Mapping class group representations from quantum character varieties

Matthieu Faitg*

On-line Johnson Seminar

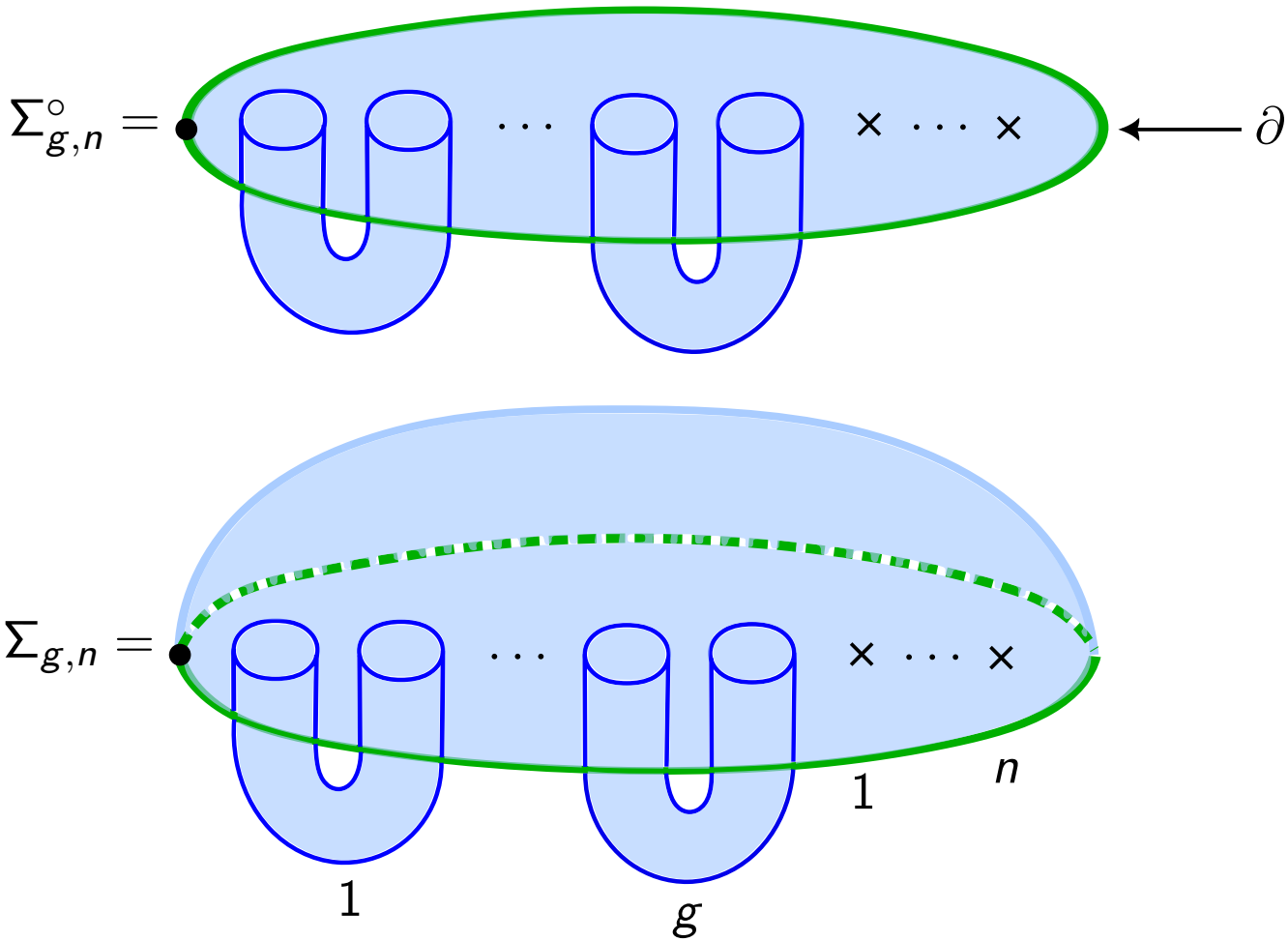
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Partially based on joint works with S. Baseilhac, F. Costantino and P. Roche

Surfaces

Notations used throughout:

- Σ = an oriented surface Σ with a basepoint $\bullet$
- Σ° = an oriented surface with one boundary component and a basepoint $\bullet$ on ∂



Part 1

The classical case

- Function algebras $\mathcal{O}[\mathrm{Rep}_G(\Sigma)]$ and $\mathcal{O}[\mathrm{Ch}_G(\Sigma)]$
- Mapping class group action on them
- Description of function algebras by labelled multicurves
- Boundary closure

"classical" = non-quantized

Representation variety

$G \subset \mathrm{GL}(V)$ a "classical group", for instance $\mathrm{SL}_n(\mathbb{C})$.

$$\mathrm{Rep}_G(\Sigma, \bullet) := \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(\Sigma, \bullet), G) \qquad \bullet = \text{basepoint}$$

Representation variety

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$$\mathrm{Rep}_G(\Sigma) := \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(\Sigma), G)$$

We mainly care of ***functions*** on this object.

Matrix coefficients: they generate the algebra $\mathcal{O}(G)$

$$c_v^\ell : G \to \mathbb{C}, \quad g \mapsto \ell(g \cdot v) \qquad\qquad (v \in V, \ell \in V^*)$$

Example $G = \mathrm{SL}_2(\mathbb{C})$

$\mathcal{O}(G) = \mathbb{C}[a, b, c, d \mid ad - bc = 1]$.

$$a = c_{v_1}^{v^1}, \quad b = c_{v_2}^{v^1}, \quad c = c_{v_1}^{v^2}, \quad d = c_{v_2}^{v^2}$$

with $\{v_1, v_2\}$ the canonical basis of $\mathbb{C}^2$ and $\{v^1, v^2\}$ the dual basis.

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Functions on $\mathrm{Rep}_G(\Sigma)$

$\mathcal{O}\big[\mathrm{Rep}_G(\Sigma)\big]$ is the subalgebra of $\{\mathrm{Rep}_G(\Sigma) \to \mathbb{C}\}$ generated by

$$F_{f,\gamma} : \mathrm{Rep}_G(\Sigma) \xrightarrow{\mathrm{ev}_\gamma} G \xrightarrow{f} \mathbb{C}, \quad \rho \mapsto f(\rho(\gamma))$$

for all $\gamma \in \pi_1(\Sigma)$ and $f \in \mathcal{O}(G)$.

In particular we have matrix coefficients for each loop:

$$c(\gamma)_v^\ell : \mathrm{Rep}_G(\Sigma) \xrightarrow{\mathrm{ev}_\gamma} G \xrightarrow{c_v^\ell} \mathbb{C}, \quad \rho \mapsto \ell(\rho(\gamma) \cdot v)$$

Character variety

Idea: identify reps which are related by change of basepoint

$$\mathrm{Ch}_G(\Sigma) := \mathrm{Rep}_G(\Sigma)/\!\!/G = \mathrm{Hom}_{\mathrm{Grp}}(\pi_1(\Sigma), G)/\!\!/G$$

where G acts on $\mathrm{Rep}_G(\Sigma)$ by $\quad (g\cdot\rho)(\gamma) = (\mathrm{ad}(g)\circ\rho)(\gamma) = g\rho(\gamma)g^{-1} \quad \forall\rho.$

"Coadjoint action" of $g \in G$ on $F \in \mathcal{O}\big[\mathrm{Rep}_G(\Sigma)\big]$:

$$\mathrm{coad}(g)(F) : \rho \mapsto F\big(\mathrm{ad}(g)\circ\rho\big) \qquad (\textit{right} \text{ action})$$

Functions on $\mathrm{Ch}_G(\Sigma)$

$$\mathcal{O}\big[\mathrm{Ch}_G(\Sigma)\big] \underset{\text{def}}{=} \mathcal{O}\big[\mathrm{Rep}_G(\Sigma)\big]^{G\text{-inv}} = \big\{F \,\big|\, \forall\, g\in G,\ \mathrm{coad}(g)(F) = F\big\}$$

N.B.: $\mathrm{coad}(g)$ is an alg. automorphism for all $g \quad\Rightarrow\quad \mathcal{O}\big[\mathrm{Ch}_G(\Sigma)\big]$ is a subalg.

Example

Recall: $F_{f,\gamma} : \rho \mapsto f\big(\rho(\gamma)\big)$

- $\mathrm{coad}(g)(F_{f,\gamma}) = F_{\mathrm{coad}(g)(f),\gamma}$.
- $f \in \mathcal{O}(G)^{G\text{-inv}} \;\Rightarrow\; F_{f,\gamma} \in \mathcal{O}\big[\mathrm{Ch}_G(\Sigma)\big]$ In particular $f = \mathrm{Tr} \;\Rightarrow$ "Wilson loop" Tr_γ
- $f \in \mathcal{O}(G)^{G\text{-inv}}$ and γ_1, γ_2 freely homotopic $\quad\Rightarrow\quad F_{f,\gamma_1} = F_{f,\gamma_2}$ (indep. of $\bullet$)

Mapping class group action

$$\mathrm{MCG}^{(\bullet)}(\Sigma) = \{\text{diffeo } \Sigma \to \Sigma \text{ fixing } \partial \text{ pointwise and } \bullet\}/\text{isotopy}$$

$\mathrm{MCG}^{(\bullet)}(\Sigma) \curvearrowright \pi_1(\Sigma)$ (left action by moving loops)

$\Rightarrow$ $\mathrm{Rep}_G(\Sigma) \curvearrowleft \mathrm{MCG}^{(\bullet)}(\Sigma)$ (right action by pullback)

$\Rightarrow$ $\mathrm{MCG}^{(\bullet)}(\Sigma) \curvearrowright \mathcal{O}\big[\mathrm{Rep}_G(\Sigma)\big]$ (left action by pullback)

$$\mathrm{MCG}^{(\bullet)}(\Sigma) \longrightarrow \mathrm{Aut}_{\mathrm{alg}}(\mathcal{O}\big[\mathrm{Rep}_G(\Sigma)\big]), \quad \tau \longmapsto \widetilde{\tau}$$

Action on generators

Recall: $F_{f,\gamma} : \rho \mapsto f\big(\rho(\gamma)\big)$

$$\forall\, \tau \in \mathrm{MCG}^{(\bullet)}(\Sigma), \quad \widetilde{\tau}(F_{f,\gamma}) = F_{f,\tau(\gamma)}$$

Recall: $\mathcal{O}\big[\mathrm{Rep}_G(\Sigma)\big] \curvearrowleft G$ by $\mathrm{coad}(g)(F_{f,\gamma}) = F_{\mathrm{coad}(g)(f),\gamma}$

$\Rightarrow$ $\mathrm{MCG}^{(\bullet)}(\Sigma) \curvearrowright \mathcal{O}\big[\mathrm{Rep}_G(\Sigma)\big] \curvearrowleft G$ commuting actions

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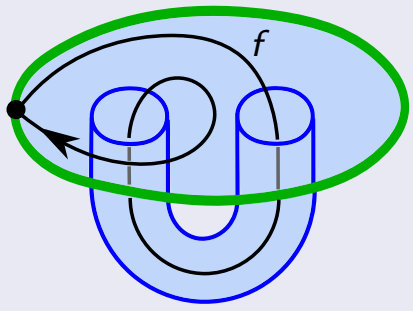
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$\Rightarrow$ $\mathrm{MCG}^{(\bullet)}(\Sigma) \curvearrowright \mathcal{O}\big[\mathrm{Rep}_G(\Sigma)\big] \curvearrowleft G$ commuting actions

$\Rightarrow$ The subalgebra $\mathcal{O}\big[\mathrm{Ch}_G(\Sigma)\big]$ is stable by $\mathrm{MCG}(\Sigma)$.

