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CATÉGORIES DÉRIVÉES, ESPACES DES MODULES.

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Introduction

Catégories dérivées et espaces de modules sont deux branches de la géométrie algébrique qui ont connu un developement remarquable entre les années 90 et le début des années 2000. Il s'agit en fait dans les deux cas d'étudier des espaces ou des catégories parametrisants les objets géometriques d'une variété donnée. Il en découle donc une étroite rélation entre les propriétés géometriques des variétés et les propriétés non seulement géométriques mais aussi algébriques des objets en question. Il est évident que deux tels sujets aussi étendus ne peuvent être traités en tant que tels dans le cadre présent. Néanmoins, on a naturellement remarqué ces derniers années l'étroit lien entre catégories dérivées et espaces de modules. En particulier, l'outils le plus performant et naturel pour l'étude des catégories dérivées est le foncteur de Fourier-Mukai. Dans le cas des surfaces K3, lorsque on considère un espace des modules de fibrés qui soit lui aussi une surface K3, le fibré universel donne naturellement une équivalence de Fourier-Mukai [**Muk84a**]. En plus, toute equivalence entre les catégories dérivées de deux variétés se réalise via un foncteur de Fourier-Mukai [**Orl97, Orl03**]. Dans le cas des surfaces K3 en particulier, si deux surfaces ont catégories dérivées équivalentes, on peut toujours réaliser l'une comme espace de modules des fibrés stables sur l'autre [**Orl97**]. On peut aussi faire correspondre dans ce cas une telle équivalence dérivée à une équivalence entre catégories abéliennes [**Huy08**].

L'étude des catégories dérivées permet aussi de lier leurs propriétés à certaines transformations birationnelles [**Rou05**]. Par exemple, un éclatement d'une sous-variété lisse induit un foncteur plein et fidèle qui immerge la catégorie dérivée de la variété de base dans celle de son éclatée [**Orl93**]. En plus, le complement orthogonal de l'image est formé par des copies de la catégorie dérivée de la sous-variété qu'on éclate dont le nombre dépend de sa codimension. De toute façon, lorsque le fibré canonique d'une variété est (anti)ample ou lorsque la variété est une courbe, la catégorie dérivée caracterise la variété: deux telles variétés sont isomorphes si est seulement si leurs catégories dérivées sont équivalentes [**BO01**]. Par conséquent, on ne peut pas esperer de faire correspondre l'équivalence dérivée à l'équivalence birationnelle. On peut tout de même conjecturer qu'une équivalence dérivée corresponde à une K-équivalence, c'est à dire une équivalence birationnelle sans discrepance [**Kaw02**]. Une telle correspondence birationnelle n'est interessante que dans le cas où le fibré canonique est trivial. Une variété irréductible avec un tel fibré canonique peut être soit abélienne, soit de Calabi-Yau, soit symplectique. Dans le cadre symplectique, on a demontré que les équivalences birationnelles les plus simples comme les flops standards [**BO95**] et les flops de Mukai [**Kaw02, Nam03**], qui sont en particulier des K-équivalences, induisent une équivalence dérivée. On a donc une nouvelle relation entre l'étude des espaces de modules et celle des catégories dérivées. En fait, les exemples les plus étudiés d'espaces de modules sont les espaces de fibrés stables avec un certain polynôme de Hilbert sur une surface K3. Ces espaces admettent une structure symplectique [**Muk84b**] et on connait de nombreux examples de correspondances birationnelles entre deux de tels espaces de modules symplectiques. L'étude particulière de certains de ces exemples pourrait donc être un premier pas vers une compréhension meilleure du lien entre K-équivalence et équivalence dérivée.

Chapitre 1: catégories dérivées. L'objet d'études de la première partie de la thèse est donc la catégorie dérivée bornée des faisceaux cohérents sur une variété projective. La définition de la catégorie dérivée d'une catégorie abélienne remonte aux années 60 [**Ver96**]. Récemment, dans les 15 derniers années, beaucoup de recherches se sont concentrées sur le sujet. D'un côté, la formulation de la conjecture de la symétrie miroir homologique par Kontsevich met les catégories dérivées au centre d'un débat actif qui intéresse mathématiques et physiques, de l'autre côté des nombreux résultat ont devoilé un lien étroit entre les propriétés géometriques d'une variété lisse projective et la structure de sa catégorie dérivée. Les deux exemples qui motivent la première partie de cette thèse sont donnés par la notion de décomposition semiorthogonale d'une catégorie triangulée et par les morphismes cohomologiques induits par un foncteur de Fourier-Mukai.

Si on se donne $\mathcal{T}$ une catégorie triangulée et deux objets A et B dans $\mathcal{T}$, on dit que A est orthogonal à gauche à B (et B orthogonal à droite à A) si $\mathrm{Hom}_{\mathcal{T}}(A, B) = 0$. Soient $\mathcal{T}_1$, $\mathcal{T}_2$ deux sous-catégories admissibles pleines et fidèles dans $\mathcal{T}$, on dit que $\mathcal{T}_1$ est orthogonale à gauche à $\mathcal{T}_2$ (et $\mathcal{T}_2$ est orthogonale à droite à $\mathcal{T}_1$) si tout objet A dans $\mathcal{T}_1$ est orthogonal à gauche à tout objet B dans $\mathcal{T}_2$. Finalement, une suite ordonnée $(\mathcal{T}_1, \ldots, \mathcal{T}_n)$ de sous-catégories admissibles pleines et fidèles de $\mathcal{T}$ est une décomposition semiorthogonale si $\mathcal{T}_i$ est orthogonale à gauche à $\mathcal{T}_j$ pour tout $j > i$ et tout objet de $\mathcal{T}$ est engendré par triangles exactes et translations d'objets de $\{\mathcal{T}_i\}_{i=1,\ldots,n}$.

Une telle structure de la catégorie dérivée reflète souvent des propriétés géométriques très concrètes d'une variété lisse projective. Par exemple, la catégorie dérivée de $\mathbb{P}^n$ admet une décomposition semiorthogonale $(\mathcal{T}_0, \ldots, \mathcal{T}_n)$, où $\mathcal{T}_i$ est la plus petite catégorie pleine et fidèle contenant $\mathcal{O}_{\mathbb{P}^n}(i)$ [**Bei84**]. Une version relative de ce résultat, due à Orlov [**Orl93**], est la suivante: soit $p : \mathbb{P}(E) \to X$ un fibré projectif de rang r, la catégorie dérivée de $\mathbb{P}(E)$ admet la décomposition semiorthogonale $(\mathcal{T}_0, \ldots \mathcal{T}_r)$ où $\mathcal{T}_i$ est la catégorie engendrée par les objets de la forme $p^*A \otimes \mathcal{O}_{\mathbb{P}(E)}(i)$, où A est un objet de la catégorie dérivée de X.

Dans ce qui suit, on généralise le résultat de Orlov aux schémas de Brauer-Severi. Soit S un schéma sur un corps k, un schéma de Brauer-Severi de dimenison relative r sur S est un S-schéma X muni d'un morphisme $f : X \to S$ qui soit plat et propre et tel que toute fibre géometrique est isomorphe à $\mathbb{P}^r$. On peut donc penser aux schémas de Brauer-Severi comme à des généralisations des fibrés projectifs. Pour généraliser le résultat d'Orlov et donner donc une décomposition semiorthogonale de X qui dépend de S, on doit introduire la notion de faisceau tordu par un élement du groupe de Brauer sur S. On considère pour cela le schéma S muni de la topologie étale. Soit α un élément de $H^2(S, \mathbb{G}_m)$, un faisceau α-tordu sur S est la donnée locale d'un faisceau, dont la condition de recollement est tordue par le cocycle α. Nous donnerons une définition plus précise dans la Section 5 du Chapitre 1. Comme on le fait pour les faisceaux (quasi) cohérents, on peut définir la catégorie dérivée des complexes bornés de faisceaux tordus sur S et vérifier que les foncteurs dérivés d'origine géometrique se définissent dans cette catégorie et ont les mêmes propriétés que leurs versions non tordues. On notera une telle catégorie par $\mathbf{D}(S, \alpha)$. On remarque tout de suite que $\mathbf{D}(S, 1) = \mathbf{D}(S)$ est la catégorie dérivée des faisceaux non tordus. En général, tout reste vrai lorsque S n'est pas lisse si on remplace la catégorie dérivée bornée des faisceaux (tordus) par la catégorie dérivée des complexes parfaits (tordus). On se placera, pour énoncer le résultat, dans ce cadre plus général.

Si $f : X \to S$ est un schéma de Brauer-Severi, on peut naturellement lui associer un élément du groupe de Brauer cohomologique $H^2(S, \mathbb{G}_m)$ qui peut être interpreté comme l'obstruction pour X d'être un fibré projectif. On pourra donc démontrer le Théorème suivant.

Theorem 1.27. *Il existe des sous-catégories pleines admissibles* $\mathbf{D}(S, X)_k$ *de* $\mathbf{D}(X)$*, telles que* $\mathbf{D}(S, X)_k$ *est équivalente à* $\mathbf{D}(S, \alpha^{-k})$ *pour tout* k *entier. La suite de sous-catégories admissibles*

$$\sigma = (\mathbf{D}(S, X)_0, \ldots, \mathbf{D}(S, X)_r)$$

est une décomposition semiorthogonale de la catégorie $\mathbf{D}(X)$ *des complexes parfaits de faisceaux cohérents sur* X*.*

Si on considère X et Y deux variétés lisses et projectives complexes et un objet $\mathcal{E}$ dans la catégorie dérivée bornée des faisceaux cohérents sur le produit $X \times Y$, on peut définir le foncteur de Fourier-Mukai

$\Phi_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(Y)$ de noyau $\mathcal{E}$ de la façon suivante

$$\Phi_{\mathcal{E}}(A) := q_*(p^* A \otimes \mathcal{E}),$$

où p et q sont les projections de $X \times Y$ sur X et Y respectivement et tout foncteur apparaissant dans la formule (tiré en arrière, produit tensoriel et poussé en avant) est dérivé. Un foncteur de Fourier-Mukai est exact en tant que composition de foncteurs exacts. La première application géométrique est due à Mukai, qui considère une surface abélienne et sa duale. Si on dénote par $\mathcal{P}$ le fibré de Poincaré sur le produit des deux surfaces, le foncteur de Fourier-Mukai $\Phi_{\mathcal{P}}$ de noyau $\mathcal{P}$ est une équivalence de catégories dérivées [**Muk81**]. En général, toute équivalence entre catégories dérivées est isomorphe en tant que foncteur à un foncteur de Fourier-Mukai. En plus, le noyau d'un tel foncteur est unique à quasi-isomorphisme près [**Orl97, Orl03**]. En général ceci est vrai pour tout foncteur plein et fidèle qui admet des adjoints à gauche et à droite.

Si on se donne un foncteur de Fourier-Mukai $\Phi_{\mathcal{E}}$ on peut étudier aussi son action sur le groupe de Grothendieck ainsi que sur la cohomologie rationnelle. En fait, si $\mathcal{E}$ est le noyau, la somme alternée de ses cohomologies donne un élément e du groupe de Grothendieck du produit des variétés et on peut définir un morphisme de groupes Φ_e de la même manière dont on a défini le foncteur de Fourier-Mukai. Cette association est fonctorielle. Si on considère le caractère de Chern ch qui est une application du groupe de Grothendieck sur l'anneau de cohomologie rationnelle d'une variété lisse projective, on peut considérer $ch(e)$, un élément de l'anneau de cohomologie rationnelle du produit des deux variétés. Pour définir un morphisme entre les anneaux de cohomologie compatible avec le foncteur de Fourier-Mukai, il faut tenir en compte le Théorème de Grothendieck-Riemann-Roch. En multipliant donc $ch(e)$ par la classe de Todd du produit des variétés, on peut donc définir, comme pour les groupes de Grothendieck, un morphisme $\Phi_{ch(e)}$ compatible avec le foncteur de Fourier-Mukai.

Soient C et C' deux courbes lisses projectives et $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$ un foncteur de Fourier-Mukai de noyau $\mathcal{E}$. Rappelons nous que on a une équivalence entre $\mathbf{D}(C)$ et $\mathbf{D}(C')$ si et seulement si les deux courbes sont isomorphes et une telle équivalence est toujours réalisée par un foncteur de Fourier-Mukai. D'un autre côté, un résultat classique tel que le Théorème de Torelli démontre que deux courbes lisses projectives sont isomorphes si et seulement si il existe un isomorphisme de variétés abéliennes principalement polarisées entre les deux jacobiennes. On peut donc se demander si deux tels résultats sont liés entre eux et de quelle façon.

Soit donc $\mathcal{E}$ le noyau d'un foncteur de Fourier-Mukai $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$. Comme on l'a décrit tout à l'heure, on peut construire un morphisme $\Phi_{ch(e)} : H^*(C,\mathbb{Q}) \to H^*(C',\mathbb{Q})$ compatible avec le Fourier-Mukai. De même on peut vérifier que il existe un unique morphisme de variétés abéliennes principalement polarisées $\phi_e : J(C) \to J(C')$ qui soit compatible avec le Fourier-Mukai. Ce morphisme est en effet la transformée de Fourier de noyau $c_1(e)$. On en déduit que $\Phi_{\mathcal{E}}$ est une équivalence si et seulement si ϕ_e est un isomorphisme de variétés abéliennes principalement polarisées. Donc il y a une parfaite correspondance entre le résultat classique - le Théorème de Torelli - et le résultats plus récents sur les catégories dérivées.

Chapitre 2: Espaces de modules. La deuxième partie de la thèse se concentre sur l'étude des espaces de modules de couples stables sur une surface elliptique K3. Les espaces de modules apparaissent naturellement dans plusieurs branches des mathématiques. Dans la géométrie algébrique et en particulier dans des questions liées à la physique théorique, on étudie les espaces des modules de fibrés ou de faisceaux semistables sur une certaine variété. La définition de stabilité et d'espace de modules ainsi que sa construction formelle ont été finalisées au début des années 90, mais l'étude de tels espaces était déjà largement présent dans la littérature depuis les années 70, au moins pour ce qui regarde les courbes.

