All-regime Lagrangian-Remap numerical schemes for the gas dynamics equations. Applications to the low Mach regime

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Joint works with
- M. Girardin and S. Kokh (first part)
- F. Bouchut and S. Guisset (second part)

Outline

Outline

Introduction

Motivation : numerical study of two-phase flows in nuclear reactors

We consider the following model

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0 \\
&\partial_t (\rho E) + \nabla \cdot [(\rho E + p)\mathbf{u}] = 0
\end{aligned}$$

where ρ, $\mathbf{u} = (u, v)^t$, E denote respectively the density, the velocity vector and the total energy of the fluid.

Let $e = E - \frac{|\mathbf{u}|^2}{2}$ be the specific and $\tau = 1/\rho$ the covolume

Introduction

We are especially interested in the design of numerical schemes when the dimensionless version of this model depends on a parameter $\epsilon > 0$ such that $\epsilon = O(1)$ (classical regime), $\epsilon << 1$ (low ϵ regime) or $\epsilon \to 0$ (limit regime)

Our objective is to propose a numerical scheme that is

- all-regime : uniform stability and uniform consistency w.r.t. ϵ
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

These requirements will be specified later on...

Outline

Low Mach regime

Introducing the characteristic and non-dimensional quantities :

$$x = \frac{x}{L}, \quad t = \frac{t}{T}, \quad \rho = \frac{\rho}{\rho_0}, \quad u = \frac{u}{u_0},$$

$$v = \frac{v}{v_0}, \quad e = \frac{e}{e_0}, \quad p = \frac{p}{p_0}, \quad c = \frac{c}{c_0}$$

with $u_0 = v_0 = \frac{L}{T}$, $e_0 = p_0\rho_0$ and $p_0 = \rho_0 c_0^2$, the non-dimensional system is

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2} \nabla p = 0 \\
&\partial_t(\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}] + \frac{M^2}{2}\big(\partial_t(\rho \mathbf{u}.\mathbf{u}) + \nabla \cdot (\rho \mathbf{u}.\mathbf{u}\mathbf{u})\big) = 0
\end{aligned}$$

where $M = \frac{u_0}{c_0}$ denotes the Mach number and plays the role of ϵ

Low Mach regime

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$
$$\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M^2} \nabla p = 0$$
$$\partial_t(\rho e) + \nabla \cdot [(\rho e + p)\mathbf{u}] + \frac{M^2}{2}\big(\partial_t(\rho \mathbf{u}.\mathbf{u}) + \nabla \cdot (\rho \mathbf{u}.\mathbf{u}\mathbf{u})\big) = 0$$

Remark 1. The flow is said to be in the low Mach regime if $M << 1$ and $\nabla p = O(M^2)$

Remark 2. Using asymptotic expansions of $\rho, \mathbf{u}, p, c$ in powers of M in the governing equations of $\rho, \mathbf{u}, p$, together with boundary conditions on a given domain $\mathcal{D}$ (global argument), we get

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{u}_0) = 0$$
$$\partial_t \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \frac{1}{\rho_0} \nabla p_2 = 0$$
$$\nabla \cdot \mathbf{u}_0 = 0$$

Numerical issue in the Low Mach regime

Accurate time-explicit computations of solutions generally require

- a mesh size $h = o(M)$
- a time step $\Delta t = O(hM)$

which is out of reach in practice

More details can be found in the large body of literature on this subject : A. Majda, E. Turkel, H. Guillard, C. Viozat, B. Thornber, S. Dellacherie, P. Omnes, P-A. Raviart, F. Rieper, Y. Penel, P. Degond, S. Jin, J.-G. Liu, P. Colella, K. Pao, E. Turkel, R. Klein, J-P Vila, M.G., B. Després, M. Ndjinga, J. Jung, M. Sun, M.-H. Vignal, G. Dimarco, R. Herbin, J.-C. Latché...

