1}{\varepsilon^2} \nabla p = 0 \\ &\partial_t p + \boldsymbol{u}.\nabla p + \rho a^2 \text{ div } \boldsymbol{u} = 0 \end{aligned}}$$

First example : Low Mach number flows

Superposition incompressible + acoustics

The incompressible limit :

$$f = f_0 + M_* f_1 + M_*^2 f_2$$

- $\mathcal{O}(1/M_*^2)$: $\nabla p_0 = 0$
  - if$\partial_t p_0 = 0 \rightarrow$ div $\boldsymbol{u}_0 = 0$
  - if$D\rho_0/Dt = 0 \rightarrow \rho_0 =$ constant
- $\mathcal{O}(1/M_*)$ same analysis
- $\mathcal{O}(1)$ $\rho_0 D\boldsymbol{u}_0/Dt + \nabla p_2 = 0$

Incompressible Euler equations

$$\boxed{\begin{array}{l} \rho D\boldsymbol{u}/Dt + \nabla p = 0 \\ \text{div } \boldsymbol{u} = 0 \end{array}}$$

First example : Low Mach number flows

Superposition incompressible + acoustics

Incompressible limit is not the unique low Mach limit of compressible eqs

- hidden assumption in incompressible asymptotic analysis
- time scale $t_* = L_*/u_*$: large time scale
- choose instead $t_* = L_*/a_*$: short time scale

$$\text{scaling becomes} \quad \begin{cases} \dfrac{1}{\varepsilon}\partial_t \rho + \text{ div } \rho \boldsymbol{u} + = 0 \\[2ex] \dfrac{1}{\varepsilon}\partial_t \rho \boldsymbol{u} + \text{ div } \rho \boldsymbol{u} \otimes \boldsymbol{u} + \dfrac{1}{M_*^2} \nabla p = 0 \\[2ex] \dfrac{1}{\varepsilon}\partial_t p + \boldsymbol{u}.\nabla p + \rho a^2 \text{ div } \boldsymbol{u} = 0 \end{cases}$$

First example : Low Mach number flows

Superposition incompressible + acoustics

Asymptotic analysis of the acoustic limit

$$f = f_0 + M_* f_1 + M_*^2 f_2$$

- $\mathcal{O}(1/M_*^2)$: $\quad \nabla p_0 = 0$
- $\mathcal{O}(1/M_*)$
  - $\partial_t p_0 = \partial_t p_0 = 0$
  - $\rho_0 \partial_t \boldsymbol{u}_0 + \nabla p_1 = 0$
- $\mathcal{O}(1)$: $\quad \partial_t p_1 + \rho_0 a_0^2 \nabla . \boldsymbol{u}_0 = 0$

Linear Acoustic equations

$$\boxed{\begin{array}{l} \rho_0 \partial_t \boldsymbol{u} + \nabla p = 0 \\[2ex] \partial_t p + \rho_0 a_0^2 \text{ div } \boldsymbol{u} = 0 \end{array}}$$

Incompressible + Acoustic superposition

- Provisional conclusion General solution = Slow (incompressible) + fast (Acoustic) component
- Does acoustic-acoustic interactions are able to modify the dynamics of the incompressible component ?

Acoustic stirring

Not only a jet can generate sound but also sound can generate a jet!

S. J., Lighthill, Acoustic streaming, J. Sound Vibr. 61, pp. 391418 (1978)

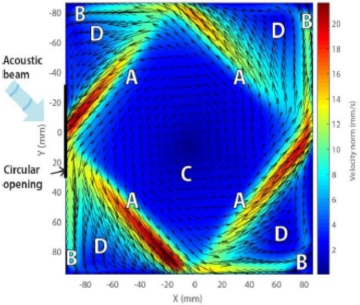
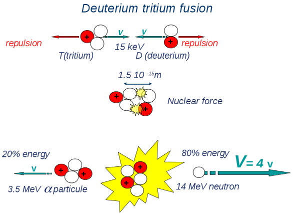
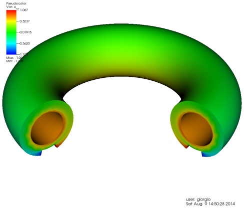


Figure: Reproduced from : V. Botton, , T. Cambonie, B. Moudjed, S. Miralles, D. Henry and H. Ben Hadid : How to drive a square flow in a liquid: acoustic stirring, 7th International Conference on Computational Methods for Coupled Problems in Science and Enginee[illegible] June 2017

