Relative entropy method for the diffusive limit of numerical schemes

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Continuous framework

- Hyperbolic system with source term:

$$\partial_t W + \partial_x f(W) = -R(W). \qquad (H)$$

- Diffusive limit $t \to +\infty$:

$$\partial_t \overline{W} - \partial_x \left(D(\overline{W}) \partial_x \overline{W} \right) = 0. \qquad (P)$$

- Rescaled system:

$$\varepsilon \partial_t W_\varepsilon + \partial_x f(W_\varepsilon) = -\frac{1}{\varepsilon} R(W_\varepsilon). \qquad (H_\varepsilon)$$

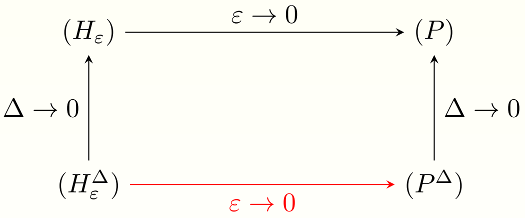
- $(H_\varepsilon) \to (P)$ as $\varepsilon \to 0$.

- **Lattanzio & Tzavaras** : Euler isentropic, p-system.
  $\hookrightarrow$ relative entropy $\Rightarrow$ convergence rate in $O(\varepsilon^4)$.

Motivations

AP schemes for the diffusive limit:

- Kinetic equations: **Jin, Pareschi & Toscani** , **Naldi & Pareschi**.
- Hyperbolic systems: **Gosse & Toscani** , **Buet & Després**, **CB & Turpault**.



Aim: Study the limit $\varepsilon \to 0$ in the discrete framework.

- Strategy: relative entropy (**Lattanzio & Tzavaras, 13**).
- Particular system: p-system with friction.

From the p-system to the porous media equation

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = -\sigma\, u, \end{cases} \tag{H}$$

pressure $p(s) = s^{-\gamma}$, $\gamma > 1$.

Rescaling: (**Naldi & Pareschi, Mei**)

$$t \to t/\varepsilon, \quad u \to \varepsilon\, u^\varepsilon, \quad \sigma \to \sigma/\varepsilon$$

$$\Rightarrow \begin{cases} \partial_t \tau^\varepsilon - \partial_x u^\varepsilon = 0, \\ \varepsilon^2 \partial_t u^\varepsilon + \partial_x p(\tau^\varepsilon) = -\sigma\, u^\varepsilon. \end{cases} \tag{H_ε}$$

Limit $\varepsilon \to 0$:

$$\tau^\varepsilon = \overline{\tau} + \varepsilon\, \tau_1 + \varepsilon^2\, \tau_2 + \ldots$$

$$u^\varepsilon = \overline{u} + \varepsilon\, u_1 + \varepsilon^2\, u_2 + \ldots$$

Order ε^0:

$$\begin{cases} \partial_t \overline{\tau} + \dfrac{1}{\sigma} \partial_{xx} p(\overline{\tau}) = 0, \\ \partial_x p(\overline{\tau}) = -\sigma\, \overline{u}. \end{cases} \tag{P}$$

Relative entropy

Entropy-entropy flux pair for (H_ε)

$$\eta^\varepsilon(\tau,u) = \varepsilon^2 \frac{u^2}{2} - P(\tau), \quad \psi(\tau,u) = u\,p(\tau),$$

where $P(\tau) = \int_0^\tau p(s)ds$, satisfying

$$\partial_t \eta^\varepsilon(\tau^\varepsilon,u^\varepsilon) + \partial_x \psi(\tau^\varepsilon,u^\varepsilon) \leq -\sigma(u^\varepsilon)^2.$$

Relative entropy

$$\begin{aligned}\eta^\varepsilon(\tau,u|\overline{\tau},\overline{u}) &:= \eta^\varepsilon(\tau,u) - \eta^\varepsilon(\overline{\tau},\overline{u}) - \nabla\eta^\varepsilon(\overline{\tau},\overline{u}) \cdot \begin{pmatrix} \tau - \overline{\tau} \\ u - \overline{u} \end{pmatrix} \\ &= \frac{\varepsilon^2}{2}(u-\overline{u})^2 - P(\tau|\overline{\tau}),\end{aligned}$$

où $P(\tau|\overline{\tau}) := P(\tau) - P(\overline{\tau}) - p(\overline{\tau})(\tau - \overline{\tau})$.

