Summary of research works

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Quantizations of character varieties of surfaces

Context and history. Let $\Sigma_{g,n}$ be the compact oriented surface of genus g with n boundary components. I am working with certain quantizations of the character variety of $\Sigma_{g,n}$. Let G be a semisimple complex algebraic Lie group and recall that the representation variety and the character variety are respectively

$$\mathcal{R}_G(\Sigma_{g,n}) = \mathrm{Hom}_{\mathrm{Grp}}\big(\pi_1(\Sigma_{g,n}), G\big), \qquad \mathcal{X}_G(\Sigma_{g,n}) = \mathcal{R}_G(\Sigma_{g,n})/G$$

where the action of G on $\mathcal{R}_G(\Sigma_{g,n})$ is by conjugation (more precisely $\mathcal{X}_G(\Sigma_{g,n})$ is a GIT quotient for this action). There is an identificaton between $\mathcal{X}_G(\Sigma_{g,n})$ and the moduli space of flat G-connections on $\Sigma_{g,n}$, which is known to have a Poisson structure [AB83]. As a result the algebra of functions $\mathcal{O}\big[\mathcal{X}_G(\Sigma_{g,n})\big]$ inherits a Poisson bracket. The quantization of this structure gave rise to many works in quantum topology and mathematical physics. Here we consider two quantizations: the quantum moduli algebras and the skein algebras.

- *Quantum moduli algebras, a.k.a. combinatorial quantization.* Here we explain the idea of this construction in the easier case of surfaces $\Sigma_{g,n+1}$ (at least one boundary component). Constructions and results in the case of closed surfaces are the subject of Theorems 1 and 5 below. Note first that $\pi_1(\Sigma_{g,n+1})$ is a free group with $2g+n$ generators. Hence

$$\mathcal{O}\big[\mathcal{R}_G(\Sigma_{g,n+1})\big] = \mathcal{O}(G)^{\otimes(2g+n)}, \qquad \mathcal{O}\big[\mathcal{X}_G(\Sigma_{g,n+1})\big] = \big(\mathcal{O}(G)^{\otimes(2g+n)}\big)^{G\text{-inv}}.$$

Taking advantage of this, Fock–Rosly [FR93] gave an explicit Poisson bracket on $\mathcal{O}\big[\mathcal{R}_G(\Sigma_{g,n+1})\big]$ which induces the Atiyah–Bott bracket on $\mathcal{O}\big[\mathcal{X}_G(\Sigma_{g,n+1})\big]$. The idea of the quantization [Ale94, AGS95, BR95] is to replace $\mathcal{O}(G)$ by the quantized algebra of functions $\mathcal{O}_q(G)$ and the action of the group G by an action of the quantum envelopping algebra $U_q(\mathfrak{g})$ on $\mathcal{O}_q(G)$[1], where $\mathfrak{g}$ is the Lie algebra of G. Then one defines the $\mathbb{C}(q)$-vector spaces

$$\mathcal{L}_{g,n}(\mathfrak{g}) = \mathcal{O}_q(G)^{\otimes(2g+n)}, \qquad \mathcal{L}_{g,n}^{\mathrm{inv}}(\mathfrak{g}) = \big(\mathcal{L}_{g,n}(\mathfrak{g})\big)^{U_q(\mathfrak{g})\text{-inv}}.$$

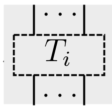
The multiplication in $\mathcal{L}_{g,n}(\mathfrak{g})$ is obtained by twisting the usual product in $\mathcal{O}_q(G)^{\otimes(2g+n)}$ thanks to the R-matrix of $U_q(\mathfrak{g})$. The definition is such that $\mathcal{L}_{g,n}(\mathfrak{g})$ is a $U_q(\mathfrak{g})$-module-algebra and hence $\mathcal{L}_{g,n}^{\mathrm{inv}}(\mathfrak{g})$ is a subalgebra, called the quantum moduli algebra. Since the definitions only use the Hopf structure

[1] For the definition of $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$, see e.g. [BG02, §I.6, §I.7]. The right coadjoint action coad is defined by $\forall\, h, x \in U_q(\mathfrak{g})$, $\forall\, \varphi \in \mathcal{O}_q(G)$, $\langle \mathrm{coad}(h)(\varphi), x\rangle = \langle \varphi, \mathrm{ad}(h)(x)\rangle$ where $\mathrm{ad}(h)(x) = \sum_{(h)} h_{(1)} x S(h_{(2)})$.

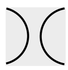
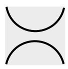
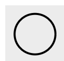
and the R-matrix of $U_q(\mathfrak{g})$, we can more generally define an algebra $\mathcal{L}_{g,n}(H)$ for any quasitriangular Hopf k-algebra H, with k a field. It is $(H^*)^{\otimes(2g+n)}$ as a k-vector space (where H^* is an appropriate dual if H is infinite-dimesional) and carries an action of H which turns it into a H-module-algebra. The subalgebra of H-invariants elements is denoted by $\mathcal{L}_{g,n}^{\mathrm{inv}}(H)$. See [BFR23, §3.1, §4.1] for detailed explanations of these definitions.

The algebras $\mathcal{L}_{g,n}(H)$ have been recovered in the context of factorization homology in [BZBJ18] and are algebra of functions in lattice gauge field theory [MW21].

- *Skein algebras.* This construction works directly for any surface $\Sigma_{g,n}$. Let H be a ribbon Hopf k-algebra [CP94, §4.2.C]. Denote by F_{RT} the Reshetikhin–Turaev functor, which associates a H-linear map to any H-colored[2] oriented ribbon graph in $[0,1]^3$ [CP94, §5.3]. The skein algebra of $\Sigma_{g,n}$ associated to H, denoted by $\mathcal{S}_H(\Sigma_{g,n})$, is the k-vector space generated by the isotopy classes of H-colored oriented ribbon links (with coupons) modulo the skein relations:

$$\sum_i \lambda_i F_{\mathrm{RT}}(T_i) = 0 \qquad \Longrightarrow \qquad \sum_i$$  $$= 0 \text{ in } \mathcal{S}_H(\Sigma). \tag{1}$$

The T_i are any ribbon graphs and the $\lambda_i \in k$ are scalars such that the linear equation on the left holds. The right hand-side represents a linear combination of links which are equal outside of the cube in $\Sigma \times [0,1]$ which is depicted in grey. The product of two links L_1, L_2 in $\mathcal{S}_H(\Sigma_{g,n})$ is obtained by putting L_1 below L_2 in $\Sigma \times [0,1]$. The case $H = U_q(\mathfrak{sl}_2)$ is special because any H-module is a direct summand of some tensor power $V_2^{\otimes N}$, where V_2 is the fundamental representation on $\mathbb{C}(q)^2$. Hence, due to the skein relations (1), it is enough to consider edges colored by V_2 and coupons colored by morphisms $V_2^{\otimes N} \to V_2^{\otimes M}$. Moreover any such coupon can be expressed by crossings, cups and caps. After these reductions, all the skein relations (1) derive from the Kauffman bracket relations:

 $= q^{1/2}$  $+\ q^{-1/2}$   $= -(q^2+q^{-2})$

Thus $\mathcal{S}_{U_q(\mathfrak{sl}_2)}(\Sigma_{g,n})$ is the Kauffman bracket skein algebra $\mathcal{S}_q(\Sigma_{g,n})$ defined by [Prz91, Tur91]. The Poisson bracket on $\mathcal{O}\big[\mathcal{X}_{\mathrm{SL}_2(\mathbb{C}}(\Sigma_{g,n})\big]$ admits a geometric description based on curves on $\Sigma_{g,n}$ [Gol86] and $\mathcal{S}_q(\Sigma_{g,n})$ is a quantization of this structure (this is well explained in [BFKB99]). One also expect this for other semisimple Lie groups G.