2nd example : reduced MHD in nuclear fusion

Goal : controlled nuclear fusion
"Lawson" criterion : $n\tau_E T > 5.10^{21} m^{-3}\ s\ keV$





Tokamaks : Toroidal chamber where a very hot plasma ($150 M\ ^\circ K$) is confined thanks to very large magnetic field (200 K x earth magnetic field)

THE (ideal) MHD MODEL

Hydrodynamics :

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\frac{\partial}{\partial t}\rho \boldsymbol{u} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = F_L$$

$$\frac{\partial}{\partial t}p + \boldsymbol{u} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{u} = 0$$

$+$ Maxwell (Maxwell-Ampère) equations :

$$\frac{\partial}{\partial t}\boldsymbol{B} + \nabla \times \boldsymbol{E} = 0$$

$$\cancel{\frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E}} + \nabla \times \boldsymbol{B} = \boldsymbol{J}$$

systems coupled by Ohm's law $E + \boldsymbol{u} \times \boldsymbol{B} = 0$
and the def of the Lorentz force $F_L = \nabla \times \boldsymbol{B} \times \boldsymbol{B}$

THE MHD MODEL

First-order Hyperbolic system intensively studied from a mathematical and numerical view point

1. Nice properties :
   - existence of a conservative form, existence of an entropy
   - symmetry form
   - hyperbolic
   - eigensystem with explicit analytic expression
2. Not so nice :
   - not strictly hyperbolic
   - some fields are neither gnl nor ld
   - existence of the involution $\nabla \cdot \boldsymbol{B} = 0$

MHD waves

Hyperbolic system with 3 different types of waves (+ material or entropy waves) If $\boldsymbol{n}$ is the direction of propagation of the wave

- Fast Magnetosonic waves : $\lambda_F = \boldsymbol{u}.\boldsymbol{n} \pm C_F$
  $C_F^2 = \frac{1}{2}(V_t^2 + v_A^2 + \sqrt{(V_t^2 + v_A^2)^2 - 4V_t^2 C_A^2})$
- Alfvén waves : $\lambda_F = \boldsymbol{u}.\boldsymbol{n} \pm C_A \;\; C_A^2 = (\boldsymbol{B}.\boldsymbol{n})^2/\rho$
- Slow Magnetosonic waves : $\lambda_S = \boldsymbol{u}.\boldsymbol{n} \pm C_S$
  $C_S^2 = \frac{1}{2}(V_t^2 + v_A^2 - \sqrt{(V_t^2 + v_A^2)^2 - 4V_t^2 C_A^2})$

$v_A^2 = |\boldsymbol{B}|^2/\rho \qquad v_A$: Alfvén speed
$V_t^2 = \gamma p/\rho \qquad V_t$: acoustic speed

Transverse MHD waves

propagation speed depends on the direction w r to the magnetic field.

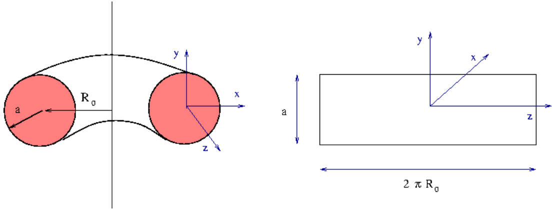
If $\boldsymbol{n} \cdot \boldsymbol{B} = 0$ (transverse waves) :

- Alfvén waves : $\lambda_F = 0$
- Slow Magnetosonic waves : $\lambda_S = 0$
- Fast Magnetosonic waves : $\lambda_F = \pm C_F$ with $C_F^2 = V_t^2 + v_A^2$

only the Fast Magnetosonic waves survive !

Limit of the MHD for small aspect ratio tokamaks

1. Tokamak geometry



2. Large dominant toroidal magnetic field $\boldsymbol{B}_\perp / \boldsymbol{B}_z = \varepsilon << 1$
3. equivalent to $\varepsilon = a/R_0$ is small
4. small parameter is here a geometrical parameter