Associated relative entropy flux: $\psi(\tau,u|\overline{\tau},\overline{u}) = (u-\overline{u})(p(\tau) - p(\overline{\tau}))$.

Convergence rate

$\phi^\varepsilon(t) := \int_{\mathbb{R}} \eta^\varepsilon(\tau^\varepsilon, u^\varepsilon | \overline{\tau}, \overline{u})\, dx.$

Theorem (Lattanzio and Tzavaras, 2013)

Let $(\overline{\tau}, \overline{u})$ be a smooth solution of (P) such that

- $\overline{\tau} \geq c > 0,$
- $\|\partial_{xx} p(\overline{\tau})\|_{L^\infty(\mathbb{R}\times[0,T))} \leq K < +\infty,$
- $\|\partial_{xt} p(\overline{\tau})\|_{L^2(\mathbb{R}\times[0,T))} \leq K < +\infty.$

Then

$$\phi^\varepsilon(t) \leq C\left(\phi^\varepsilon(0) + \varepsilon^4\right).$$

Sketch of proof (1/2)

- **1st step:** write an equation verified by the relative entropy.

$$\partial_t \eta^\varepsilon(\tau^\varepsilon, u^\varepsilon | \overline{\tau}, \overline{u}) + \partial_x \psi(\tau^\varepsilon, u^\varepsilon | \overline{\tau}, \overline{u}) =$$
$$- \sigma(u^\varepsilon - \overline{u})^2 + \frac{1}{\sigma} \partial_{xx} p(\overline{\tau}) p(\tau^\varepsilon | \overline{\tau}) + \frac{\varepsilon^2}{\sigma} \partial_{xt} p(\overline{\tau})(u^\varepsilon - \overline{u}).$$

- **2nd step:** integrate on $Q_t = \mathbb{R} \times [0, t)$.

$$\phi^\varepsilon(t) - \phi^\varepsilon(0) \leq -\sigma \int_{Q_t} (u^\varepsilon - \overline{u})^2 + \underbrace{\frac{1}{\sigma} \int_{Q_t} \partial_{xx} p(\overline{\tau}) p(\tau^\varepsilon | \overline{\tau})}_{T_1}$$
$$+ \underbrace{\frac{\varepsilon^2}{\sigma} \int_{Q_T} \partial_{xt} p(\overline{\tau})(u^\varepsilon - \overline{u})}_{T_2}.$$

Sketch of proof (2/2)

- **Control of T_1:** $\exists C>0$ such that $\forall \tau,\, \overline{\tau} \geq c > 0$, $|p(\tau|\overline{\tau})| \leq -C\,P(\tau|\overline{\tau})$

$$\Rightarrow \frac{1}{\sigma}\int_{Q_t} |\partial_{xx}p(\overline{\tau})p(\tau^\varepsilon|\overline{\tau})|\,dx\,ds \leq -\frac{C}{\sigma}\|\partial_{xx}p(\overline{\tau})\|_\infty \int_{Q_t} P(\tau^\varepsilon|\overline{\tau})\,dx\,ds$$

$$\leq \frac{C}{\sigma}\int_0^t \phi^\varepsilon(s)\,ds.$$

- **Control of T_2:** Cauchy-Schwarz and Young inequalities

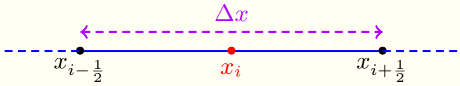
$$\Rightarrow \frac{\varepsilon^2}{\sigma}\int_{Q_t} |\partial_{xt}p(\overline{\tau})(u^\varepsilon - \overline{u})| \leq \frac{\sigma}{2}\int_{Q_t}(u^\varepsilon - \overline{u})^2 + \frac{\varepsilon^4}{2\sigma^3}\int_{Q_t}|\partial_{xt}p(\overline{\tau})|^2$$

$$\leq \frac{\sigma}{2}\int_{Q_t}(u^\varepsilon - \overline{u})^2\,dx\,ds + C\,\varepsilon^4.$$

