The stated skein TQFT

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Based on arXiv:2505.16909, joint work with F. Costantino

Algebraic, Topological and Probabilistic approaches in CFT

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Overview

(2+1)-TQFT in general

$$\begin{array}{rcl} \text{3-cobordism category} & \xrightarrow{\text{br. mon. functor}} & \text{“algebraic” category} \\ \text{surface } \Sigma & \longmapsto & \text{object } V(\Sigma) \\ \text{3-cobordism } \Sigma_- \to \Sigma_+ & \longmapsto & \text{morphism } V(\Sigma_-) \to V(\Sigma_+) \end{array}$$

Stated skein TQFT (given a ribbon Hopf algebra H)

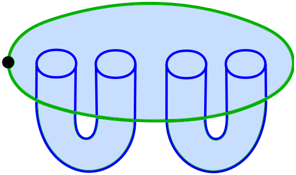
$$\begin{array}{rcl} \mathcal{S}_H : \mathrm{3Cob}_{\mathrm{arc}} & \xrightarrow{\text{br. mon. functor}} & \mathrm{Bim}_{H,\mathrm{lf}}^{\mathsf{QMM}} \\ \text{surface } \Sigma & \longmapsto & \text{algebra } \mathcal{S}_H(\Sigma) \\ \text{3-cobordism } \Sigma_- \to \Sigma_+ & \longmapsto & (\mathcal{S}_H(\Sigma_+), \mathcal{S}_H(\Sigma_-))\text{-bimodule} \end{array}$$

Summary:

1. Surfaces, 3-cobordisms and their $\mathcal{S}_H$
2. Algebraic properties of the stated skein spaces
3. Relation with the Kerler–Lyubashenko TQFT

The monoidal category $3\mathrm{Cob}_{\mathrm{arc}}$

Objects: connected oriented surfaces Σ such that $\partial\Sigma = S^1$ and there is a marked point on $\partial\Sigma$.

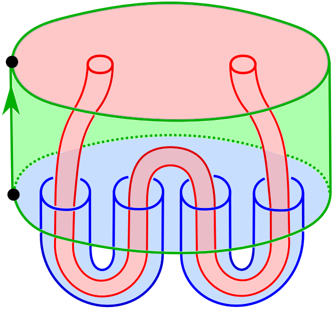


Morphisms $\Sigma^- \to \Sigma^+$: connected oriented 3-manifolds M such that

$$\partial M = \partial^- M \cup \partial^s M \cup \partial^+ M$$

with:

$$\partial^\pm M \cong \Sigma^\pm, \quad \partial^s M \cong S^1 \times [-1, 1],$$
$$\partial^\pm M \cap \partial^s M \cong S^1 \times \{\pm 1\},$$
$$\partial^- M \cap \partial^+ M = \emptyset$$

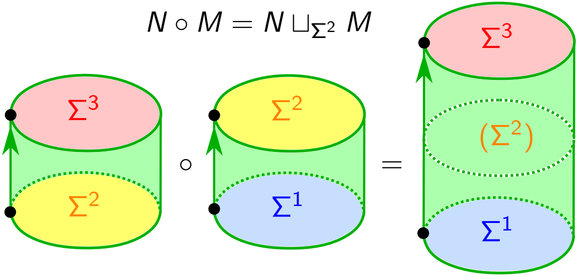


Moreover an oriented arc $\mathfrak{a}$ is fixed in $\partial^s M$, going from marked point of Σ^- to marked point of Σ^+.

The monoidal category $3\mathrm{Cob}_{\mathrm{arc}}$

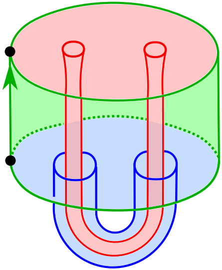
Composition: glue along the common surface.

$$N \circ M = N \sqcup_{\Sigma^2} M$$

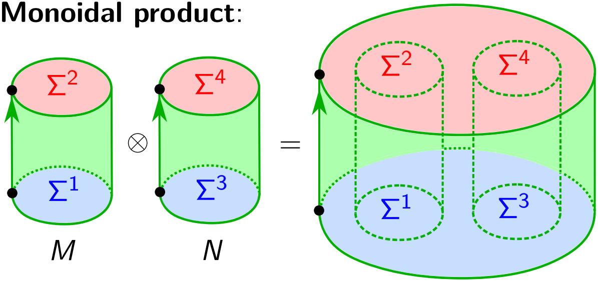


Identity morphisms

$$\mathrm{Id}_\Sigma = \Sigma \times [-1, 1]$$

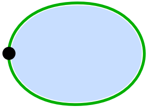


Monoidal product:



Monoidal unit

the disk



The monoidal category $3\mathrm{Cob}_{\mathrm{arc}}$

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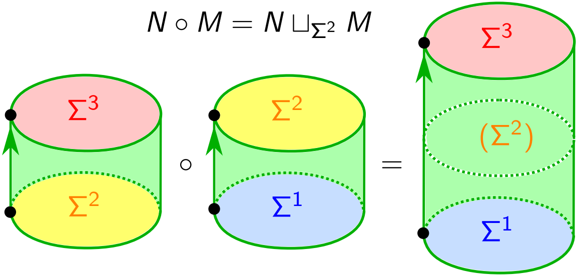

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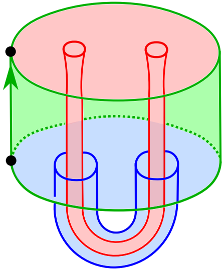
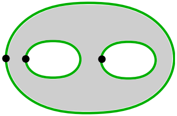
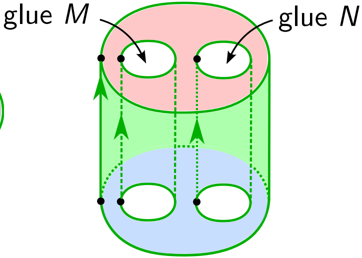


Monoidal product:



$\times[-1, 1]$



Monoidal unit

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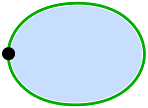
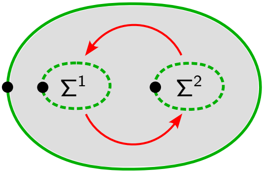
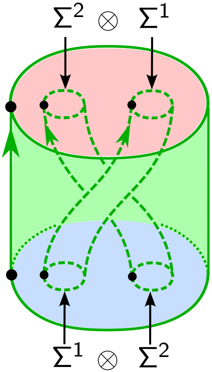


The monoidal category $3\mathrm{Cob}_{\mathrm{arc}}$ is braided

Braiding:



$\Longrightarrow \mathrm{Br}_{\Sigma^1,\Sigma^2} =$



(and forget inner arcs and marked points)

Remarks. 1. Forgetting the arc and marked points yields the category $3\mathrm{Cob}_{\mathrm{CY}}$ in Marco's lectures.

2.With small variations, this category appears in many works: [Kerler–Lyubashenko 01], [Kerler 01], [Asaeda 11] (citing Habiro), [Bobtcheva–Piergallini 12], [Beliakova–Bobtcheva–De Renzi–Piergallini 23].

Stated skeins in 3-cobordisms

Definition
(H ribbon Hopf algebra over a field $\mathbb{k}$)

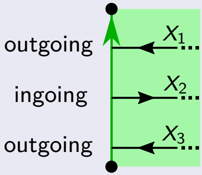
A H-colored **skein** Γ in a 3-cobordism $(M, \mathfrak{a})$ is an isotopy class of framed and oriented embedding of copies of S^1 and $[0,1]$ such that

- $\partial\Gamma \subset \mathfrak{a}$ and framing is vertical at these points
- each connected component of Γ is labelled by a fin.-dim. H-module.

$+$ Γ is allowed to contain *coupons*, little rectangles labelled by H-linear maps.

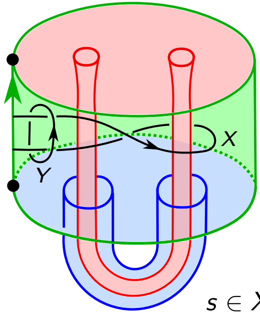
State for a skein

Example:



$\Longrightarrow$ A **state** is any element s in $X_1 \otimes X_2^* \otimes X_3$

A **stated skein** is a pair (Γ, s).

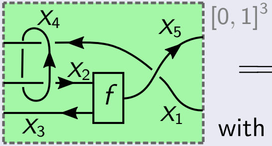


$s \in X^* \otimes X$

Stated skein space of a 3-cobordism

H is a ribbon Hopf $\mathbb{k}$-alg. $\implies$ H-mod is a ribbon category.