Remark: If $\partial \neq \varnothing$ then $\bullet \in \partial$ by convention, and no need to worry for $\bullet$.

Diagrams for functions on $\mathrm{Rep}_G(\Sigma)$



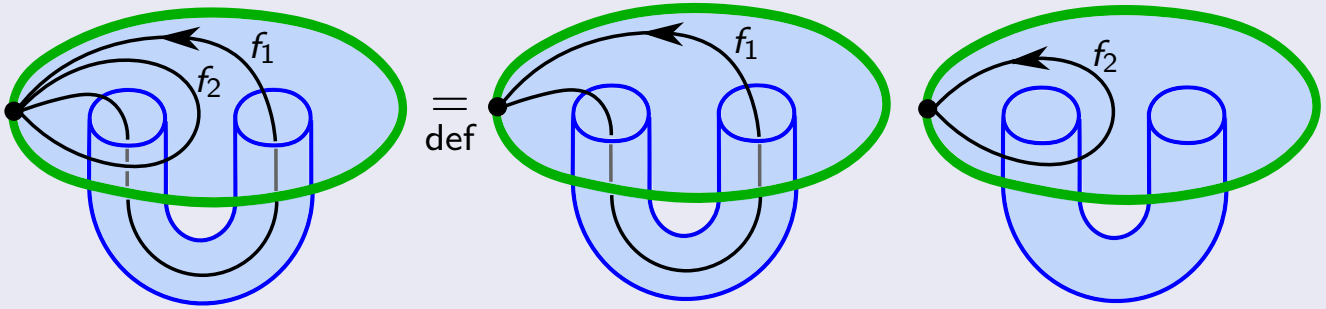
$$\underset{\text{def}}{=} \left[\begin{array}{rl} F_{f,\gamma} : \mathrm{Rep}_G(\Sigma) & \longrightarrow \mathbb{C} \\ \rho & \longmapsto f\left(\rho(\gamma)\right) \end{array} \right]$$

with $f \in \mathcal{O}(G) \subset \{G \to \mathbb{C}\}$

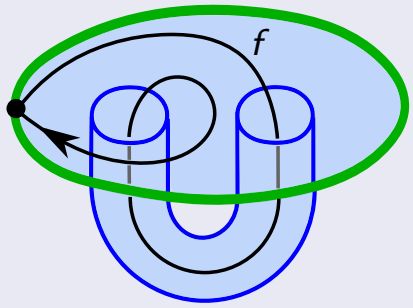
$\gamma = ba$ and $\Sigma = \Sigma^{\circ}_{1,0}$ in this example

Rules of the game (an algebra of labelled multicurves)

- Labelled multicurves represent products: ($\Rightarrow$ product is union)



Diagrams for functions on $\mathrm{Rep}_G(\Sigma)$



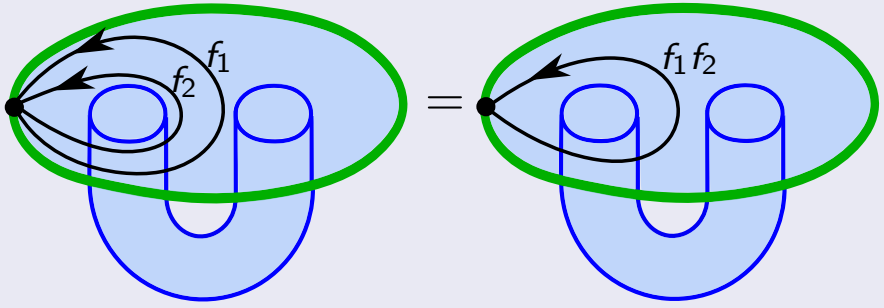
$$\underset{\text{def}}{=} \left[\begin{array}{rl} F_{f,\gamma} : \mathrm{Rep}_G(\Sigma) & \longrightarrow \mathbb{C} \\ \rho & \longmapsto f\big(\rho(\gamma)\big) \end{array} \right]$$

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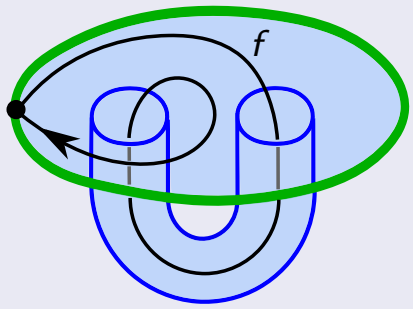
- "Fusion relation":



Indeed, for all $\rho \in \mathrm{Rep}_G(\Sigma)$,

$$\begin{aligned} \big(F_{f_1,\gamma} F_{f_2,\gamma}\big)(\rho) = F_{f_1,\gamma}(\rho) F_{f_2,\gamma}(\rho) &= f_1\big(\rho(\gamma)\big) f_2\big(\rho(\gamma)\big) \\ &= (f_1 f_2)\big(\rho(\gamma)\big) = F_{f_1 f_2,\gamma}(\rho). \end{aligned}$$

Diagrams for functions on $\mathrm{Rep}_G(\Sigma)$



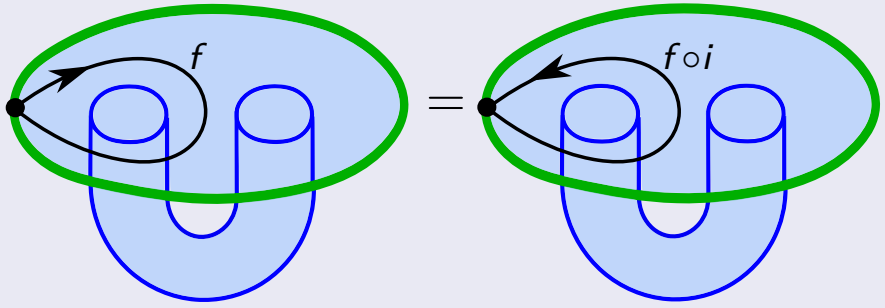
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with $f \in \mathcal{O}(G) \subset \{G \to \mathbb{C}\}$

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Rules of the game (an algebra of labelled multicurves)

- Change of orientation:

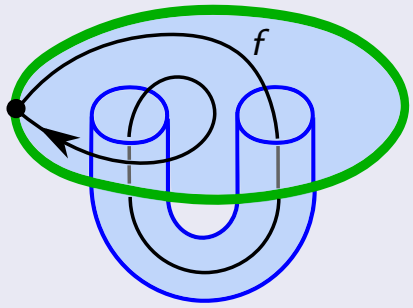


where $i : G \to G$, $g \mapsto g^{-1}$

Indeed, for all $\rho \in \mathrm{Rep}_G(\Sigma)$,

$$F_{f,\gamma^{-1}}(\rho) = f\big(\rho(\gamma^{-1})\big) = f\big(\rho(\gamma)^{-1}\big) = (f \circ i)\big(\rho(\gamma)\big) = F_{f\circ i,\gamma}(\rho).$$

Diagrams for functions on $\mathrm{Rep}_G(\Sigma)$



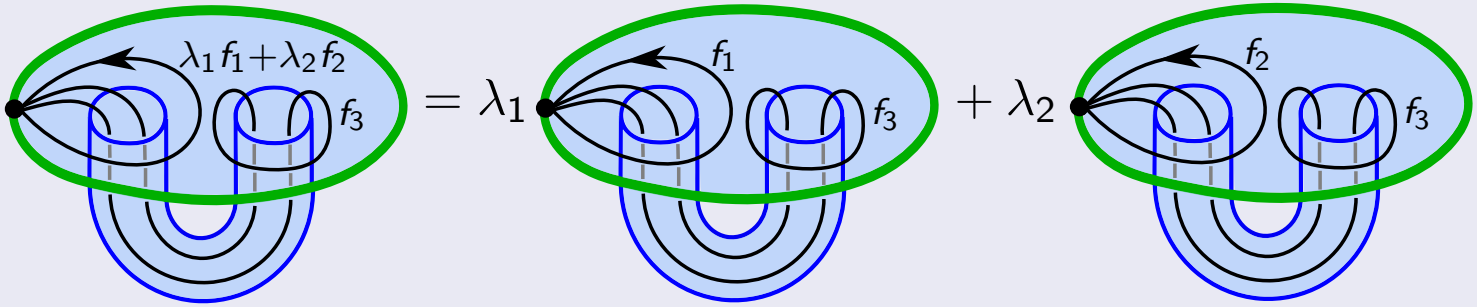
$$\underset{\text{def}}{=} \left[\begin{array}{rcl} F_{f,\gamma} : \mathrm{Rep}_G(\Sigma) & \longrightarrow & \mathbb{C} \\ \rho & \longmapsto & f(\rho(\gamma)) \end{array} \right]$$

with $f \in \mathcal{O}(G) \subset \{G \to \mathbb{C}\}$

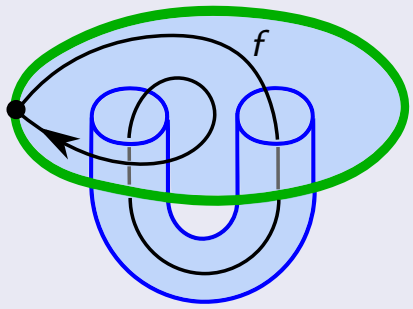
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Rules of the game (an algebra of labelled multicurves)

- Linearity for each label:



Diagrams for functions on $\mathrm{Rep}_G(\Sigma)$



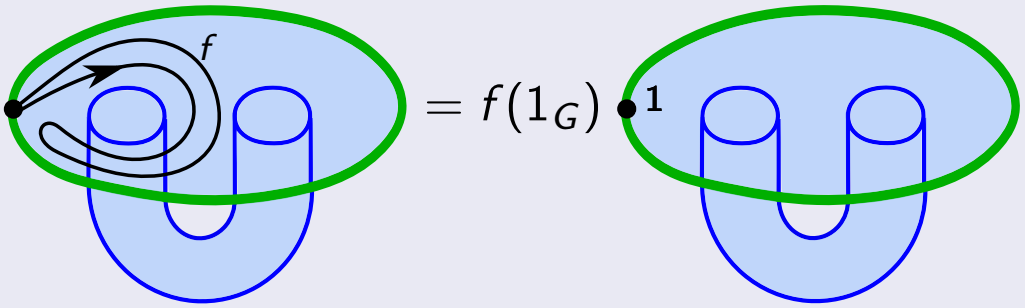
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Rules of the game (an algebra of labelled multicurves)

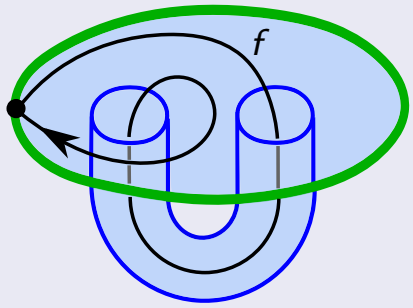
- Contractible loops can be erased (or "$\supset$-relation"):



Indeed, for all $\rho \in \mathrm{Rep}_G(\Sigma)$,

$$F_{f,\mathrm{triv}}(\rho) = f\big(\rho(\mathrm{triv})\big) = f\big(\rho(1_{\pi_1(\Sigma)})\big) = f(1_G) = f(1_G)\mathbf{1}(\rho).$$

Diagrams for functions on $\mathrm{Rep}_G(\Sigma)$



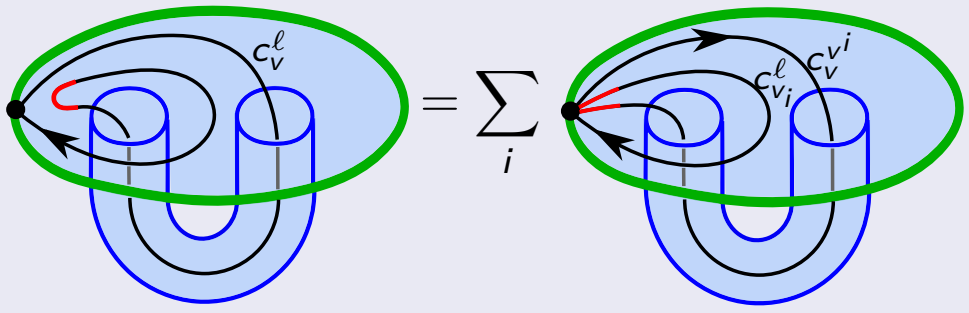
$$\underset{\text{def}}{=} \left[\begin{array}{rl} F_{f,\gamma} : \mathrm{Rep}_G(\Sigma) & \longrightarrow \mathbb{C} \\ \rho & \longmapsto f(\rho(\gamma)) \end{array} \right]$$

with $f \in \mathcal{O}(G) \subset \{G \to \mathbb{C}\}$

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Rules of the game (an algebra of labelled multicurves)

- "$\subset$-relation":



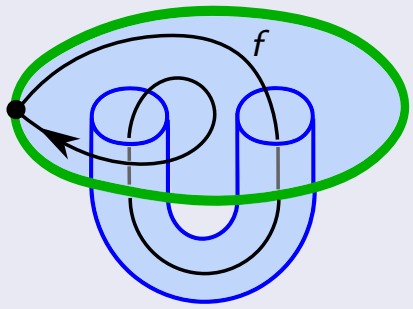
(v_i) basis of V
(v^i) dual basis

Indeed, for all $\rho \in \mathrm{Rep}_G(\Sigma)$,

$$c(\gamma_1\gamma_2)^{v^j}_{v_k}(\rho) = \rho(\gamma_1\gamma_2)^j_k = \big(\rho(\gamma_1)\rho(\gamma_2)\big)^j_k = \sum_i \rho(\gamma_1)^j_i \rho(\gamma_2)^i_k$$

$$= \sum_i c(\gamma_1)^{v^j}_{v_i}(\rho)\, c(\gamma_2)^{v^i}_{v_k}(\rho) = \Big(\sum_i c(\gamma_1)^{v^j}_{v_i} c(\gamma_2)^{v^i}_{v_k}\Big)(\rho).$$

Diagrams for functions on $\mathrm{Rep}_G(\Sigma)$



$$\underset{\text{def}}{=} \left[\begin{array}{rl} F_{f,\gamma} : \mathrm{Rep}_G(\Sigma) & \longrightarrow \mathbb{C} \\ \rho & \longmapsto f(\rho(\gamma)) \end{array} \right]$$

with $f \in \mathcal{O}(G) \subset \{G \to \mathbb{C}\}$

$\gamma = ba$ and $\Sigma = \Sigma^{\circ}_{1,0}$ in this example

Rules of the game (an algebra of labelled multicurves)

- Action of $g \in G$: act by $\mathrm{coad}(g)$ on the labels
- Action of $\tau \in \mathrm{MCG}(\Sigma)$: apply τ to the multicurve

Boundary closure

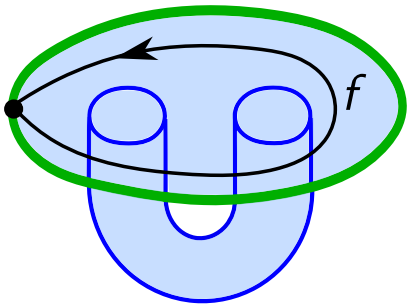
Let Σ without boundary and $\Sigma^\circ = \Sigma \setminus \mathbb{D}^2$.

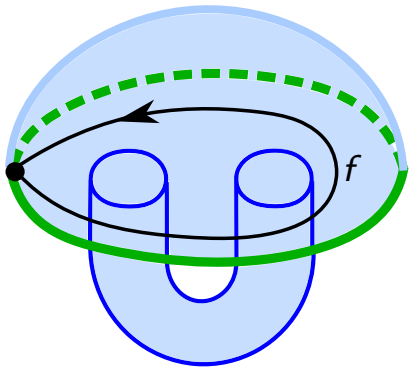
$$\pi_1(\Sigma) = \pi_1(\Sigma^\circ)/\langle\partial\rangle \quad \Rightarrow \quad \mathrm{Rep}_G(\Sigma) = \{\rho \in \mathrm{Rep}_G(\Sigma^\circ) \,|\, \rho(\partial) = 1_G\}$$

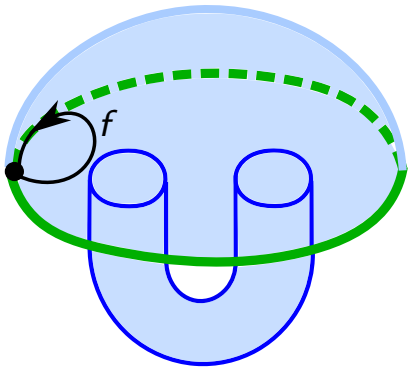
Boundary map

$$\mathfrak{d} : \mathcal{O}(G) \longrightarrow \mathcal{O}[\mathrm{Rep}_G(\Sigma^\circ)], \quad \mathfrak{d}(f) = F_{f,\partial} : \rho \mapsto f(\rho(\partial))$$

G-equivariant (for coad) and algebra morphism

$\mathfrak{d}(f) =$ 



$=$ 

$= f(1_G)\mathbf{1}$

$$\mathcal{O}^+(G) := \{f \in \mathcal{O}(G) \,|\, f(1_G) = 0\} \quad \text{(“augmentation ideal”)}$$

At least heuristically

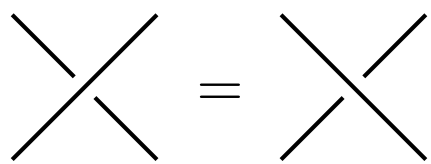
$$\mathcal{O}[\mathrm{Rep}_G(\Sigma)] = \mathcal{O}[\mathrm{Rep}_G(\Sigma^\circ)]/\langle \mathfrak{d}(f) \,|\, f \in \mathcal{O}^+(G)\rangle$$

$$\mathcal{O}[\mathrm{Ch}_G(\Sigma)] = \mathcal{O}[\mathrm{Ch}_G(\Sigma^\circ)]/\langle \mathfrak{d}(f) \,|\, f \in \mathcal{O}^+(G)^{G\text{-inv}}\rangle$$

Part 2

Towards quantization (from 2d to 3d)

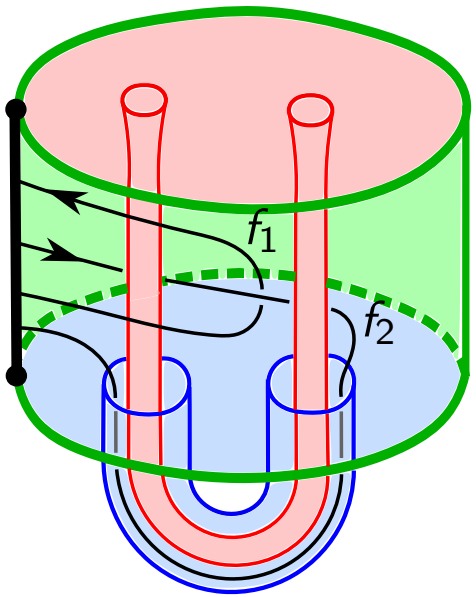
- A free algebra $\mathcal{SK}_G(\Sigma^\circ)$ of "stated skeins" in $\Sigma^\circ \times [0,1]$
- Projection $\mathcal{SK}_G(\Sigma^\circ) \twoheadrightarrow \mathcal{O}[\mathrm{Rep}_G(\Sigma^\circ)]$
- Description of its kernel and resulting quotient $\mathrm{Sk}_G(\Sigma^\circ)$



Skeins in thickened surfaces

$\mathcal{O}(G)$-*labelled skein* = collection of arcs in $\Sigma^\circ \times [0,1]$

- No intersections (and no self-intersections)
- The end-points are in $\{\bullet\} \times [0,1]$
- Each arc is labelled by an element of $\mathcal{O}(G)$
- Considered up to isotopy (you can deform them)