Scaled full MHD equations

$$\frac{\partial}{\partial \tau}\boldsymbol{v}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp + \nabla_\perp(\mathcal{B}^2/2 + q) - \partial_z \mathcal{B}_\perp + \boxed{\frac{1}{\varepsilon}\nabla_\perp \mathcal{B}_z} = \mathcal{O}(\varepsilon) \qquad (3.2)$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_z + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_z + \mathcal{B}_z \nabla_\perp \cdot \boldsymbol{v}_\perp + \boxed{\frac{1}{\varepsilon}\nabla_\perp \cdot \boldsymbol{v}_\perp} = \mathcal{O}(\varepsilon) \qquad (3.1)$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp + \mathcal{B}_\perp \nabla_\perp \cdot \boldsymbol{v}_\perp - \partial_z \boldsymbol{v}_\perp = \mathcal{O}(\varepsilon) \qquad (3.3)$$

$$\frac{1}{\gamma p}\left(\frac{\partial}{\partial \tau} p + \boldsymbol{v}_\perp \cdot \nabla_\perp p\right) + \nabla_\perp . \boldsymbol{u} = 0$$

Indeed of the form :

$$\partial_t \boldsymbol{W} + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W} + \frac{1}{\varepsilon}\mathbb{L}\boldsymbol{W} = 0 \qquad \text{with} \qquad \mathbb{L}\boldsymbol{W} = \begin{pmatrix} 0 & \nabla_\perp & 0 & 0 \\ \nabla_\perp \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{W}$$

Reduced MHD model

Plug these assumptions into the full compressible MHD system and obtain (after some calculus) :

$$\frac{\partial}{\partial \tau}\boldsymbol{v}_\perp + \boxed{(\boldsymbol{v}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp + \nabla_\perp \pi} \quad - \boxed{\partial_z \mathcal{B}_\perp}$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_\perp + \boxed{(\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp} \quad - \boxed{\partial_z \boldsymbol{v}_\perp} = 0$$

$$\boxed{\nabla_\perp \cdot \boldsymbol{v}_\perp = \nabla_\perp \cdot \mathcal{B}_\perp = 0}$$

In the presence of a large dominant magnetic field, the dynamic can be described by

- 2D incompressible MHD in the transverse direction and
- Alfvén waves propagating in the direction of the dominant magnetic field.

Reduced MHD and fast transverse magnetosonic wave

- Fast transverse magnetosonic waves are absent from reduced MHD
- incompressible 2D model in the transverse direction !
- Aerodynamics,
  - small parameter $\varepsilon =$ Mach number
  - Acoutics vs incompressible
- MHD
  - small parameter $\varepsilon =$ Tokamak aspect ratio
  - Fast transverse magnetosonic waves vs reduced MHD

Singular limit of hyperbolic PDEs

Let $\boldsymbol{W} \in \boldsymbol{R}^N$ solution of the hyperbolic system with a large operator

$$\begin{cases} \partial_t \boldsymbol{W} + \sum_j [A_j(\boldsymbol{W}, \varepsilon) + \frac{1}{\varepsilon} C_j] \partial_{x_j} \boldsymbol{W} = 0 \\ \boldsymbol{W}(0, \boldsymbol{x}, \varepsilon) = \boldsymbol{W}_0(\boldsymbol{x}, \varepsilon) \end{cases}$$

What is the behavior of the solutions when $\varepsilon \to 0$?

How Slow and Fast waves co-exist ?
Why do we think that we can split the fast and slow phenomena ?

An Explicit linear example I

Consider the linear system

$$
\begin{aligned}
\frac{\partial r}{\partial t} \quad & + \boldsymbol{a}.\nabla r \quad && + \frac{1}{\varepsilon} div \boldsymbol{u} = 0 \\
\frac{\partial \boldsymbol{u}}{\partial t} \quad & + \boldsymbol{a}.\nabla \boldsymbol{u} \quad && + \frac{1}{\varepsilon} \nabla r = 0
\end{aligned}
$$

Warm-up : Explicit linear example II

Compact form :

$$\partial_t \boldsymbol{v} + \ \mathbb{H}\boldsymbol{v} + \frac{1}{\varepsilon} \ \mathbb{L}\boldsymbol{v} = 0$$

Notations :

$$\boldsymbol{v} = \begin{pmatrix} r \\ \boldsymbol{u} \end{pmatrix} \quad \mathbb{L}\boldsymbol{v} = \begin{pmatrix} \nabla \cdot \boldsymbol{u} \\ \nabla r \end{pmatrix}$$

$\mathbb{H}\boldsymbol{v} = \boldsymbol{a}.\nabla \boldsymbol{v}$ is a constant velocity linear advection operator