- **Conclusion:** using the Gronwall Lemma,

$$\phi^\varepsilon(t) \leq \phi^\varepsilon(0) + C\varepsilon^4 + \frac{C}{\sigma}\int_0^t \phi^\varepsilon(s)ds \quad \Rightarrow \quad \phi^\varepsilon(t) \leq \left(\phi^\varepsilon(0) + C\,\varepsilon^4\right)e^{CT/\sigma}.$$

From the continuous to the semidiscrete framework



- **Finite volume scheme**, with HLL fluxes:

$$\frac{d}{dt}\tau_i = \frac{1}{2\Delta x}(u_{i+1} - u_{i-1}) + \frac{\lambda}{2\Delta x}(\tau_{i+1} - 2\tau_i + \tau_{i-1}), \qquad (H_\varepsilon^\Delta)$$
$$\frac{d}{dt}u_i = -\frac{1}{2\varepsilon^2\Delta x}(p(\tau_{i+1}) - p(\tau_{i-1})) + \frac{\lambda}{2\Delta x}(u_{i+1} - 2u_i + u_{i-1}) - \frac{\sigma}{\varepsilon^2}u_i,$$

where $\lambda = \max_{i\in\mathbb{Z}}(\sqrt{-p'(\tau_i)})$.

- **Limit scheme** $(\varepsilon \to 0)$:

$$\frac{d}{dt}\overline{\tau}_i = \frac{1}{2\Delta x}(\overline{u}_{i+1} - \overline{u}_{i-1}) + \frac{\lambda}{2\Delta x}(\overline{\tau}_{i+1} - 2\overline{\tau}_i + \overline{\tau}_{i-1}),$$
$$\sigma\overline{u}_i = -\frac{1}{2\Delta x}(p(\overline{\tau}_{i+1}) - p(\overline{\tau}_{i-1})). \qquad (P^\Delta)$$

Convergence rate

$$\eta_i^\varepsilon := \eta^\varepsilon(\tau_i, u_i | \overline{\tau}_i, \overline{u}_i), \quad \phi^\varepsilon(t) := \sum_{i\in\mathbb{Z}} \Delta x\, \eta_i^\varepsilon(t).$$

Theorem

We assume that $(\overline{\tau}_i, \overline{u}_i)_{i\in\mathbb{Z}}$ solution of (P^Δ) satisfies:

- $\overline{\tau}_i \geq c > 0, \quad \|D_{xx}p(\overline{\tau})\|_\infty, \|D_{xt}p(\overline{\tau})\|_2 \leq K < +\infty,$
- $\|\tilde{D}_{xx}\overline{\tau}\|_\infty \leq K, \quad \|D_x\overline{\tau}\|_\infty \leq K, \quad \|D_{xx}\overline{u}\|_2 \leq K.$

Définitions

Then

$$\phi^\varepsilon(t) \leq B(\phi^\varepsilon(0) + \varepsilon^4),$$

where B only depends on p, T, K and c.

Sketch of proof (1/3)

- **1st step:** write an equation verified by the semidiscrete relative entropy.

$$\frac{d\eta_i^\varepsilon}{dt} + \frac{1}{\Delta x}(\psi_{i+\frac{1}{2}} - \psi_{i-\frac{1}{2}}) =$$
$$-\sigma(u_i - \overline{u}_i)^2 + \frac{1}{\sigma}\frac{p(\overline{\tau}_{i+2}) - 2p(\overline{\tau}_i) + p(\overline{\tau}_{i-2})}{(2\Delta x)^2} p(\tau_i|\overline{\tau}_i)$$
$$+ \frac{\varepsilon^2}{\sigma}(u_i - \overline{u}_i)\frac{d}{dt}\left(\frac{p(\overline{\tau}_{i+1}) - p(\overline{\tau}_{i-1})}{2\Delta x}\right) + R_i^u + R_i^\tau,$$

where

$$\psi_{i+\frac{1}{2}} := \frac{1}{2}(u_i - \overline{u}_i)(p(\tau_{i+1}) - p(\overline{\tau}_{i+1})) + \frac{1}{2}(p(\tau_i) - p(\overline{\tau}_i))(u_{i+1} - \overline{u}_{i+1}),$$

and

$$R_i^u := \frac{\lambda\varepsilon^2}{2\Delta x}(u_i - \overline{u}_i)(u_{i+1} - 2u_i + u_{i-1}),$$
$$R_i^\tau := -\frac{\lambda}{2\Delta x}\left[(p(\tau_i) - p(\overline{\tau}_i))(\tau_{i+1} - 2\tau_i + \tau_{i-1}) - (\tau_i - \overline{\tau}_i)p'(\overline{\tau}_i)(\overline{\tau}_{i+1} - 2\overline{\tau}_i + \overline{\tau}_{i-1})\right].$$