Soit X une variété projective lisse. De façon très synthétique, un faisceau cohérent F sur X est semistable lorsque le polynôme de Hilbert réduit de tout sous-faisceau est inférieur ou égal au polynôme de Hilbert réduit de F. On rappelle que le polynôme de Hilbert réduit est obtenu en divisant le polynôme de Hilbert par le rang. Dans la pratique pour donner une définition consistante il faut se poser des questions à propos de la pureté et de la dimension du faisceau. Pour cela on renvoi à la Section 1 du Chapitre 2 pour un bref tour ou à [**HL98**] pour une présentation approfondie.

Le premières questions de stabilité de fibrés regardaient des fibrés sur des courbes. Naturellement le degré de difficulté augmente lorsque on considère des variétés de dimension supérieure. Néanmoins, dans le cas des surfaces, on dispose désormais d'un grand nombre d'exemples et d'une théorie forte qui permet un approche assez général. En particulier, les surfaces K3 representent une classe sur laquelle se concentrent les efforts des chercheurs. En fait, depuis Mukai [**Muk84b**], on sait qu'un espace de modules de fibrés stables sur une surface K3 admet une structure symplectique, ce qui le rend en soi même un objet d'étude intéressant. Dans quelques exemples [**GH96, Fri95**], on a pu établir une correspondance birationnelle entre l'espace de modules de certains fibrés semistables de rang 2 sur une surface K3 et le schéma de Hilbert des sous-schémas de la surface de codimension 2. Ceci est obtenu grâce à la construction de Serre, qui à partir d'un tel sous-schéma et d'un fibré en droites, permet de construire un faisceau de rang 2 et d'en étudier les propriétés et les invariants. Dans le cadre de la deuxième partie de la thèse, on cherche à mieux comprendre la correspondance birationnelle décrite par Friedman [**Fri95**]. Pour cela, on utilise la notion de stabilité de couple.

Soit X une variété lisse projective et E_0 un faisceau cohérent sur X. Nous pouvons définir la stabilité d'un couple (V, α) où V est un faisceau cohérent et α est un morphisme $\alpha : V \to E_0$ par rapport à un polynôme $\delta(z)$ à coefficients rationnels positifs. De façon synthétique, le polynôme $\delta(z)$ perturbe la condition de stabilité en la renforçant pour les sous-faisceaux contenus dans le noyau de α et en l'assouplissant pour tout autre sous-faisceau. Tels couples ont été introduites en differentes formes dans des cas spécifiques au début des années 90 [**Bra91, GP93, Lüb93, Tha94**]. La formalisation et l'étude de l'existence et des propriétés locales des espaces de modules de tels couples sont detaillés dans [**HL95a, HL95b**]. Dans le cas des courbes, ces espaces ont étés utilisés par Thaddeus pour demontrer la formule de Verlinde, [**Tha94**]. Dans le cas des 3-variétés de Calabi-Yau, Pandharipandhe and Thomas ont utilisé des couples stables dans le calcul des invariants BPS pour l'énumeration des courbes rationnelles [**PT07a, PT07b, PT07c**]. Pour ce qui regarde les surfaces K3, Göttsche and Huybrechts ont utilisé un espace de modules de couples semistables pour résoudre une correspondance birationnelle entre le schéma de Hilbert et un espace de modules de fibrés semistables de rang deux [**GH96**].

Soit donc $\pi : S \to \mathbb{P}^1$ une surface elliptique lisse projective, nous disposons grâce à Friedman d'une description explicite des fibrés semistables de rang deux [**Fri95**]. En particulier, lorsque la surface admet une section σ et on demande que le degré de la restriction à une fibre soit 1 pour les fibrés en question, on peut décrire très explictement, via des extensions, tout fibré semistable sur S de rang 2 et donner une correspondance birationnelle entre l'espace de modules de tels fibrés semistables et le schéma de Hilbert des sous-schémas de S de codimension 2 et de longueur determiné par les invariants des fibrés. Dans le cas où ces deux variétés sont de dimension 2 ou 4, cette correspondance est en effet un isomorphisme.

Soit donc $\pi : S \to \mathbb{P}^1$ une surface elliptique K3 lisse avec section σ. Nous denoterons par f la fibre de π. Nous pouvons donc, en s'appuyant sur la construction de Friedman, se fixer le faisceau $E_0 := \mathcal{O}_S(\sigma - f)$ et définir la condition de stabilité pour un couple (V, α), où V est un faisceau de rang 2, déterminant $\sigma - tf$ et $c_2(V) = 1$ et $\alpha : V \to \mathcal{O}_S(\sigma - f)$. Tout tel faisceau semistable admet au moins un tel morphisme. La condition de stabilité de tels couples dépend d'un paramètre rationnel positif δ. On dispose donc d'une famille d'espaces de modules $\mathcal{M}_\delta$ dans laquelle on observe des phénomènes de wall crossing. En fait, on a des valeurs critiques pour $\delta = n + \frac{1}{2}$, pour n entier, tandis que pour tout

δ dans tout intervalle $(n - \frac{1}{2}, n + \frac{1}{2})$ les espaces de modules $\mathcal{M}_\delta$ sont tous isomorphes. Finalement, pour $\delta < 0$ la condition n'est pas définie et pour $\delta > t + \frac{1}{2}$ la condition devient trop stricte et l'espace est donc vide. On peut donc se ramener à l'étude d'une famille finie d'espaces de modules $\mathcal{M}_i$ pour i entier compris entre 0 et t.

La première propriété qu'on observe est que le premier espace $\mathcal{M}_0$ de la famille admet une fibration en fibres projectives au dessus de l'espace de modules $M(2, \sigma - tf, 1)$, car les deux conditions de stabilité dans la définition donnent exactement la définition de stabilité pour le faisceau V par rapport à la polarisation choisie. La seule nouveauté est donc representé par la possibilité de choisir un morphisme α. Ceci nous dit donc que pour chaque V stable la fibre es donnée par l'espace projectif $\mathbb{P}\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$. Dans le cas en question, un tel espace est réduit à un point pour un V générique et donc on a un morphisme birationnel $\mathcal{M}_0 \to M(2, \sigma - tf, 1)$.

De l'autre côté, il existe un sous-schéma fermé $\tilde{\mathcal{M}}_t$ dans $\mathcal{M}_t$ qui admet une fibration en fibres projectives au dessus du schéma de Hilbert $\mathrm{Hilb}^t(S)$. En fait lorsque on considère un couple (V, α) dans $\mathcal{M}_t$ tel que le noyau de α est localement libre, on peut décrire V comme extension de $\mathcal{O}_S(\sigma - f) \otimes I_Z$ par $\mathcal{O}_S((1-t)f)$, où Z est le conoyau de α, ce qui donne un sous-schéma localement intersection complète dans S de codimension 2 et longueur t. Dans le cas en question, pour un Z générique, il existe une seule telle extension, ce qui donne donc un morphisme birationnel $\tilde{\mathcal{M}}_t \to \mathrm{Hilb}^t(S)$. En plus, si $t = 2$, un tel morphisme est un éclatement.

On dispose finalement d'une suite d'espaces de modules $\mathcal{M}_i$ pour i entier compris entre 0 et t et tels que le premier et le dernier espaces de la suite admettent un morphisme birationnel respectivement sur l'espace de modules des fibrés stables de rang deux et sur le schéma de Hilbert. En étudiant donc les transformations birationnelles induites par les phénomènes de wall crossing dans la famille, on peut esperer résoudre la correspondence birationnelle donnée par Friedman [**Fri95**]. Lorsque on fixe $t = 2$, les transformation birationnelles correspondant aux wall crossings dans la famille peuvent être décrites explicitement en appliquant des tranformations élémentaires au faisceau universel $\mathcal{V}$ de l'espace de modules $\tilde{\mathcal{M}}_2$. Ceci nous permet de retracer, à travers une suite de correspondances birationnelles, l'isomorphisme décrit par Friedman dans le cas $t = 2$.

Le but de cette recherche est donc de donner un regard nouveau sur les correspondances birationnelles décrites par Friedman [**Fri95**], tout en expliquant dans le cadre le plus simple où la surface elliptique est K3 et on fixe $t = 2$ comment retracer avec un nouveau langage l'isomorphisme déjà connu. Comme la définition et les propriétés fondamentales des espaces de couples (V, α) peuvent être étendues au cas plus général d'une surface elliptique avec section sans contraintes sur le degré, nous esperons ainsi pouvoir ouvrir une voie pour résoudre les correspondances birationnelle en suites de correspondances plus simples liées aux phénomènes de wall crossing.

CHAPTER 1

Derived categories of coherent sheaves

1. Introduction

This chapter gives a fast introduction to some topics about derived categories of coherent sheaves on a smooth projective manifold. This is a wide subject, which has grown fast in the last years. Here we just recall the definitions and the result we need, such as Fourier-Mukai functors, semiorthogonal decompositions and the generalization to twisted sheaves. Some new result is proved, such as the comparison between derived equivalence and Torelli theorem for smooth projective curves and the generalization of the decomposition of a derived category of a projective bundle to the case of a Brauer-Severi variety. To have a complete introduction to the subject and a discussion of the most important topics on derived categories and Fourier-Mukai functors, see [**Huy06**].

Let X a smooth projective variety over a field k, we consider the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X and its bounded derived category $\mathbf{D}^b(\mathrm{Coh}(X))$, which we will denote simply by $\mathbf{D}(X)$. We skip its construction and we just recall that it is a triangulated k-linear category.

Derived categories are the right setting to perform derived functors. In general, given abelian categories $\mathcal{A}$ and $\mathcal{B}$ with enough injective (projective) objects and a left (right) exact functor $F : \mathcal{A} \to \mathcal{B}$, we can use injective (projective) resolutions to define the right (left) derived functor $RF : \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{B})$ (resp. $LF : \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{B})$). In algebraic geometry, given two smooth projective varieties X and Y and a morphism $f : X \to Y$, we consider the abelian categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}(Y)$. We then have some relevant functors which are left (right) exact, which give rise to some relevant derived functors, such as Rf_*, Lf^*, $R\mathrm{Hom}$, $R\underline{\mathrm{Hom}}$ and $\otimes^L$ for example. Remark once again that it is not straightforward to define such derived functors on $\mathbf{D}(X)$. For details, see [**Huy06**].

Derived categories encode geometrical information about the variety. In particular, two smooth projective varieties with equivalent derived categories have the same dimension, their canonical bundles have the same order, the same Kodaira dimension. Recall that the order of a line bundle L is the smallest integer m such that $L^{\otimes r} \simeq \mathcal{O}$ and could be infinite. Moreover, if the canonical or anticanonical bundle is ample, the varieties are isomorphic, as shown by Bondal and Orlov

THEOREM 1.1. [**BO01**]. *Let X be a smooth projective complex variety whose canonical (or anticanonical) bundle is ample. Suppose we have an exact, linear equivalence $\mathbf{D}(X) \cong \mathbf{D}(Y)$, then Y is isomorphic to X.*

Remark that as we consider the derived category with its triangulated and k-linear structure, functors and especially equivalences are always tacitly considered exact and k-linear from so on.

There are examples in which a derived equivalence does not yield an isomorphism. One of the simplest example is when X is a K3 projective surface and Y is the moduli space of stable vector bundles on X with isotropic Mukai vector. We will discuss moduli spaces of K3 surfaces in Chapter 2. It is conjectured anyway that for any smooth projective variety X there is a finite number, up to isomorphism of smooth projective varieties Y such that $\mathbf{D}(X) \cong \mathbf{D}(Y)$.

Derived categories carry information about the birational class of the variety. Anyway, it is clear by Theorem 1.1 that derived equivalence does not correspond to birational equivalence, but it should correspond to some stronger birational equivalence taking in account canonical bundles.

DEFINITION 1.2. Two varieties X and Y are *K-equivalent* if there exists a birational correspondence

$$X \xleftarrow{p_X} Z \xrightarrow{p_Y} Y$$

such that $p_X^*\omega_X \simeq p_Y^*\omega_Y$.

CONJECTURE 1.3. *Let X and Y be smooth projective varieties. If X and Y are K-equivalent, then $\mathbf{D}(X) \cong \mathbf{D}(Y)$.*

The conjecture is true in many cases: for standard flops [**BO95**], for Mukai flops and then for symplectic fourfolds [**Kaw02**] and [**Nam03**], for Calabi-Yau threefolds [**Bri02**], for $\mathrm{Gr}(2,4)$ stratified Mukai flops [**Kaw**]. Nice surveys of the argument are [**Kaw02**], [**Rou05**].

2. Fourier-Mukai functors

In this section we introduce the notion of a Fourier-Mukai functor between derived categories. It is a central notion in this theory and it allows to prove plenty of theorems of deep geometrical meaning. Moreover, it is the derived version of a well-known correspondence and hence it involves many geometric invariants of the varieties, such as K-theory and rational cohomology.

Let X and Y be smooth projective varieties, let us consider their product $X \times Y$ and the projections $p : X \times Y \to X$ and $q : X \times Y \to Y$.

DEFINITION 1.4. Let $\mathcal{E}$ be an object in $\mathbf{D}(X \times Y)$. The *Fourier-Mukai functor with kernel* $\mathcal{E}$ is the functor

$$\begin{array}{rccc} \Phi_{\mathcal{E}}: & \mathbf{D}(X) & \longrightarrow & \mathbf{D}(Y) \\ & A & \longmapsto & q_*(p^*A \otimes \mathcal{E}). \end{array}$$

A Fourier-Mukai functor is exact and always admits a right and a left adjoint, with kernels respectively $\mathcal{E}_R$ and $\mathcal{E}_L$, given by the following formulas

$$\begin{aligned} \mathcal{E}_R &:= \mathcal{E}^\vee \otimes p^*\omega_X[\dim(X)], \\ \mathcal{E}_L &:= \mathcal{E}^\vee \otimes q^*\omega_Y[\dim(Y)]. \end{aligned}$$

The adjoint property is a direct consequence of Serre duality.

The composition of two Fourier-Mukai functor is still a Fourier-Mukai functor and we can write down explicitly the kernel. For more details, see [**Huy06**].

Let us state without proving it the celebrated Theorem by Orlov which tells that any fully faithful exact functor, with left and right adjoint, between derived categories is, up to isomorphism, a Fourier-Mukai functor.