A couple of definitions

Uniform stability

A scheme is said to be stable in the uniform sense if the CFL condition is uniform with respect to $\epsilon = M$

Goal : to avoid stringent CFL restrictions $\Delta t = O(h\epsilon)$

Uniform consistency

A scheme is said to be consistent in the uniform sense if the truncation error is uniform with respect to $\epsilon = M$

Goal : to avoid mesh size restrictions $h = o(\epsilon)$

All-regime scheme

A scheme is said to be all-regime if it is able to compute accurate solutions with a mesh size h and a time step Δt independent of ϵ

Objectives

Our objective is to propose a numerical scheme that is

- all-regime : uniform stability and uniform consistency w.r.t. ϵ
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

How to do that...

Outline

How to reach these objectives

How to get the uniform stability ?

- implicit treatment of the fast phenomenon
- explicit treatment of the slow phenomenon (sake of accuracy)

$\rightarrow$ Lagrange-Projection strategy Coquel-Nguyen-Postel-Tran

How to get the uniform consistency ?

- modify the numerical fluxes to reduce the numerical diffusion

$\rightarrow$ Truncation errors in equivalent equations

How to deal with any (possibly strongly nonlinear) pressure law p ?

- overcome the non linearities, "linearization"

$\rightarrow$ Relaxation strategy Suliciu, Jin-Xin, Bouchut, C.-Coquel, C.-Coulombel

How to deal with unstructured meshes in multi-D ?

- work on a fixed mesh (no need to deform unstructured meshes)

$\rightarrow$ Operator splitting strategy and rotational invariance

Lagrange-Projection strategy

Let us first focus on the 1D system

$$\begin{cases} \partial_t \varrho + \partial_x \varrho u = 0 \\ \partial_t \varrho u + \partial_x (\varrho u^2 + p) = 0 \\ \partial_t (\varrho E) + \partial_x (\varrho E u + p u) = 0 \end{cases}$$

Using chain rule arguments, we also have

$$\begin{cases} \partial_t \varrho + u \partial_x \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + u \partial_x \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + u \partial_x \varrho E + \varrho E \partial_x u + \partial_x p u = 0 \end{cases}$$

so that splitting the transport part leads to

$$\begin{cases} \partial_t \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + \varrho E \partial_x u + \partial_x p u = 0 \end{cases} \qquad \begin{cases} \partial_t \varrho + u \partial_x \varrho = 0 \\ \partial_t \varrho u + u \partial_x \varrho u = 0 \\ \partial_t \varrho E + u \partial_x \varrho E = 0 \end{cases}$$

Lagrangian-step Transport-step

Lagrange-Projection strategy

The Lagrangian-step

$$\begin{cases} \partial_t \varrho + \varrho \partial_x u = 0 \\ \partial_t \varrho u + \varrho u \partial_x u + \partial_x p = 0 \\ \partial_t \varrho E + \varrho E \partial_x u + \partial_x pu = 0 \end{cases} \quad \text{also writes} \quad \begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

with $\tau = 1/\varrho$ and $\tau \partial_x = \partial_m$.

- Eigenvalues are given by $-\rho c,\ 0,\ \rho c$
- Usual CFL conditions for time-explicit schemes write

$$\Delta t \leq \frac{h}{2\max(\rho c)} \quad \text{or equivalently} \quad \Delta t \leq \frac{hM}{2\max(\rho c)}$$

The idea is to propose a time-implicit scheme to avoid the time-step restriction $\Delta t = O(hM)$ in the low Mach regime

Lagrange-Projection strategy

The Transport-step is

$$\left\{\begin{array}{l} \partial_t \varrho + u \partial_x \varrho = 0 \\ \partial_t \varrho u + u \partial_x \varrho u = 0 \\ \partial_t \varrho E + u \partial_x \varrho E = 0 \end{array}\right. \quad \text{also writes} \quad \left\{\begin{array}{l} \partial_t \varrho + \partial_x \varrho u - \varrho \partial_x u = 0 \\ \partial_t \varrho u + \partial_x \varrho u^2 - \varrho u \partial_x u = 0 \\ \partial_t \varrho E + \partial_x \varrho E u - \varrho E \partial_x u = 0 \end{array}\right.$$

- Eigenvalues are given by u
- Usual CFL conditions for time-explicit schemes write

$$\Delta t \leq \frac{h}{2 \max(|u|)}$$

The idea is then to propose a standard time-explicit scheme to keep accuracy on the slow phenomenon and $\Delta t = O(h)$ in all regime