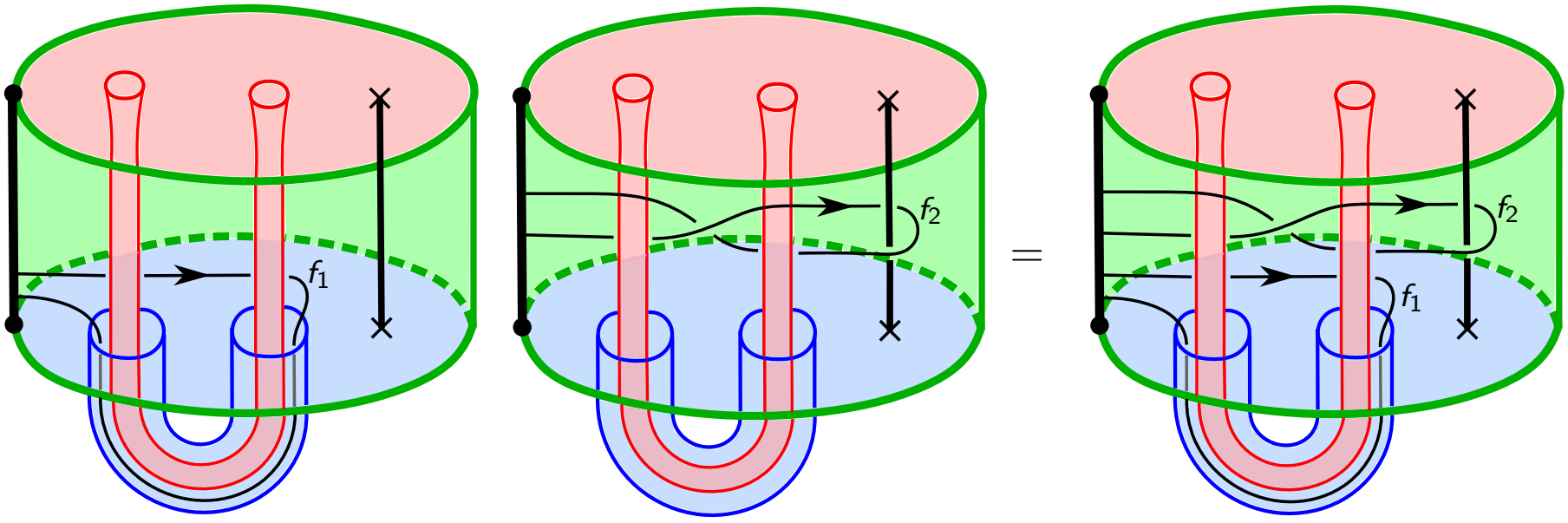


Boundary points can be moved but their order with respect to height must be preserved

Remark for experts

We could allow non-colored closed components (circles), but they can be transformed into arcs by the skein relations.

Skeins in thickened surfaces



Stacking product of two $\mathcal{O}(G)$-colored skeins S_1, S_2

- Isotope S_1 in $\Sigma^\circ \times \left[0, \frac{1}{2}\right[$
- Isotope S_2 in $\Sigma^\circ \times \left]\frac{1}{2}, 1\right]$
- Take their union in $\Sigma^\circ \times [0, 1]$

In search of relations: projection

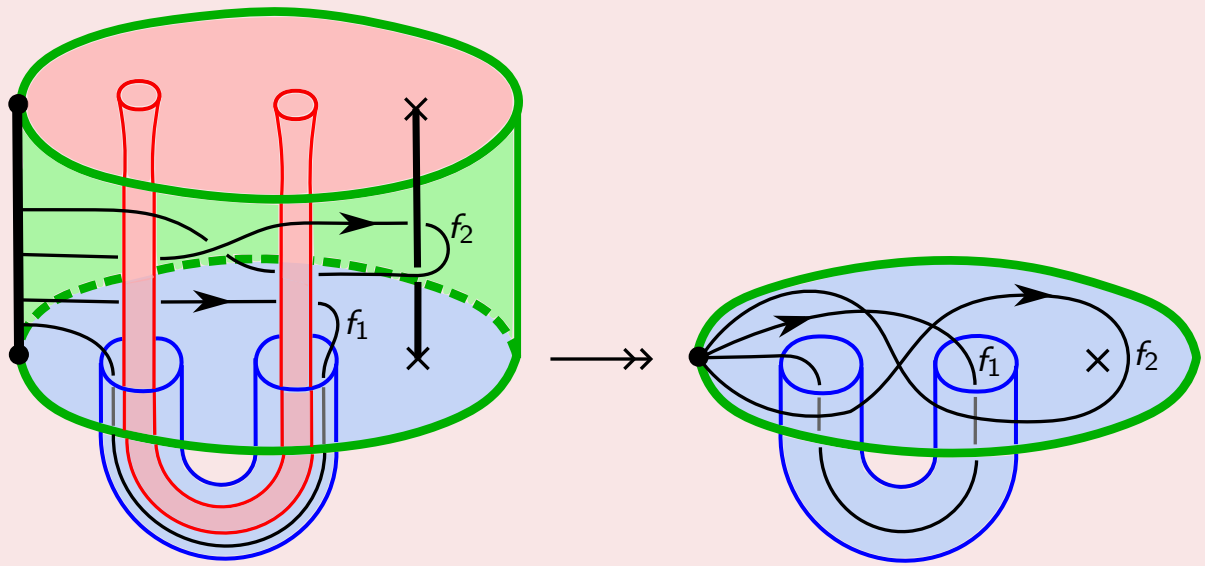
Free algebra of $\mathcal{O}(G)$-labelled skeins

$\mathcal{SK}_G(\Sigma^\circ) =$ the $\mathbb{C}$-vector space generated by the $\mathcal{O}(G)$-labelled skeins in $\Sigma^\circ \times [0,1]$, equipped with the bilinearly-extended stacking product.

Projection

$\Sigma^\circ \times [0,1] \twoheadrightarrow \Sigma^\circ \times \{0\} \cong \Sigma^\circ$ induces an algebra morphism

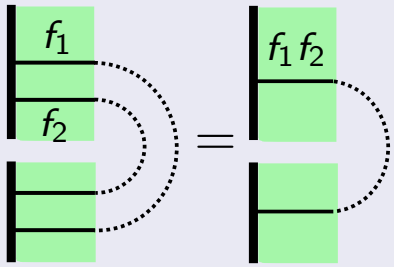
$$\mathrm{proj} : \mathcal{SK}_G(\Sigma^\circ) \twoheadrightarrow \mathcal{O}\big[\mathrm{Rep}_G(\Sigma^\circ)\big]$$

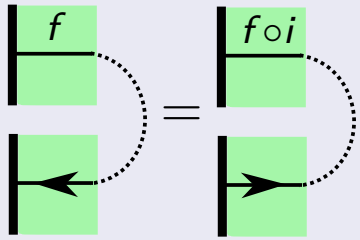


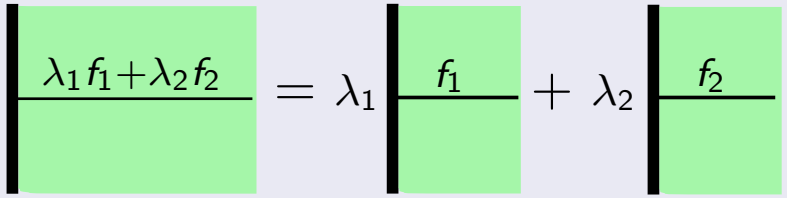
What are the relations defined by the kernel of the projection?

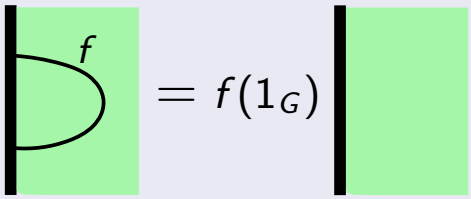
A 3d description of $\mathcal{O}\left[\mathrm{Rep}_G(\Sigma^\circ)\right]$

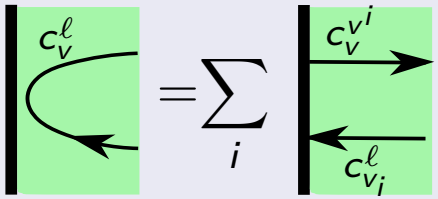
Relations coming from $\mathcal{O}\left[\mathrm{Rep}_G(\Sigma^\circ)\right]$

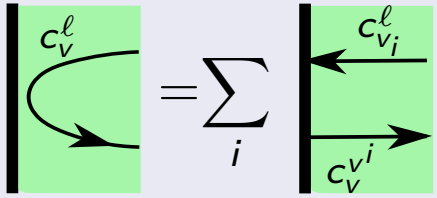




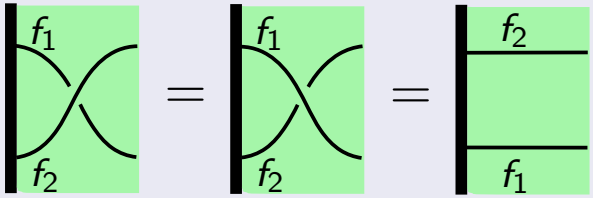








Relations coming from the projection: crossing relations

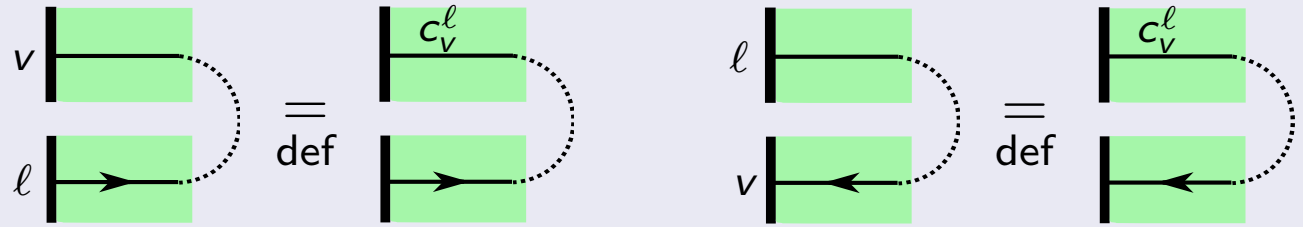


$$\mathrm{Sk}_G(\Sigma^\circ) \underset{\text{def}}{=} \mathcal{SK}_G(\Sigma^\circ)\,/\,(\text{these relations}) \xrightarrow[\text{proj}]{\sim} \mathcal{O}\left[\mathrm{Rep}_G(\Sigma^\circ)\right]$$