THEOREM 1.5. [**Orl97, Orl03**] *Let X and Y be two smooth projective varieties. Let*

$$F : \mathbf{D}(X) \to \mathbf{D}(Y)$$

be a fully faithful exact functor admitting right and left adjoint functors. Then there exists an object $\mathcal{E}$ in $\mathbf{D}(X \times Y)$ such that F is isomorphic to $\Phi_{\mathcal{E}}$. Moreover, such an object is unique up to isomorphism.

Remark that we can weaken the hypothesis of this Theorem, since the existence of both adjoint functors is ensured by [**BVdB03**].

A Fourier-Mukai transform is defined in the derived context, but it always induces a morphism between the rational cohomology rings.

The first step in making such a descent is going from derived categories to Grothendieck groups. Given X a smooth projective variety, to any object $\mathcal{E}$ in $\mathbf{D}(X)$, we can associate an element $[\mathcal{E}]$ in the Grothendieck group $K(X)$ by the alternate sum of the classes of cohomology sheaves of $\mathcal{E}$. We thus obtain a map $[\ ]$ from the isomorphism classes of $\mathbf{D}(X)$ to the Grothendieck group $K(X)$. Given $f : X \to Y$ a projective morphism between smooth projective varieties, the pull back $f^* : K(Y) \to K(X)$ defines a ring homomorphism. The generalized direct image $f_! : K(X) \to K(Y)$, defined by $f_!\mathcal{F} = \sum(-1)^i R^i f_*\mathcal{F}$ for any coherent sheaf $\mathcal{F}$ on X, defines a group homomorphism.

We can define a K-theoretic Fourier-Mukai transform. Given e a class in $K(X \times Y)$, let us define

$$\begin{array}{rccc} \Phi_e^K: & K(X) & \longrightarrow & K(Y) \\ & a & \longmapsto & q_!(p^*a \otimes e). \end{array}$$

If we are now given a Fourier-Mukai functor $\Phi_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(Y)$ with kernel $\mathcal{E}$ in $\mathbf{D}(X \times Y)$, we obtain the corresponding K-theoretic

Fourier-Mukai $\Phi_e^K : K(X) \to K(Y)$ by using the kernel $e := [\mathcal{E}]$. By the compatibility of $f_!$ and f^* with $[\,]$, we get the following commutative diagram (see [**Huy06**], 5.2):

$$\begin{tikzcd} \mathbf{D}(X) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "{[\,]}"] & \mathbf{D}(Y) \arrow[d, "{[\,]}"] \\ K(X) \arrow[r, "\Phi_{[\mathcal{E}]}"'] & K(Y). \end{tikzcd} \tag{1}$$

We want to make a step further and consider rational Chow rings. Consider the exponential Chern character:

$$ch : \;\; K(X) \longrightarrow \;\; CH^*_{\mathbb{Q}}(X)$$

which maps a class of the Grothendieck group to a cycle in the Chow ring with rational coefficients. For a given $f : X \to Y$ we can define the pull-back $f^* : CH^*_{\mathbb{Q}}(Y) \to CH^*_{\mathbb{Q}}(X)$ and the direct image $f_* : CH^*_{\mathbb{Q}}(X) \to CH^*_{\mathbb{Q}}(Y)$. Anyway, in order to get a compatibility with the Chern character ch, the Grothendieck-Riemann-Roch Theorem has to be taken into account.

Theorem 1.6. **(Grothendieck-Riemann-Roch).** *Let $f : X \to Y$ a projective morphism of smooth projective varieties. Then for any e in $K(X)$*

$$ch(f_!(e)) = f_*(ch(e).Td(f)),$$

where $Td(f)$ is the relative Todd class of f.

Given an element e in $K(X \times Y)$, let us define the Chow-theoretic Fourier-Mukai

$$\begin{array}{rlcl} \Phi_e^{CH} : & CH^*_{\mathbb{Q}}(X) & \longrightarrow & CH^*_{\mathbb{Q}}(Y) \\ & M & \longmapsto & q_*(p^*M.ch(e).Td(q)). \end{array}$$

Given a Fourier-Mukai functor $\Phi_{\mathcal{E}}$ and $e := [\mathcal{E}]$, the Chow-theoretic Fourier-Mukai Φ_e^{CH} fits a functorial compatibility.

On the cohomological side, it is possible to show in the same way that given a Fourier-Mukai transform with kernel $\mathcal{E}$, the map induced between the rational cohomology rings is given by the following formula

$$\begin{array}{rlcl} \Phi_e^{H} : & H^*(X, \mathbb{Q}) & \longrightarrow & H^*(Y, \mathbb{Q}) \\ & M & \longmapsto & q_*(p^*M.v(e)), \end{array}$$

where $v(e)$ is the Mukai vector of the class $e = [\mathcal{E}]$. This vector encodes the correction given by the Todd class. The cohomological Fourier-Mukai does not respect the usual grading of the cohomology ring, but it respects the parity, sending odd (resp. even) cohomology to odd (resp. even) cohomology, and Mukai pairing. Moreover it preserves the Hodge diamond columnwise. Indeed, if $\Phi_{\mathcal{E}}$ is an equivalence, Φ_e^H yields an isomorphism

$$\bigoplus_{p-q=i} H^{p,q}(X) \simeq \bigoplus_{p-q=i} H^{p,q}(Y) \tag{2}$$

for all $i = -\mathrm{dim}X, \ldots, \dim(X)$.

With such a result, we are able to prove that the derived category characterizes a smooth projective curve up to isomorphism.

THEOREM 1.7. *Let C be a smooth projective curve and Y a smooth projective variety. Then there is a derived equivalence $\mathbf{D}(C) \cong \mathbf{D}(Y)$ if and only if there is an isomorphism $C \simeq Y$.*

PROOF. If the curve is not elliptic, then this is a special case of Theorem 1.1. In the case C is elliptic, Y has to be a smooth elliptic curve as well. Moreover, there exists an object $\mathcal{E}$ such that the equivalence is realized as the Fourier-Mukai transform $\Phi_{\mathcal{E}}$ with kernel $\mathcal{E}$. The induced cohomological transform Φ_e^H preserves parity and then yields an isomorphism

$$H^1(C) \simeq H^1(Y)$$

which respects the Hodge decomposition $H^1 = H^{0,1} \oplus H^{1,0}$ by (2). Since $C \simeq H^{0,1}(C)/H^1(C, \mathbb{Z})$, we just have to show that Φ_e^H is defined on integers. Indeed, we have $Td(C \times Y) = 1$ and $ch(e) = r + c_1(e) + \frac{1}{2}(c_1^2 - 2c_2)(e)$ and then the only term which could be non integer has degree four and does not contribute to $H^1(C) \to H^1(Y)$. □

For more details on these topics, see [**Huy06**], chapter 5.

3. Fourier-Mukai functors of curves and principal polarizations

In the case of curves, we can see how the characterization of a curve by its derived category corresponds to the Torelli Theorem.

Let C and C' be two smooth projective curves. Suppose we are given a Fourier-Mukai functor $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$, with kernel $\mathcal{E}$. If $\Phi_{\mathcal{E}}$ is an equivalence, then we get an isomorphism between the two curves. We can ask ourselves if this derived Fourier Mukai transform (DFM for short) does carry an isomorphism between the Jacobian varieties preserving the principal polarizations.

In order to do that, recall the definition of the Jacobian variety as $\mathrm{Pic}^0(C)$, the degree zero part of the Picard group. What we are actually going to do is to make the DFM descend to an affine map $\Phi_e^P : \mathrm{Pic}_{\mathbb{Q}}(C) \to \mathrm{Pic}_{\mathbb{Q}}(C')$ between the rational Picard groups.

We want to define a morphism $\phi_e^J : J(C) \to J(C')$ compatible with Φ_e^P. This is done in three steps. Firstly, we describe the morphism induced by $\Phi_{\mathcal{E}}$ on the rational Picard group, that is the degree one part of the rational Chow ring. What we find is actually an affine map between rational vector spaces and not a linear morphism. Secondly, we define a morphism between the Jacobian varieties and we consider its restriction to the Jacobians with rational coefficients, that is $\mathrm{Pic}^0_{\mathbb{Q}}$. This can be done in a unique way. Finally we show that linearizing the affine map given on $\mathrm{Pic}^0_{\mathbb{Q}}$ by the DFM we obtain the classical Fourier transform on Jacobian varieties with rational coefficients. The correspondence is functorial.

3.1. From derived Fourier-Mukai to an affine Map on $\mathrm{Pic}_{\mathbb{Q}}$. Define the affine map

$$\begin{array}{rlcl} \Phi_e^P: & \mathrm{Pic}_{\mathbb{Q}}(C) & \longrightarrow & \mathrm{Pic}_{\mathbb{Q}}(C') \\ & M & \longmapsto & q_*(p^*M.c_1(e) - \frac{1}{2}c_1(e).p^*K_C + \frac{1}{2}(c_1^2(e) + 2c_2(e))). \end{array}$$

The map Φ_e^P is the one induced by the DFM $\Phi_{\mathcal{E}}$.

LEMMA 1.8. *Let C and C' be smooth projective curves and $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$ be a Fourier-Mukai transform with kernel $\mathcal{E}$ in $\mathbf{D}(C \times C')$. The diagram*

$$\begin{tikzcd} \mathbf{D}(C) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "c_1\circ[\ ]"] & \mathbf{D}(C') \arrow[d, "c_1\circ[\ ]"] \\ \mathrm{Pic}_{\mathbb{Q}}(C) \arrow[r, "\Phi_e^P"'] & \mathrm{Pic}_{\mathbb{Q}}(C') \end{tikzcd} \tag{3}$$

is commutative.

PROOF. Let us denote by M both an element of the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(C)$ and its class in the Grothendieck group $K(C)$. We want to calculate the first Chern class $(ch(\Phi_e^K(M))_1$.

We have the following chain of equalities:

$$(ch(q_!(p^*M \otimes e)))_1 = (q_*(ch(p^*M \otimes e)(1 - \frac{1}{2}p^*K_C)))_1,$$

by Grothendieck-Riemann-Roch and $Td(q) = 1 - \frac{1}{2}p^*K_C$.

$$(q_*(ch(p^*M \otimes e).(1 - \frac{1}{2}p^*K_C)))_1 = q_*(ch(p^*M).ch(e).(1 - \frac{1}{2}p^*K_C))_2.$$

Now let us make it more explicit

$$\begin{gathered} ch(p^*M).ch(e).(1 - \tfrac{1}{2}p^*K_C) = \\ = (1 + p^*M).(r + c_1(e) + \tfrac{1}{2}(c_1^2(e) + 2c_2(e)).(1 - \tfrac{1}{2}p^*K_C), \end{gathered} \tag{4}$$

where r is the rank of e.

We take the degree two part of (4) and we obtain

$$(ch(q_!(p^*M \otimes e)))_1 = p^*M.c_1(e) - \frac{1}{2}c_1(e).p^*K_C + \frac{1}{2}(c_1^2(e) + 2c_2(e)). \tag{5}$$

The morphism Φ_e^P between the Picard groups with rational coefficients commutes with the K-theoretic transform with kernel e. Combining this with the commutative diagram (1) we get (3). $\square$

It is clear that the affine map Φ_e^P restricted to $\mathrm{Pic}^0_{\mathbb{Q}}(C)$ does not give a group morphism to $\mathrm{Pic}^0_{\mathbb{Q}}(C')$. Remark anyway that only the first term of $\Phi_e^P(M)$ depends on M, while the other terms are constant with respect to it.

3.2. From Jacobian Fourier to a Morphism on $\mathrm{Pic}^0_{\mathbb{Q}}$. Let us define

$$\begin{array}{cccc} \phi_e^J: & J(C) & \longrightarrow & J(C') \\ & M & \longmapsto & q_*(p^*(M-\mathcal{O}_C).c_1(e)), \end{array}$$

where e is a class in the Grothendieck group $K(C\times C')$ and $\mathcal{O}_C$ is the unity in $J(C)$. This is the classical Fourier transform with kernel $c_1(e)$ between the Jacobian varieties, and we are referring to that by JF.

There is a unique morphism $\phi_e^{J_{\mathbb{Q}}}: J_{\mathbb{Q}}(C)\to J_{\mathbb{Q}}(C')$ that gives ϕ_e^J on $J(C)$.

3.3. They go together. So far we can say that the DFM with kernel $\mathcal{E}$ uniquely induces on $\mathrm{Pic}_{\mathbb{Q}}(C)$ the map Φ_e^P. Linearizing this map and restricting it to $\mathrm{Pic}^0_{\mathbb{Q}}(C)$, we get exactly the morphism $\phi_e^{J_{\mathbb{Q}}}$, induced by the JF with kernel $c_1(e)$. We can then conclude that the JF $\phi_e^J: J(C)\to J(C')$ is the only morphism compatible with the DFM $\Phi_{\mathcal{E}}: \mathbf{D}(C)\to\mathbf{D}(C')$. Moreover, the correspondence between DFMs and JFs is functorial.

LEMMA 1.9. *The correspondence between derived Fourier-Mukai functors and Fourier transforms on the Jacobian varieties associating ϕ_e^J to $\Phi_{\mathcal{E}}$ is functorial.*

PROOF. Given a smooth projective curve C, the identity on $\mathbf{D}(C)$ is given by the DFM with kernel $\mathcal{O}_\Delta$, the structure sheaf of the diagonal in $C\times C$. The identity on $J(C)$ clearly corresponds to it.

By Lemma 1.8 the correspondence between DFMs and the affine maps is functorial. Indeed if we consider two composable DFMs $\Phi_{\mathcal{E}_1}$ and $\Phi_{\mathcal{E}_2}$ and their composition $\Phi_{\mathcal{R}}$, the affine maps $\Phi_{e_1}^P$ and $\Phi_{e_2}^P$ are composable and their composition is given by the affine map Φ_r^P.