Reshetikhin–Turaev ribbon graph invariant (example)

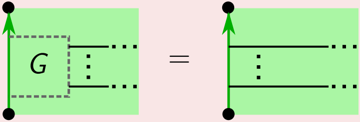
$G =$ 

$\implies \mathrm{RT}_H(G) \in \mathrm{Hom}_H(X_1 \otimes X_2^* \otimes X_3, X_5^* \otimes X_1)$

with $f \in \mathrm{Hom}_H(X_2^* \otimes X_3, X_5^*)$

Definition

For $M = (M, \mathfrak{a})$ a 3-cobordism, $\mathcal{S}_H(M)$ is the $\mathbb{k}$-vector space spanned by stated skeins modulo:



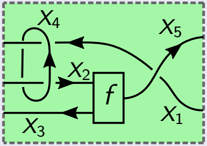
with state s with state $\mathrm{RT}_H(G)(s)$

and also $(\Gamma, \lambda_1 s_1 + \lambda_2 s_2) = \lambda_1(\Gamma, s_1) + \lambda_2(\Gamma, s_2)$

Stated skein space of a 3-cobordism

H is a ribbon Hopf $\Bbbk$-alg. $\Longrightarrow$ H-mod is a ribbon category.

Reshetikhin–Turaev ribbon graph invariant (example)

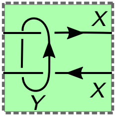
$G =$ 

$[0,1]^3$

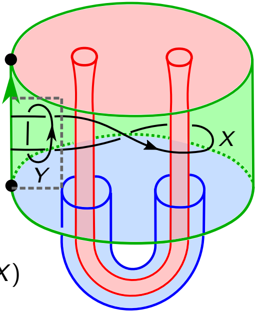
$\Longrightarrow \quad \mathrm{RT}_H(G) \in \mathrm{Hom}_H(X_1 \otimes X_2^* \otimes X_3, X_5^* \otimes X_1)$

with $f \in \mathrm{Hom}_H(X_2^* \otimes X_3, X_5^*)$

Example of a relation:

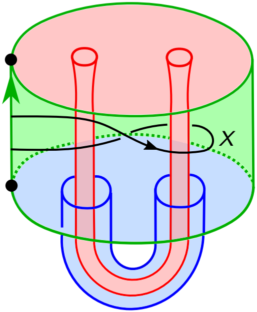
$G =$ 

$\mathrm{RT}_H(G) \in \mathrm{End}_H(X^* \otimes X)$



with state $s \in X^* \otimes X$

$=$



with state $\mathrm{RT}_H(G)(s)$

Algebra and bimodule structures

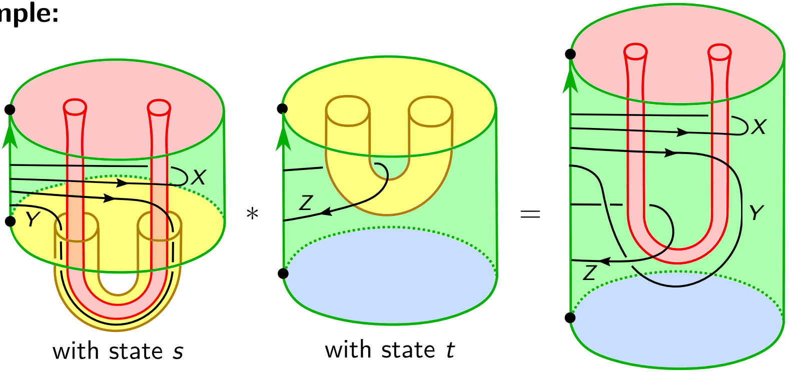
Stacking

$M : \Sigma^1 \to \Sigma^2, \qquad N : \Sigma^2 \to \Sigma^3, \qquad N \circ M = N \sqcup_{\Sigma_2} M : \Sigma^1 \to \Sigma^3.$

$$* : \mathcal{S}_H(N) \times \mathcal{S}_H(M) \to \mathcal{S}_H(N \circ M), \quad (\Gamma, s) * (\Lambda, t) = (\Gamma \sqcup \Lambda, s \otimes t)$$

(extended by $\mathbb{k}$-bilinearity). Associative operation.

Example:



with state s with state t with state $s \otimes t$

Algebra and bimodule structures

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(extended by $\Bbbk$-bilinearity). Associative operation.