Further remarks

Remember: $\mathcal{O}(G)$ is generated by matrix coefficients c_v^ℓ

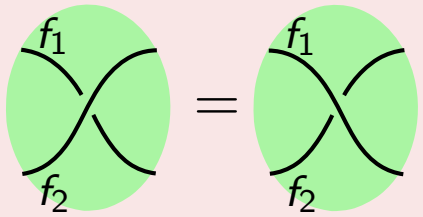
Linearity and fusion relations $\Rightarrow$ labels can be decomposed into matrix coefficients



ℓ and v are called "states"

$\rightsquigarrow$ *stated skeins*

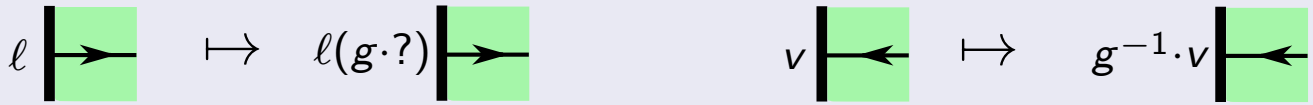
Isotopy + $\subset$-relation + crossing relation $\Rightarrow$



anywhere in $\Sigma^{\circ,\bullet} \times [0,1]$

- Right action of $g \in G$: apply $\mathrm{coad}(g)$ to each label

  or act on each state as follows:

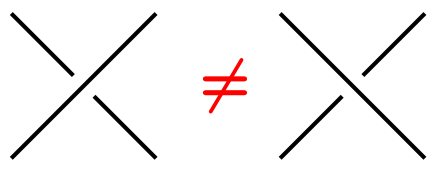


- Action of $\tau \in \mathrm{MCG}(\Sigma^\circ)$: apply τ to each layer $\Sigma^\circ \times \{h\}$.

Part 3

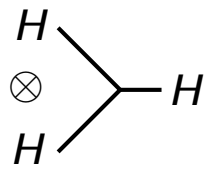
The "quantized" case

- $\mathcal{O}(G) \xrightarrow{\text{replaced by}} H^*$, with H a ribbon Hopf algebra
- Stated skein algebra $\mathrm{Sk}_H(\Sigma^\circ)$
- Algebraic properties of $\mathrm{Sk}_H(\Sigma^\circ)$
- $\rightsquigarrow$ **Projective representation of the mapping class group**

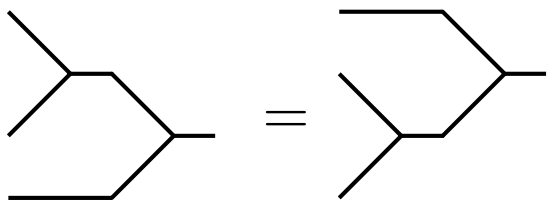


Ribbon Hopf algebra H

- H is a $\mathbb{C}$-algebra, *assumed fin-dim. to simplify the exposition*



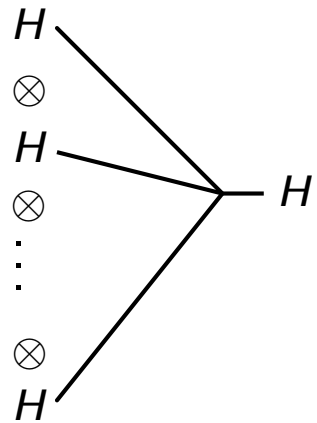
$H \otimes H \xrightarrow[m]{} H$



$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$

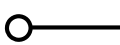
associativity

$\Rightarrow$

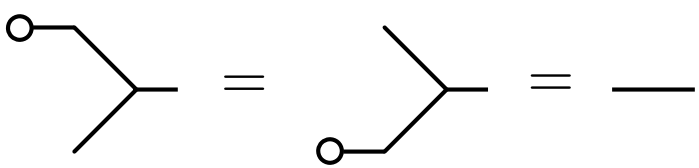


$H^{\otimes n} \longrightarrow H$

iterated product



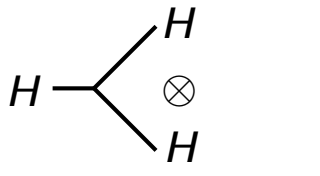
$\mathbb{C} \xrightarrow[1_H]{} H$



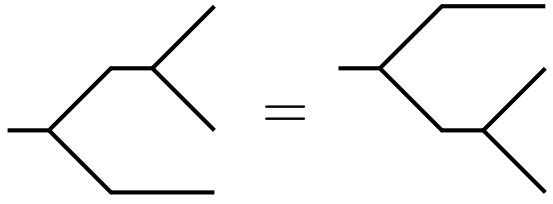
unitality

Ribbon Hopf algebra H

- H is a $\mathbb{C}$-algebra, *assumed fin-dim. to simplify the exposition*
- **coproduct**: a co-associative algebra morphism $\Delta : H \to H \otimes H$



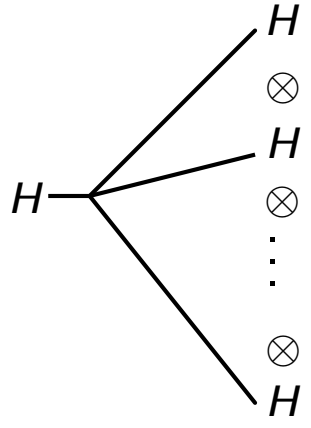
$$H \xrightarrow[\Delta]{} H \otimes H$$



$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

coassociativity

$\Rightarrow$

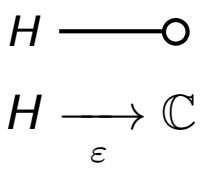


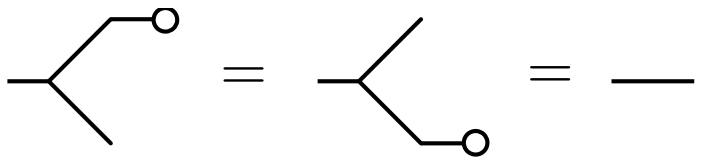
$$H \longrightarrow H^{\otimes n}$$

iterated coproduct

Ribbon Hopf algebra H

- H is a $\mathbb{C}$-algebra, *assumed fin-dim. to simplify the exposition*
- **coproduct**: a co-associative algebra morphism $\Delta : H \to H \otimes H$
- **counit** $\varepsilon : H \to \mathbb{C}$ and **antipode** $S : H \to H$ satisfying various properties





$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$$

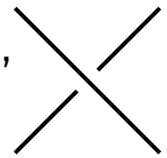
co-unitality

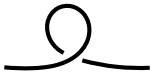
S is analogous to group inversion $G \to G$, $g \mapsto g^{-1}$

Ribbon Hopf algebra H

- H is a $\mathbb{C}$-algebra, *assumed fin-dim. to simplify the exposition*
- **coproduct**: a co-associative algebra morphism $\Delta : H \to H \otimes H$
- **counit** $\varepsilon : H \to \mathbb{C}$ and **antipode** $S : H \to H$ satisfying various properties
- **R-matrix**: an invertible element $R \in H \otimes H$ satisfying various properties
- **ribbon element**: an invertible element $\theta \in \mathcal{Z}(H)$ satisfying various properties

R "corresponds to" 

R^{-1} "corresponds to" 

θ "corresponds to" 

θ^{-1} "corresponds to" 

Ribbon Hopf algebra H

- H is a $\mathbb{C}$-algebra, *assumed fin-dim. to simplify the exposition*
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Property: H^* is a ***non-commutative*** algebra with unit ε

Given $f_1, f_2 \in H^*$, define $f_1 f_2 : H \xrightarrow{\Delta} H \otimes H \xrightarrow{f_1 \otimes f_2} \mathbb{C} \otimes \mathbb{C} = \mathbb{C}$.

Writing $R = \sum_j R_j^{(1)} \otimes R_j^{(2)}$, let $R' = \sum_j R_j^{(2)} \otimes R_j^{(1)}$ be the *flip* of R.

A kind of "non-degeneracy" for R

H is called ***factorizable*** if the map

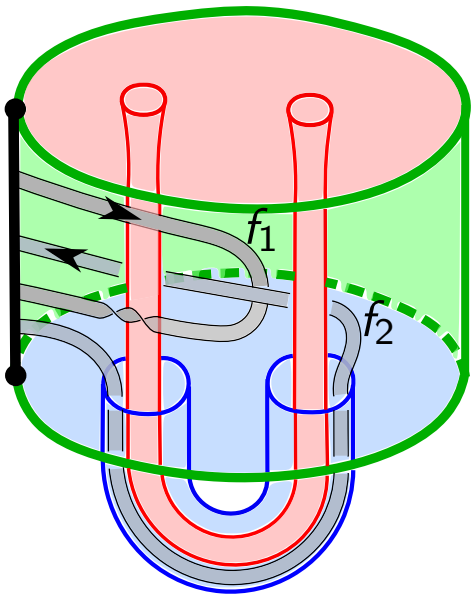
$$H^* \longrightarrow H, \qquad f \longmapsto (f \otimes \mathrm{id})(RR')$$

is a bijection (linear isomorphism).

Stated (ribbon) skeins

H^*-labelled skein = collection of ***ribbon*** arcs in $\Sigma^{\circ,\bullet} \times [0,1]$

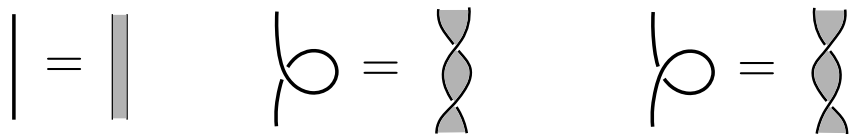
- No intersections (and no self-intersections) + half-twists not allowed
- The end-segments are in $\{\bullet\} \times [0,1]$
- Each ribbon arc is labelled by an element of H^*
- Considered up to isotopy (you can deform them)



Boundary segments can be moved but their order with respect to height must be preserved

Remark for experts
We could allow non-labelled closed components (annuli), but they can be transformed into arcs by the skein relations.