In Fourier space

$$\frac{\partial \hat{\boldsymbol{v}}(\boldsymbol{k})}{\partial t} + i[\hat{\mathbb{H}}(\boldsymbol{k}) + \frac{1}{\varepsilon}\hat{\mathbb{L}}\ (\boldsymbol{k})]\hat{\boldsymbol{v}}(\boldsymbol{k}) = 0 \quad \text{for} \quad \boldsymbol{k} \in Z^2 \tag{5}$$

where the matrix $\hat{\mathbb{H}}(\boldsymbol{k}) + 1/\varepsilon\hat{\mathbb{L}}(\boldsymbol{k})$ is equal to :

$$\begin{pmatrix} \boldsymbol{a}.\boldsymbol{k} & k_1/\varepsilon & k_2/\varepsilon \\ k_1/\varepsilon & \boldsymbol{a}.\boldsymbol{k} & 0 \\ k_2/\varepsilon & 0 & \boldsymbol{a}.\boldsymbol{k} \end{pmatrix} \tag{6}$$

This matrix is diagonalizable, its eigenvectors are :

$$s_1(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -k_1/\mid \boldsymbol{k} \mid \\ -k_2/\mid \boldsymbol{k} \mid \end{pmatrix}, \quad s_2(\boldsymbol{k}) = \frac{1}{\mid \boldsymbol{k} \mid} \begin{pmatrix} 0 \\ -k_2 \\ k_1 \end{pmatrix}$$
$$, \quad s_3(\boldsymbol{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ k_1/\mid \boldsymbol{k} \mid \\ k_2/\mid \boldsymbol{k} \mid \end{pmatrix} \tag{7}$$

with associated eigenvalues $\lambda_1 = \boldsymbol{a}.\boldsymbol{k} - \dfrac{\mid \boldsymbol{k} \mid}{\varepsilon}$, $\lambda_2 = \boldsymbol{a}.\boldsymbol{k}$ and $\lambda_3 = \boldsymbol{a}.\boldsymbol{k} + \dfrac{\mid \boldsymbol{k} \mid}{\varepsilon}$.

Note : $\boxed{\hat{\mathbb{L}} s_2(\boldsymbol{k}) = 0}$;in physical space $s_2(\boldsymbol{k})$ corresponds to constant density ($\nabla r = 0$) and div free vectors ($\nabla \,.\, \boldsymbol{u} = 0$)

Explicit linear example III

$$\hat{\boldsymbol{v}}(\boldsymbol{k},t) =$$

$$\left\{ \begin{array}{l} \dfrac{1}{\sqrt{2}}(\hat{r}(\boldsymbol{k},0) - \dfrac{k_1}{\mid \boldsymbol{k} \mid}\hat{u}(\boldsymbol{k},0) - \dfrac{k_2}{\mid \boldsymbol{k} \mid}\hat{v}(\boldsymbol{k},0))e^{-i(\boldsymbol{a}.\boldsymbol{k}-|\boldsymbol{k}|/\varepsilon)t}s_1(\boldsymbol{k}) \\ + \dfrac{1}{\mid \boldsymbol{k} \mid}(-k_2\hat{u}(\boldsymbol{k},0) + k_1\hat{v}(\boldsymbol{k},0))e^{-i\boldsymbol{a}.\boldsymbol{k}t}s_2(\boldsymbol{k}) \\ + \dfrac{1}{\sqrt{2}}(\hat{r}(\boldsymbol{k},0) + \dfrac{k_1}{\mid \boldsymbol{k} \mid}\hat{u}(\boldsymbol{k},0) + \dfrac{k_2}{\mid \boldsymbol{k} \mid}\hat{v}(\boldsymbol{k},0))e^{-i(\boldsymbol{a}.\boldsymbol{k}+|\boldsymbol{k}|/\varepsilon)t}s_3(\boldsymbol{k}) \end{array} \right.$$

Explicit linear example IV

Fast oscillatory component $\hat{\boldsymbol{v}}_f(\boldsymbol{k}, t, t/\varepsilon)$

$$\frac{1}{\sqrt{2}}\left\{\begin{array}{l}(\hat{r}(\boldsymbol{k},0) - \frac{k_1}{\mid \boldsymbol{k} \mid}\hat{u}(\boldsymbol{k},0) - \frac{k_2}{\mid \boldsymbol{k} \mid}\hat{v}(\boldsymbol{k},0))e^{-i(\boldsymbol{a}.\boldsymbol{k} - \frac{\mid \boldsymbol{k} \mid}{\varepsilon})t} s_1(\boldsymbol{k}) \\ + \\ (\hat{r}(\boldsymbol{k},0) + \frac{k_1}{\mid \boldsymbol{k} \mid}\hat{u}(\boldsymbol{k},0) + \frac{k_2}{\mid \boldsymbol{k} \mid}\hat{v}(\boldsymbol{k},0))e^{-i(\boldsymbol{a}.\boldsymbol{k} + \frac{\mid \boldsymbol{k} \mid}{\varepsilon})t} s_3(\boldsymbol{k})\end{array}\right. \tag{8}$$