Corollary

- $\mathcal{S}_H(\Sigma) \underset{\text{def}}{=} \mathcal{S}_H(\Sigma \times [-1, 1])$ is an associative algebra.
- $\mathcal{S}_H(M)$ is a $(\mathcal{S}_H(\Sigma^2), \mathcal{S}_H(\Sigma^1))$-bimodule. $\qquad$ ($M : \Sigma^1 \to \Sigma^2$ cobordism).

H-equivariance

H acts on $\mathcal{S}_H(M)$ by $\quad h \cdot (\Gamma, s) = (\Gamma, h \cdot s)$ $\qquad$ Rmk: this action is locally finite.

Property: $h \cdot (X * Y) = (h_{(1)} \cdot X) * (h_{(2)} \cdot Y)$.

$\Longrightarrow$ Algebras and bimodules internal to $H\text{-}\mathrm{Mod}_{\mathrm{lf}}$.

Stated skein functor

Monoidal category Bim_H

Objects: Associative algebras in H-Mod.

Morphisms: $\mathrm{Hom}_{\mathrm{Bim}_H}(A_1, A_2) = \{$iso classes of (A_2, A_1)-bim in H-$\mathrm{Mod}\}$

Composition: V (A_2, A_1)-bim, W (A_3, A_2)-bim, $\quad W \circ V = W \otimes_{A_2} V$.

Monoidal product: Tensor product of algebras and bimodules in H-Mod .

$$H \text{ quasitriangular} \implies H\text{-}\mathrm{Mod}\ \textbf{braided}\ \text{category}$$

algebras A, A' $\implies$ algebra $A \mathbin{\widetilde{\otimes}} A'$:

$$(A \otimes A') \otimes (A \otimes A') \xrightarrow{\mathrm{id} \otimes \mathrm{br} \otimes \mathrm{id}} A \otimes A \otimes A' \otimes A' \xrightarrow{\text{mult} \otimes \text{mult}} A \otimes A'$$

(A_2, A_1)-bim B, (A'_2, A'_1)-bim B' $\implies$ $(A_2 \mathbin{\widetilde{\otimes}} A'_2, A_1 \mathbin{\widetilde{\otimes}} A'_1)$-bim $B \mathbin{\widetilde{\otimes}} B'$:

$$(A_2 \otimes A'_2) \otimes (B \otimes B') \xrightarrow{\mathrm{id} \otimes \mathrm{br} \otimes \mathrm{id}} A_2 \otimes B \otimes A'_2 \otimes B' \xrightarrow{\text{act} \otimes \text{act}} B \otimes B'$$

and similarly for right action.

Stated skein functor

Monoidal category Bim_H

Objects: Associative algebras in H-Mod.

Morphisms: $\mathrm{Hom}_{\mathrm{Bim}_H}(A_1, A_2) = \{$iso classes of (A_2, A_1)-bim in H-$\mathrm{Mod}\}$

Composition: V (A_2, A_1)-bim, W (A_3, A_2)-bim, $\quad W \circ V = W \otimes_{A_2} V$.

Monoidal product: Tensor product of algebras and bimodules $\boxed{\text{in } H\text{-}\mathrm{Mod}}$.

Theorem 1 [Costantino–Lê] for "functor", [Costantino–F.] for "monoidal"

$\mathcal{S}_H$ is a strict monoidal functor $3\mathrm{Cob}_{\mathrm{arc}} \to \mathrm{Bim}_H$.

It means:

- $\mathcal{S}_H(N \sqcup_{\Sigma^2} M) = \mathcal{S}_H(N) \otimes_{\mathcal{S}_H(\Sigma^2)} \mathcal{S}_H(M)$ for $M : \Sigma^1 \to \Sigma^2$, $N : \Sigma^2 \to \Sigma^3$
- $\mathcal{S}_H(C \otimes C') = \mathcal{S}_H(C) \,\widetilde{\otimes}\, \mathcal{S}_H(C')$

What about the braiding in $3\mathrm{Cob}_{\mathrm{arc}}$?

$$\mathscr{L} = \int^{X \in H\text{-mod}} X^* \otimes X, \qquad \left(i_X : X^* \otimes X \to \mathscr{L}\right)_{X \in H\text{-mod}} \text{ univ. dinat. transfo.}$$