Drawing convention in the sequel :



Stated (ribbon) skeins

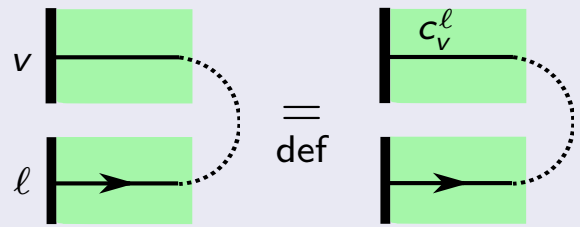
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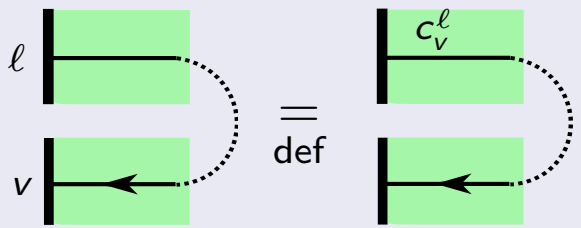
- No intersections (and no self-intersections) + half-twists not allowed
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Notation for states

Regular representation $\Rightarrow$ $H \hookrightarrow \mathrm{End}_{\mathbb{C}}(V)$ with $V = H$

$\Rightarrow$ Matrix coefficients $c_v^\ell : H \to \mathbb{C}, \quad h \mapsto \ell(hv).$ $\quad (\ell \in V^*, v \in V)$





ℓ and v are called "states" $\rightsquigarrow$ *stated skeins*

Remark: state-label correspondence



Stated (ribbon) skeins

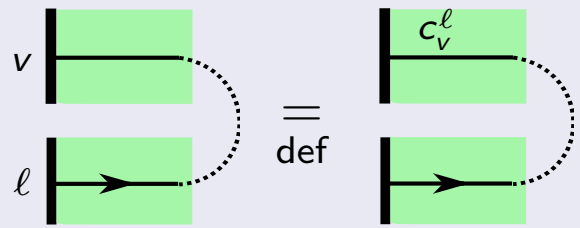
H^*-labelled skein = collection of ***ribbon*** arcs in $\Sigma^{\circ,\bullet} \times [0,1]$

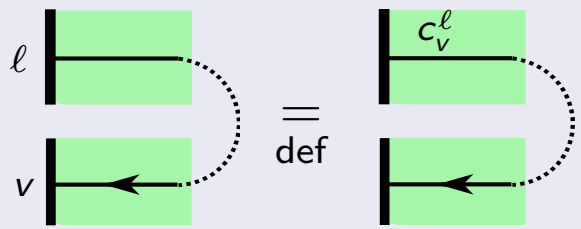
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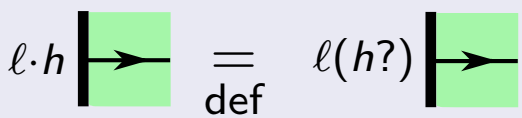


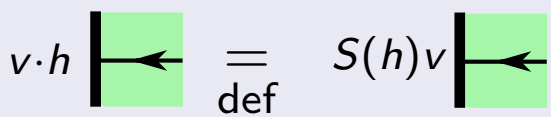


ℓ and v are called "states" $\rightsquigarrow$ *stated skeins*

Right action of H on a state

For all $h \in H$, define

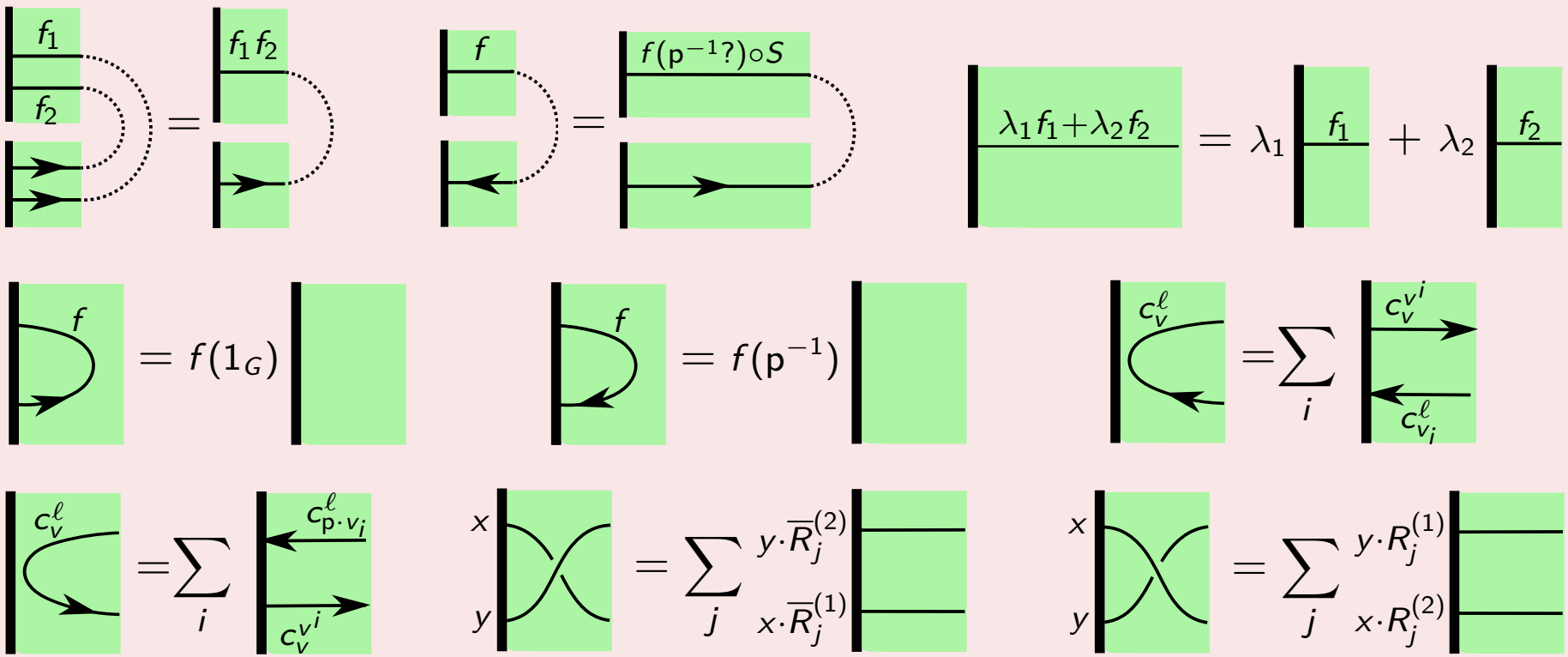




The stated skein algebra

Definition

$\mathrm{Sk}_H(\Sigma^\circ)$ is the $\mathbb{C}$-vector space generated by the H^*-labelled ribbon skeins in $\Sigma^\circ \times [0,1]$, modulo the ***stated skein relations***:



and equipped with the stacking product.

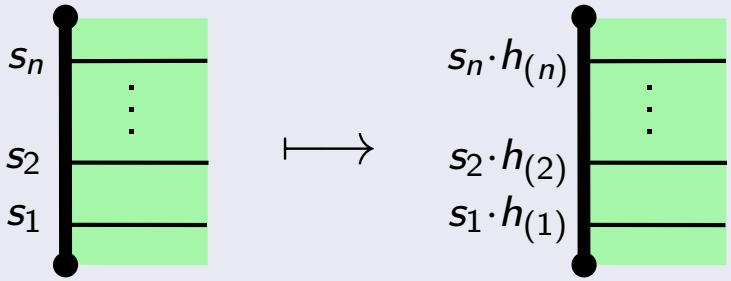
Notations: $R = \sum_j R_j^{(1)} \otimes R_j^{(2)}$, $R^{-1} = \sum_j \overline{R}_j^{(1)} \otimes \overline{R}_j^{(2)}$

$\mathrm{p} = u\theta^{-1}$ the *pivotal element*

where $u = \sum_j S(R_j^{(2)}) R_j^{(1)}$ (Drinfeld element)

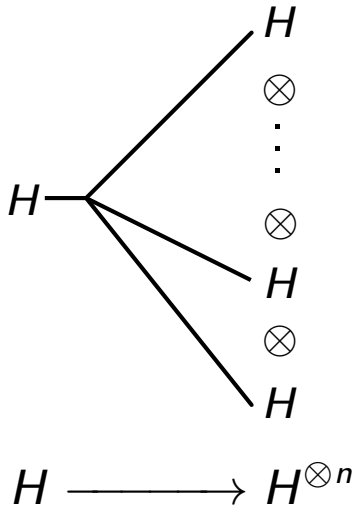
Action of H

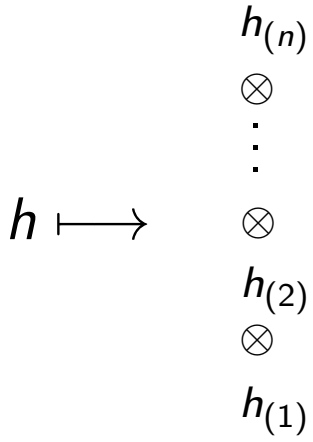
Right action of $h \in H$ on a stated skein = diagonal action on its states



Extend linearly $\Rightarrow$ $\mathrm{Sk}_H(\Sigma^\circ) \curvearrowleft H$

Remember 1: iterated coproduct

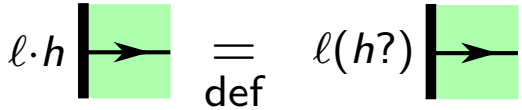


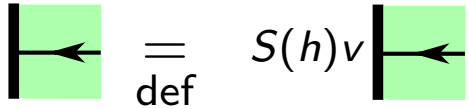


Sweedler's notation

(implicit summation)

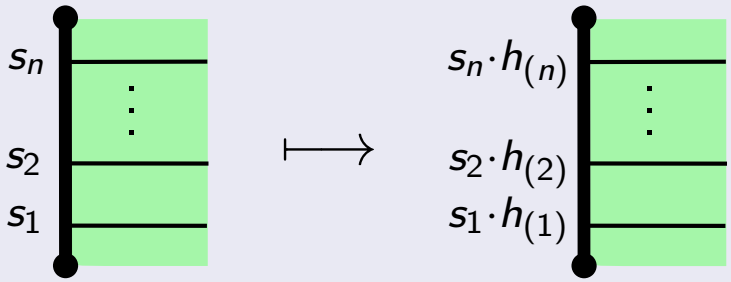
Remember 2: right action on states





Action of H

Right action of $h \in H$ on a stated skein = diagonal action on its states



Extend linearly $\Rightarrow$ $\mathrm{Sk}_H(\Sigma^\circ) \curvearrowleft H$

Compatibility: $\mathrm{Sk}_H(\Sigma^\circ)$ *is a H-module algebra*

By co-associativity and def. of the stacking product and of the action:

$$(XY) \cdot h = (X \cdot h_{(1)})(Y \cdot h_{(2)})$$

$\Rightarrow$ the subspace of H-invariant elements

$$\mathrm{Sk}_H^{\mathrm{inv}}(\Sigma^\circ) \underset{\text{def}}{=} \{X \in \mathrm{Sk}_H(\Sigma^\circ) \,|\, \forall\, h \in H,\ \ X \cdot h = \varepsilon(h)X\}$$

is a subalgebra.