Projective representations of mapping class groups. Let $\mathrm{MCG}(\Sigma_{g,n})$ be the mapping class group of $\Sigma_{g,n}$, *i.e.* the group of isotopy classes of homeomorphisms which preserve the orientation and fix the boundary pointwise. There is a natural action of $\mathrm{MCG}(\Sigma_{g,n})$ on $\mathcal{R}_G(\Sigma_{g,n})$ and $\mathcal{X}_G(\Sigma_{g,n})$, and hence on their algebras of functions. Does this action survives quantization? The answer is yes: assuming that H is a ribbon Hopf algebra [CP94, §4.2.C], Alekseev–Schomerus [AS96] provided a morphism of groups

$$\mathrm{MCG}(\Sigma_{g,n+1}) \longrightarrow \mathrm{Aut}_{\mathrm{Alg}}\big(\mathcal{L}_{g,n}(H)\big). \tag{2}$$

But there is more: under the assumption that H is a finite-dimensional semisimple and modular, [AS96] used the algebraic properties of $\mathcal{L}_{g,n}(H)$ to construct a projective representation of $\mathrm{MCG}(\Sigma_{g,n})$ (note that one boundary component disappeared). They also proved that this representation is equivalent to the one given by the Reshetikhin–Turaev TQFT for H-mod [RT91].

During my PhD thesis I generalized this construction for closed surfaces, under the assumptions that H is finite-dimensional and factorizable[3], but not necessarily semisimple ([Fai19] for the case of

[2] This means that every ribbon edge in the graph is labelled by a finite-dimensional H-module and any coupon is labelled by a H-morphism compatible with the labels of its edges. For $H = U_q(\mathfrak{g})$ we restrict the colors to type 1 modules [CP94, §10.1].

[3] A quasitriangular Hopf algebra H with R-matrix $R = a_i \otimes b_i$ is factorizable if the element $R_{21}R \in H^{\otimes 2}$ induces an isomorphism of vectors spaces $H^* \xrightarrow{\sim} H$, where $R_{21} = b_i \otimes a_i$.

the torus and [Fai20b] for the general case). The starting point is that certain morphisms of algebras defined in [Ale94] can be combined into an isomorphism

$$\rho : \mathcal{L}_{g,0}(H) \xrightarrow{\sim} \mathrm{End}_k\big((H^*)^{\otimes g}\big).$$

Hence $\mathcal{L}_{g,0}(H)$ is isomorphic to a matrix algebra, and any automorphism is determined by an invertible element unique up to scalar. As a result we get a projective representation

$$Z : \mathrm{MCG}(\Sigma_{g,1}) \longrightarrow \mathrm{Aut}_{\mathrm{Alg}}\big(\mathcal{L}_{g,0}(H)\big) \xrightarrow{\sim} \mathcal{L}_{g,0}(H)^\times/k \xrightarrow{\rho} \mathrm{PGL}\big((H^*)^{\otimes g}\big). \tag{3}$$

Now we want to close the boundary component, *i.e.* pass from $\Sigma_{g,1}$ to $\Sigma_{g,0}$. This is based on three facts. The first fact is that the composition of the first two maps in (3) takes values in $\mathcal{L}^{\mathrm{inv}}_{g,0}(H)^\times/k$. The second fact is the existence of a subspace $\mathrm{Inv}_g \subset (H^*)^{\otimes g}$[4] which is stable by $\rho(a)$ for all $a \in \mathcal{L}^{\mathrm{inv}}_{g,0}(H)$, which gives a group morphism:

$$Z_{\mathrm{Inv}} : \mathrm{MCG}(\Sigma_{g,1}) \longrightarrow \mathrm{Aut}_{\mathrm{Alg}}\big(\mathcal{L}_{g,0}(H)\big) \xrightarrow{\sim} \mathcal{L}^{\mathrm{inv}}_{g,0}(H)^\times/k \xrightarrow{\rho} \mathrm{PGL}(\mathrm{Inv}_g).$$

The third fact is that $\mathrm{MCG}(\Sigma_{g,0})$ is the quotient of $\mathrm{MCG}(\Sigma_{g,1})$ by an explicit relation [Waj83], [FM12, §5.2]. It remains to show that Z_{Inv} preserves this relation:

Theorem 1. ([Fai20b], [Fai19] in the case of the torus)
1. Z_{Inv} pass to the quotient and gives a projective representation of $\mathrm{MCG}(\Sigma_{g,0})$.
2. There are explicit formulas for the representation of certain Dehn twists which generate $\mathrm{MCG}(\Sigma_{g,0})$, *see [Fai20b, §5.5].*

For the torus $\Sigma_{1,0}$ we have $\mathrm{MCG}(\Sigma_{1,0}) = \mathrm{SL}_2(\mathbb{Z})$. I described explcitly the representation of $\mathrm{SL}_2(\mathbb{Z})$ obtained when $H = \bar{U}_\varepsilon(\mathfrak{sl}_2)$, the small quantum group of $\mathfrak{sl}_2$ at an even root of unity ε [Fai19, §6.3]. The representation space in this case is the subspace $\mathrm{SLF}\big(\bar{U}_\varepsilon(\mathfrak{sl}_2)\big) \subset \bar{U}_\varepsilon(\mathfrak{sl}_2)^*$ of symmetric linear forms, which I studied in [Fai20a] (similar results on the structure of $\mathrm{SLF}\big(\bar{U}_\varepsilon(\mathfrak{sl}_2)$ were obtained independently in [GT07]). A motivation for this choice of H is the relation with logarithmic CFTs [FGST06].

There is a famous construction by Lyubashenko [Lyu95] of projective representations of mapping class groups and 3-manifolds invariants, which uses a finite tensor category $\mathcal{C}$ as algebraic input. If $\mathcal{C}$ is semisimple this construction recovers the Reshetikhin–Turaev representation and invariants. A key ingredient in Lyubashenko's construction is the coend $\int^{X\in\mathcal{C}} X^* \otimes X$. Actually $\mathcal{L}_{0,1}(H)$ is the coend of $\mathcal{C} = H$-mod [Fai20b, Prop. 6.3]. This is not the only relation between the two theories:

Theorem 2. ([Fai20b, Th 6.4], [Fai19, Th. 5.2] in the case of the torus) *The projective representation Z_{Inv} is equivalent to the Lyubashenko projective representation for $\mathcal{C} = H$-mod (Lyubashenko–Majid [LM94] in the case of the torus).*

The Lyubashenko invariant of 3-manifolds has recently been extended into a TQFT [DGGPR22]. The projective representations of mapping class groups associated to this TQFT are equivalent to those of Lyubashenko [DGGPR23] and hence to those obtained from $\mathcal{L}_{g,n}(H)$ when $\mathcal{C} = H$-mod.