- $\mathscr{L}$ is a Hopf algebra in H-Mod [Lyubashenko, Majid]
- There is a half-braiding $\sigma : \mathscr{L} \otimes - \xrightarrow{\sim} - \otimes \mathscr{L}$ defined by

$$\begin{tikzcd}
X^* \otimes X \otimes V \arrow[r, "\mathrm{id}_{X^*} \otimes \mathrm{br}_{X,V}"] \arrow[d, "i_X \otimes \mathrm{id}_V"'] & X^* \otimes V \otimes X \arrow[r, "\mathrm{br}_{V,X^*}^{-1} \otimes \mathrm{id}_X"] \arrow[d, phantom, "\circlearrowleft"] & V \otimes X^* \otimes X \arrow[d, "\mathrm{id}_V \otimes i_X"] \\
\mathscr{L} \otimes V \arrow[rr, dashed, "\exists!\, \sigma_V"'] & & V \otimes \mathscr{L}
\end{tikzcd}$$

Definition: Quantum Moment Map (in the sense of Safronov)

A algebra in H-Mod. A QMM for A is an algebra morphism $\mathfrak{d} : \mathscr{L} \to A$ in H-Mod such that

$$\begin{tikzcd}
\mathscr{L} \otimes A \arrow[rr, "\mathfrak{d} \otimes \mathrm{id}_A"] \arrow[d, "\sigma_A"'] & \arrow[d, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "\mathrm{mult}"] \\
A \otimes \mathscr{L} \arrow[r, "\mathrm{id}_A \otimes \mathfrak{d}"'] & A \otimes A \arrow[r, "\mathrm{mult}"'] & A
\end{tikzcd}$$

What about the braiding in $3\mathrm{Cob}_{\mathrm{arc}}$?

Subcategory $\mathrm{Bim}_H^{\mathsf{QMM}} \subset \mathrm{Bim}_H$

Objects: Associative algebras in H-Mod **equipped with a QMM**.

Morphisms $(A_1, \mathfrak{d}_1) \to (A_2, \mathfrak{d}_2)$ = iso classes of (A_2, A_1)-bim in H-Mod **compatible** with the QMMs $\mathfrak{d}_1$, $\mathfrak{d}_2$.

$$\begin{tikzcd}
\mathscr{L} \otimes B \arrow[rr, "\mathfrak{d}_2 \otimes \mathrm{id}_A"] \arrow[d, "\sigma_B"'] \arrow[drr, phantom, "\circlearrowleft"] & & A_2 \otimes B \arrow[d, "\mathsf{act}"] \\
B \otimes \mathscr{L} \arrow[r, "\mathrm{id}_A \otimes \mathfrak{d}_1"'] & B \otimes A_1 \arrow[r, "\mathsf{act}"'] & A
\end{tikzcd}$$

What about the braiding in $3\mathrm{Cob}_{\mathrm{arc}}$?

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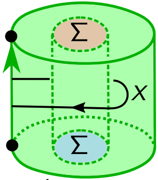
Theorem 2 [Costantino–F] (important to work with loc. fin. H-modules here)

$\mathrm{Bim}_{H,\mathrm{lf}}^{\mathsf{QMM}}$ is a **braided** monoidal subcategory of Bim_H (the braiding is defined by means of the QMMs)

Theorem 1 bis [Costantino–F]

$\mathcal{S}_H$ takes values in $\mathrm{Bim}_{H,\mathrm{lf}}^{\mathsf{QMM}}$ and is a **braided** strict monoidal functor.

$d_X(\varphi \otimes x) =$ 

with state $\varphi \otimes x$

dinat transfo $\Longrightarrow$

$$\begin{tikzcd} X^* \otimes X \arrow[r, "i_X"] \arrow[dr, "d_X"'] & \mathscr{L} \arrow[d, "\mathsf{QMM}"', "\exists!\,\mathfrak{d}"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathcal{S}_H(\Sigma) \end{tikzcd}$$

$(\forall\, X \in H\text{-}\mathrm{mod})$

Kerler–Lyubashenko TQFT (cf. Marco's lectures for more details)

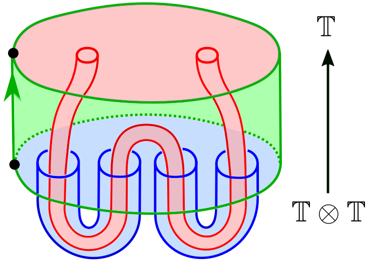
$\mathbb{T}$ = the torus with one boundary component. $\mathbb{D}$ = the disk.
Any object (=surface) in $3\mathrm{Cob}_{\mathrm{arc}}$ is of the form $\mathbb{T}^{\otimes g}$, $g \geq 0$.