For suitable G and H

- $\mathrm{Sk}_H(\Sigma^\circ)$ is a deformation-quantization of $\mathcal{O}\big[\mathrm{Rep}_G(\Sigma^\circ)\big]$
- $\mathrm{Sk}_H^{\mathrm{inv}}(\Sigma^\circ)$ is a def.-quant. of $\mathcal{O}\big[\mathrm{Ch}_G(\Sigma^\circ)\big] = \mathcal{O}\big[\mathrm{Rep}_G(\Sigma^\circ)\big]^{G\text{-inv}}$

Basis of the stated tangle algebra

In the classical case

$$\pi_1(\Sigma^\circ_{g,n}) = \mathrm{Free}\big(b_1, a_1, \ldots, b_g, a_g, p_1, \ldots, p_n\big)$$

$$\Rightarrow \quad \mathrm{Rep}_G(\Sigma^\circ_{g,n}) \cong G^{\times(2g+n)} \quad \Rightarrow \quad \mathcal{O}\big[\mathrm{Rep}_G(\Sigma^\circ_{g,n})\big] \cong \mathcal{O}(G)^{\otimes(2g+n)}$$

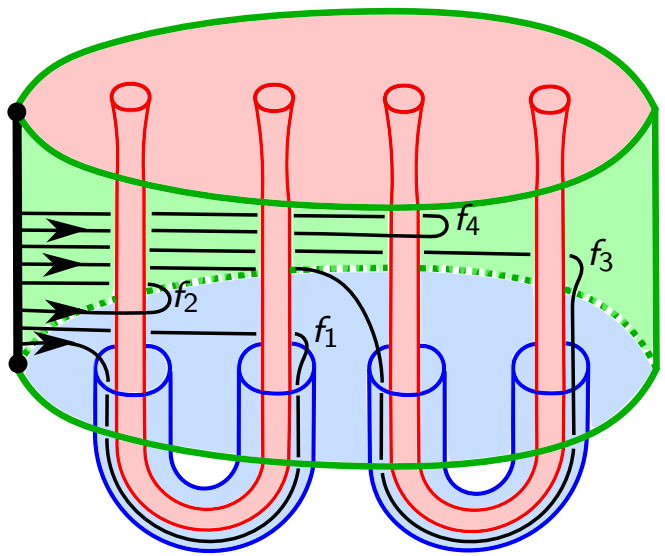
Example: $\mathcal{O}(G)^{\otimes 4} \xrightarrow{\sim} \mathcal{O}\big[\mathrm{Rep}_G(\Sigma^\circ_{2,0})\big]$, $f_1 \otimes \ldots \otimes f_4 \longmapsto F_{f_1,b_1} F_{f_2,a_1} F_{f_3,b_2} F_{f_4,a_2}$

Theorem

(§6.2 in [arXiv:2302.00396])

$\mathrm{Sk}_H(\Sigma^\circ_{g,n}) \cong (H^*)^{\otimes(2g+n)}$ as $\mathbb{C}$-vector spaces.

$f_1 \otimes \ldots \otimes f_4 \longmapsto$



$= \widetilde{F}_{f_1,b_1} \widetilde{F}_{f_2,a_1} \widetilde{F}_{f_3,b_2} \widetilde{F}_{f_4,a_2}$

Differences with the classical case:

- $\widetilde{F}_{f_1,\gamma_1} \widetilde{F}_{f_2,\gamma_2} \neq \widetilde{F}_{f_2,\gamma_2} \widetilde{F}_{f_1,\gamma_1}$

(instead we have "braided exchange rules")

- $\widetilde{F}_{f_1,\gamma} \widetilde{F}_{f_2,\gamma} \neq \widetilde{F}_{f_1 f_2,\gamma}$

(instead we have "braided fusion")

See §4.1 in [arXiv:2302.00396] or §4.1 in [arXiv:2502.04267] for the explicit formulas

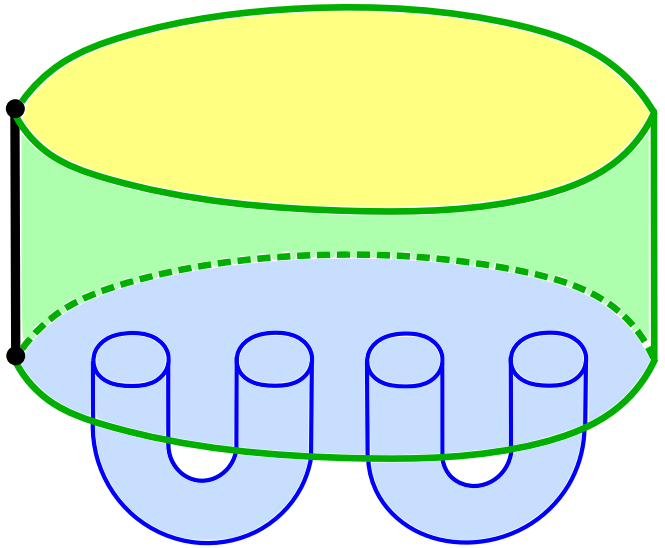
Standard representation on a handlebody

$\mathbb{H}_g$ = handlebody of genus g equipped with an arc on its boundary.

Stated skein module

$\mathrm{Sk}_H(\mathbb{H}_g)$ = the $\mathbb{C}$-vector space generated by the H^*-labelled ribbon skeins in $\mathbb{H}_g$, modulo the stated skein relations listed above.

$\mathbb{H}_2^\bullet =$



Standard representation on a handlebody

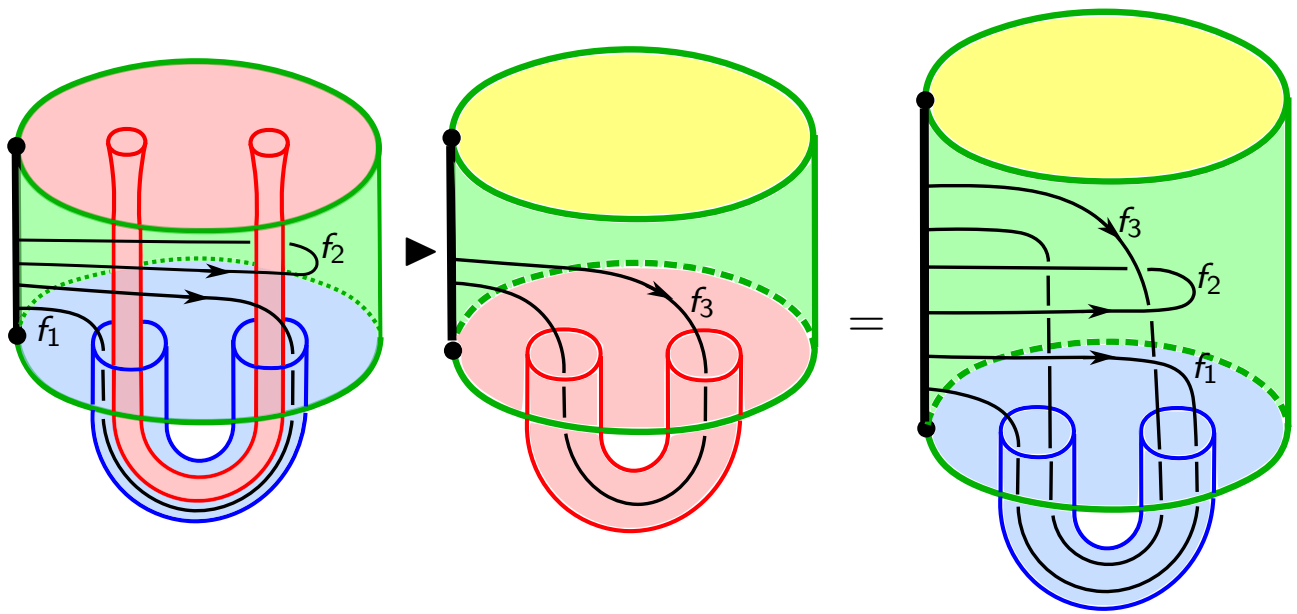
$\mathbb{H}_g$ = handlebody of genus g equipped with an arc on its boundary.

Stated skein module

$\mathrm{Sk}_H(\mathbb{H}_g)$ = the $\mathbb{C}$-vector space generated by the H^*-labelled ribbon skeins in $\mathbb{H}_g$, modulo the stated skein relations listed above.

Stacking action $\blacktriangleright : \mathrm{Sk}_H(\Sigma^\circ_{g,0}) \otimes \mathrm{Sk}_H(\mathbb{H}_g) \to \mathrm{Sk}_H(\mathbb{H}_g)$

Defined by gluing $\Sigma^\circ_{g,0} \times [0,1]$ and $\mathbb{H}_g$ along $\Sigma^\circ_{g,0}$ (think in terms of cobordisms)



This defines the action on skeins and we extend it bilinearly

Standard representation on a handlebody

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Remarks

- We have $\mathrm{Sk}_H(\mathbb{H}_g) \cong (H^*)^{\otimes g}$ as a vector space.
- There are explicit formulas for the action $\blacktriangleright$ of $\mathrm{Sk}_H(\Sigma^\circ_{g,0}) \cong (H^*)^{\otimes 2g}$ on $\mathrm{Sk}_H(\mathbb{H}_g) \cong (H^*)^{\otimes g}$ (see Prop. 4.10 in [arXiv:2502.04267])

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Defined by gluing $\Sigma^\circ_{g,0} \times [0,1]$ and $\mathbb{H}_g$ along $\Sigma^\circ_{g,0}$ (think in terms of cobordisms)

Let $\Psi : \mathrm{Sk}_H(\Sigma^\circ_{g,0}) \longrightarrow \mathrm{End}_{\mathbb{C}}\big(\mathrm{Sk}_H(\mathbb{H}_g)\big) \cong \mathrm{End}_{\mathbb{C}}\big((H^*)^{\otimes g}\big)$

be the representation morphism, *i.e.* $\Psi(X) : Y \mapsto X \blacktriangleright Y$.

Theorem (§3.3 in [arXiv:1812.00446], also see App. A in [arXiv:2502.04267])

If H is finite-dimensional and factorizable then Ψ is an isomorphism.

Hence the stated skein algebra $\mathrm{Sk}_H(\Sigma^\circ_{g,0})$ is iso to a matrix algebra.