Explicit linear example V

Slow component belonging to the kernel of $\mathbb{L}$

$$\hat{\boldsymbol{v}}_s(\boldsymbol{k},\tau) = \frac{1}{\mid \boldsymbol{k} \mid}(-k_2\hat{u}(\boldsymbol{k},0) + k_1\hat{v}(\boldsymbol{k},0))e^{-i\boldsymbol{a}.\boldsymbol{k}t} s_2(\boldsymbol{k})$$

This component belongs to the kernel of $\mathbb{L}$ and satisfies the incompressible system

$$\begin{cases} \dfrac{\partial \boldsymbol{v}_s}{\partial t} + \mathbb{H}\boldsymbol{v}_s = 0 \\ \\ \mathbb{L}\boldsymbol{v}_s = 0 \end{cases}$$

Explicit linear example VI

What is the behavior of the solutions when $\varepsilon \to 0$?

For any ε the solution is composed of a superposition of fast and slow waves.

Does the solution converge toward something when $\varepsilon \to 0$?

- In a point-wise : NO : faster and faster oscillations
- In a weak sense (average or distribution) YES

  $e^{\pm i(\frac{|\boldsymbol{k}|}{\varepsilon})t} \to 0$

  thus the oscillatory part of the solution $\to 0$

  and the solutions converge (weakly) toward $\boldsymbol{v}_0$ that satisfies the incompressible system :

$$\begin{cases} \dfrac{\partial \boldsymbol{v}_0}{\partial t} + \mathbb{H}\boldsymbol{v}_0 = 0 \\[2ex] \mathbb{L}\boldsymbol{v}_0 = 0 \end{cases}$$

Is it true also for non-linear systems ?

Can we discard the fast component of the solution ?

- How to deal with non-linear interactions of the fast waves : non linear system contain quadratic terms e.g : $\mathcal{Q}(U,U) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$

$$\boldsymbol{W} = \boldsymbol{W}_{\text{Slow}} + \boldsymbol{W}_{\text{Fast}}$$

thus

$$\begin{aligned}\mathcal{Q}(\boldsymbol{W},\boldsymbol{W}) &= \mathcal{Q}(\boldsymbol{W}_{\text{Slow}},\boldsymbol{W}_{\text{Slow}}) + \mathcal{Q}(\boldsymbol{W}_{\text{Slow}},\boldsymbol{W}_{\text{Fast}}) + \mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Slow}}) \\ &+\mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Fast}})\end{aligned}$$

Can we prove that non-linear interaction of fast waves : $\mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Fast}})$ is not important for the slow dynamics of the system ?

Is it true also for non-linear systems ?

Can we discard the fast component of the solution ?

- How to deal with non-linear interactions of the fast waves :
  non linear system contain quadratic terms e.g : $\mathcal{Q}(U,U) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$

$$\boldsymbol{W} = \boldsymbol{W}_{\text{Slow}} + \boldsymbol{W}_{\text{Fast}}$$

thus

$$\begin{aligned}\mathcal{Q}(\boldsymbol{W},\boldsymbol{W}) &= \mathcal{Q}(\boldsymbol{W}_{\text{Slow}},\boldsymbol{W}_{\text{Slow}}) + \mathcal{Q}(\boldsymbol{W}_{\text{Slow}},\boldsymbol{W}_{\text{Fast}}) + \mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Slow}}) \\ &+ \mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Fast}})\end{aligned}$$

Can we prove that non-linear interaction of fast waves : $\mathcal{Q}(\boldsymbol{W}_{\text{Fast}},\boldsymbol{W}_{\text{Fast}})$ is not important for the slow dynamics of the system ?

Counter-example : Turbulence and Reynolds stresses !