Operator splitting strategy

We will consider the following two-step numerical scheme :

First step $(t^n \rightarrow t^{Lag})$: solve implicitly the acoustic system with the solution at time t^n as initial solution

Second step $(t^{Lag} \rightarrow t^{n+1})$ solve explicitly the transport system with the solution at time t^{Lag} as initial solution

A few words about the relaxation approach

The gas dynamics in Lagrangian coordinates

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m p = 0 \\ \partial_t E + \partial_m pu = 0 \end{cases}$$

The relaxation system

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t E + \partial_m \Pi u = 0 \\ \partial_t \Pi + a^2 \partial_m u = \lambda(p - \Pi) \end{cases}$$

At least formally, observe that

$$\lim_{\lambda \to +\infty} \Pi = p \quad (\text{if} \quad a > \rho c(\tau, e))$$

(see e.g. Chalons-Coulombel for a rigorous proof)

A few words about the relaxation approach

The time-explicit Godunov scheme applied to the relaxation system with initial data at equilibrium writes

$$\begin{cases} \tau_j^{Lag} = \tau_j^n + \dfrac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\ u_j^{Lag} = u_j^n - \dfrac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) \\ \Pi_j^{Lag} = \Pi_j^n - a^2\dfrac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\ E_j^{Lag} = E_j^n - \dfrac{\Delta t}{\Delta m}(p^*_{j+1/2}u^*_{j+1/2} - p^*_{j-1/2}u^*_{j-1/2}) \end{cases}$$

with $\Pi_j^n = p(\tau_j^n, e_j^n)$ and

$$u^*_{j+1/2} = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{2a}(\Pi_{j+1}^n - \Pi_j^n)$$

$$p^*_{j+1/2} = \frac{1}{2}(\Pi_j^n + \Pi_{j+1}^n) - \frac{a}{2}(u_{j+1}^n - u_j^n)$$

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with $\Pi_j^n = p(\tau_j^n, e_j^n)$ and

$$u^*_{j+1/2} = \frac{1}{2}(u_j^{Lag} + u_{j+1}^{Lag}) - \frac{1}{2a}(\Pi_{j+1}^{Lag} - \Pi_j^{Lag})$$

$$p^*_{j+1/2} = \frac{1}{2}(\Pi_j^{Lag} + \Pi_{j+1}^{Lag}) - \frac{a}{2}(u_{j+1}^{Lag} - u_j^{Lag})$$

A few words about the relaxation approach

The time-implicit scheme

- deals with (possibly strongly nonlinear) pressure laws
- is free of CFL restriction !
- is cheap in the sense that only a linear problem w.r.t. u and Π has to be solved

In 1D, the following two equations are decoupled

$$\begin{cases} \partial_t(\Pi + au) + a\partial_x(\Pi + au) = 0 \\ \partial_t(\Pi - au) - a\partial_x(\Pi - au) = 0 \end{cases}$$

Formulation on unstructured meshes

On unstructured meshes, the time-explicit ($\sharp = n$) and time-implicit ($\sharp = Lag$) schemes write

$$
\begin{aligned}
\mathbf{u}_j^{Lag} &= \mathbf{u}_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} \Pi_{jk}^* \mathbf{n}_{jk} \\
\Pi_j^{Lag} &= \Pi_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} (a_{jk})^2 u_{jk}^* \\
\tau_j^{Lag} &= \tau_j^n + \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^* \\
E_j^{Lag} &= E_j^n - \tau_j^n \Delta t \sum_{k \in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} p_{jk}^* u_{jk}^*
\end{aligned}
$$

$$
u_{jk}^* = \frac{1}{2}\mathbf{n}_{jk}^T(\mathbf{u}_j^\sharp + \mathbf{u}_k^\sharp) - \frac{1}{2a_{jk}}(\Pi_k^\sharp - \Pi_j^\sharp), \quad p_{jk}^* = \frac{1}{2}(\Pi_j^\sharp + \Pi_k^\sharp) - \frac{a_{jk}}{2}\mathbf{n}_{jk}^T(\mathbf{u}_k^\sharp - \mathbf{u}_j^\sharp)
$$