Relating quantum moduli algebras and skein algebras. This is partly joint work with S. Baseilhac and P. Roche. In this section $\Sigma_{g,n}$ is the compact surface of genus g with n punctures (points removed) and let $D \subset \Sigma_{g,n}$ be an open disk. We denote by $\Sigma^{\circ,\bullet}_{g,n}$ the surface $\Sigma^{\circ}_{g,n} = \Sigma_{g,n} \setminus D$ with a puncture ($\bullet$) on the boundary. We consider H-colored oriented ribbon tangles with coupons T in $\Sigma^{\circ,\bullet}_{g,n} \times [0,1]$ such that $\partial T \subset \partial(\Sigma^{\circ,\bullet}_{g,n}) \times [0,1]$ and the boundary points of T have increasing heights

[4] Inv_g is the subspace of invariant elements for a certain "coadjoint action" of H on $(H^*)^{\otimes g}$. For instance if $g = 1$ the action is $h \cdot \varphi = \sum_{(h)} \varphi\big(S^{-1}(h')?h''\big)$, where $\Delta(h) = \sum_{(h)} h' \otimes h''$, and the invariant linear forms are the symmetric ones, *i.e.* $\varphi(xy) = \varphi(yx)$. To understand why this action is related to the boundary of $\Sigma_{g,1}$, see [Fai20b, §4].

when one goes through the boundary curve of $\Sigma_{g,n}^{\circ,\bullet}$ starting from the puncture $\bullet$. In [Fai20c, §4.1] I defined a "holonomy map"

$$\mathrm{hol} : \left\{ \begin{matrix} H\text{-colored oriented ribbon tangles} \\ \text{in } \Sigma_{g,n}^{\circ,\bullet} \times [0,1] \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Tensor with coefficients} \\ \text{in } \mathcal{L}_{g,n}(H) \end{matrix} \right\}. \tag{4}$$

It is an extension to surfaces of the Reshetikhin–Turaev tangle graph invariant [RT90]. We have $\mathrm{hol}(T) \in \mathcal{L}_{g,n}(H) \otimes V_T$ where V_T is a H-module which depends on the number of boundary points of the ribbon graph T and of the orientations of the strands at these points. Actually this generalizes the "Wilson loop map" defined in [BR96, BFK98]

$$W : \left\{ \begin{matrix} H\text{-colored oriented ribbon links} \\ \text{in } \Sigma_{g,n}^{\circ} \times [0,1] \end{matrix} \right\} \longrightarrow \mathcal{L}_{g,n}^{\mathrm{inv}}(H) \tag{5}$$

in the sense that if we restrict hol to ribbon links (*i.e.* ribbon tangles without boundary points) we recover W.

There are two natural operations for ribbon tangles in thickened surfaces:

- stacking product: $T_1 * T_2$ means that we put T_1 below T_2 in $\Sigma_{g,n}^{\circ,\bullet} \times [0,1]$ using isotopy.
- action of the mapping class group: the action of $f \in \mathrm{MCG}(\Sigma_{g,n}^{\circ}) = \mathrm{MCG}(\Sigma_{g,n+1})$ on a ribbon tangle $T \subset \Sigma_{g,n}^{\circ,\bullet} \times [0,1]$ is $f(T) = (f \times \mathrm{id}_{[0,1]})(T)$.

The holonomy map is compatible with these operations:

Theorem 3. [Fai20c, Th. 4.4, Th. 4.5]
1. $\mathrm{hol}(T_1 * T_2)$ *is the Kronecker product*[5] *of* $\mathrm{hol}(T_1)$ *and* $\mathrm{hol}(T_2)$*.*
2. For any $f \in \mathrm{MCG}(\Sigma_{g,n+1})$*,* $\mathrm{hol}\big(f(T)\big) = (\widetilde{f} \otimes \mathrm{id}_{V_T})\big(\mathrm{hol}(T)\big)$*, where* $\widetilde{f} \in \mathrm{Aut}_{\mathrm{Alg}}\big(\mathcal{L}_{g,n}(H)\big)$ *is the image of* f *by the map* (2)*.*

The motivation to introduce the holonomy map was the problem of relating $\mathcal{L}_{g,n}(H)$ with the *stated skein algebra* $\mathcal{S}_H^{\mathrm{st}}(\Sigma_{g,n}^{\circ,\bullet})$. This is a generalization of the skein algebra $\mathcal{S}_H(\Sigma_{g,n}^{\circ})$ whose main features are

- one uses ribbon tangles in $\Sigma_{g,n}^{\circ,\bullet} \times [0,1]$ instead of just using ribbon links,
- there are *boundary skein relations* which again come from the Reshetikhin–Turaev invariant of ribbon tangles for H-mod,
- each boundary point of a ribbon tangle is labelled by a *state*, *i.e.* a vector in the H-module (or its dual, depending on orientation) which colors the strand ending at this point,
- the product is again by stacking, *i.e.* putting a ribbon tangle below another one.

Actually the definition works for more general surfaces than $\Sigma_{g,n}^{\circ,\bullet}$. Stated skein algebras were introduced by Lê [Le16] and further studied by Costantino–Lê [CL19] for $H = U_q(\mathfrak{sl}_2)$; in this case they are a generalization of the Kauffman bracket skein algebras. For $H = U_q(\mathfrak{sl}_n)$ they were defined and studied in [LS21]. For a general ribbon Hopf algebra H they will be defined in the forthcoming paper [CKL], see [BFR23, §6.1] for the case of $\Sigma_{g,n}^{\circ,\bullet}$.

Let $T \subset \Sigma_{g,n}^{\circ,\bullet} \times [0,1]$ be a ribbon tangle. By contracting the states of the boundary points of T with the tensor $\mathrm{hol}(T)$, we get an element $\mathrm{hol}^{\mathrm{st}}(T) \in \mathcal{L}_{g,n}(H)$ ([BFR23, §6.2] and [Fai20c, §5] for the case $H = U_q(\mathfrak{sl}_2)$). This defines a "stated holonomy map"

$$\mathrm{hol}^{\mathrm{st}} : \left\{ \begin{matrix} H\text{-colored oriented stated} \\ \text{ribbon tangles in } \Sigma_{g,n}^{\circ,\bullet} \times [0,1] \end{matrix} \right\} \longrightarrow \mathcal{L}_{g,n}(H).$$

We have $\mathrm{hol}^{\mathrm{st}}(T_1 * T_2) = \mathrm{hol}^{\mathrm{st}}(T_1)\,\mathrm{hol}^{\mathrm{st}}(T_2)$ (this follows easily from Theorem 3). Moreover since both the skein relations in $\mathcal{S}_H^{\mathrm{st}}(\Sigma_{g,n}^{\circ,\bullet})$ and the holonomy map are based on the Reshetikhin–Turaev functor, $\mathrm{hol}^{\mathrm{st}}$ descends to $\mathcal{S}_H^{\mathrm{st}}(\Sigma_{g,n}^{\circ,\bullet})$. Hence we get a morphism of algebras $\mathrm{hol}^{\mathrm{st}} : \mathcal{S}_H^{\mathrm{st}}(\Sigma_{g,n}^{\circ,\bullet}) \to \mathcal{L}_{g,n}(H)$.