Now the rational linear map $\phi_e^{J_{\mathbb{Q}}}$ is the linearization of Φ_e^P restricted to $\mathrm{Pic}^0_{\mathbb{Q}}$. Consider in general two composable affine maps $F_1: V_1\to V_2$ and $F_2: V_2\to V_3$ between vector spaces and their linearizations f_i. The linearization of the composition $F_2\circ F_1$ is $f_2\circ f_1$, the composition of f_1 and f_2. This allows us to state that the correspondence associating the linear map $\phi_e^{J_{\mathbb{Q}}}$ to the DFM $\Phi_{\mathcal{E}}$ is functorial. Just remark now that the functoriality for $\phi_e^{J_{\mathbb{Q}}}$ implies the functoriality for ϕ_e^J. $\square$

Remark 1.10. Let us observe what happens to the kernel of the JF when we modify the kernel of the DFM.

The DFM with kernel $\mathcal{E}[1]$ induces the JF with kernel $-c_1(e)$. This is computed by the definition of $[\ ] : \mathbf{D}(C) \to K(C)$.

The DFM with kernel $\mathcal{E}^\vee$ induces the JF with kernel $-c_1(e)$. This is computed remarking that $c_t(e^\vee) = c_{-t}(e)$.

Given line bundles F on C and F' on C', the DFM with kernel $\mathcal{E} \otimes p^*F \otimes q^*F'$ induces the JF with kernel $c_1(e)$. This is computed remarking that $p^*F.q^*M = q^*F'.p^*M = p^*F.q^*F' = 0$ for any element M in $J(C)$. In the terminology of [**BL92**], Chapter 11, we would say that we have two equivalent correspondences.

3.4. Preservation of the Principal Polarization. A principal polarization on an abelian variety A defines an isomorphism $\theta_A : A \to \hat{A}$. Given an isogeny $\phi : A \to B$ between two abelian varieties, we can define the dual isogeny $\hat{\phi} : \hat{B} \to \hat{A}$ between the dual varieties. If both A and B have a principal polarization, the isogeny ϕ respects them if the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\theta_A"'] & B \arrow[d, "\theta_B"] \\ \hat{A} & \hat{B} \arrow[l, "\hat{\phi}"] \end{tikzcd}$$

is commutative.

In the case of a smooth projective curve C we know the principal polarization $\theta_C : J(C) \to \hat{J}(C)$. We identify, by means of the isomorphisms θ_C and $\theta_{C'}$ the Jacobian varieties $J(C)$ and $J(C')$ with their respective duals. We then have to check that the composition $\hat{\phi}_e^J \circ \phi_e^J$ is the identity map on $J(C)$. The dual isomorphism $\hat{\phi}_e^J$ can be obtained as the JF in the opposite way with the same kernel as ϕ_e^J. Namely

$$\begin{array}{rccl} \hat{\phi}_e^J : & J(C') & \longrightarrow & J(C) \\ & M' & \longmapsto & p_*(q^*(M' - \mathcal{O}_{C'}).c_1(e)), \end{array} \tag{6}$$

see for example [**BL92**], Chapter 11, Proposition 5.3.

Theorem 1.11. *Given two smooth projective curves C and C' of positive genus, a Fourier Mukai functor $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$ is an equivalence if and only if the morphism $\phi_e^J : J(C) \to J(C')$ is an isomorphism preserving principal polarization.*

Proof. Given a DFM $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$ with kernel $\mathcal{E}$, we can describe the kernels $\mathcal{E}_L$ and $\mathcal{E}_R$ of its left and right adjoint. If $\Phi_{\mathcal{E}}$ is an equivalence, its adjoints are its quasi-inverses. The left adjoint of $\Phi_{\mathcal{E}}$ is the DFM $\Phi_{\mathcal{E}_L} : \mathbf{D}(C') \to \mathbf{D}(C)$ with kernel

$$\mathcal{E}_L := \mathcal{E}^\vee \otimes q^*K_{C'}[1]. \tag{7}$$

We know by remark 1.10 that the JF isomorphism induced by the DFM $\Phi_{\mathcal{E}_L}$ on the Jacobian varieties is given by

$$\begin{array}{rccl} \phi^J_{e_L}: & J(C') & \longrightarrow & J(C) \\ & M' & \longmapsto & p_*(q^*(M' - \mathcal{O}_{C'}).c_1(e)) \end{array}$$

Then if $\Phi_{\mathcal{E}}$ induces on the Jacobian varieties the isomorphism ϕ^J_e, its quasi inverse $\Phi_{\mathcal{E}_L}$ induces the dual isomorphism $\hat{\phi}^J_e$. The proof follows by Lemma 1.9. □

Remark 1.12. Recall Theorem 1.7, which states that for smooth projective curves a derived equivalence always corresponds to an isomorphism. Theorem 1.11 just states the correspondence between the Torelli Theorem (see for example [**GH78**], page 359) and the characterization of a curve by its derived category in the positive genus case.

4. Semiorthogonal decompositions

Let k be a field and $\mathbf{D}$ a k-linear triangulated category. Recall that the bounded derived category of coherent sheaves on a smooth projective variety over k is triangulated and k-linear. Hence everything in this section applies to it.

Definition 1.13. A full triangulated subcategory $\mathbf{D}' \subset \mathbf{D}$ is *admissible* if the inclusion functor $i : \mathbf{D}' \to \mathbf{D}$ admits a right adjoint.

Definition 1.14. The *orthogonal complement* $\mathbf{D}'^{\perp}$ of $\mathbf{D}'$ in $\mathbf{D}$ is the full subcategory of all objects $A \in \mathbf{D}$ such that $\mathrm{Hom}(B, A) = 0$ for all $B \in \mathbf{D}'$.

Remark that the orthogonal complement of an admissible subcategory is a triangulated subcategory.

It can be shown that a full triangulated subcategory $\mathbf{D}' \subset \mathbf{D}$ is admissible if and only if for all object A of $\mathbf{D}$, there exists a distinguished triangle $B \to A \to C$ where $B \in \mathbf{D}'$ and $C \in \mathbf{D}'^{\perp}$, see [**Bon90**]. We also have the following Theorem.

Theorem 1.15. [**BK90**, Proposition 1.5], *or* [**Bon90**, Lemma 3.1]. *Let* $\mathbf{D}'$ *be a full triangulated subcategory of a triangulated category* $\mathbf{D}$. *Then* $\mathbf{D}'$ *is admissible if and only if* $\mathbf{D}$ *is generated by* $\mathbf{D}'$ *and* $\mathbf{D}'^{\perp}$.

Admissible subcategories occur when we have a fully faithful exact functor $F : \mathbf{D}' \to \mathbf{D}$ which admits a right adjoint. To be precise, this functor defines an equivalence between $\mathbf{D}'$ and an admissible subcategory of $\mathbf{D}$.

Definition 1.16. A sequence of admissible triangulated subcategories $\sigma = (\mathbf{D}_1, \ldots, \mathbf{D}_n)$ is *semiorthogonal* if, for all $i > j$, one has $\mathbf{D}_j \subset \mathbf{D}_i^{\perp}$. If σ generates the category $\mathbf{D}$, we call it a *semiorthogonal decomposition* of $\mathbf{D}$.

LEMMA 1.17. *Let* $\sigma = (\mathbf{D}_1, \dots, \mathbf{D}_n)$ *be a sequence of full subcategories of* $\mathbf{D}$ *such that* $\mathbf{D}_j \subset \mathbf{D}_i^{\perp}$ *for all* $i > j$ *and* σ *generates* $\mathbf{D}$. *Then* $\mathbf{D}_i$ *is admissible for* $i = 1, \dots, n$, *and* σ *is a semiorthogonal decomposition of* $\mathbf{D}$.

PROOF. Consider $\mathbf{D}_n$ and $\mathbf{D}_n^{\perp}$: they generate the category $\mathbf{D}$ and then they are admissible. In general, consider $\mathbf{D}_i$ and $\mathbf{D}_i^{\perp}$ for $1 \leq i < n$: they generate the category $\mathbf{D}_{i+1}^{\perp}$ and then they are admissible. □

For further information about admissible subcategories and semiorthogonal decomposition, see [**Bon90, BK90, BO95**].

4.1. Some examples. Let X be a smooth projective variety. A semiorthogonal decomposition of the derived category $\mathbf{D}(X)$ often reflects geometric properties of X. Let us give some examples in order to give some evidence.

EXAMPLE 1.18 (Projective bundles). Let S be a smooth projective variety, E a vector bundle of rank $r+1$ over S. We consider its projectivization $p : X = \mathbb{P}(E) \to S$. We then have the following semiorthogonal decomposition for the bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X.

THEOREM 1.19. [**Orl93**] *Let* $\mathbf{D}(S)_k$ *be the full subcategory of* $\mathbf{D}(X)$ *whose objects are all objects of the form* $p^*A \otimes \mathcal{O}_X(k)$ *for an object* A *of* $\mathbf{D}(S)$. *Then the set of admissible subcategories*

$$(\mathbf{D}(S)_0, \dots, \mathbf{D}(S)_r)$$

is a semiorthogonal decomposition for $\mathbf{D}(X)$.

In Section 6, we will give a generalization of this result to the case of Brauer-Severi schemes.

EXAMPLE 1.20 (Blow-ups). Let Y be a smooth subvariety of a projective smooth variety X of codimension c. Let $\pi : \tilde{X} \to X$ be the blow-up of X along Y and E the exceptional divisor. Let $i : E \hookrightarrow \tilde{X}$ be the embedding.

THEOREM 1.21. [**Orl93**] *Let* $\mathbf{D}_k$ *be the full subcategory of* $\mathbf{D}(\tilde{X})$ *whose objects are all objects of the form* $i_*(\pi^*A \otimes \mathcal{O}_E(k))$ *for an object* A *in* $\mathbf{D}(Y)$. *The set of admissible subcategories*

$$(\mathbf{D}_{-c+1}, \dots, \mathbf{D}_{-1}, \pi^*\mathbf{D}(X))$$

is a semiorthogonal decomposition for $\mathbf{D}(\tilde{X})$.

5. Twisted sheaves

In this section, we give the definition of twisted sheaves and we state the relationship between them and Brauer-Severi schemes.

We are working in the étale topology, but all can be defined and stated in analytic topology as well (see [**Cal00**], I, 1). All schemes considered are locally noetherian and we suppose that any pair of points has an affine open neighborhood.

Let S be such a scheme. We are using the following notations. Given $U \to S$ an open étale cover of S, we denote U'' the fibered product $U \times_S U$ and U''' the fibered product $U \times_S U \times_S U$. We call p_1 and p_2 the projections $U'' \to U$ and $q_{i,j}$ the projections $U''' \to U''$. If $f : X \to S$ is a morphism, X_U denotes $f^{-1}(U)$. We use notations X''_U, X'''_U, $p_{i,X}$ and $q_{i,j,X}$ in the natural way. Notice that $X''_U = X_{U''}$.

Definition 1.22. Let S be a scheme with étale topology, $U \to S$ an open étale cover, $\alpha \in \Gamma(U''', \mathbb{G}_m)$ a 2-cocycle.

An *α-twisted sheaf* on S is given by a sheaf E over U and an isomorphism $\phi : p_1^*E \to p_2^*E$, such that

$$(q_{2,3}^*\phi) \circ (q_{1,2}^*\phi) = \alpha(q_{1,3}^*\phi)$$

We say that such a sheaf is *(quasi)-coherent* if E is a (quasi)-coherent sheaf on U, and we denote $\mathrm{Mod}(S, \alpha)$ the category of α-twisted sheaves on S, $\mathrm{Coh}(S, \alpha)$ the category of coherent α-twisted sheaves on S and $\mathbf{D}(S, \alpha)$ the category of perfect complexes of such sheaves.

The category $\mathrm{Mod}(S, \alpha)$ does not change neither by refining the open cover $U \to X$, nor by changing α by a cochain.

Lemma 1.23. *If α and α' represent the same element of $H^2(S, \mathbb{G}_m)$, the categories $\mathrm{Mod}(S, \alpha)$ and $\mathrm{Mod}(S, \alpha')$ are equivalent.*

Proof. This is [**Cal00**], Lemma 1.2.8. Indeed if α and α' are in the same cohomology class they differ by a 1-cochain: $\alpha = \alpha' + \delta\gamma$. But then sending any α'-twisted sheaf (E, ϕ) to the α-twisted sheaf $(E, \gamma\phi)$ gives the required equivalence. □

Remark 1.24. Notice that in general the choice of the cochain γ matters: different choices give different equivalences. Since we are just interested in the existence of such equivalences and not in a special one, in what follows this choice will not matter.

5.1. Twisted Sheaves and Brauer-Severi schemes. Now we can see how twisted sheaves arise naturally when we consider Brauer-Severi schemes. Let $f : X \to S$ be a flat and proper morphism between schemes such that each geometric fiber is isomorphic to $\mathbb{P}^r$. Then we call X a Brauer-Severi scheme of relative dimension r over S.

We can find an étale covering $U \to S$, such that $X_U = f^{-1}(U)$ is a projective bundle over U and $X_U \to X$ is an étale covering. Then we have a local picture $\mathbb{P}(E_U) \to U$, where E_U is a locally free sheaf of rank $r + 1$ on U and we have an isomorphism $\rho : \mathbb{P}(E_U) \tilde{\to} X_U$. This fact is a classical application of descent theory ([**Gro**], I, 8).

Consider the cartesian diagram

$$X'''_U \overset{\textstyle\rightarrow}{\underset{\textstyle\rightarrow}{\rightarrow}} X''_U \rightrightarrows X_U$$

and call the projections $p_{i,X}$ and $q_{i,j,X}$. We have an isomorphism

$$\psi := p_{1,X}^*\rho^{-1} \circ p_{2,X}^*\rho : \mathbb{P}(p_1^*E_U) \xrightarrow{\sim} \mathbb{P}(p_2^*E_U).$$

We would like to lift it to an isomorphism $\phi : p_1^*E_U \tilde{\rightarrow} p_2^*E_U$.

Consider U such that $p_1^*E_U$ and $p_2^*E_U$ can be trivialized. This implies that ψ is an automorphism of $U'' \times \mathbb{P}^r$ and then it gives a section of $PGL(r+1, U'')$. We can again refine U in order to obtain from it a section of $GL(r+1, U'')$, which will give us the required isomorphism $\phi : p_1^*E_U \tilde{\rightarrow} p_2^*E_U$. Notice that this is not canonical since it can be done up to a choice of an element of $\Gamma(U'', \mathbb{G}_m)$. This can be done since any pair of point on S has an affine open neighborhood (see [**Sch03**]).