Transport step

In order to approximate the solutions of the transport step

$$\begin{aligned} \partial_t\rho + (\mathbf{u}\cdot\nabla)\rho &= 0 \\ \partial_t(\rho\mathbf{u}) + (\mathbf{u}\cdot\nabla)\rho\mathbf{u} &= 0 \\ \partial_t(\rho E) + (\mathbf{u}\cdot\nabla)\rho E &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) - \rho\nabla\cdot\mathbf{u} &= 0 \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) - \rho\mathbf{u}\nabla\cdot\mathbf{u} &= 0 \\ \partial_t\rho E + \nabla\cdot(\rho E\mathbf{u}) - \rho E\nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

we simply use the time-explicit upwind finite-volume scheme

$$\varphi_j^{n+1} = \varphi_j^{n+1-} - \Delta t \sum_{k\in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^* \varphi_{jk}^{n+1-} + \Delta t \varphi_j^{n+1-} \sum_{k\in N(j)} \frac{|\Gamma_{jk}|}{|\Omega_j|} u_{jk}^*$$

where $\varphi = \rho, \rho\mathbf{u}, \rho E$ and $\varphi_{jk}^{n+1-} = \begin{cases} \varphi_j^{n+1-} & \text{if } u_{jk}^* > 0 \\ \varphi_k^{n+1-} & \text{if } u_{jk}^* \leq 0 \end{cases}$

This scheme is stable under a material CFL condition $\Delta t = O(h)$

Objectives

Our objective is to propose a numerical scheme that is

- all-regime : uniform stability and uniform consistency w.r.t. ϵ
- able to deal with any equation of state
- multi-dimensional on (possibly) unstructured meshes

What about the first objective ?

Uniform consistency in the low Mach regime

Let us focus on the first step of the time-explicit scheme

$$\begin{aligned}
\tau_j^{n+1-} &= \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2}) \\
u_j^{n+1-} &= u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2}) \\
E_j^{n+1-} &= E_j^n - \frac{\Delta t}{\Delta m}((pu)^*_{j+1/2} - (pu)^*_{j-1/2})
\end{aligned}$$

with

$$u^*_{j+1/2} = \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2a}(p_{j+1} - p_j)$$

$$p^*_{j+1/2} = \frac{1}{2}(p_j + p_{j+1}) - \frac{a}{2}(u_{j+1} - u_j)$$

Uniform consistency in the low Mach regime

In dimensionless form we get

$$\tau_j^{n+1-} = \tau_j^n + \frac{\Delta t}{\Delta m}(u^*_{j+1/2} - u^*_{j-1/2})$$
$$u_j^{n+1-} = u_j^n - \frac{\Delta t}{\Delta m}(p^*_{j+1/2} - p^*_{j-1/2})$$
$$E_j^{n+1-} = E_j^n - \frac{\Delta t}{\Delta m}((pu)^*_{j+1/2} - (pu)^*_{j-1/2})$$

with, since $p_{j+1} - p_j = \mathcal{O}(M^2)$

$$u^*_{j+1/2} = \frac{u_j + u_{j+1}}{2} - \frac{M\Delta m}{2aM^2}\frac{(p_{j+1} - p_j)}{\Delta m} = \frac{u_j + u_{j+1}}{2} + \mathcal{O}(M\Delta m)$$

$$p^*_{j+1/2} = \frac{p_j + p_{j+1}}{2M^2} - \frac{a\Delta m}{2M}\frac{(u_{j+1} - u_j)}{\Delta m} = \frac{p_j + p_{j+1}}{2M^2} + \mathcal{O}(\frac{\Delta m}{M})$$

Uniform consistency in the low Mach regime

We note that

- the numerical diffusion (or consistency error) is extremely small on the first equation

- the numerical diffusion (or consistency error) is extremely large on the second equation

$$u^*_{j+1/2} = \frac{u_j + u_{j+1}}{2} - \frac{M\Delta m}{2aM^2}\frac{(p_{j+1} - p_j)}{\Delta m} = \frac{u_j + u_{j+1}}{2} + \mathcal{O}(M\Delta m)$$

$$p^*_{j+1/2} = \frac{p_j + p_{j+1}}{2M^2} - \frac{a\Delta m}{2M}\frac{(u_{j+1} - u_j)}{\Delta m} = \frac{p_j + p_{j+1}}{2M^2} + \mathcal{O}(\frac{\Delta m}{M})$$