Some notations

$$\text{The variables} \quad : \quad \mathscr{V}^\varepsilon = (\mathcal{B}_z^\varepsilon, \boldsymbol{v}_\perp^\varepsilon, \mathcal{B}_\perp^\varepsilon)^t \text{ or } = (p^\varepsilon, \boldsymbol{v})^t$$
$$\text{The equations} \quad : \quad \partial_t \mathscr{V}^\varepsilon + \mathbb{H}(\mathscr{V}^\varepsilon, \mathscr{V}^\varepsilon) + \frac{1}{\varepsilon} \mathbb{L}\mathscr{V}^\varepsilon = \mathcal{O}(\varepsilon)$$

$\mathbb{H}(\mathscr{V}, \mathscr{V})$ is a non-linear operator (at most quadratic)

$$\mathbb{H}(\mathscr{V}, \mathscr{V}) = \begin{pmatrix} (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_z + \mathcal{B}_z \nabla_\perp \cdot \boldsymbol{v}_\perp \\ (\boldsymbol{v}_\perp \cdot \nabla_\perp)\boldsymbol{v} - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp + \nabla_\perp(\mathcal{B}^2/2 + q) - \partial_z \mathcal{B}_\perp \\ (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp + \mathcal{B}_\perp \nabla_\perp \cdot \boldsymbol{v}_\perp - \partial_z \boldsymbol{v}_\perp \end{pmatrix}$$

$\mathbb{L}\mathscr{V}$ is the constant coefficient linear operator

$$\mathbb{L}\mathscr{V} = \begin{pmatrix} \nabla \cdot \boldsymbol{v}_\perp \\ \nabla \mathcal{B}_z \\ 0 \end{pmatrix}$$

The proof strategy

S. Schochet, E. Grenier, P.L.Lions-N.Masmoudi, B. Desjardins...

1. Introduce a filtered variable $\tilde{\mathscr{V}}^{\varepsilon} = \mathcal{F}\mathscr{V}^{\varepsilon}$ to remove the oscillations
2. Prove that the filtered variable $\tilde{\mathscr{V}}^{\varepsilon} \to \tilde{\mathscr{V}}^{0}$ satisfying some equation $\partial_t \mathscr{V}^0 + \mathcal{H}(\mathscr{V}^0, \mathscr{V}^0) = 0$ with $\mathcal{H}$ time-independent.
3. Prove that the original variable $\mathscr{V}^{\varepsilon} \to \mathcal{F}^{-1}\tilde{\mathscr{V}}^{0}$
4. Since $\mathcal{F}^{-1}\tilde{\mathscr{V}}^{0} \to P\tilde{\mathscr{V}}^{0}$ where P is the L^2 projection on the kernel of $\mathbb{L}$

Result

$$\mathscr{V}^{\varepsilon} \to \overline{\mathscr{V}} = P\tilde{\mathscr{V}}^{0} \text{ and } \overline{\mathscr{V}} \text{ satisfies :}$$
$$\partial_t \overline{\mathscr{V}} + P\mathcal{H}(\mathscr{V}^0, \mathscr{V}^0) = 0$$

The wave operator $\mathbb{L}$

$$\mathbb{L}\mathscr{V} = \begin{pmatrix} \nabla \cdot \boldsymbol{v} \\ \nabla \mathcal{B}_z \\ 0 \end{pmatrix}$$

- $L^2(\Omega) \times (L^2(\Omega))^2 = \mathrm{Ker}\mathbb{L} \oplus \mathrm{Im}\mathbb{L}$
  $\mathrm{Ker}\mathbb{L} = \{(\mathcal{B}_z, \boldsymbol{v}); \mathcal{B}_z = cte, \nabla \cdot \boldsymbol{v} = 0\}$
  $\mathrm{Im}\mathbb{L} = \{(\mathcal{B}_z, \boldsymbol{v}); \int \mathcal{B}_z = 0, \exists \Phi \boldsymbol{v} = \nabla \Phi\}$
- Spectrum of $\mathbb{L}$ on $\mathrm{Im}\mathbb{L}$
  Let $\{\psi_k, k \geq 1\}$ the eigenvectors of the Laplace operator

$$-\Delta \psi_k = \lambda_k^2 \psi_k \qquad \lambda_k > 0$$

then the eigenvectors of $\mathbb{L}$ are :

$$\Phi_k^{\pm} = \begin{bmatrix} \psi_k \\ \\ \pm \dfrac{\nabla \psi_k}{i\lambda_k} \end{bmatrix} \qquad \text{with} \quad \mathbb{L}\Phi_k^{\pm} = \pm i \lambda_k \Phi_k^{\pm}$$