For this reason, we have $(q_{1,2}^*\phi) \circ (q_{2,3}^*\phi) = \alpha_U(q_{1,3}^*\phi)$, where $\alpha_U \in \Gamma(U''', \mathbb{G}_m)$. We can see that α_U gives a cocycle and then (E_U, ϕ) is an α-twisted sheaf.

From now on, given a Brauer-Severi scheme $f : X \to S$, we will consider the α-twisted sheaf (E_U, ϕ) described above. Everything depends just on the cohomology class α, which represents the obstruction to $f : X \to S$ to be a projective bundle. To express this via cohomology, recall the exact sequence of sheaves over S:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow GL(r+1) \longrightarrow PGL(r+1) \longrightarrow 1.$$

It gives a long cohomology sequence:

$$\cdots \to H^1(S, GL(r+1)) \to H^1(S, PGL(r+1)) \xrightarrow{\delta} H^2(S, \mathbb{G}_m)$$

and especially a connecting homomorphism δ.

Let $[X]$ be the cohomology class of X in $H^1(S, PGL(r+1))$ and $\alpha' := \delta([X])$ in $H^2(S, \mathbb{G}_m)$. If $\alpha' = 0$, the class $[X]$ would lift to an element of $H^1(S, GL(r+1))$, that is a rank $r+1$ vector bundle on S. Since X is not a projective bundle, α' is a nonzero element of the cohomological Brauer group $\mathrm{Br}'(S) := H^2(S, \mathbb{G}_m)$ and it is exactly the cohomology class α of the α_U described above.

As a projective bundle $\mathbb{P}(E_U)$ over U, on X_U there exists a tautological line bundle $\mathcal{O}_{X_U}(1)$. We will also write $\mathcal{O}_{X_U}(k)$ for $k \in \mathbb{Z}$.

Notice that the choice of the bundle $\mathcal{O}_{X_U}(1)$ over X_U depends on the choice of E_U, moreover $\mathcal{O}_{X_U}(1)$ does not glue as a global untwisted sheaf $\mathcal{O}_X(1)$ on X. However, the existence of a section for the morphism f ensures the existence of a global $\mathcal{O}_X(1)$.

LEMMA 1.25. *Let $f : X \to S$ be a Brauer-Severi scheme. If $s : S \to X$ is a section of f, then there exists a vector bundle G on S such that $\mathbb{P}(G) \cong X \to S$.*

PROOF. The result is known, but since it is hard to find a reference, we give a proof.

Consider the diagram

$$\begin{tikzcd}
X'''_U \arrow[r, shift left=2, "q_{i,j,X}"] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left, "f"] & X''_U \arrow[r, shift left, "p_{1,X}"] \arrow[r, shift right, "p_{2,X}"'] \arrow[d, shift left, "f"] & X_U \arrow[r] \arrow[d, shift left, "f"] & X \arrow[d, shift left, "f"] \\
U''' \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2, "q_{i,j}"'] \arrow[u, shift left, "s"] & U'' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[u, shift left, "s"] & U \arrow[r] \arrow[u, shift left, "s"] & S. \arrow[u, shift left, "s"]
\end{tikzcd}$$

Here s and f are improperly used to mean their pull-backs to U, U'' and U''' in order to keep a clearer notation.

We can choose $\mathcal{O}_{X_U}(1)$ such that $s^*\mathcal{O}_{X_U}(1) = \mathcal{O}_U$.

Consider now $p_{1,X}^*\mathcal{O}_{X_U}(1)$ and $p_{2,X}^*\mathcal{O}_{X_U}(1)$, the two pull-backs of $\mathcal{O}_{X_U}(1)$ to $X_{U''}$. There exists an invertible sheaf L on U'' such that $p_{1,X}^*\mathcal{O}_{X_U}(1) \cong p_{2,X}^*\mathcal{O}_{X_U}(1) \otimes f^*L$. Since

$$s^*p_{i,X}^*\mathcal{O}_{X_U}(1) = \mathcal{O}_{U''}$$

we have L trivial. We choose an isomorphism

$$\phi : p_{1,X}^*\mathcal{O}_{X_U}(1) \longrightarrow p_{2,X}^*\mathcal{O}_{X_U}(1)$$

such that $s^*\phi = \mathrm{Id}_{\mathcal{O}_{U''}}$.

The isomorphism ϕ satisfies an untwisted cocycle condition. Indeed,

$$s^*((q_{1,2,X}^*\phi) \circ (q_{2,3,X}^*\phi) \circ (q_{1,3,X}^*\phi)^{-1}) = \mathrm{Id}_{\mathcal{O}_{U'''}}.$$

This shows that $\mathcal{O}_{X_U}(1)$ gives a global untwisted sheaf $\mathcal{O}_X(1)$ and that means X is a projective bundle over S. □

5.2. Triangulated categories of twisted sheaves. Recall we are using the notations $\mathbf{D}(S)$ and $\mathbf{D}(S,\alpha)$ to denote the triangulated categories of perfect complexes of quasi-coherent and α-twisted quasi-coherent sheaves on S. A perfect complex of quasi-coherent sheaves is a complex whose cohomology sheaves are quasi-coherent and which has finite global Tor-dimension. Equivalently, it is quasi isomorphic, over any affine open set, to a bounded complex of locally free sheaves of finite rank in any degree. A complete treatment of perfect complexes on a site is given in [**Gro71**]. Everything is defined in the very general context of fibered categories, hence all definitions fit for twisted sheaves. In general, $\mathbf{D}(S)$ is just a full triangulated subcategory of the derived category, but if S is smooth the two categories are equivalent. In the case S is non smooth, we need to restrict to perfect complexes to perform our constructions.

Let us briefly recall what happens to most common derived functors when we consider the category of perfect complexes of twisted sheaves on a scheme. A more satisfying description can be found in [**Cal00**]. It is in fact an adaptation to twisted case of the results of [**Har66**].

THEOREM 1.26. [**Cal00**, Theorem 2.2.6] *Let $f : X \to S$ be a morphism between schemes, let α, α' be in $H^2(S, \mathbb{G}_m)$, and $\mathcal{AB}$ be the category of abelian groups. Then the following derived functors are defined:*

$$\begin{array}{rl} \underline{R\mathrm{Hom}} & : \mathbf{D}(S,\alpha)^\circ \times \mathbf{D}(S,\alpha') \longrightarrow \mathbf{D}(S,\alpha^{-1}\alpha') \\ R\mathrm{Hom} & : \mathbf{D}(S,\alpha)^\circ \times \mathbf{D}(S,\alpha) \longrightarrow \mathbf{D}^b(\mathcal{AB}) \\ \bigotimes_S & : \mathbf{D}(S,\alpha) \times \mathbf{D}(S,\alpha') \longrightarrow \mathbf{D}(S,\alpha\alpha') \\ Lf^* & : \mathbf{D}(S,\alpha) \longrightarrow \mathbf{D}(X, f^*\alpha) \end{array}$$

If $f : X \to S$ is a projective lci (locally complete intersection) morphism, then we can define:

$$Rf_* : \mathbf{D}(X, f^*\alpha) \longrightarrow \mathbf{D}(S,\alpha).$$

Let us recall without explicit statements that Projection Formula, Adjoint Property of Rf_* and Lf^* and Flat Base Change are still valid in the α-twisted context. The only thing to care of is the choice of the right twist. All this and much more is detailed in [**Cal00**] and can easily be generalized to categories of perfect complexes in a nonsmooth case.

6. A semiorthogonal decomposition for Brauer-Severi schemes

Let $f : X \to S$ be a Brauer-Severi scheme of relative dimension r and α in $\mathrm{Br}(S)$ the element associated to it as explained in section 5. This section is dedicated to the proof of the following Theorem.

THEOREM 1.27. *There exist admissible full subcategories $\mathbf{D}(S,X)_k$ of $\mathbf{D}(X)$, such that $\mathbf{D}(S,X)_k$ is equivalent to the category $\mathbf{D}(S,\alpha^{-k})$ for all k in $\mathbb{Z}$. The set of admissible subcategories*

$$\sigma = (\mathbf{D}(S,X)_0, \ldots, \mathbf{D}(S,X)_r)$$

is a semiorthogonal decomposition for the category $\mathbf{D}(X)$ of perfect complexes of coherent sheaves on X.

Recall that there exists a rank $r+1$ locally free sheaf E_U on U, such that $X_U = \mathbb{P}(E_U)$ and that E_U gives an α-twisted sheaf on S. Moreover on X_U we have a tautological line bundle $\mathcal{O}_{X_U}(1)$. In this case, we consider α as a cocycle chosen once for all in the cohomology class $[\alpha]$. By Lemma 1.23 this choice does not affect the category $\mathbf{D}(S,\alpha)$ up to equivalence.

We split the proof in three parts: in the first one we define the full subcategories $\mathbf{D}(S,X)_k$ of $\mathbf{D}(X)$ and we show the equivalence between $\mathbf{D}(S,X)_k$ and $\mathbf{D}(S,\alpha^{-k})$; this is inspired by a construction by Yoshioka [**Yos06**]. It will be clear in the proof of the theorem that the construction of the full admissible subcategories $\mathbf{D}(S,X)_k$ is closely related to the definition of the full admissible subcategories $\mathbf{D}(S)_k$ in Orlov's proof of theorem 1.19. In the second one we show that the sequence σ is indeed a semiorthogonal decomposition. In the third one we give a simple example.

6.1. Construction of $\mathbf{D}(S,X)_k$.

DEFINITION 1.28. We define $\mathbf{D}(S,X)_k$, for $k \in \mathbb{Z}$, to be the full subcategory of $\mathbf{D}(X)$ generated by objects A such that

$$A_{|X_U} \simeq_{\text{q.iso}} f^*A_U \otimes \mathcal{O}_{X_U}(k) \tag{8}$$

where A_U is an object in $\mathbf{D}(U)$.

LEMMA 1.29. *For all k in $\mathbb{Z}$, there is a functor*

$$f_k^* : \mathbf{D}(S, \alpha^{-k}) \longrightarrow \mathbf{D}(S,X)_k$$

given by the association

$$A_{|U} \mapsto f^*A_{|U} \otimes \mathcal{O}_{X_U}(k). \tag{9}$$

PROOF. Firstly, X_U is the projective bundle $\mathbb{P}(E_U)$ over U. We then have on X_U the surjective morphism $f^*E_U \twoheadrightarrow \mathcal{O}_{X_U}(1)$. Given F an α^{-1}-twisted sheaf on S, we have the surjective morphism:

$$f^*(F_U \otimes E_U) = f^*F_U \otimes f^*E_U \twoheadrightarrow f^*F_U \otimes \mathcal{O}_{X_U}(1).$$

Since F_U and E_U give respectively an α^{-1}-twisted and an α-twisted sheaf on S, their tensor product $F_U \otimes E_U$ gives an untwisted sheaf on S. We can naturally see $f^*F_U \otimes f^*E_U$ as an untwisted sheaf on X: the gluing isomorphism is obtained by pull-back with f and this makes naturally $f^*F_U \otimes \mathcal{O}_{X_U}(1)$ an untwisted sheaf as well. It is now clear that given an object A in $\mathbf{D}(S,\alpha^{-1})$, the object given locally by (9) belongs to $\mathbf{D}(S,X)_1$.

The proof is similar for any k in $\mathbb{Z}$. $\square$

THEOREM 1.30. *The functor f_k^* defined in Lemma 1.29 is an equivalence between the category $\mathbf{D}(S,\alpha^{-k})$ and the category $\mathbf{D}(S,X)_k$.*

PROOF. Given A in $\mathbf{D}(S,X)_1$, consider the association over U

$$A_{|X_U} \mapsto Rf_*(A_{|X_U} \otimes \mathcal{O}_{X_U}(-1)).$$

We show that it gives a functor Λ from $\mathbf{D}(S,X)_1$ to $\mathbf{D}(S,\alpha^{-1})$ and that this one is the quasi-inverse functor of f_1^*.

Firstly, since A is in $\mathbf{D}(S,X)_1$, on X_U we have $A_{|X_U} = f^*A_U \otimes \mathcal{O}_{X_U}(1)$, with A_U in $\mathbf{D}(U)$. Evaluating Λ on $A_{|X_U}$ we get

$$Rf_*(A_{|X_U} \otimes \mathcal{O}_{X_U}(-1)) = Rf_*f^*A_U.$$

Now use projection formula:

$$Rf_*f^*A_U = Rf_*\mathcal{O}_X \otimes A_U.$$

We have $R^if_*\mathcal{O}_X = 0$ for $i > 0$ and $f_*\mathcal{O}_X = \mathcal{O}_S$, and then

$$Rf_*f^*A_U \simeq_{\text{q.iso}} A_U. \tag{10}$$

It follows that Λ associates to $A_{|X_U}$ the object A_U in $\mathbf{D}(U)$.

Let F_U be a coherent sheaf on U. By the same reasoning used in Lemma 1.29, we have the surjective morphism

$$f^*(F_U \otimes E_U) \twoheadrightarrow f^*F_U \otimes \mathcal{O}_{X_U}(1).$$

Since E_U is an α-twisted sheaf on S, we can give to F_U the structure of α^{-1}-twisted sheaf over S. This shows that Λ is actually a functor from the subcategory $\mathbf{D}(S,X)_1$ to the category $\mathbf{D}(S,\alpha^{-1})$.

It is now an evidence by (10) that Λ and f_1^* are quasi-inverse to each other.

The proof for $k \in \mathbb{Z}$ is similar. □

We then have constructed full subcategories $\mathbf{D}(S,X)_k$ of $\mathbf{D}(X)$, each one equivalent to a category of perfect complexes of suitably twisted sheaves on S.

Notice that we have $f_0^* = Lf^* = f^*$ since f is flat and the full subcategory of $\mathbf{D}(X)$ which is the image of $\mathbf{D}(S)$ under the functor f^* is in fact the category $\mathbf{D}(S,X)_0$ defined earlier.

6.2. σ is a semiorthogonal decomposition.