Sketch of proof (1/3)

- **1st step:** write an equation verified by the semidiscrete relative entropy.

$$\frac{d\eta_i^\varepsilon}{dt} + \frac{1}{\Delta x}(\psi_{i+\frac{1}{2}} - \psi_{i-\frac{1}{2}}) =$$
$$-\sigma(u_i - \overline{u}_i)^2 + \frac{1}{\sigma}(\tilde{D}_{xx}p(\overline{\tau}))_i\, p(\tau_i|\overline{\tau}_i)$$
$$+\frac{\varepsilon^2}{\sigma}(u_i - \overline{u}_i)\frac{d}{dt}(D_x p(\overline{\tau}))_i + R_i^u + R_i^\tau,$$

where

$$\psi_{i+\frac{1}{2}} \approx (u-\overline{u})(p(\tau) - p(\overline{\tau}))_{|x_{i+\frac{1}{2}}} = \psi(\tau, u|\overline{\tau}, \overline{u})_{|x_{i+\frac{1}{2}}},$$

and

$$R_i^u := \frac{\lambda\varepsilon^2}{2}\Delta x(u_i - \overline{u}_i)(D_{xx}u)_i,$$
$$R_i^\tau := -\frac{\lambda}{2}\Delta x\left[(p(\tau_i) - p(\overline{\tau}_i))(D_{xx}\tau)_i - (\tau_i - \overline{\tau}_i)p'(\overline{\tau}_i)(D_{xx}\overline{\tau})_i\right].$$

Sketch of proof (2/3)

- **2nd step:** integrate over $[0,t)$ and sum over $i \in \mathbb{Z}$.

$$\phi^\varepsilon(t) - \phi^\varepsilon(0) \leq -\sigma \int_0^t \sum_{i\in\mathbb{Z}} \Delta x (u_i - \overline{u}_i)^2(s)\, ds$$

$$+ \underbrace{\frac{1}{\sigma} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x \left(\tilde{D}_{xx} p(\overline{\tau})\right)_i p(\tau_i | \overline{\tau}_i)\, ds}_{T_1}$$

$$+ \underbrace{\frac{\varepsilon^2}{\sigma} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x (u_i - \overline{u}_i) \frac{d}{dt} \left(D_x p(\overline{\tau})\right)_i ds}_{T_2}$$

$$+ \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^u\, ds + \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^\tau\, ds$$

T_1 et T_2

Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x \, R_i^u \, ds = \frac{\varepsilon^2 \lambda \Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x \, (D_{xx}u)_i \, (u_i - \overline{u}_i) \, ds$$

Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^u\, ds = \frac{\varepsilon^2\,\lambda\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\,(D_{xx}(u-\overline{u}))_i\,(u_i - \overline{u}_i)\,ds$$
$$+ \frac{\varepsilon^2\,\lambda\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\,(D_{xx}\overline{u})_i\,(u_i - \overline{u}_i)\,ds.$$

Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^u\, ds = \frac{\varepsilon^2\,\lambda\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, (D_{xx}(u-\overline{u}))_i\, (u_i - \overline{u}_i)\, ds + \frac{\varepsilon^2\,\lambda\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, (D_{xx}\overline{u})_i\, (u_i - \overline{u}_i)\, ds.$$

  - 1st term: we perform a discrete integration by parts.

$$\frac{\varepsilon^2\,\lambda\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, (D_{xx}(u-\overline{u}))_i\, (u_i - \overline{u}_i)\, ds = -\frac{\varepsilon^2\,\lambda\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x \left[(D_x(u-\overline{u}))_{i+\frac{1}{2}}\right]^2 \leq 0.$$

Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^u\, ds \leq \frac{\varepsilon^2\,\lambda\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, (D_{xx}\overline{u})_i\, (u_i - \overline{u}_i)\, ds.$$

  - 2nd term: we use the Cauchy-Schwarz and Young inequalities.