For experts: Ψ corresponds to the "Alekseev morphism" in the theory of moduli algebras

Mapping class group action

- Obvious action of $\mathrm{MCG}(\Sigma^\circ)$ on skeins.
  (apply τ in each layer $\Sigma^\circ \times \{h\}$ of $\Sigma^\circ \times [0,1]$)
- Obviously compatible with the stacking product: $\tau \cdot (T_1 T_2) = (\tau \cdot T_1)(\tau \cdot T_2)$.
- Linear extension $\Rightarrow$ action on linear combinations of skeins.
- Compatible with the stated skein relations.
  (because all relations appear in a small neighborhood of $\partial \times [0,1]$)

$$\Rightarrow \quad \mathrm{MCG}(\Sigma^\circ) \longrightarrow \mathrm{Aut}_{\mathrm{alg}}(\mathrm{Sk}_H(\Sigma^\circ_{g,0})).$$

- For $\Sigma^\circ_{g,0}$ the stated skein algebra is iso to a matrix algebra.

Skolem–Noether $\Rightarrow$ $\mathrm{Aut}_{\mathrm{alg}}(\mathrm{Sk}_H(\Sigma^\circ_{g,0})) \xrightarrow{\sim} \mathrm{Sk}_H(\Sigma^\circ_{g,0})^\times/\mathbb{C}^\times$

$f \longmapsto \widehat{f}$

$\widehat{f}$ = invertible element ***unique up to scalar*** such that $f(X) = \widehat{f} X \widehat{f}^{-1}$.

$$\Rightarrow \quad \mathrm{MCG}(\Sigma^\circ_{g,0}) \longrightarrow \mathrm{Sk}_H(\Sigma^\circ_{g,0})^\times/\mathbb{C}^\times, \quad \tau \mapsto \widehat{\tau}.$$

Proj. rep. of the mapping class group

- $\exists$ group morphism $\mathrm{MCG}(\Sigma^{\circ}_{g,0}) \longrightarrow \mathrm{Sk}_H(\Sigma^{\circ}_{g,0})^{\times}/\mathbb{C}^{\times}$, $\tau \longmapsto \widehat{\tau}$.
- $\exists$ representation Ψ of $\mathrm{Sk}_H(\Sigma^{\circ}_{g,0})$ on $(H^*)^{\otimes g} \cong \mathrm{Sk}_H(\mathbb{H}_g)$.

Theorem (Th. 5.10 in [arXiv:1812.00446], also see §5.2 in [arXiv:2502.04267])

$\exists$ projective representation $\mathrm{MCG}(\Sigma^{\circ}_{g,0}) \longrightarrow \mathrm{PGL}((H^*)^{\otimes g})$, $\tau \longmapsto \Psi(\widehat{\tau})$.

Remarks

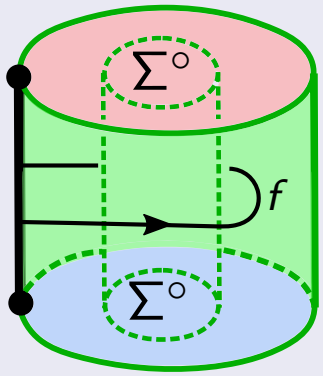
- The ***only*** non-trivial point in the above construction is the fact

  "$\mathrm{Sk}_H(\Sigma^{\circ}_{g,0})$ is iso to a matrix algebra".
- One can show that actually $\widehat{\tau} \in \mathrm{Sk}_H^{\mathrm{inv}}(\Sigma^{\circ}_{g,0})^{\times}/\mathbb{C}^{\times}$
- There are explicit formulas for the Lickorish generators.
  (see e.g. Prop. 5.22 in [arXiv:2502.04267])

What about the closed surface $\Sigma_{g,0}$?

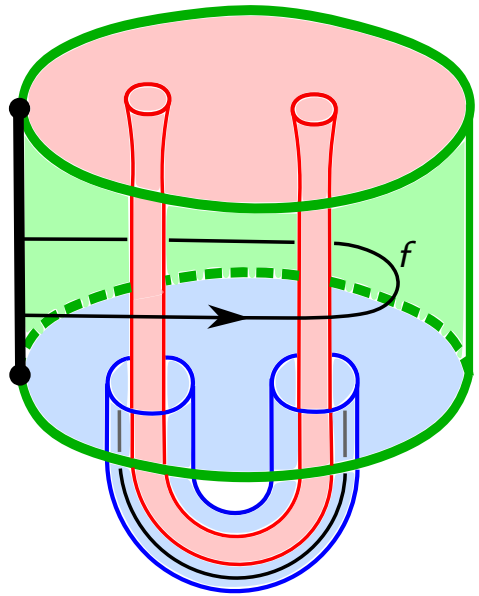
Killing the boundary

Boundary elements

For all $f \in H^*$, let $\mathfrak{d}(f) =$ 

$\in \mathrm{Sk}_H(\Sigma^\circ)$

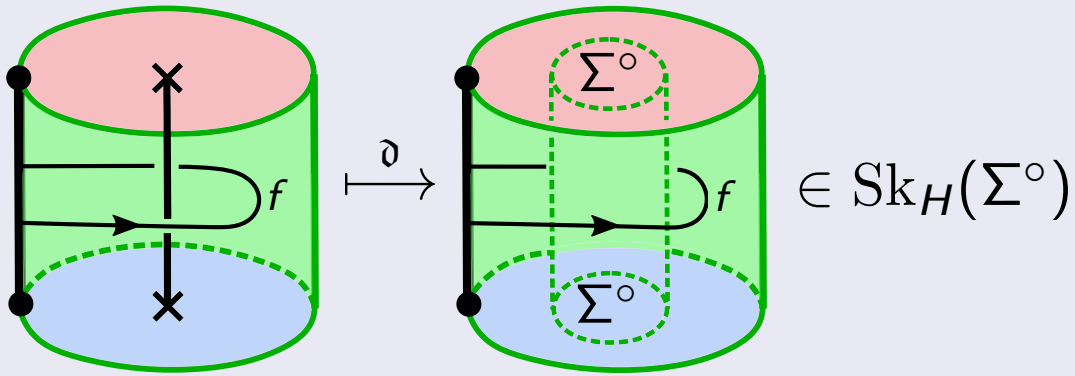
Example for $\Sigma^\circ_{1,0}$:

$\mathfrak{d}(f) =$ 

Killing the boundary

Boundary elements

For all $f \in H^*$, let



$\in \mathrm{Sk}_H(\Sigma^\circ)$

Subspace of elements which kill the boundary

$$\mathbb{V}_g \underset{\text{def}}{=} \left\{ v \in (H^*)^{\otimes g} \,\middle|\, \forall\, f \in H^*,\ \mathfrak{d}(f) \blacktriangleright v = f(1_H)v \right\}.$$

Remarks

- $\mathfrak{d}$ is an algebra morphism $\mathrm{Sk}_H(\Sigma^\circ_{0,1}) \to \Sigma^\circ$.
- We have $\mathrm{Sk}_H(\Sigma^\circ_{0,1}) \cong H$ as algebras (by factorizability)
- $\Rightarrow$ $\mathbb{V}_g$ can be described as invariant elements for a "coadjoint action" of H on $(H^*)^{\otimes g}$. (see Prop. 4.10 in [arXiv:2502.04267])

Proj. rep. of the mapping class group

Remember:

- $\exists$ representation Ψ of $\mathrm{Sk}_H(\Sigma^\circ_{g,0})$ on $(H^*)^{\otimes g} \cong \mathrm{Sk}_H(\mathbb{H}_g)$.
- $\exists$ group morphism $\mathrm{MCG}(\Sigma^\circ_{g,0}) \longrightarrow \mathrm{Sk}_H(\Sigma^\circ_{g,0})^\times/\mathbb{C}^\times$, $\tau \longmapsto \widehat{\tau}$.

Facts

- $\mathbb{V}_g \subset (H^*)^{\otimes g}$ is stable by $\mathrm{Sk}^{\mathrm{inv}}_H(\Sigma^\circ_{g,0})$.
- $\widehat{\tau} \in \mathrm{Sk}^{\mathrm{inv}}_H(\Sigma^\circ_{g,0})^\times/\mathbb{C}^\times$.

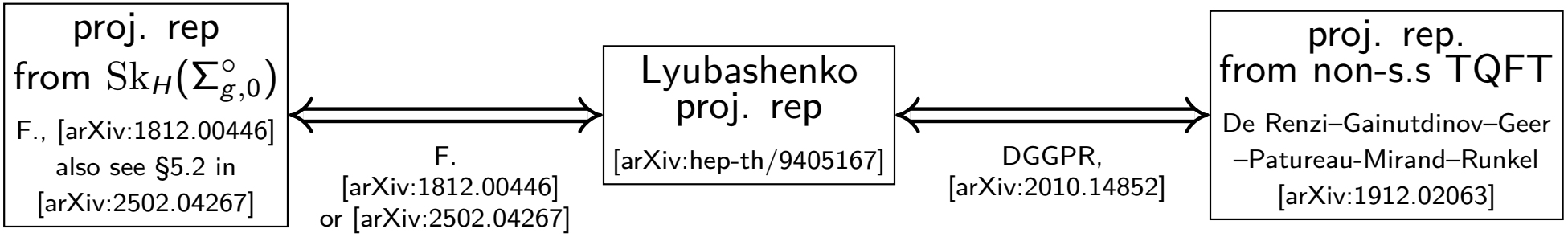
$$\Rightarrow \quad \text{proj. rep.} \quad \mathrm{MCG}(\Sigma^\circ_{g,0}) \longrightarrow \mathrm{PGL}(\mathbb{V}_g), \quad \tau \longmapsto \Psi(\widehat{\tau})|_{\mathbb{V}_g}.$$

Theorem (Th. 5.10 in [arXiv:1812.00446], also see §5.2 in [arXiv:2502.04267])

This projective representation descends to $\mathrm{MCG}(\Sigma_{g,0})$.

Concluding remarks

- The above construction can be generalized to $\mathrm{MCG}(\Sigma^{\circ}_{g,n})$ and $\mathrm{MCG}(\Sigma_{g,n})$. (J. Moulai, [arXiv:2606.24378])
- We have equivalences of projective representations of $\mathrm{MCG}(\Sigma^{\circ}_{g,0})$ and $\mathrm{MCG}(\Sigma_{g,0})$:



- The proj. rep. of $\mathrm{MCG}(\Sigma^{\circ}_{g,0})$ and $\mathrm{MCG}(\Sigma_{g,0})$ have "derived versions" where the representation space is an Ext space
  - Construction from Lyubashenko's rep: Lentner–Mierach–Schweigert–Sommerhaüser [arXiv:2003.06527]
  - Construction from $\mathrm{Sk}_H(\Sigma^{\circ}_{g,0})$: F. [arXiv:2502.04267]
  - These derived versions are again equivalent.

Thank you for your attention