Lemma 1.31. *For any A in $\mathbf{D}(S,X)_k$ and B in $\mathbf{D}(S,X)_n$ we have $R\mathrm{Hom}(A,B) = 0$ for $r \geq k-n > 0$.*

Proof. We have locally $A_{|X_U} = f^*A_U \otimes \mathcal{O}_{X_U}(k)$ and $B_{|X_U} = f^*B_U \otimes \mathcal{O}_{X_U}(n)$.

We have:

$$\underline{R\mathrm{Hom}}(A_{|X_U}, B_{|X_U}) = \underline{R\mathrm{Hom}}(f^*A_U \otimes \mathcal{O}_{X_U}(k), f^*B_U \otimes \mathcal{O}_{X_U}(n)) =$$

$$= \underline{R\mathrm{Hom}}(f^*A_U, f^*B_U \otimes \mathcal{O}_{X_U}(n-k)).$$

We now use the adjoint property of f^* and Rf_*:

$$\underline{R\mathrm{Hom}}(f^*A_U, f^*B_U \otimes \mathcal{O}_{X_U}(n-k)) = \underline{R\mathrm{Hom}}(A_U, Rf_*(f^*B_U \otimes \mathcal{O}_{X_U}(n-k)).$$

Now by projection formula

$$Rf_*(f^*B_U \otimes \mathcal{O}_{X_U}(n-k)) = B_U \otimes Rf_*(\mathcal{O}_{X_U}(n-k)).$$

We have $Rf_*(\mathcal{O}_{X_U}(n-k)) = 0$ for $-r \leq n-k < 0$ and hence the sheaves $\underline{R\mathrm{Hom}}(A,B)$ are zero.

Using the local to global Ext spectral sequence, we get the proof. □

We thus have an ordered set $\sigma = (\mathbf{D}(S,X)_0, \ldots, \mathbf{D}(S,X)_r)$ of subcategories of $\mathbf{D}(X)$. Last step in proving Theorem 1.27 is to show that it generates the whole category, and then it gives a semiorthogonal decomposition of $\mathbf{D}(X)$.

Consider the fiber square over S:

$$\begin{tikzcd} P := X \times_S X \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "g"] \\ X \arrow[r, "f"'] & S. \end{tikzcd}$$

We have $g = f : X \to S$. We call P the product $X \times_S X$.

Consider the diagonal embedding $\Delta : X \to P$. It is a section for the projection morphism $p : P \to X$. By Lemma 1.25, there exists a vector bundle G on X such that $P \cong \mathbb{P}(G) \to X$.

We can choose $\mathcal{O}_P(1)$ such that $\Delta^*\mathcal{O}_P(1) \simeq \mathcal{O}_\Delta$, and then we can consider on P the surjective morphism: $p^*G \twoheadrightarrow \mathcal{O}_P \to 0$. We also have the Euler short exact sequence on P:

$$0 \longrightarrow \Omega_{P/X}(1) \longrightarrow p^*G \longrightarrow \mathcal{O}_P(1) \to 0.$$

Combining the exact sequence and the surjective morphism, we get a section of $\mathrm{Hom}(\Omega_{P/X}(1), \mathcal{O}_P)$ whose zero locus is the diagonal Δ of P. Remark that $\Omega_{P/X}(1) = p^*\Omega_{X/S} \otimes \mathcal{O}_P(1)$ and

$$\Lambda^k(p^*\Omega_{X/S} \otimes \mathcal{O}_P(1)) = p^*\Omega^k_{X/S} \otimes \mathcal{O}_P(k).$$

We get a Koszul resolution:

$$0 \to p^*\Omega^r_{X/S} \otimes \mathcal{O}_P(r) \to \ldots \to p^*\Omega_{X/S} \otimes \mathcal{O}_P(1) \to \mathcal{O}_P \to \mathcal{O}_\Delta \to 0.$$

By this complex we deduce that $\mathcal{O}_\Delta$ belongs, as an element of the category $\mathbf{D}(P)$, to the subcategory generated from

$$\{p^*\Omega^r_{X/S} \otimes \mathcal{O}_P(r), \ldots, p^*\Omega_{X/S} \otimes \mathcal{O}_P(1), \mathcal{O}_X \boxtimes \mathcal{O}_X\} \tag{11}$$

by exact triangles and shifting.

Given A an element of $\mathbf{D}(X)$, we remark that $A = Rq_*(p^*A \otimes \mathcal{O}_\Delta)$. Since all involved functors (pull-back, direct image and tensor product) are exact functors, A belongs to the subcategory of $\mathbf{D}(X)$ generated by

$$\{Rq_*(p^*(A \otimes \Omega^r_{X/S}) \otimes \mathcal{O}_P(r)), \ldots, Rq_*(p^*(A \otimes \Omega_{X/S}) \otimes \mathcal{O}_P(1)), Rq_*p^*A\}.$$

LEMMA 1.32. *The object $Rq_*(p^*(A \otimes \Omega^k_{X/S}) \otimes \mathcal{O}_P(k))$ in $\mathbf{D}(X)$ belongs to the subcategory $\mathbf{D}(S, X)_k$.*

PROOF. We look at it in a local situation. In this case X_U is a projective bundle over U, and we have

$$q^*\mathcal{O}_{X_U}(k) = \mathcal{O}_P(k)_{|X_{U''}}.$$

This leads us to write locally:

(12)

$$Rq_*(p^*(A \otimes \Omega^k_{X/S}) \otimes \mathcal{O}_P(k))_{|X_{U''}} = Rq_*((p^*(A \otimes \Omega^k_{X/S}))_{|X_U} \otimes q^*\mathcal{O}_{X_U}(k)) =$$

$$= Rq_*(p^*(A \otimes \Omega^k_{X/S}))_{|X_U} \otimes \mathcal{O}_{X_U}(k) = f^*Rg_*((A \otimes \Omega^k_{X/S})_{|X_U}) \otimes \mathcal{O}_{X_U}(k)$$

where we used projection formula and flat base change in the last two equalities. Then we have an object locally of the form finally given in (12), and then it is an object in $\mathbf{D}(S,X)_k$. □

We have shown that all objects A in $\mathbf{D}(X)$ belong to the subcategory generated by the orthogonal sequence σ. This implies, by Lemma 1.17, that the subcategories $\mathbf{D}(S,X)_k$ are admissible and then σ is in fact a semiorthogonal decomposition of $\mathbf{D}(X)$. This completes the proof of Theorem 1.27.

6.3. An example. We finally treat the simplest example of a Brauer-Severi scheme. Let K be a field and X a Brauer-Severi variety over the scheme $Spec(K)$. In this case Theorem 1.27 gives a very explicit semiorthogonal decomposition of the bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X in terms of central simple algebras over K.

The cohomological Brauer group of $Spec(K)$ is indeed the Brauer group $\mathrm{Br}(K)$ of the field K. The elements of $\mathrm{Br}(K)$ are equivalence classes of central simple algebras over K and its composition law is tensor product. To each α in $\mathrm{Br}(K)$ corresponds the choice of a central simple algebra over K.

Given the α corresponding to the Brauer-Severi variety X, an α^{-1}-twisted sheaf is then a module over a properly chosen central simple algebra A, and it is coherent if it is finitely generated. The category $\mathbf{D}(Spec(K),\alpha^{-1})$ is the bounded derived category of finitely generated modules over the algebra A. Concerning the element α^{-k} in $\mathrm{Br}(K)$, just recall that the composition law is tensor product, to see that we can choose $A^{\otimes k}$ to represent it. The construction of $\mathbf{D}(Spec(K),\alpha^{-k})$ is then straightforward. We can state the following Corollary of the Theorem 1.27.

Corollary 1.33. *Let K be a field, X a Brauer-Severi variety over $Spec(K)$ of dimension r. Let α be the class of X in $\mathrm{Br}(K)$ and A a central simple algebra over K representing α^{-1}.*

The bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X admits a semiorthogonal decomposition $\sigma = (\mathbf{D}(K,X)_0,\dots,\mathbf{D}(K,X)_r)$, where $\mathbf{D}(K,X)_i$ is equivalent to the bounded derived category of finitely generated $A^{\otimes i}$-modules.

CHAPTER 2

Moduli spaces of stable sheaves and pairs on K3 surfaces

1. Introduction

Let us consider a smooth projective elliptic K3 surface $\pi : S \to \mathbb{P}^1$. Friedman studied in [**Fri95**] stability of rank 2 vector bundles on S with fixed determinant in order to calculate certain Donaldson polynomials. For any $t > 0$, fixing a suitable Chern class and then the dimension $2t$ of the moduli space, he especially constructed a birational correspondence between the moduli space and the t-th symmetric product of a Jacobian of S. Moreover, in the case π admits a section σ and we denote the fiber by f, fixing the determinant to be $\sigma - tf$ and the second Chern class to be 1, he describes explicitly all semistable sheaves as extensions. This gives an explicit correspondence between the moduli space and the Hilbert scheme of codimension 2 length t subschemes of S, which turns out to be an isomorphism if t is 1 or 2.

We want to use such a construction to study the behavior of moduli spaces of pairs (V, α) on such surfaces. Such a tool could be very useful if we want to resolve the birational correspondence or at least decompose it in simple steps. In the case of four dimensional moduli spaces, this turns out to be a good frame to understand how to construct the isomorphism step by step. Stable pairs have already been used to resolve birational correspondences between Hilbert schemes and moduli spaces on rank 2 vector bundles on a K3 surface as for example in [**GH96**].

The plan is as follows: we briefly recall the necessary definitions of stability of sheaves and of pairs, then we recall the construction of Friedman. This leads to a natural choice for the definition of stable pairs on S and to a strong base over which perform constructions to describe moduli spaces of pairs, birational maps between them and to recover Friedman's isomorphism.

2. Stable sheaves and stable pairs on a projective variety

2.1. Stable sheaves: a quick tour. Let us briefly recall the main definitions and results about moduli spaces of (semi)stable sheaves on a projective scheme. In such a short introduction, there will be not enough time to develop any argument, for this and for proofs see [**HL98**].

Let X be a projective scheme over a field k and $\mathcal{O}_X(1)$ a fixed ample line bundle on it. We will often refer to it as the *polarization* of X. Consider a coherent sheaf E on X. The Hilbert polynomial $P_E(m)$ of E is given by

$$m \mapsto \chi(E \otimes \mathcal{O}_X(m))),$$

where, for F a sheaf, $\chi(F)$ is the Euler characteristic, given by the alternate sum of the dimensions of the cohomology groups of E. The Hilbert polynomial can be uniquely written in the form

$$P_E(m) = \sum_{i=0}^{\dim(E)} \alpha_i(E) \frac{m^i}{i!}$$

with coefficients α_i. We define the reduced Hilbert polynomial to be

$$p_E(m) := \frac{P_E(m)}{\alpha_d(E)}.$$

In what follows we always consider pure sheaves in order to avoid more complicated definitions we do not need. Moreover, we will consider sheaves of dimension $\dim(X)$ in order to define degree and slope and make free use of them.

Definition 2.1. Let E be a pure coherent sheaf. The *rank* of E is defined by

$$\mathrm{rk}(E) := \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}.$$

The *degree* of E is defined by

$$\deg(E) := \alpha_{d-1}(E) - \mathrm{rk}(E) \cdot \alpha_{d-1}(\mathcal{O}_X)$$

and the *slope* of E is defined by

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

On a smooth projective variety, the degree is actually the intersection number $\deg(E) = c_1(E).H^{d-1}$, where H is the class represented by the ample divisor. It is clear that the definitions of Hilbert polynomial, degree, rank and slope depend on the chosen polarization.

We can now define (semi)stability for a coherent sheaf on X with respect to the ample line bundle $\mathcal{O}_X(1)$.

Definition 2.2. A coherent sheaf E of dimension d is *semistable* if E is pure and

$$p_F \ \leq \ p_E$$

for any proper subsheaf $F \subset E$. A semistable sheaf is *stable* if the strict inequality holds.

We can also define μ-(semi)stability, which is a stronger property.

Definition 2.3. A pure coherent sheaf E of dimension d is *μ-semistable* if $\mu(F) \leq \mu(E)$ for all subsheaves $F \subset E$ with rank $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$. A μ-semistable sheaf is *μ-stable* if the strict inequality holds.

The moduli space of (semi)stable sheaves on a projective variety, if it exists, is a scheme whose points represent equivalence classes of stable sheaves and which has nice universal and functorial properties. In order to define it, we have to introduce the notion of a scheme representing a functor. Indeed, a moduli space is the scheme representing a moduli functor. Let us consider $\mathcal{C}$ a category and denote by $\mathcal{C}^\circ$ the opposite category. If F is an object of $\mathcal{C}$, we denote by $\underline{F}$ the functor from $\mathcal{C}^\circ$ to $\underline{Sets}$ associating to an object A the set $\mathrm{Mor}_{\mathcal{C}}(A, F)$.

Definition 2.4. A functor $\mathcal{F} : \mathcal{C}^\circ \to \underline{Sets}$ is *corepresented* by an object F of $\mathcal{C}$ if there is a natural transformation of functors $\alpha : \mathcal{F} \to \underline{F}$ such that any natural transformation $\alpha' : \mathcal{F} \to \underline{F'}$ factors uniquely through some $\beta : \underline{F} \to \underline{F'}$. The functor $\mathcal{F}$ is *represented* by F if the natural transformation is an isomorphism of functors.

The moduli functor will associate to any scheme S the set of S-flat families of (semi)stable sheaves on X with a fixed Hilbert polynomial. Let us fix a rational polynomial P and define

$$\mathcal{M}'^{(s)s}(P) : \underline{k - sch}^\circ \to \underline{Sets}$$

by associating to a k-scheme S the set $\mathcal{M}'^{(s)s}(P, S)$ of isomorphism classes of S-flat families of (semi)stable sheaves on X with Hilbert polynomial P. The functor associates to a morphism $f : S' \to S$ the map obtained by pulling back sheaves via $f_X := f \times \mathrm{Id}_X$.