[5] The Kronecker product of $\mathrm{hol}(T_1) = \sum_i x_i \otimes v_i \in \mathcal{L}_{g,n}(H) \otimes V_{T_1}$ and $\mathrm{hol}(T_2) = \sum_j y_j \otimes w_j \in \mathcal{L}_{g,n}(H) \otimes V_{T_2}$ is $\sum_{i,j} x_i y_j \otimes v_i \otimes w_j \in \mathcal{L}_{g,n}(H) \otimes V_{T_1} \otimes V_{T_2}$.

Theorem 4. ([BFR23, Th. 6.5, Th. 6.9] and [Fai20c, Th. 5.3] for the case $H = U_q(\mathfrak{sl}_2)$)
1. $\mathrm{hol}^{\mathrm{st}} : \mathcal{S}^{\mathrm{st}}_H(\Sigma^{\circ,\bullet}_{g,n}) \xrightarrow{\sim} \mathcal{L}_{g,n}(H)$ *is an isomorphism of algebras.*
2. If H-mod *is a semisimple category, the Wilson loop map* (5) *yields an isomorphism of algebras* $W : \mathcal{S}_H(\Sigma^{\circ}_{g,n}) \xrightarrow{\sim} \mathcal{L}^{\mathrm{inv}}_{g,n}(H)$

For $H = U_q(\mathfrak{sl}_2)$ it was already known that W is an isomorphism [BFK98]. To sum up:

$$\begin{tikzcd}
\text{Stated skein algebra } \mathcal{S}^{\mathrm{st}}_H(\Sigma^{\circ,\bullet}_{g,n}) \arrow[r, "\sim", "\text{Stated holonomy } \mathrm{hol}^{\mathrm{st}}"'] & \mathcal{L}_{g,n}(H) \\
\text{Skein algebra } \mathcal{S}_H(\Sigma^{\circ}_{g,n}) \arrow[u, "I"] \arrow[r, "\sim\ (\text{if } H\text{-mod semisimple})", "\text{Wilson loop map } W"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{L}^{H}_{g,n}(H) \arrow[u, hook, "(\text{subalgebra})"']
\end{tikzcd}$$

The morphism I is obtained by seeing a ribbon link as a ribbon tangle without boundary points. Note that it follows from this commutative diagram that I is injective if H is semisimple; this is a non-trivial fact since there are more skein relations in $\mathcal{S}^{\mathrm{st}}_H(\Sigma^{\circ,\bullet}_{g,n})$ than in $\mathcal{S}_H(\Sigma^{\circ}_{g,n})$, because of the boundary skein relations.

A question remains to be answered. The skein algebras can be defined for any surface $\Sigma_{g,n}$ and in particular for $\Sigma^{\circ}_{g,n} \approx \Sigma_{g,n+1}$. On the other hand the algebras $\mathcal{L}_{g,n}(H)$ are only associated to the surfaces $\Sigma^{\circ}_{g,n}$ with one boundary component. Can we "truncate" $\mathcal{L}_{g,n}(H)$ in order to get an algebra associated to $\Sigma_{g,n}$? The answer is yes and uses *quantum reduction*. Let us recall this construction, due to Lu [Lu93] and Varagnolo–Vasserot [VV10]. Let A be a H-module-algebra and assume that there is a *quantum moment map* $\mu : H \to A$, *i.e.* a morphism of algebras[6] such that

$$\forall\, h \in H,\ \forall\, a \in A, \qquad \mu(h)a = \sum_{(h)} (h'' \cdot a)\, \mu(h')$$

where $\Delta(h) = \sum_{(h)} h' \otimes h''$. Consider the left ideal $I_\varepsilon = A\mu(\ker \varepsilon)$, where $\varepsilon : H \to k$ is the counit. I_ε is stable by the action of H, hence the action of H descends to A/I_ε. It follows that we can consider the subspace of invariant elements $\big(A/I_\varepsilon\big)^{H\text{-inv}}$ which we denote by A^{qr}. A remarkable fact is that the product of A descends to A^{qr}, despite I_ε is just a left ideal. The associative algebra A^{qr} is called the quantum reduction of A.

Now recall that $\mathcal{L}_{g,n}(H)$ is a H-module-algebra. There is a quantum moment map $\mu : H \to \mathcal{L}_{g,n}(H)$ ([Jor14, Prop. 7.21], also see [BFR23, Th. 7.14]) which is intimately related to the boundary circle $\partial(\Sigma^{\circ}_{g,n})$, see formula in [BFR23, Def. 7.11]. Hence we can expect that the associated quantum reduction $\mathcal{L}^{\mathrm{qr}}_{g,n}(H)$ will be related to the closed surface $\Sigma_{g,n}$. For the next result we note that $\mathcal{S}_H(\Sigma_{g,n})$ is a quotient of $\mathcal{S}_H(\Sigma^{\circ}_{g,n})$.

Theorem 5. [BFR23, Lem. 7.6, Prop. 7.20, Th. 7.22] *This theorem requires suitable assumptions on* H*, which are in particular satisfied for* $H = U_q(\mathfrak{g})$.
1. Let $\pi : \mathcal{L}^{\mathrm{inv}}_{g,n}(H) \to \mathcal{L}^{\mathrm{qr}}_{g,n}(H) = \big(\mathcal{L}_{g,n}(H)/I_\varepsilon\big)^{H\text{-inv}}$ *be the restriction of the canonical projection* $\mathcal{L}_{g,n}(H) \twoheadrightarrow \mathcal{L}_{g,n}(H)/I_\varepsilon$*. Then* π *is a surjective morphism of algebras.*
2. The Wilson loop map W (5) *pass to the quotients as follows:*

$$\begin{tikzcd}
\mathcal{S}_H(\Sigma^{\circ}_{g,n}) \arrow[r, "W"] \arrow[d, twoheadrightarrow] & \mathcal{L}^{\mathrm{inv}}_{g,n}(H) \arrow[d, twoheadrightarrow, "\pi"] \\
\mathcal{S}_H(\Sigma_{g,n}) \arrow[r, "\exists!\, W^{\mathrm{qr}}"'] & \mathcal{L}^{\mathrm{qr}}_{g,n}(H)
\end{tikzcd}$$

3. $W^{\mathrm{qr}} : \mathcal{S}_H(\Sigma_{g,n}) \to \mathcal{L}^{\mathrm{qr}}_{g,n}(H)$ *is an isomorphism of algebras.*

[6]In practice it might be necessary to restrict to a subspace $H' \subset H$ which is a subalgebra and a right coideal. When $H = U_q(\mathfrak{g})$ we use the subspace of locally finite elements for the adjoint action.

Structure results for quantum moduli algebras. Joint work with S. Baseilhac and P. Roche. Write $\mathcal{L}_{g,n}(\mathfrak{g})$ for $\mathcal{L}_{g,n}\big(U_q(\mathfrak{g})\big)$. Recall that it is $\mathcal{O}_q(G)^{\otimes(2g+n)}$ as a $\mathbb{C}(q)$-vector space, and is endowed by a "twisted multiplication" based on the R-matrix $R \in U_q(\mathfrak{g})^{\otimes 2}$, see [BFR23, Prop. 4.4] for explicit formulas.