$$\frac{\varepsilon^2\,\lambda\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \sqrt{\Delta x}\, (D_{xx}\overline{u})_i\, \sqrt{\Delta x}(u_i - \overline{u}_i)\, ds \leq$$

$$\frac{\lambda\,\Delta x}{2} \frac{\varepsilon^4}{\theta} \|D_{xx}\overline{u}\|_2^2 + \frac{\lambda\,\Delta x}{2}\theta \int_0^t \sum_{i\in\mathbb{Z}} \Delta x (u_i - \overline{u}_i)^2.$$

Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\Rightarrow \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^u\, ds \leq C\,\varepsilon^4 + \frac{\lambda\,\theta\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x (u_i - \overline{u}_i)^2\, ds.$$

Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\Rightarrow \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^u\, ds \leq C\,\varepsilon^4 + \frac{\lambda\,\theta\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x (u_i - \overline{u}_i)^2\, ds.$$

In the same way,

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^\tau\, ds \leq C \int_0^t \phi^\varepsilon(s)\, ds.$$

Sketch of proof (3/3)

- **3rd step:** control of the numerical error terms.

$$\Rightarrow \int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^u\, ds \leq C\,\varepsilon^4 + \frac{\lambda\,\theta\,\Delta x}{2} \int_0^t \sum_{i\in\mathbb{Z}} \Delta x (u_i - \overline{u}_i)^2\, ds.$$

In the same way,

$$\int_0^t \sum_{i\in\mathbb{Z}} \Delta x\, R_i^\tau\, ds \leq C \int_0^t \phi^\varepsilon(s)\, ds.$$

- **Conclusion:**

$$\phi^\varepsilon(t) \leq \left(-\frac{\sigma}{2} + \frac{\lambda\,\Delta x\,\theta}{2}\right) \int_0^t \sum_{i\in\mathbb{Z}} \Delta x (u_i - \overline{u}_i)^2 + \phi^\varepsilon(0) + C\,\varepsilon^4 + C \int_0^t \phi^\varepsilon(s)\, ds.$$

We choose $\theta = \dfrac{\sigma}{\lambda\,\Delta x}$, and we conclude by using the Gronwall Lemma.

Fully discrete scheme

Jin, Pareschi & Toscani, 98 : splitting scheme based on the reformulation

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = \dfrac{1}{\varepsilon^2}\left(\sigma\, u + (1-\varepsilon^2)\partial_x p(\tau)\right). \end{cases}$$

- **Step 1: transport**

$$\tau_i^{n+\frac{1}{2}} = \tau_i^n + \frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{\lambda \Delta t}{2\Delta x}(\tau_{i+1}^n - 2\tau_i^n + \tau_{i-1}^n),$$
$$u_i^{n+\frac{1}{2}} = u_i^n - \frac{\Delta t}{2\Delta x}(p(\tau_{i+1}^n) - p(\tau_{i-1}^n)) + \frac{\lambda \Delta t}{2\Delta x}(u_{i+1}^n - 2u_i^n + u_{i-1}^n).$$

- **Step 2: relaxation**

$$\tau_i^{n+1} = \tau_i^{n+\frac{1}{2}},$$
$$u_i^{n+1} = u_i^{n+\frac{1}{2}} - \Delta t \frac{\sigma}{\varepsilon^2} u_i^{n+1} - \Delta t \left(\frac{1-\varepsilon^2}{\varepsilon^2}\right) \frac{p(\tau_{i+1}^{n+1}) - p(\tau_{i-1}^{n+1})}{2\Delta x}.$$

Fully discrete scheme

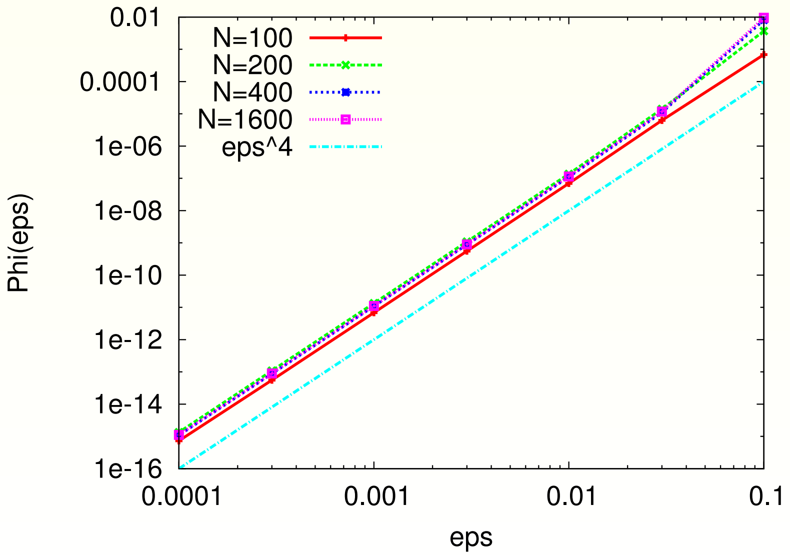
Asymptotic discretization $\varepsilon \to 0$