Uniform consistency in the low Mach regime

The main problem comes from the numerical diffusion in $p^*_{j+1/2}$

We get the uniform consistency with respect to M by introducing a parameter $\theta_{j+1/2}$ and setting

$$p^*_{j+1/2} = \frac{1}{2}(p^n_j + p^n_{j+1}) - \theta_{j+1/2}\frac{a}{2}(u^n_{j+1} - u^n_j)$$

which gives the uniform consistency if $\theta_{j+1/2} = \mathcal{O}(M)$

$$p^*_{j+1/2} = \frac{p_j + p_{j+1}}{2M^2} + \mathcal{O}(\frac{\theta_{j+1/2}\Delta m}{M})$$

Note that the numerical diffusion in $u^*_{j+1/2}$ is still very small...

Remarks

The modifications give the uniform consistency and we recover the classical scheme provided that $\theta_{j+1/2} = 1$

The modifications apply directly on unstructured meshes

Considering the time-implicit treatment of the Lagrangian step gives the uniform stability

The relaxation approach allows to consider any given pressure law

Recall that the unstructured mesh is fixed (not moving)

All the objectives are reached

Remarks

How does the modifications affect the stability properties ?

The whole scheme is

- conservative
- positive
- entropy satisfying under a suitable definition of θ NOT compatible in the asymptotic limit

$\theta = 0$ is also possible ! (numerical diffusion in the transport step)

How to get the entropy inequality in the asymptotic limit ?
By adding numerical diffusion on the first equation...

A two-speed relaxation approach

The former relaxation system

$$\begin{cases} \partial_t \tau - \partial_m u = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t \Pi + a^2 \partial_m u = \lambda(p - \Pi) \end{cases}$$

The two-speed relaxation system

$$\begin{cases} \partial_t \tau - \partial_m v = 0 \\ \partial_t u + \partial_m \Pi = 0 \\ \partial_t v + (a/a_v) \partial_m \Pi = \lambda(u - v) \\ \partial_t \Pi + a a_v \partial_m v = \lambda(p - \Pi) \end{cases}$$

At least formally, observe that

$$\lim_{\lambda \to +\infty} \Pi = p \quad \text{and} \quad \lim_{\lambda \to +\infty} v = u \text{ if } \quad a > a_v \text{ and } a a_v > \rho^2 c^2$$

A two-speed relaxation system

The former sub-characteristic condition

$$a > \rho c$$

which gives $a = \mathcal{O}(1/M)$

The new sub-characteristic condition

$$a > a_v \quad \text{and} \quad a a_v > \rho^2 c^2$$

Here, the idea will be to take $a_v = \mathcal{O}(1)$ and $a = \mathcal{O}(1/M^2)$!

A two-speed relaxation system

The former eigenvalues

$$-a, \quad 0, \quad a$$

The new eigenvalues

$$-a, \quad 0, \quad a$$

Therefore, the explicit CFL condition behaves like $\mathcal{O}(M^2\Delta x)$ and the Lagrangian step must be time-implicit again !

Remark. We are also able to design a time-explicit scheme, the CFL of which behaves like $\mathcal{O}(\Delta x^2)$! The key idea is to equal the errors of the numerical scheme in $\mathcal{O}(\Delta x)$ with the error to the incompressible limit in $\mathcal{O}(M^2)$ by replacing M with $\sqrt{\Delta x}$

A two-speed relaxation system

The former numerical fluxes

$$v^*_{j+1/2} = \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2a}(p_{j+1} - p_j)$$

$$p^*_{j+1/2} = \frac{1}{2}(p_j + p_{j+1}) - \frac{a}{2}(u_{j+1} - u_j)$$

The new numerical fluxes

$$u^*_{j+1/2} = \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2a_v}(p_{j+1} - p_j) = \frac{1}{2}(u_j + u_{j+1}) + \mathcal{O}(\Delta x)!$$

$$p^*_{j+1/2} = \frac{1}{2}(p_j + p_{j+1}) - \frac{a_v}{2}(u_{j+1} - u_j) = \frac{1}{2}(p_j + p_{j+1}) + \mathcal{O}(\Delta x)!$$