Remark that if F is an S-flat family of semistable sheaves and L a line bundle on S, then $F \otimes p^*L$ is an S-flat family of semistable sheaves and for any $s \in S$ the fiber F_s is isomorphic to the fiber $(F \otimes p^*L)_s$. Let us introduce an equivalence relation $\sim$ between S-flat families as follows:

$$F \sim F' \text{ if and only if } F \cong F' \otimes p^*L \text{ for some } L \in \mathrm{Pic}(S).$$

The moduli functor is then defined to be the quotient functor

$$\mathcal{M}^{(s)s}(P) := \frac{\mathcal{M}'^{(s)s}(P)}{\sim}.$$

It is clear that the moduli functor of stable sheaves is a subfunctor of the moduli functor of semistable sheaves.

Definition 2.5. A scheme $M^{(s)s}(P)$ is called a *moduli space of (semi)stable sheaves* if it corepresents the functor $\mathcal{M}^{(s)s}(P)$.

We can show that the moduli functor of semistable sheaves cannot be represented if there exists a properly semistable sheaf.

DEFINITION 2.6. A scheme $M^s(P)$ is called a *fine moduli space of stable sheaves* if it represents the functor $\mathcal{M}^s(P)$.

In the case $M^s(P)$ corepresents the moduli functor of stable sheaves, the natural question is whether $M^s(P)$ is fine or not. The geometrical meaning of such question is related to the existence of a universal family.

DEFINITION 2.7. A flat family $\mathcal{E}$ of stable sheaves on X parametrized by $M^s(P)$ is called *universal* if the following holds: for any S-flat family F of stable sheaves on X with Hilbert polynomial P with induced morphism $\Phi_F : S \to M^s(P)$ there is a line bundle L on S such that $F \otimes p^*L \cong \Phi_F^*\mathcal{E}$. A flat family $\mathcal{E}$ is called *quasi-universal* if the same holds replacing L by a locally free $\mathcal{O}_S$-module W.

It is clear by the definition of universal family that $M^s(P)$ is fine if and only if such a family exists.

2.2. Stable pairs. Let us recall the notion of stability of a pair. The definition we are giving and the consequent study of such objects is given in [**HL95b, HL95a**], to which we will refer in this section. This notion of pair is intended to comprise various other notions of pairs appeared at the beginning of the 90s in different papers [**Bra91, GP93, Lüb93, Tha94**].

Let X be a smooth projective variety over an algebraically closed field k of characteristic zero, and E_0 a fixed coherent sheaf on X. Let us consider pairs (V, α) consisting of a coherent sheaf V and a homomorphism $\alpha : V \to E_0$. Let $\delta(z)$ be a polynomial with rational coefficients such that $\delta(z) > 0$.

DEFINITION 2.8. A pair (V, α) is *semistable with respect to* δ if the following two conditions are satisfied:

(i) $\mathrm{rk}(V)P_G(z) \leq \mathrm{rk}(G)(P_V(z) - \delta(z))$ for all nontrivial submodules $G \subset \ker(\alpha)$.
(ii) $\mathrm{rk}(V)P_G(z) \leq \mathrm{rk}(G)(P_V(z) - \delta(z)) + \mathrm{rk}(V)\delta(z)$ for all nontrivial submodules $G \subset V$.

A semistable pair is *stable with respect to* δ if both inequalities hold strictly.

In the case E_0 is a line bundle $\mathcal{O}_S(-D)$, where D is a divisor on X, we will often refer to the map $\alpha : V \to \mathcal{O}_S(-D)$ as the *framing*.

LEMMA 2.9. *Suppose* (V, α) *is semistable.*

(i) $\ker(\alpha)$ *is torsion free.*
(ii) *Unless* α *is injective,* $\delta(z)$ *has degree smaller than the dimension of* X.

PROOF. This is Lemma 1.2 in [**HL95b**]. □

Remark moreover that relevant stability conditions for a pair arise only with respect to a polynomial $\delta(z)$ of degree strictly smaller then $\dim(X)$. Indeed, for bigger values we just consider semistable submodules of E_0 with fixed Hilbert polynomial. We can then consider $\delta(z)$ as a degree $\dim(X)-1$ polynomial with leading coefficient δ_1. We also define μ-(semi)stability with respect to δ_1.

Definition 2.10. A pair (V,α) is called *μ-semistable with respect to a positive rational number δ_1,* if the following two conditions are satisfied:

(i) $\mathrm{rk}(V)\deg(G) \leq \mathrm{rk}(G)(\deg(V)-\delta_1)$ for all nontrivial submodules $G \subset \ker(\alpha)$.
(ii) $\mathrm{rk}(V)\deg(G) \leq \mathrm{rk}(G)(\deg(V)-\delta_1)+\mathrm{rk}(V)\delta_1$ for all nontrivial submodules $G \subset V$.

A μ-semistable pair is *μ-stable with respect to* δ_1 if both inequalities hold strictly.

It is clear that, as in the theory of stability of sheaves, μ-stability is stronger than stability.

A flat family of pairs, parametrized by a scheme T of finite type over k, is a T-flat coherent $\mathcal{O}_{X\times T}$-module $\mathcal{E}$, together with a homomorphism $\alpha : \mathcal{E} \to \mathcal{O}_T \otimes E_0$ such that $\alpha_t \neq 0$ for all t in T. In order to construct moduli spaces of stable pairs, we define the following moduli functor

$$\underline{\mathcal{M}}^{(s)s}_{\delta(z)}(P, E_0) : \underline{k-sch}^{\circ} \to \underline{sets},$$

which associates to any k-scheme T the set of isomorphism classes of families of pairs (V,α) which are (semi)stable with respect to $\delta(z)$ and V has fixed Hilbert polynomial P. Remark that the notation we used for the moduli functor would be a priori not correct, since it suggests that there exists a scheme representing the functor. Indeed this is the case, as the following Theorem states. We then avoided using a special notation for the functor which would be used just once.

Theorem 2.11. *Let X be a smooth projective variety, let $\delta(z)$ be a positive rational polynomial of degree strictly smaller then* $\dim(X)$. *Then there is a projective k-scheme $\mathcal{M}^{ss}_{\delta(z)}(P, E_0)$ and a natural transformation of functors*

$$\phi : \underline{\mathcal{M}}^{ss}_{\delta(z)}(P, E_0) \longrightarrow \mathrm{Hom}_{\mathrm{Spec}k}(-, \mathcal{M}^{ss}_{\delta(z)}(P, E_0)),$$

such that ϕ is surjective on rational points and $\mathcal{M}^{ss}_{\delta(z)}(P, E_0)$ is minimal with this property. Moreover, there is an open subscheme $\mathcal{M}^{s}_{\delta(z)}(P, E_0)$ of $\mathcal{M}^{ss}_{\delta(z)}(P, E_0)$ such that ϕ induces an isomorphism of subfunctors

$$\phi : \underline{\mathcal{M}}^{s}_{\delta(z)}(P, E_0) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Spec}k}(-, \mathcal{M}^{s}_{\delta(z)}(P, E_0)),$$

which means that $\mathcal{M}^s_{\delta(z)}(P, E_0)$ is a fine moduli space of stable pairs. A closed point in $\mathcal{M}^{ss}_{\delta(z)}(P, E_0)$ represents an S-equivalence class of semistable framed modules.

PROOF. This is Theorem 0.1 of [**HL95a**]. □

We then have a fine moduli space for stable pairs on smooth projective varieties. Moreover, a deformation theory leading to study dimension and smoothness of such moduli spaces is studied in [**HL95a**]. In what follows, we will not need such a theory, since smoothness question will be tackled by effective arguments. Anyway, the picture of the geometry of moduli spaces of pairs would be incomplete without it.

THEOREM 2.12. *Let X and $\delta(z)$ be as in Theorem 2.11. Let (V, α) be a point in $\mathcal{M}^s_{\delta(z)}(P, E_0)$ and consider V and $V \xrightarrow{\alpha} E_0$ as objects of $\mathbf{D}(X)$ concentrated in dimensions zero and (zero,one) respectively.*

(i) *The Zariski tangent space of $\mathcal{M}^s_{\delta(z)}(P, E_0)$ at a point (V, α) is naturally isomorphic to the hyper-Ext group $\mathrm{Ext}^1_{\mathbf{D}(X)}(V, V \xrightarrow{\alpha} E_0)$.*

(ii) *If the second hyper-Ext group $\mathrm{Ext}^2_{\mathbf{D}(X)}(V, V \xrightarrow{\alpha} E_0)$ vanishes, then $\mathcal{M}^s_{\delta(z)}(P, E_0)$ is smooth at the point (V, α).*

PROOF. This is Theorem 4.1 of [**HL95a**]. □

In what follows we are considering framed rank 2 vector bundles on an elliptic surface. Before getting into the description of stable vector bundles on an elliptic surface, let us just rewrite definitions 2.8 and 2.10 for rank 2 vector bundles on a smooth projective surface with respect to a degree one polynomial.

Suppose X is a polarized surface, V has rank 2, E_0 is a line bundle and $\delta(z) = 2\delta z$ with $\delta > 0$.

DEFINITION 2.13. Let (V, α) be a pair with $\mathrm{rk}(V) = 2$. Let δ be a positive rational number. The pair is *semistable with respect to* $2\delta(z)$ if the following two conditions are satisfied:

(i) $\deg G \leq \frac{1}{2}\deg V - \delta$ for all nontrivial subline bundles $G \subset \ker(\alpha)$.

(ii) $\deg G \leq \frac{1}{2}\deg V + \delta$ for all nontrivial subline bundles $G \subset V$.

Such a pair is *stable with respect to* $2\delta z$ if both inequalities hold strictly.

Let $n \in \mathbb{Z}$. If the degree is an odd integer, which will be the case in the next, purely semistable sheaves exist if and only if $\delta = (2n+1)/2$. For any δ in any interval $(n - \frac{1}{2}, n + \frac{1}{2})$ the stability condition with respect to it does not change. This is a typical setting in which wall crossing phenomena happen and this is what we are going to study in the specific case of an elliptic surface. Before defining stability of pairs in that specific case, we have to recall Friedman's description of stable rank 2 vector bundles on elliptic surfaces. This will give us a natural choice for the framing.

2.3. Elementary transformations. This Subsection deals with elementary transformations of vector bundles. This construction is often performed to construct stable bundles on surfaces. Indeed, this is one of the main ingredients in Friedman's construction of birational correspondences between certain moduli spaces and Hilbert schemes on an elliptic surface [**Fri95**] and we are going to make a large use of it in the study of stable pairs in Section 5.3. Anyway, we introduce it as a part of this Section, recalling it as a general tool for managing vector bundles on surfaces, as explained in Section 5.2 of [**HL98**]. Finally, we will recall a Theorem by Friedman [**Fri95**] which give concrete results about elementary transformations for families of sheaves.

In what follows, let X be a smooth projective variety.

DEFINITION 2.14. Let C be an effective divisor on X and $i : C \hookrightarrow X$ the embedding. If F and G are vector bundles on X and C respectively, then a vector bundle E on X is obtained by an *elementary transformation* of F along G if there exists an exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow i_*G \longrightarrow 0.$$

If we are handling locally free sheaves on X and C respectively, such a construction gives a locally free sheaf (that is, a vector bundle) on X whose invariants can be easily calculated in the case X is a surface.

PROPOSITION 2.15. *Let X be a smooth projective surface. If F and G are locally free on X and C respectively, then the kernel E of any surjection $\phi : F \to i_*G$ is locally free. Moreover, if ρ denotes the rank of G, one has* $\det(E) \cong \det(F) \otimes \mathcal{O}_X(-\rho \cdot C)$ *and* $c_2(E) = c_2(F) - \rho C.c_1(F) + \frac{1}{2}\rho C.(\rho C + K_X) + \chi(G)$.

PROOF. Proposition 5.2.2 of [**HL98**]. □

In the next sections, we will perform elementary transformations on the universal sheaf of a given moduli space. We need then some results about the behavior of the single fibers. In the last part of this Section, we recall a result explained in the Appendix of [**Fri95**].

Let T be a smooth scheme and D a reduced divisor on T. Suppose $\mathcal{F}$ is a rank 2 vector bundle over $X \times T$ and L is a line bundle on X. Let $i : X \times D \hookrightarrow X \times T$ be the embedding, π_j the projection from $X \times T$ onto the j-th factor. Suppose there exists a surjection $\mathcal{F} \to i_*\pi_1^*L$ defining $\mathcal{E}$ as the elementary transformation

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow i_*\pi^*L \longrightarrow 0.$$

For t a point of T, we denote by $E_t = \mathcal{E}_{|X\times\{t\}}$ and by $F_t = \mathcal{F}_{|X\times\{t\}}$ the fibers of $\mathcal{E}$ and $\mathcal{F}$ respectively. Let us fix a point 0 of D. We have the two following extensions

$$0 \longrightarrow M \longrightarrow F_0 \longrightarrow L \longrightarrow 0,$$
$$0 \longrightarrow L \longrightarrow E_0 \longrightarrow M \longrightarrow 0.$$

We then can express the fibers E_t and F_t as extensions under some condition.

PROPOSITION 2.16. *Let $\mathcal{F}$ be a rank 2 reflexive sheaf over $X \times T$, flat over T. Let D be a reduced non necessarily smooth divisor on T. Suppose L is a line bundle on X and $\mathcal{Z}$ is a codimension two subscheme of $X \times D$, flat over D. Suppose further we have a surjection $\mathcal{F} \to i_*\pi_1^*L \otimes I_{\mathcal{Z}}$ and let $\mathcal{E}$ be its kernel*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow i_*\pi_1^*L \otimes I_{\mathcal{Z}} \longrightarrow 0.$$

(i) *$\mathcal{E}$ is reflexive and flat over T.*
(ii) *For each t in D, there are exact sequences*

$$0 \longrightarrow M \otimes I_{Z'} \longrightarrow F_t \longrightarrow L \otimes I_Z \longrightarrow 0,$$
$$0 \longrightarrow L \otimes I_Z \longrightarrow E_t \longrightarrow M \otimes I_{Z'} \longrightarrow 0,$$

where Z is the subscheme of X defined by $\mathcal{Z}$ for the slice $X \times \{t\}$ and Z' is a subscheme of codimension ≤ 2 of X.
(iii) *If D is smooth, then the extension class corresponding to E_t in $\mathrm{Ext}^1(M \otimes I_{Z'}, L \otimes I_Z)$ is defined by the image of the normal vector to D at t under the composition of the Kodaira-Spencer map from the tangent space of T at t to $\mathrm{Ext}^1(F_t, F_t)$, followed by the natural map $\mathrm{Ext}^1(F_t, F_t) \to \mathrm{Ext}^1(M \otimes I_{Z'}, L \otimes I_Z)$.*

The same holds if we drop the ideal sheaf $I_{\mathcal{Z}}$.