$$
\begin{aligned}
\overline{\tau}_i^{n+1} &= \overline{\tau}_i^n + \frac{\Delta t}{2\Delta x}(\overline{u}_{i+1}^n - \overline{u}_{i-1}^n) + \frac{\lambda \Delta t}{2\Delta x}(\overline{\tau}_{i+1}^n - 2\overline{\tau}_i^n + \overline{\tau}_{i-1}^n), \\
\overline{u}_i^{n+1} &= -\frac{1}{2\sigma\Delta x}(p(\overline{\tau}_{i+1}^n) - p(\overline{\tau}_{i-1}^n)).
\end{aligned}
$$

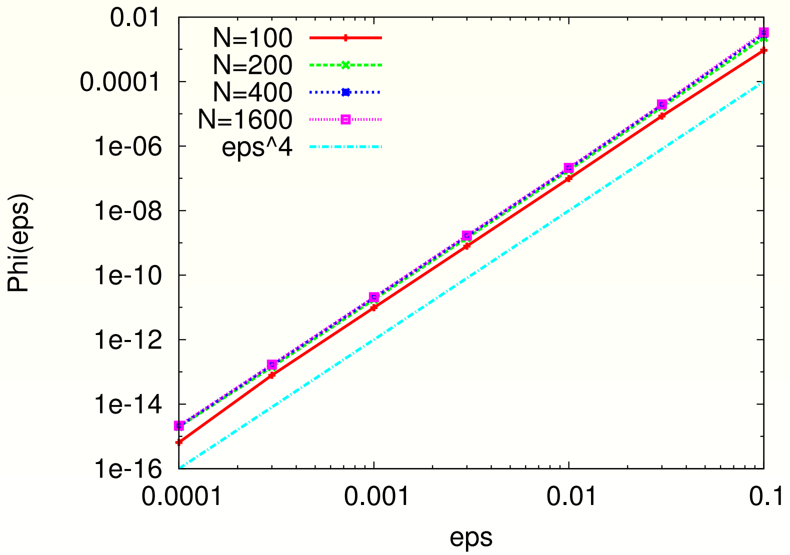
Data

- Final time: $T = 10^{-2}$.
- Spatial domain: $[-4, 4]$, cell number $= 100, 200, 400, 1600$.
- Pressure law: $p(s) = s^{-1.4}$.
- $\sigma = 1$, $\varepsilon \in \{10^{-1}, 3.10^{-2}, 10^{-2}, 3.10^{-3}, 10^{-3}, 3.10^{-4}, 10^{-4}\}$.
- Boundary conditions: homogeneous Neumann.
- Two initial data:
  - discontinuous: $\tau_0(x) = \begin{cases} 2 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$,
  - smooth: $\tau_0(x) = \exp(-100x^2) + 1$.

Discontinuous I.C.



Smooth I.C.


N=100
N=200
N=400
N=1600
eps^4
Phi(eps)
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
1e-14
1e-16
0.0001
0.001
0.01
0.1
eps

Convergence rate for the Jin, Pareschi and Toscani scheme

Theorem 1/3

Let $(\tau_i^n, u_i^n)_{i\in\mathbb{Z}}$ given by the Jin-Pareschi-Toscani scheme and $(\overline{\tau}_i^n, \overline{u}_i^n)_{i\in\mathbb{Z}}$ given by the asymptotic scheme.
We assume the existence of a positive constant K such that

$$\|\delta_t \overline{u}^{n+1/2}\|_{L_x^2}^2 \leq K, \qquad \|\delta_{xx}\overline{u}^n\|_{L_x^2}^2 \leq K, \qquad \|\delta_{tx} p(\overline{\tau}^{n+1/2})\|_{L_x^2}^2 \leq K,$$

$$\|\delta_t \overline{\tau}^{n+1/2}\|_{L_x^\infty} \leq K \qquad \|\tilde{\delta}_x \overline{\tau}^n\|_{L_x^\infty} \leq K.$$