New properties

The whole scheme is now

- conservative
- positive
- uniformly stable and uniformly consistent
- entropy satisfying and of order 1

Outline

Vortex in a box : test case

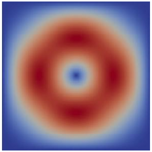
The fluid is equipped with a perfect gas equation of state

$$p = (\gamma - 1)\rho e, \quad \gamma = 1.4$$

We consider the domain $\Omega = (0,1)^2$.
The initial condition is given by

$$\begin{cases} \rho_0(x,y) = 1 - \frac{1}{2} tanh\left(y - \frac{1}{2}\right), & u_0(x,y) = 2sin^2(\pi x)sin(\pi y)cos(\pi y)), \\ p_0(x,y) = 1000, & v_0(x,y) = -2sin(\pi x)cos(\pi x)sin^2(\pi y). \end{cases}$$

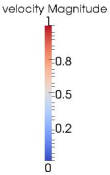
We impose a no-slip boundary condition.

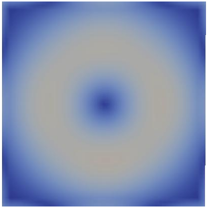


This configuration leads to a Mach number of order 0.026, so that we are in the low Mach regime.

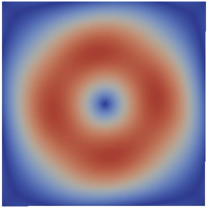
Vortex in a box ($M\#0.026$) : explicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.

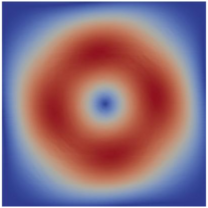




explicit scheme
($\theta = 1$)
Cartesian Mesh
$50 * 50 cells$



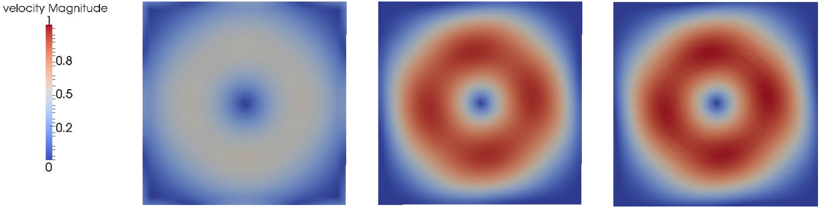
explicit scheme
($\theta = 1$)
Cartesian Mesh
$400 * 400 cells$



reference solution
explicit scheme
($\theta = 1$)
Triangular Mesh

Vortex in a box ($M\#0.026$) : modified explicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.



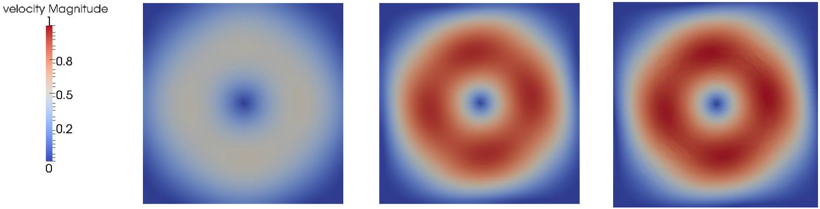
explicit scheme
$(\theta = 1)$
Cartesian Mesh
$50 * 50 cells$

explicit scheme
$(\theta_{ij} = M_{ij}^n)$
Cartesian Mesh
$50 * 50 cells$

reference solution
explicit scheme
$(\theta = 1)$
Triangular Mesh

Vortex in a box ($M\#0.026$) : modified implicit scheme

We plot the flow speed magnitude at time $T = 0.125s$.



implicit-explicit scheme ($\theta = 1$)
Cartesian Mesh
$50 * 50 cells$

implicit-explicit scheme ($\theta_{ij} = M_{ij}^n$)
Cartesian Mesh
$50 * 50 cells$

reference solution
explicit scheme
($\theta = 1$)
Triangular Mesh

Vortex in a box ($M\#0.026$) : CPU Time

EX : $\beta = n$,   IMEX : $\beta = Lag$.