The solution operator $\mathcal{L}$ of the wave equation

Let $\mathcal{L}(t)$ be the semi-group $(\mathcal{L}(t), t \in \boldsymbol{R})$ defined by

$$\mathcal{L}(t) = \exp(-\mathbb{L}t) \tag{9}$$

In other words

$$\mathscr{V}(t,x) = \mathcal{L}(t)\mathscr{V}_0(x) \qquad \text{means that} \quad \frac{\partial \mathscr{V}}{\partial t} + \mathbb{L}\mathscr{V} = 0 \qquad \text{with} \quad \mathscr{V}(t=0,x) = \mathscr{V}_0(x)$$

Using the expression of the spectrum of $\mathbb{L}$ we can have an explicit representation of the solution operator $\mathcal{L}(t)$: Let P be the L^2 projection on $\mathrm{Ker}\mathbb{L}$

on the velocity component $\mathcal{L}_v(t)\mathscr{V}$

$$\text{if} \quad \mathscr{V} - P\mathscr{V} = \sum_{k,\pm} a_k^{\pm}\Phi_k^{\pm} \quad \text{then} \quad \mathcal{L}_v(t)\mathscr{V} = \pi \boldsymbol{v}_{\perp} + \sum_{k,\pm} \pm a_k^{\pm} e^{\pm i\lambda_k t}\frac{\nabla \psi_k}{i\lambda_k}$$

$a_k^- = (a_k^+)^*$ conjugate (real functions)

Step 1 : Equation satisfied by the filtered variable $\tilde{\mathscr{V}}^\varepsilon$

$$\partial_t \mathscr{V}^\varepsilon + \quad \mathbb{H}(\mathscr{V}^\varepsilon, \mathscr{V}^\varepsilon) + \frac{1}{\varepsilon} \quad \mathbb{L}\mathscr{V}^\varepsilon = \mathcal{O}(\varepsilon)$$

$$\text{introduce the filtered variable } \quad \tilde{\mathscr{V}}^\varepsilon = \mathcal{L}(-t/\varepsilon)\mathscr{V}^\varepsilon$$

with

$$\mathcal{L}(t) = \exp(-\mathbb{L}t)$$

From the definition of $\mathcal{L}$, we deduce that

$$\begin{aligned}
\frac{\partial \tilde{\mathscr{V}}^\varepsilon}{\partial t} \quad &= \frac{\mathbb{L}}{\varepsilon}\tilde{\mathscr{V}}^\varepsilon + \mathcal{L}(-t/\varepsilon)\frac{\partial \mathscr{V}^\varepsilon}{\partial t} \\
&= \frac{\mathbb{L}}{\varepsilon}\tilde{\mathscr{V}}^\varepsilon - \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon) - \mathcal{L}(-t/\varepsilon)\frac{\mathbb{L}}{\varepsilon}\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon + \mathcal{O}(\varepsilon) \\
&= -\mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon) + \mathcal{O}(\varepsilon)
\end{aligned}$$

since $\mathcal{L}(t/\varepsilon)$ and $\mathbb{L}$ commute.

Initial data : $\tilde{\mathscr{V}}^\varepsilon(t=0) = \mathscr{V}^\varepsilon(t=0)$ since $\mathcal{L}(0)$ is the identity

Limit Equation

Step 2 : Limit Equation for the filtered variable $\tilde{\mathscr{V}}^\varepsilon$

$$\frac{\partial \tilde{\mathscr{V}}^\varepsilon}{\partial t} + \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon) = \mathcal{O}(\varepsilon)$$

$\tilde{\mathscr{V}}^0 = \lim_{\varepsilon\to 0} \tilde{\mathscr{V}}^\varepsilon$

$$\frac{\partial \tilde{\mathscr{V}}^0}{\partial t} + \mathcal{H}(\tilde{\mathscr{V}}^0, \tilde{\mathscr{V}}^0) = 0$$

where $\mathcal{H}(\tilde{\mathscr{V}}^0, \tilde{\mathscr{V}}^0) = \lim_{\varepsilon\to 0} \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0)$ is a time-independent operator whose expression can be computed explicitly (see next slides)

Step 3 : Go back to the unfiltered variable $\mathscr{V}^\varepsilon$

$$\mathscr{V}^\varepsilon \to \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0$$

Limit for the original variable

But we have

$$\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0 \rightarrow P\tilde{\mathscr{V}}^0$$

since

$$\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0 = \mathcal{L}(t/\varepsilon)(P\tilde{\mathscr{V}}^0 + \sum_{k,\pm} \pm a_k^{\pm} e^{\pm i\lambda_k t/\varepsilon} \Phi_k^{\pm}) \rightharpoonup P\tilde{\mathscr{V}}^0$$