Theorem 6. [BFR23, Th. 4.11, Th. 4.17, Th. 5.8]
1. The algebra $\mathcal{L}_{g,n}(\mathfrak{g})$ is finitely generated, Noetherian and without zero divisors.
2. The algebra $\mathcal{L}^{\mathrm{inv}}_{g,n}(\mathfrak{g})$ is Noetherian and finitely generated (and of course without zero divisors).

The case $g = 0$ of this theorem was obtained in [BR22, BR21]. The case $g > 0$ requires many non-trivial generalizations and new computations. Let us give a few words on the ideas of the proof.

To prove that $\mathcal{L}_{g,n}$ is Noetherian, we use filtrations. A filtration for $\mathcal{L}_{0,1}(\mathfrak{g})$ has been introduced in [VY20, §3.14.4]. We first modify it in order to define a filtration of $\mathcal{L}_{1,0}(\mathfrak{g})$. Then combining these two filtrations in a non-trivial way we define a filtration on $\mathcal{L}_{g,n}(\mathfrak{g})$ and prove that the associated graded algebra is Noetherian, which implies that $\mathcal{L}_{g,n}(\mathfrak{g})$ itself is Noetherian.

The proof of the second item in Theorem 6 is based on a generalization of the Hilbert–Nagata theorem [DC70]. Let K be a group acting on a graded algebra A in such a way that the action is compatible with the multiplication and the grading. The Hilbert–Nagata theorem gives sufficient conditions for the subalgebra of K-invariant elements of A to be Noetherian and finitely generated. We generalize this theorem to the case where A is a graded module-algebra over a Hopf algebra H [BFR23, Th. 4.13]. We then apply this general result to the case where $H = U_q(\mathfrak{g})$ and A is a "graded truncation" of $\mathcal{L}_{g,n}(\mathfrak{g})$.

The fact that $\mathcal{L}_{g,n}$ does not have non-trivial zero divisors is a consequence of an important morphism
$$\Phi_{g,n} : \mathcal{L}_{g,n}(H) \to \mathcal{HH}(H^*)^{\otimes g} \otimes H^{\otimes n}$$
which we call the Alekseev morphism. The algebra $\mathcal{HH}(H^*)$ is a generalization of the Heisenberg double $\mathcal{H}(H^*) = H^* \# H$ [BFR23, §5.1]; in general for $g > 1$ it is necessary to use $\mathcal{HH}(H^*)$ instead of $\mathcal{H}(H^*)$ to make sense of the formulas in [Ale94]. For $H = U_q(\mathfrak{g})$ we prove that $\Phi_{g,n}$ is injective [BFR23, Th. 5.8] and that the algebra $\mathcal{HH}\big(\mathcal{O}_q(G)\big)^{\otimes g} \otimes U_q(\mathfrak{g})^{\otimes n}$ does not have non-trivial zero divisors [BFR23, Prop. 5.7]. It then follows that $\mathcal{L}_{g,n}(\mathfrak{g})$ does not have non-trivial zero divisors.

We deduce from Theorems 4 and 6 that:

Corollary 7. *1. The stated skein algebra $\mathcal{S}^{\mathrm{st}}_{U_q(\mathfrak{g})}(\Sigma^{\circ,\bullet}_{g,n})$ is finitely generated, Noetherian and without zero divisors.*
2. The skein algebra $\mathcal{S}_{U_q(\mathfrak{g})}(\Sigma^{\circ,\bullet}_{g,n})$ is finitely generated, Noetherian and without zero divisors.

Let us insist that this result is true for any semisimple complex Lie algebra $\mathfrak{g}$. The analysis of the structure of $\mathcal{S}_{U_q(\mathfrak{g})}(\Sigma^{\circ,\bullet}_{g,n})$ beyond $\mathfrak{sl}_2$ is very difficult because there exists no explicit skein description of this space (except for $\mathfrak{sl}_n$, in terms of webs). It is the algebraic nature of $\mathcal{L}_{g,n}(\mathfrak{g})$ which allowed us to obtain the above structure results for skein algebras, thanks to tools from quantum group theory.

Deformation of monoidal structures and cohomology

Context and history. A deformation theory of monoidal structures has been introduced and studied by Davydov, Crane and Yetter [Dav97, CY98, Yet98]; it describes deformations of the monoidal structure of a k-linear monoidal functor or the associator of a k-linear monoidal category (where k is a field), without changing the underlying functor and categories. This theory is the first step to the classification problem of monoidal structures [Dav97] but is also related to quantum algebra and low-dimensional topology. Within this deformation theory, it was shown in [DE19] that the category of all modules over the enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ admits a one-parameter family of non-trivial deformations. One therefore expects to recover the category of modules over the quantum group $U_q(\mathfrak{g})$. Also, this theory allows to deform the braiding of a tensor category and this can be used to produce link invariants; see [Yet98] where a relation with Vassiliev invariants was established.

We recall a bit more precisely this deformation theory, which is often called Davydov–Yetter (DY) theory as e.g. in [EGNO15, §7.22]. Let $\mathcal{C}, \mathcal{D}$ be k-linear monoidal categories, assumed strict for simplicity, and let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor, *i.e.* a k-linear monoidal functor. By definition, F comes with a natural isomorphism $\theta_{X,Y} : F(X \otimes Y) \xrightarrow{\sim} F(X) \otimes F(Y)$ such that the diagram

$$\begin{tikzcd} F(X \otimes Y \otimes Z) \arrow[r, "\theta_{X\otimes Y,Z}"] \arrow[d, "\theta_{X,Y\otimes Z}"'] & F(X \otimes Y) \otimes F(Z) \arrow[d, "\theta_{X,Y}\otimes \mathrm{id}_{F(Z)}"] \\ F(X) \otimes F(Y \otimes Z) \arrow[r, "\mathrm{id}_{F(X)}\otimes\theta_{Y,Z}"'] & F(X) \otimes F(Y) \otimes F(Z) \end{tikzcd} \tag{6}$$

is commutative. To simplify notation assume that F is strict, *i.e.* $\theta = \mathrm{id}$. In DY theory we consider infinitesimal deformations $\theta_h = \mathrm{id} + hf$ with $h^2 = 0$, where f is a natural transformation $f_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$, such that the diagram (6) remains commutative with θ_h instead of θ. Then the condition (6) on θ_h implies

$$\mathrm{id}_{F(X_1)} \otimes f_{X_2,X_3} - f_{X_1\otimes X_2,X_3} + f_{X_1,X_2\otimes X_3} - f_{X_1,X_2} \otimes \mathrm{id}_{F(X_3)} = 0. \tag{7}$$

This motivates the following definition (we continue to assume that F is strict for simplicity):

- The space of DY cochains in degree n is $C^n_{\mathrm{DY}}(F) = \{\text{natural transformations } F^{\otimes n} \Rightarrow F^{\otimes n}\}$, where the functor $F^{\otimes n} : \mathcal{C}^n \to \mathcal{D}$ is $(X_1, \ldots, X_n) \mapsto F(X_1) \otimes \ldots \otimes F(X_n)$.
- The DY differential $\delta^n : C^n_{\mathrm{DY}}(F) \to C^{n+1}_{\mathrm{DY}}(F)$ is defined by

$$\begin{aligned} \delta^n(f)_{X_1,\ldots,X_{n+1}} = \mathrm{id}_{F(X_1)} \otimes f_{X_2,\ldots,X_{n+1}} + \sum_{i=1}^{n} f_{X_1,\ldots,X_i\otimes X_{i+1},\ldots,X_{n+1}} \\ + (-1)^{n+1} f_{X_1,\ldots,X_n} \otimes \mathrm{id}_{F(X_{n+1})}. \end{aligned} \tag{8}$$

- We denote by $H^n_{\mathrm{DY}}(F) = \ker(\delta^n)/\mathrm{im}(\delta^{n-1})$ the associated cohomology groups.