Numerical scheme	EX($\theta = 1$) (Mesh $400 * 400$)	EX($\theta = 1$) (Mesh $50 * 50$)	EX($\theta_{ij} = M_{ij}$) (Mesh $50 * 50$)
Number of iterations	18 457	2 306	2 305
CPU time (s)	9 263.04 ($2h34min$)	17.14	19.3

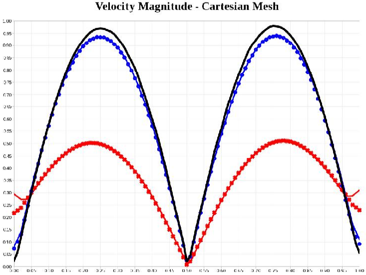
Speed up $(\theta = 1 \rightarrow \theta_{ij} = M_{ij}) = 480$

Numerical scheme	IMEX($\theta = 1$) (Mesh $50 * 50$)	IMEX($\theta_{ij} = M_{ij}$) (Mesh $50 * 50$)
Number of iterations	43	56
CPU time (s)	3.75	5.77

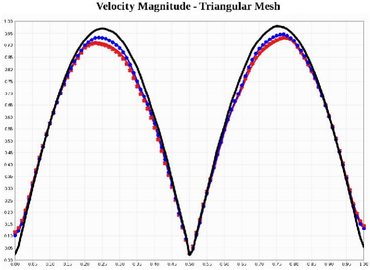
Speed up (explicit$\rightarrow$ implicit-explicit)$= 3.3$

Vortex in a box ($M\#0.026$) : Influence of the cell geometry

We plot a 1D-cut at $x = 0.5$ of the flow speed magnitude at time $T = 0.125s$.



Cartesian Mesh



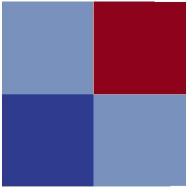
Triangular Mesh

2D-Riemann problem : test case

The fluid is equipped with a perfect gas equation of state

$$p = (\gamma - 1)\rho e, \quad \gamma = 1.4$$

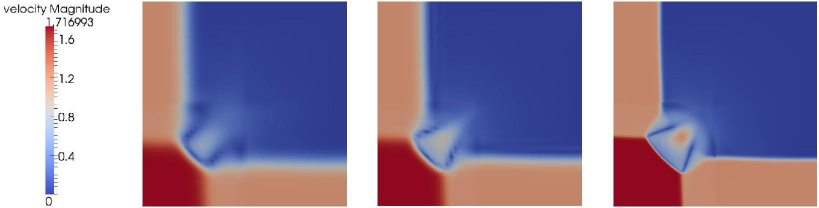
We consider the domain $\Omega = (0,1)^2$.
The initial condition corresponds to a 2D Riemann problem that consists of 4 shock waves. We impose Neumann boundary conditions.



This configuration leads to a Mach number that ranges from 10^{-5} to 3.15, so that we have both low Mach and order 1 Mach values.

2D-Riemann problem $M \in (10^{-5}, 3.15)$: modified explicit scheme

We plot the flow speed magnitude at time $T = 0.4s$.



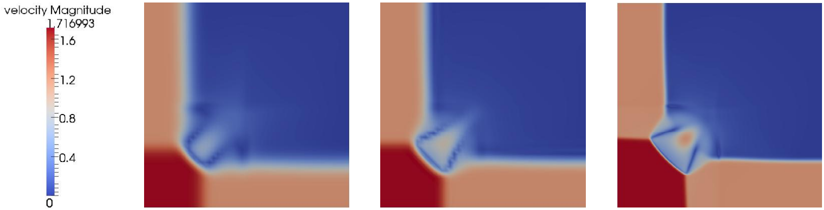
explicit scheme
$(\theta = 1)$
Cartesian Mesh
$50 * 50 cells$

explicit scheme
$(\theta = 0)$
Cartesian Mesh
$50 * 50 cells$

reference solution
explicit scheme
$(\theta = 1)$
Triangular Mesh

2D-Riemann problem $M \in (10^{-5}, 3.15)$: modified implicit scheme

We plot the flow speed magnitude at time $T = 0.4s$.