Final result : weak limit of $\mathscr{V}^\varepsilon = P\tilde{\mathscr{V}}^0$ that satisfies

$$\frac{\partial P\tilde{\mathscr{V}}^0}{\partial t} + P\mathcal{H}(\tilde{\mathscr{V}}^0, \tilde{\mathscr{V}}^0) = 0$$

Explicit form of the limit equation for $P\tilde{\mathscr{V}}^0$

example : computation of the quadratic term $\mathcal{Q}(\boldsymbol{W}_{\mathrm{Fast}}, \boldsymbol{W}_{\mathrm{Fast}}) = (\boldsymbol{v}_\perp \cdot \nabla)\boldsymbol{v}_\perp = (\boldsymbol{v}_\perp)_j \partial_j \boldsymbol{v}_\perp$

$$
\begin{aligned}
&(\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} \cdot \nabla)\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} = \\
&\{\sum_k (a_k^+ e^{i\lambda_k t/\varepsilon} - a_k^- e^{-i\lambda_k t/\varepsilon})\frac{\nabla\psi_k}{i\lambda_k}\}_j \partial_j \{\sum_l (a_l^+ e^{i\lambda_l t/\varepsilon} - a_l^- e^{-i\lambda_l t/\varepsilon})\frac{\nabla\psi_l}{i\lambda_l}\} = \\
&\sum_{k,l}[-a_k^+ a_l^+ e^{i(\lambda_k+\lambda_l)t/\varepsilon} - a_k^- a_l^- e^{-i(\lambda_k+\lambda_l)t/\varepsilon}]\frac{1}{\lambda_k\lambda_l}(\nabla\psi_k)_j\partial_j(\nabla\psi_l) \\
&+\sum_{k,l}[a_k^- a_l^+ e^{i(\lambda_l-\lambda_k)t/\varepsilon} + a_k^+ a_l^- e^{i(\lambda_k-\lambda_l)t/\varepsilon}]\frac{1}{\lambda_k\lambda_l}(\nabla\psi_k)_j\partial_j(\nabla\psi_l)
\end{aligned}
$$

$\lim_{\varepsilon\to 0}$ (distribution) of all the terms is 0 except when $k = l$ and we get :

$$
(\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} \cdot \nabla)\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} \to \sum_k [a_k^- a_k^+ + a_k^+ a_k^-]\frac{1}{\lambda_k^2}(\nabla\psi_k)_j\partial_j(\nabla\psi_k) = \sum_k \frac{|a_k^+|^2}{\lambda_k^2}\nabla(|\nabla\psi_k|^2/2)
$$

On the average (weak limit) fast k-waves interact with l-waves only if $k = l$ and the result is a gradient

the result of the interaction between fast waves and slow dynamics is a gradient !

Summary

When it goes well :
Weak limit of the solutions of compressible systems :

$$\begin{cases} \partial_t \boldsymbol{W} + \sum_j A_j(\boldsymbol{W}, \varepsilon)\partial_{x_j} \boldsymbol{W} + \dfrac{1}{\varepsilon}\mathbb{L}\boldsymbol{W} = 0 \\ \boldsymbol{W}(0, \boldsymbol{x}, \varepsilon) = \boldsymbol{W}_0(\boldsymbol{x}, \varepsilon) \end{cases}$$

are the solutions of the incompressible system

$$\begin{cases} \partial_t \boldsymbol{W} + \mathbb{P}\sum_j A_j(\boldsymbol{W}, 0)\partial_{x_j} \boldsymbol{W} = 0 \\ \mathbb{L}\boldsymbol{W} = 0 \\ \boldsymbol{W}(0, \boldsymbol{x}) = \mathbb{P}\boldsymbol{W}_0(\boldsymbol{x}) \end{cases}$$

where $\mathbb{P}$ is the projection on $\ker(\mathbb{L})$.

In general for these systems :

decoupling between fast waves and slow dynamics

Comments and perspectives

- Understanding of the interactions between fast and slow dynamics
- Some implications for numerical methods :
  - compressible solvers are usually inaccurate when computing low Mach flows
  - modification are required : this workshop !
- At present, modification of compressible solvers allows to compute near incompressible flows
- I do not know if they can compute low Mach number interaction of acoustic and incompressible phenomena