We assume the existence of a positive constant L_τ such that specific volumes are bounded as follows:

$$\frac{1}{L_\tau} \leq \tau_i^n, \overline{\tau}_i^n \leq L_\tau \ \forall\ i \in \mathbb{Z},\ 0 \leq n \leq N.$$

Theorem 2/3

We assume the existence of a positive constant L_p such that the pressure p and its three first derivate are bounded as follows:

$$\begin{aligned} \frac{1}{L_p} &\leq p(\tau) \leq L_p, & -L_p \leq p'(\tau) \leq -\frac{1}{L_p}, \\ \frac{1}{L_p} &\leq p''(\tau) \leq L_p, & -L_p \leq p^{(3)}(\tau) \leq -\frac{1}{L_p}, \end{aligned} \qquad \forall\, \tau \in [1/L_\tau, L_\tau]\,.$$

We assume the following parabolic CFL condition on Δx and Δt:

$$\frac{\Delta t}{\Delta x^2} \leq C_p,$$

where

$$C_p = \frac{\sigma}{8\left(2 + L_p^3 + 15L_p^2/2 + 4L_\tau\left(L_p/2 + 2/3 + 4L\tau L_p/3\right)\right)}.$$

Theorem 3/3

We assume that ε and Δt are a small enough according to:

$$\varepsilon^2 \leq \frac{\sigma \Delta x}{2\left(2\lambda + \Delta t/2 + 5L_p^2 C_p\left(1 + K^2\Delta t^2/4\right)/2\right)},$$

and

$$\Delta t \leq \min\left(\frac{\sigma}{16}, \frac{\sqrt{\sigma}}{\sqrt{30 C_p} K L_p}\right).$$

Then the following convergence rate holds:

$$\phi_\varepsilon^{N+1} \leq M\left(\phi_\varepsilon^0 + \|u^0 - \overline{u}^0\|_{L_x^2}^2 + \varepsilon^4\right),$$

where M is a positive constant only depending on on the final time $T = N\Delta t$ and the parameters σ, λ, K, L_τ and L_p.

Work in progress

- In the continuous framework:
  - generalization to other models
  - general formalism ?
- In the discrete framework, generalization of the method:
  - for general IMEX schemes
  - for high order schemes
  - Kinetic models
- Low Mach extension

Thank you for your attention !

Definition of the semidiscrete norms

Let $v : Q_T = \mathbb{R} \times [0,T) \to \mathbb{R}$ defined by $v(x,t) = v_i(t), \quad \forall x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$.

- $\|D_x v\|_\infty := \sup_{t\in[0,T)} \sup_{i\in\mathbb{Z}} \left| \dfrac{v_{i+1} - v_i}{\Delta x} \right|,$
- $\|\tilde{D}_{xx} v\|_\infty := \sup_{t\in[0,T)} \sup_{i\in\mathbb{Z}} \left| \dfrac{v_{i+2} - 2v_i + v_{i-2}}{(2\Delta x)^2} \right|,$
- $\|D_{xx} v\|_\infty := \sup_{t\in[0,T)} \sup_{i\in\mathbb{Z}} \left| \dfrac{v_{i+1} - 2v_i + v_{i-1}}{(\Delta x)^2} \right|,$
- $\|D_{tx} v\|_2 := \left(\displaystyle\int_0^T \sum_{i\in\mathbb{Z}} \Delta x \left| \dfrac{d}{dt} \left(\dfrac{v_{i+1} - v_{i-1}}{2\Delta x} \right)(t) \right|^2 dt \right)^{\frac{1}{2}},$
- $\|D_{xx} v\|_2 := \left(\displaystyle\int_0^T \sum_{i\in\mathbb{Z}} \Delta x \left| \left(\dfrac{v_{i+1} - 2v_i + v_{i-1}}{(\Delta x)^2} \right)(t) \right|^2 dt \right)^{\frac{1}{2}}.$

Retour