The infinitesimal deformations of the monoidal structure of F are classified up to equivalence by $H^2_{\mathrm{DY}}(F)$, and it was shown in [Yet98] that the obstructions are contained in $H^3_{\mathrm{DY}}(F)$. In particular if $H^3_{\mathrm{DY}}(F) = 0$ an infinitesimal deformation can be extended to all orders in h.

We note that the identity functor $F = \mathrm{Id}_{\mathcal{C}}$ deserves a special attention because $H^3_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$ classifies the infinitesimal deformations of the associator of $\mathcal{C}$. Such a deformation is an expression $a_h = \mathrm{id}+hg$[7] over $k[h]/\langle h^2\rangle$ which satisfies the pentagon equation, where g is a natural transformation $g_{X,Y,Z} : X \otimes Y \otimes Z \to X \otimes Y \otimes Z$. The obstructions are contained in $H^4_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$, at least for the extension to the order 2 in h [BD20, Prop. 3.21]. We will denote $H^n_{\mathrm{DY}}(\mathcal{C})$ instead of $H^n_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$.

Let $\mathcal{Z}(F)$ be the centralizer of F. It is the category whose objects are pairs (V, λ) where $V \in \mathcal{D}$ and λ is a natural isomorphism $V \otimes F(-) \xrightarrow{\sim} F(-) \otimes V$ such that $\lambda_{X\otimes Y} = (\mathrm{id}_{F(X)} \otimes \lambda_Y) \circ (\lambda_X \otimes \mathrm{id}_{F(Y)})$ for all $X, Y \in \mathcal{C}$; morphisms in $\mathcal{Z}(F)$ are morphisms in $\mathcal{D}$ which commute with the half-braidings. In all the sequel *we assume that $\mathcal{C}$, $\mathcal{D}$ are finite tensor categories and that $F : \mathcal{C} \to \mathcal{D}$ is exact.* Then there is an adjunction

$$\begin{tikzcd} \mathcal{Z}(F) \arrow[d, bend left, "\mathcal{U}"] \arrow[d, phantom, "\dashv"] \\ \mathcal{D} \arrow[u, bend left, "\mathcal{F}"] \end{tikzcd} \tag{9}$$

where $\mathcal{U}$ is the forgetful functor $(V, \lambda) \mapsto V$. In [GHS23] a version with coefficients of DY theory was introduced. The coefficients are objects $\mathsf{V} = (V, \lambda), \mathsf{W} = (W, \rho) \in \mathcal{Z}(F)$, the cochain spaces are

[7] Note that the degree 0 term is the identity because for simplicity we take $\mathcal{C}$ strict.

$C^n_{\mathrm{DY}}(F;\mathsf{V},\mathsf{W}) = \{\text{natural transformations } F^{\otimes n}\otimes V \Rightarrow W\otimes F^{\otimes n}\}$ and the half-braidings λ,ρ are used to modify the boundary terms in (8). We denote by $H^\bullet_{\mathrm{DY}}(F;\mathsf{V},\mathsf{W})$ the resulting cohomology. In particular $C^n_{\mathrm{DY}}(F;\mathbf{1},\mathbf{1}) = C^n_{\mathrm{DY}}(F)$ (trivial coefficients). Coefficients are an important technical tool, which might be useful even if one is interested in computing the DY cohomology only for trivial coefficients, see e.g. the comment after Theorem 11 below.

The main result of [GHS23] is that $H^\bullet_{\mathrm{DY}}(F;\mathsf{V},\mathsf{W})$ is isomorphic to the cohomology of the comonad $G=\mathcal{FU}$ on $\mathcal{Z}(F)$ with coefficients V and $\mathrm{Hom}_{\mathcal{Z}(F)}(-,\mathsf{W})$[8]. Even better, there is an explicit isomorphism of complexes between $C^n_{\mathrm{DY}}(F;\mathsf{V},\mathsf{W})$ and the image through $\mathrm{Hom}_{\mathcal{Z}(F)}(-,\mathsf{W})$ of the bar resolution of V for the comonad G. This gives a way to compute the dimension of the DY cohomology spaces, by finding a well-chosen G-projective resolution of V (of course the bar resolution is in general not a good choice).

DY cohomology and relative Ext groups. Joint work with A. Gainutdinov and C. Schweigert. Recall that we assume that $\mathcal{C}$, $\mathcal{D}$ are finite tensor categories and that the k-linear monoidal functor $F:\mathcal{C}\to\mathcal{D}$ is exact (and k is a field). The aim is to make progress towards the computation of DY cohomology. This can be divided into two problems:

- Compute the dimension of the Davydov–Yetter cohomology groups.
- Determine explicit cocycles. This question is especially relevant for 2-cocycles (or 3-cocycles for the identity functor) since they give rise to infinitesimal deformations.

Recall a few notions of relative homological algebra. Let

$$\begin{tikzcd} \mathcal{A} \arrow[d, bend left, "\mathcal{U}"] \\ \mathcal{B} \arrow[u, bend left, "\mathcal{F}"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \tag{10}$$

be an adjunction between abelian categories such that $\mathcal{U}$ is additive, exact and faithful. A *relatively projective object* is a direct summand of $\mathcal{F}(X)$ for some $X\in\mathcal{B}$. A *relatively projective resolution* of $V\in\mathcal{A}$ is an exact sequence $0\leftarrow V\leftarrow P_0\leftarrow P_1\leftarrow\ldots$ such that each P_i is relatively projective and the sequence splits when we apply $\mathcal{U}$ to it. For such a resolution, the cohomology groups of the complex $0\to\mathrm{Hom}_{\mathcal{A}}(P_0,W)\to\mathrm{Hom}_{\mathcal{A}}(P_1,W)\to\ldots$ are denoted by $\mathrm{Ext}^\bullet_{\mathcal{A},\mathcal{B}}(V,W)$ and called relative Ext groups. For more details see [ML75, Chap. IX] (or [BFR23, §2.1, §2.2] for a review). If the categories $\mathcal{A},\mathcal{B}$ are k-linear then $\mathrm{Ext}_{\mathcal{A},\mathcal{B}}(V,W)$ are k-vector spaces.