implicit-explicit scheme $(\theta = 1)$
Cartesian Mesh
$50 * 50 cells$

implicit-explicit scheme $(\theta = 0)$
Cartesian Mesh
$50 * 50 cells$

reference solution
explicit scheme
$(\theta = 1)$
Triangular Mesh

2D-Riemann problem $M \in (10^{-5}, 3.15)$: CPU time

Numerical scheme	EX($\theta = 1$) (Mesh $50 * 50$)	EX($\theta = 0$) (Mesh $50 * 50$)
Number of iterations	323	343
CPU time (s)	2.59	2.79

Speed up $(\theta = 1 \rightarrow \theta = 0) \approx 1$

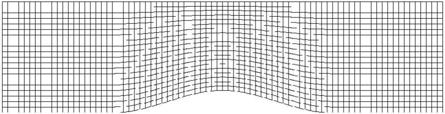
Numerical scheme	IMEX($\theta = 1$) (Mesh $50 * 50$)	IMEX($\theta = 0$) (Mesh $50 * 50$)
Number of iterations	216	218
CPU time (s)	10.28	10.33

Speed up (explicit$\rightarrow$ implicit-explicit)$= 0.25$

flow in a channel with bump

The fluid is equipped with a mixture of two perfect gas with different adiabatic coefficients equation of state : $\gamma_1 = 2$, $\gamma_2 = 1.4$.

We consider for the domain a channel with a 20% sinusoidal bump.



The initial condition corresponds to a constant state

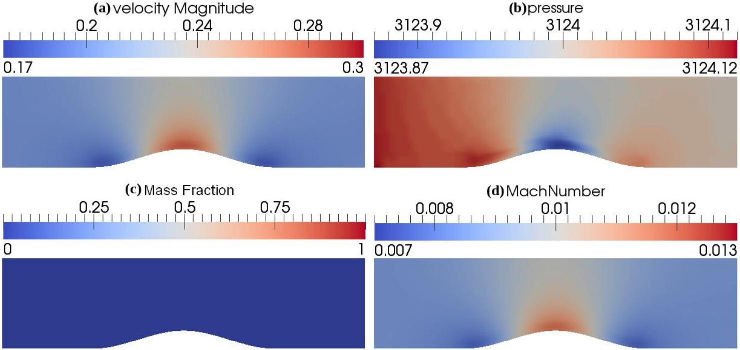
$$(\rho, Y, p, u, v) = (7.81, 0, 3124, 0, 0).$$

We impose inlet/outlet and Wall boundary conditions.

This configuration leads to a subsonic flow for $u_{in} = 0.2$ and a transonic flow for $u_{in} = 12$.

flow in a channel with bump : subsonic flow

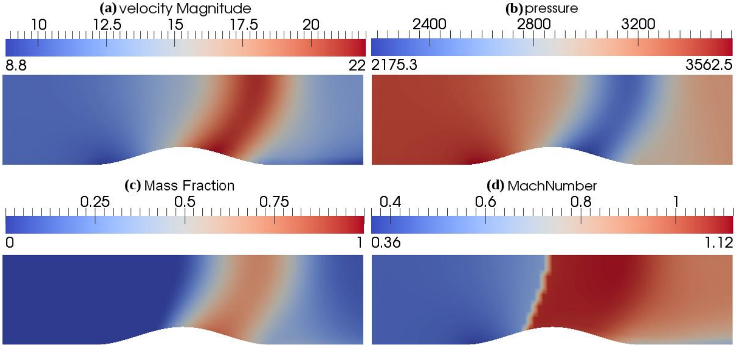
We plot the results obtained for the subsonic test case $(u_{in} = 0.2)$ on a 80×20 quadrangular mesh at time $T = 2s$ with $\beta = Lag$ and $\theta_{ij} = M_{ij}$



Flow speed animation

flow in a channel with bump : transonic flow

We plot the results obtained for the transonic test case $(u_{in} = 12)$ on a 80×20 quadrangular mesh at time $T = 2s$ with $\beta = n$ and $\theta_{ij} = 0$



Flow speed animation