Theorem: generators and relations for $3\mathrm{Cob}_{\mathrm{arc}}$

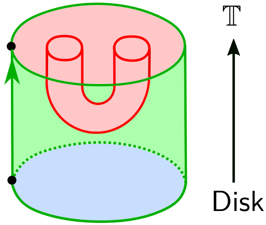
$\mathbb{T}$ is a Hopf algebra object in $3\mathrm{Cob}_{\mathrm{arc}}$. Any morphism (=3-cobordism) can be built from the Hopf data of $\mathbb{T}$ and morphisms $\lambda : \mathbb{T} \to \mathbb{D}$, $\Lambda : \mathbb{D} \to \mathbb{T}$, $w : \mathbb{D} \to \mathbb{T} \otimes \mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$ satisfying a certain list of relations.

Biblio: partly in [Kerler 01], announce in [Asaeda 11] citing Habiro, first proof in [Bobtcheva–Piergallini 12], new proof in [B–Beliakova–De Renzi–P 23].

Example:



Multiplication of $\mathbb{T}$



Unit of $\mathbb{T}$

Kerler–Lyubashenko TQFT (cf. Marco's lectures for more details)

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Said differently

Braided monoidal functor $F : 3\mathrm{Cob}_{\mathrm{arc}} \to \mathcal{C}$
$\iff$ Hopf alg in $\mathcal{C}$ with morphisms $\lambda', \Lambda', w', \tau'$ satisfying the relations.

Consequence: Kerler–Lyubashenko TQFT [Beliakova–De Renzi 21]

Let $\mathcal{C}$ be a ribbon factorizable finite tensor category and $\mathscr{L} = \int^{X \in \mathcal{C}} X^* \otimes X$.
There is a br. mon. functor $\mathrm{KL}_{\mathcal{C}} : 3\mathrm{Cob}^{\sigma} \to \mathcal{C}$ sending $\mathbb{T}$ to $\mathscr{L}$.

(σ means "extra datum on cobordisms")

KL vs. $\mathcal{S}_H$

Assume that the ribbon Hopf alg H is **fin-dim and factorizable**.
$\mathscr{L} = \int^{X \in H\text{-mod}} X^* \otimes X \cong H^*_{\text{coad}}$ as a H-module.

Theorem 3 [Costantino–F.]

$$\begin{tikzcd} 3\mathrm{Cob}^\sigma \arrow[r, "\mathrm{KL}_H"] \arrow[d, "\text{Forget}"'] \arrow[dr, phantom, "\circlearrowleft"] & H\text{-mod} \arrow[d] \\ 3\mathrm{Cob}_{\mathrm{arc}} \arrow[r, "\mathcal{S}_H"'] & \mathrm{Bim}^{\mathsf{QMM}}_{H,\mathrm{fin}} \end{tikzcd} \longleftarrow - - \begin{cases} V \mapsto \underline{\mathrm{End}}(V) \\ \mathrm{Hom}_H(V,W) \ni f \mapsto \underline{\mathrm{Hom}}(V,W) \quad (\forall f!!!) \end{cases}$$

Said differently:

- $\forall$ surface $\mathbb{T}^{\otimes g}$, $\mathcal{S}_H(\mathbb{T}^{\otimes g}) \cong \underline{\mathrm{End}}(\mathscr{L}^{\otimes g})$ as algebras in H-mod.
- $\forall$ cob $M : \mathbb{T}^{\otimes g_1} \to \mathbb{T}^{\otimes g_2}$, $\mathcal{S}_H(M) \cong \underline{\mathrm{Hom}}(\mathscr{L}^{\otimes g_1}, \mathscr{L}^{\otimes g_2})$ as bim in H-mod.

Idea of the proof

$\mathcal{S}_H(\mathbb{T}^{\otimes g}) \cong \underline{\mathrm{End}}(\mathscr{L}^{\otimes g})$ is from [F.20], [Baseilhac–F.–Roche 23] ("Alekseev iso")

$\Rightarrow$ Any $\big(\mathcal{S}_H(\mathbb{T}^{\otimes g_2}), \mathcal{S}_H(\mathbb{T}^{\otimes g_1})\big)$-bim is iso to $\bigoplus \underline{\mathrm{Hom}}(\mathscr{L}^{\otimes g_1}, \mathscr{L}^{\otimes g_2})$.

Then count dimension to show there is a unique summand.