The starting point of our work is the relation between DY cohomology and relative Ext groups:

Theorem 8. [FGS22, Prop. 2.17, Cor. 4.7]
1. Given an adjunction as in (10), the relative Ext groups $\mathrm{Ext}^\bullet_{\mathcal{A},\mathcal{B}}(V,W)$ are isomorphic to the cohomology groups of the comonad $G=\mathcal{FU}$ on $\mathcal{A}$ with coefficients V and $\mathrm{Hom}_{\mathcal{A}}(-,W)$.
2. In particular for the adjunction (9) we get $H^\bullet_{\mathrm{DY}}(F;\mathsf{V},\mathsf{W})\cong\mathrm{Ext}^\bullet_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V},\mathsf{W})$ thanks to the result of [GHS23].

Here is a first consequence:

Proposition 9. [FGS22, Prop. 4.8] *If the ground field k has characteristic 0 and is algebraically closed then $H^1_{\mathrm{DY}}(F)=0$.*

This is because $H^1_{\mathrm{DY}}(F)\cong\mathrm{Ext}_{\mathcal{Z}(F),\mathcal{D}}(\mathbf{1},\mathbf{1})\subset\mathrm{Ext}_{\mathcal{Z}(F)}(\mathbf{1},\mathbf{1})=0$. The last equality is because the unit object does not trivial self-extensions in a finite tensor category under our assumptions on k [EGNO15, Th. 4.4.1]. The inclusion is only true in degree 1.

We can derive results on DY cohomology from the general theorems on relative Ext groups [ML75, Chap. IX]. Using the long exact sequence for relative Ext groups we derived a formula for the dimension of the DY cohomology spaces:

[8] For comonad cohomology see [BB96].

Proposition 10. [FGS22, §3.1, Cor. 4.10] *Let* $\mathsf{P} \xrightarrow{\pi} \mathbf{1} \longrightarrow 0$ *be the first step of a relatively projective resolution of* $\mathbf{1} \in \mathcal{Z}(F)$ *and let* $\mathsf{K} = \ker(\pi)$. *Then for* $n \geq 2$

$$\dim H^n_{\mathrm{DY}}(F) = \dim \mathrm{Hom}_{\mathcal{Z}(F)}\big(\mathsf{K}, (\mathsf{K}^\vee)^{\otimes(n-1)}\big) - \dim \mathrm{Hom}_{\mathcal{Z}(F)}\big(\mathsf{P}, (\mathsf{K}^\vee)^{\otimes(n-1)}\big) + \dim \mathrm{Hom}_{\mathcal{Z}(F)}\big(\mathbf{1}, (\mathsf{K}^\vee)^{\otimes(n-1)}\big).$$

In particular if P *is the relatively projective cover*[9] *of* $\mathbf{1}$ *and* k *is algebraically closed and has characteristic* 0, *then* $\dim H^2_{\mathrm{DY}}(F) = \dim \mathrm{Hom}_{\mathcal{Z}(F)}\big(\mathsf{K}, \mathsf{K}^\vee\big) - \dim \mathrm{Hom}_{\mathcal{Z}(F)}\big(\mathsf{P}, \mathsf{K}^\vee\big)$.

Finding a relatively projective resolution can be hard, especially if we want it to be simple enough to allow the computation of the relative Ext groups. The formula above replaces the computation of a relatively projective resolution and of the associated cohomology by the computation of a relatively projective cover and the computation of certain Hom spaces, which is a purely representation-theoretic problem. It is very efficient for $n = 2$, as we have demonstrated on examples (see e.g. [FGS22, §6.4]).

There is an operation $\circ : \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W) \times \mathrm{Ext}^m_{\mathcal{A},\mathcal{B}}(U, V) \to \mathrm{Ext}^{m+n}_{\mathcal{A},\mathcal{B}}(U, W)$ called the *Yoneda product*. By Theorem 8 we get a Yoneda product on DY cohomology and we computed its formula:

Theorem 11. [FGS22, Th. 4.12] *Let* $f \in H^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W})$ *and* $g \in H^m_{\mathrm{DY}}(F; \mathsf{U}, \mathsf{V})$. *Then the components of the natural transformation* $f \circ g \in H^{n+m}_{\mathrm{DY}}(F; \mathsf{U}, \mathsf{W})$ *are*

$$(f \circ g)_{X_1,\ldots,X_m,Y_1,\ldots,Y_n} = (-1)^{nm}\big(\mathrm{id}_{F(X_1)\otimes\ldots\otimes F(X_m)} \otimes f_{Y_1,\ldots,Y_n}\big)\big(g_{X_1,\ldots,X_m} \otimes \mathrm{id}_{F(Y_1)\otimes\ldots\otimes F(Y_n)}\big).$$

We note that this formula already makes sense at the level of cochains. It follows from a result in relative homological algebra that each DY cocyle can be written as the Yoneda product of 1-cocyles with different coefficients. This is useful in practice, see e.g. [FGS22, §5.4].

Finally the long exact sequence theorem for relative Ext groups gives a long exact sequence theorem for DY cohomology [FGS22, §4.5].

The usefullness of the above results is demonstrated in [FGS22, §6] to compute the DY cohomology of H-mod for certain finite-dimensional Hopf algebras H: $\Lambda\mathbb{C}^k \rtimes \mathbb{C}[\mathbb{Z}_2]$ (bosonization of the exterior algebra), Taft algebra and small quantum groups associated to $\mathfrak{sl}_2$. Moreover we give a method to compute explicitly DY cocycles for H-mod [FGS22, §5.4]; the Yoneda product (Theorem 11) is a key ingredient of this method. We apply it to compute explicit DY cocycles for the above examples. For $A = \Lambda\mathbb{C}^k \rtimes \mathbb{C}[\mathbb{Z}_2]$ and A=Taft, we obtained the full description of $H^\bullet_{\mathrm{DY}}(A\text{-mod})$ and its algebra structure under the Yoneda product.

References

[Ale94] A. Yu. Alekseev, *Integrability in the Hamiltonian Chern-Simons theory*, Alg. i Anal. 6 (2) (1994) 53–66.

[AGS95] A.Y. Alekseev, H. Grosse, V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern-Simons theory, I*, Commun. Math. Phys. **172**(2), 317–358 (1995).

[AS96] A.Y. Alekseev, V. Schomerus, *Representation theory of Chern-Simons observables*, Duke Math. J. **85**(2), 447–510 (1996).

[AB83] M. F. Atiyah, R. Bott, *The Yang-Mills Equations over Riemann Surfaces*, Philos. Trans. R. Soc. A **308**, 523-615 (1983).

[BB96] M. Barr, J. Beck, *Homology and Standard Constructions*, Seminar on Triples and Categorical Homology Theory, Vol. 80, Springer, 245–336 (1996).

[9] This is a minimality condition, see [FGS22, §2.3].

[BR19] S. Baseilhac, P. Roche, *Unrestricted quantum moduli algebras, I: the case of punctured spheres*, SIGMA **18** (2022), 025, 78 pages.

[BR22] S. Baseilhac, P. Roche, *Unrestricted Quantum Moduli Algebras, I: The Case of Punctured Spheres*, SIGMA 18 (2022), 025, 78 pages.

[BR21] S. Baseilhac, P. Roche, *Unrestricted Quantum Moduli Algebras, II: Noetherianity and simple fraction rings at roots of* 1, arXiv:2106.04136, (2021).

[BFR23] S. Baseilhac, M. Faitg, P. Roche, *Unrestricted quantum moduli algebras, III: surfaces of arbitrary genus*, arXiv:2302.00396, 75 pages (2023).

[BD20] M. Batanin, A. Davydov, *Cosimplicial monoids and deformation theory of tensor categories*, arXiv:2003.13039.

[BZBJ18] D. Ben-Zvi, A. Brochier, D. Jordan, *Integrating quantum groups over surfaces*, J. Topology **11**(4), 874–917 (2018).

[BG02] K. A. Brown, K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics, CRM Barcelona, Birkhaüser Verlag, Basel (2002).

[BR95] E. Buffenoir, P. Roche, *Two dimensional lattice gauge theory based on a quantum group*, Commun. Math. Phys. **170**(3), 669–698 (1995).

[BR96] E. Buffenoir, P. Roche, *Link invariants and combinatorial quantization of Hamiltonian Chern Simons theory*, Commun. Math. Phys. **181**(2), 331–365 (1996).

[BFK98] D. Bullock, C. Frohman, J. Kania-Bartoszynska, *Topological interpretations of lattice gauge field theory*, Commun. Math. Phys. **198**(1), 47–81 (1998).

[BFKB99] D. Bullock, C. Frohman, J. Kania-Bartoszynska, *Understanding the Kauffman bracket skein module*, J. Knot Theory Ramifications **8**(3), 265–277 (1999).

[CP94] V. Chari, A. Pressley, *A guide to quantum groups*, Cambridge University Press, 1994.

[CKL] F. Costantino, J. Korinman, T. T. Q. Lê, *Stated skein algebras for Tannakian ribbon categories*, to appear, private communication.

[CL19] F. Costantino, T.T.Q. Lê, *Stated skein algebras of surfaces*, 74 pages, arXiv:1907.11400.

[CY98] L. Crane, D. N. Yetter, *Deformations of (bi)tensor categories*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 1998.

[Dav97] A. Davydov, *Twisting of monoidal structures*, 37 pages, arXiv:q-alg/9703001.

[DE19] A. Davydov, M. Elbehiry, *Deformation cohomology of Schur-Weyl categories. Free symmetric categories*, arXiv:1908.09192.

[DGGPR22] M. De Renzi, A.M. Gainutdinov, N. Geer, B. Patureau–Mirand, I. Runkel, *3-Dimensional TQFTs from non-semisimple modular categories*, Selecta Math. **28**, paper n°42 (2022).

[DGGPR23] M. De Renzi, A.M. Gainutdinov, N. Geer, B. Patureau–Mirand, I. Runkel, *Mapping class group representations from non-semisimple TQFTs*, Commun. Contemp. Math. **25**, No. 01, 2150091 (2023).

[DC70] J. A. Dieudonné, J. B. Carrell, *Invariant theory, Old and New*, Adv. Math. 4(1), 1–80 (1970).

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*, Mathematical surveys and monographs **205**, American Mathematical Society.

[Fai20a] M. Faitg, *A note on symmetric linear forms and traces on the restricted quantum group* $\bar{U}_q(\mathfrak{sl}(2))$, Osaka J. Math. **57**, no. 3, 575–595 (2020).

[Fai19] M. Faitg, *Modular group representations in combinatorial quantization with non-semisimple Hopf algebras*, SIGMA **15**, 077, 39 pages (2019).

[Fai20b] M. Faitg, *Projective representations of mapping class groups in combinatorial quantization*, Commun. Math. Phys. **377**, 161–198 (2020).

[Fai] M. Faitg, *Mapping class groups, skein algebras and combinatorial quantization*, PhD thesis, 163 pages. University of Montpellier, 2019. arXiv:1910.04110.

[Fai20c] M. Faitg, *Holonomy and (stated) skein algebras in combinatorial quantization*, 44 pages, arXiv:2003.08992. Accepted in Quantum Topology.

[FGS22] M. Faitg, A.M. Gainutdinov, C. Schweigert, *Relative homological algebra and Davydov–Yetter cohomology*, 63 pages, arXiv:2202.12287 (2022). Accepted in Selecta Math.

[FM12] B. Farb, D. Margalit, *A Primer on Mapping Class Groups*, Princeton University Press, Princeton (2012).

[FGST06] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, *Modular Group Representations and Fusion in Logarithmic Conformal Field Theories and in the Quantum Group Center*, Commun. Math. Phys. **265**, 47–93 (2006).

[FR93] V. V. Fock, A. A. Rosly, *Flat connections and polyubles*, Theor. Math. Phys. **95**(2), 526–534 (1993).

[GHS23] A.M. Gainutdinov, J. Haferkamp, C. Schweigert, *Davydov-Yetter cohomology, comonads and Ocneanu rigidity*, Advances in Math. **414**, 108853 (2023).

[GT07] A.M. Gainutdinov , I.Yu. Tipunin, *Radford, Drinfeld, and Cardy boundary states in $(1, p)$ logarithmic conformal field models*, J. Phys. A **42**, 315207 (2009).

[Gol86] W.M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. math. **85**(2), 263–302 (1986).

[Jor14] D. Jordan, *Quantized multiplicative quiver varieties*, Adv. Math. 250 (2014), 420–466.

[Le16] T.T.Q. Lê, *Triangular decomposition of skein algebras*, Quantum Topology **9**, 591-632 (2018).

[LS21] T.T.Q. Lê, A.S. Sikora, *Stated* $\mathrm{SL}(n)$*-skein modules and algebras*, arXiv:2201.00045.

[Lu93] J.-H. Lu, *Moment maps at the quantum level*, Comm. Math. Phys. 157(2) (1993) 389–404.

[LM94] V. Lyubashenko, S. Majid, *Braided Groups and Quantum Fourier Transform*, Journal of Algebra **166**(3), 506–528 (1994).

[Lyu95] V. Lyubashenko, *Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Commun. Math. Phys. **172**(3), 467–516 (1995).

[ML75] S. Mac Lane, *Homology*, Classics in Mathematics, Springer, 1975.

[MW21] C. Meusburger, D.K. Wise, *Hopf algebra gauge theory on a ribbon graph*, Rev. Math. Phys. 33, no. 5 (2021), 2150016.

[Prz91] J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Polish Acad. Sci. Math. **39**, 91–100 (1991).

[RT90] N. Yu. Reshetikhin, V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127**(1), 1–26 (1990).

[RT91] N. Reshetikhin, V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103**, 547–597 (1991).

[Tur91] V.G. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. **24**, 635–704 (1991).

[VV10] M. Varagnolo, E. Vasserot, *Double affine Hecke algebras at roots of unity*, Representation Theory 14 (2010), 510–600.

[VY20] C. Voigt, R. Yuncken, *Complex Semisimple Quantum Groups and Representation Theory*, Lecture Notes in Mathematics 2264, Springer, 2020.

[Waj83] B. Wajnryb, *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. **45**(2–3), 157–174 (1983).

[Yet98] D. N. Yetter, *Braided deformations of monoidal categories and Vassiliev invariants*, Higher category theory, Contemp. Math. **230**, AMS, 117–134 (1998).