PROOF. Proposition A.2 of [**Fri95**].

Remark that in the original proof of Proposition A.2, if we drop the ideal sheaf of a codimension 2 subscheme and we suppose we have a surjection $\mathcal{F} \to i_*\pi_1^*L$, nothing changes. $\square$

3. Moduli spaces of rank 2 vector bundles on K3 elliptic surfaces

In this Section, we recall the construction performed by Friedman [**Fri95**] of stable rank 2 vector bundles on elliptic surfaces. With this in mind, we will then be able to fix a framing for a convenient choice of pairs.

3.1. Birational correspondence between the moduli space and the Hilbert scheme. Let $\pi : S \to \mathbb{P}^1$ be an elliptic K3 surface and denote by f a general fiber of π. Let us assume $\pi : S \to \mathbb{P}^1$ to have singular fibers with at most nodal singularities.

Definition 2.17. Let Δ be a divisor on S and c an integer. An ample line bundle L on S is (Δ, c)-*suitable* if for all divisors D on S such that $-D^2 + D \cdot \Delta \leq c$, either $f \cdot (2D - \Delta) = 0$ or

$$sign f \cdot (2D - \Delta) = sign L \cdot (2D - \Delta).$$

Lemma 2.18. *For all pairs (Δ, c), (Δ, c)-suitable ample line bundles exist.*

Proof. This is Lemma 2.3. in [**Fri95**], I. □

Let Δ be a divisor on S and c an integer. Fix a (Δ, c)-suitable ample line bundle L. Since we fixed $rk(V) = 2$ once and for all, we denote by $M(\Delta, c)$ the moduli space of equivalence classes of L-stable rank 2 vector bundles V on S with $c_1(V) = \Delta$ and $c_2(V) = c$. The scheme $M(\Delta, c)$ does not depend on the choice of L. Moreover, if we denote $w = \Delta \mod 2$ and $p = \Delta^2 - 4c$, the moduli space $M(\Delta, c)$ depends only on (w, p).

Let us fix the determinant of V to have fiber degree 1, we are then in the framework of part III of [**Fri95**]. In this case, the dimension $2t$ is given by $2t = 4c - \Delta^2 - 3\chi(\mathcal{O}_S)$. Since $\chi(\mathcal{O}_S) = 2$, we have $2t = 4c - \Delta^2 - 6$. In the case $t = 0$ the only element V_0 of $M(\Delta, c)$ is explicitly constructed.

The most important issue in having chosen a suitable polarization, is that in this case a vector bundle V is stable if and only if its restriction to the generic fiber is stable. This allows to show that any stable rank 2 vector bundle with the same restriction to the generic fiber as V_0 is obtained from V_0 by a finite chain of elementary transformations along the fibers by line bundles of degree e at each step.

Theorem 2.19. *In the above notation, $M(\Delta, c)$ is nonempty, smooth and irreducible, and it is birational to* $\mathrm{Sym}^t(J^{e+1}(S))$, *where* $2t =$ $\dim M(\Delta, c)$.

Proof. Theorem 3.14 of [**Fri95**], III. □

3.2. Explicit construction of stable rank 2 vector bundles. Let us consider the case where π has a section σ. In this case, the construction of stable vector bundles with appropriate Chern classes is explicit. This will give a natural choice for the framing in the definition of stable pairs. We are going to describe stable bundles on S such that $\det V$ has the same restriction to the generic fiber as the section σ of the elliptic fibration and $c_2(V) = 1$. Thus $\det V = \sigma - tf$, where $2t$ is the dimension of the moduli space. Let us first write down explicitly the vector bundle V_0.

Proposition 2.20. *There is a unique nonsplit extension*

$$0 \longrightarrow \mathcal{O}_S(f) \longrightarrow V_0 \longrightarrow \mathcal{O}_S(\sigma - f) \longrightarrow 0$$

and $\det V_0 = \sigma$, $c_2(V_0) = 1$ *and the restriction of* V_0 *to every fiber is stable.*

PROOF. Proposition 4.2 of [**Fri95**], III. □

The bundle V_0 has also been described independently by Kametani and Sato [**KS94**].

Let us now consider the case where the moduli space has positive dimension $2t$, that is the space $M(\sigma - tf, 1)$.

PROPOSITION 2.21. *Let* V *be a stable rank 2 vector bundle over* S *such that* $\det V = \sigma - tf$, $c_2(V) = 1$. *Then there exist an integer* s, $0 \le s \le t$ *and an exact sequence*

$$0 \longrightarrow \mathcal{O}_S((1-s)f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma + (-1+s-t)f) \otimes I_Z \longrightarrow 0,$$

where Z *is a codimension 2 local complete intersection subscheme of length* s. *Moreover, any nonzero map* $\mathcal{O}_S(af) \to V$ *factors through the inclusion* $\mathcal{O}_S((1-s)f) \to V$.

PROOF. Proposition 4.3 of [**Fri95**], III. □

We can also characterize semistable torsion free sheaves.

PROPOSITION 2.22. *Let* V *be a rank 2 torsion free sheaf with* $c_1(V) = \sigma - tf$ *and* $c_2(V) = 1$ *such that* V *is semistable with respect to a* (Δ, c)*-suitable line bundle. Suppose* $4c - 6 - c_1^2(V) = 2t$. *Then there are zero-dimensional subschemes* Z_1 *and* Z_2 *of* S, *not necessarily local complete intersections, an integer* s *with* $0 \le s \le t$ *and an exact sequence*

$$0 \to \mathcal{O}_S((1-s)f) \otimes I_{Z_1} \to V \to \mathcal{O}_S(\sigma + (-1+s-t)f) \otimes I_{Z_2} \to 0.$$

Moreover, $l(Z_1) + l(Z_2) = s$.

PROOF. Proposition 4.3', [**Fri95**], III. □

Let us consider when such an extension can be unstable.

PROPOSITION 2.23. *Suppose* V *is an extension of the form*

$$0 \longrightarrow \mathcal{O}_S((1-s)f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma + (-1+s-t)f) \otimes I_Z \longrightarrow 0,$$

where $l(Z) = s$. *Let* s_0 *be the smallest integer such that* $h^0(\mathcal{O}_S(s_0 f) \otimes I_Z) \neq 0$. *Thus* $0 \le s_0 \le s$ *and* $s_0 = 0$ *if and only if* $s = 0$. *If* V *is unstable then the maximal destabilizing subbundle is equal to* $\mathcal{O}_S(\sigma - af)$ *where*

$$t + 1 - (s - s_0) \le a \le t + 1.$$

PROOF. Proposition 4.4 of [**Fri95**], III. □

3.3. The generic extension. Let us consider extensions of the form

$$0 \longrightarrow \mathcal{O}_S((1-t)f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - f) \otimes I_Z \longrightarrow 0,$$

where $l(Z) = t$. These are indeed the extensions which give the generic vector bundle of $M(\sigma - tf, 1)$.

PROPOSITION 2.24. *Suppose V is an extension of the form*

$$0 \longrightarrow \mathcal{O}_S((1-t)f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - f) \otimes I_Z \longrightarrow 0,$$

where $l(Z) = t > 0$.

(1) *A locally free extension V as above exists if and only if Z has the Cayley-Bacharach property with respect to $|\sigma + (t-2)f|$.*
(2) *Suppose that $s_0 = t$ or $t - 1$ and* $\mathrm{Supp}Z \cap \sigma = \emptyset$. *Then* $\dim\mathrm{Ext}^1(\mathcal{O}_S(\sigma - f) \otimes I_Z, \mathcal{O}_S((1-t)f)) = 1$. *A locally free extension exists in this case if $s_0 = t$.*
(3) *Suppose that Z consists of t points lying on distinct fibers, exactly one of which lies on σ. Then* $\dim\mathrm{Ext}^1(\mathcal{O}_S(\sigma - f) \otimes I_Z, \mathcal{O}_S((1-t)f)) = 1$. *A locally free extension exists in this case if and only if $t = 1$*
(4) *If $s_0 \leq t - 1$, then V is unstable.*
(5) *If $s_0 = t$ then V is stable if no point of Z lie on σ.*

PROOF. Proposition 4.6 of [**Fri95**], III. □

So far, we know how a stable vector bundle in $M(\sigma - tf, 1)$ looks like: the generic one can be uniquely associated to a codimension 2 local complete intersection subscheme of length t, that is a point Z in $\mathrm{Hilb}^t(S)$. However, only a case by case analysis could lead us to establish if this correspondence could or not be extended to an isomorphism. This is the case for $t = 1$ and $t = 2$, see [**Fri95**], section 4 of part III. This leads Friedman to state the following conjecture (Conjecture 4.13 of [**Fri95**], III).

CONJECTURE 2.25. *There is an isomorphism* $\mathrm{Hilb}^t(S) \xrightarrow{\simeq} M(\sigma - tf, 1)$.

In what follows, we are going to define stability for pairs $(V, \alpha : V \to \mathcal{O}_S(\sigma - f))$. The choice of the framing is naturally suggested by Proposition 2.21, which implies that any stable vector bundle V admits such a map. Moduli space of such pairs turns then out to be, at least in the case $t = 2$, a natural setting in which recover the isomorphism $\mathrm{Hilb}^2(S) \cong M(\sigma - 2f, 1)$ by considering wall crossing phenomena. We hope this could give a hint for the solution of the conjecture.

4. Stable pairs on elliptic surfaces

We are now ready to define a framing and a stability condition for pairs on K3 elliptic surfaces. In the four dimensional case, we will construct explicitly the moduli spaces and birational maps between them. This will allow us to recover the isomorphism described by Friedman through these birational maps.

Consider an elliptic K3 surface $\pi : S \to \mathbb{P}^1$ with a section σ and a stable rank 2 sheaf V on it with Chern class $(\sigma - tf, 1)$. We have seen in Theorem 2.21 that there is a map $V \to \mathcal{O}_S(\sigma + (-1 + s - t)f)$, with $0 \leq s \leq t$. Then we can fix the framing to be $\sigma - f$ and consider stability of pairs (V, α) with $\alpha : V \to \mathcal{O}_S(\sigma - f)$.

We have already defined in Definition 2.13 the stability condition for a pair on a surface in the rank 2 case. In order to make it as explicit as possible, we have to fix a suitable polarization with respect to which we calculate the degree. Given t a nonnegative integer let us fix the suitable polarization to be $\mathcal{O}_S(\sigma + (t + 5)f)$. If V has Chern class $(\sigma - tf, 1)$, then $\deg(V) = 3$ and hence we can define the following stability condition for the pair (V, α).

Definition 2.26. Let V be a rank 2 vector bundle over S with Chern class $(\sigma - tf, 1)$, let $\mathcal{O}_S(\sigma + (t+5)f)$ be the suitable polarization and δ a positive rational number. Let $\alpha : V \to \mathcal{O}_S(\sigma - f)$. The pair (V, α) is *semistable with respect to* $2\delta z$ if the following two conditions are satisfied

(i) $\deg G \leq \frac{3}{2} - \delta$ for all nontrivial submodules $G \subset \ker(\alpha)$.
(ii) $\deg G \leq \frac{3}{2} + \delta$ for all nontrivial submodules $G \subset V$.

Such a pair is *stable with respect to* $2\delta z$ if both inequalities hold strictly.

Theorem 2.11 tells us that for any δ positive the fine moduli space of stable pairs with respect to $2\delta z$ exists. Let us denote it by $\mathcal{M}_\delta$. We are using Friedman's results exposed in Section 3 to study the behavior of these moduli spaces when δ varies. Let us spell out in details what happens.

Let $n \in \mathbb{N}$, the moduli spaces $\mathcal{M}_\delta$ are all isomorphic for δ in $(n - \frac{1}{2}, n + \frac{1}{2})$. For critical values of δ, that is for $\delta = (2n+1)/2$, semistable pairs exist. We will not consider such critical cases. It will be enough to consider the moduli spaces $\mathcal{M}_n$ for n integer, and to consider the critical value as a wall. To investigate wall crossing, our first analysis will focus on the first element of the pair (V, α). Remark that this will be not enough to detail the structure of $\mathcal{M}_n$, since for a given V there could be many different maps α.

If (V, α) is a point of $\mathcal{M}_0$, then V is a stable sheaf. Indeed conditions (i) and (ii) in the definition simply give the definition of stability in this case. Then $\mathcal{M}_0$ fibers with projective fibers over $M(\sigma - tf, 1)$. The fiber over a point V is given by $\mathbb{P}\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$.

On the other side, we have a relevant subscheme of $\mathcal{M}_t$ fibering with projective fibers over $\mathrm{Hilb}^t(S)$, as it will be shown in Propostion 2.27. Then moving from 0 to t we get a chain of wall crossings giving rise to a decomposition of the birational correspondence between $M(\sigma - tf, 1)$ and $\mathrm{Hilb}^2(S)$.

Before stating Proposition 2.27, we need to describe the wall crossing phenomena. We will then study the locus of pairs (V, α) belonging to $\mathcal{M}_i$ and not to $\mathcal{M}_{i+1}$ and the locus of the ones belonging to $\mathcal{M}_{i+1}$ and not to $\mathcal{M}_i$. This kind of question was asked and answered by Thaddeus in the case of curves [**Tha94**], in which case wall crossing induces a Mukai flip between the moduli spaces, since the two loci are parametrized by dual projective spaces. Unfortunately, in our case these loci do not fit such a nice parametrization. Anyway, we can use Friedman's description of vector bundles to make conditions (i) and (ii) of Definition 2.26 the most explicit as possible. This will lead in the case $t = 2$ to an explicit construction of the birational correspondences arising under wall crossing phenomena.

Passing from $\mathcal{M}_0$ to $\mathcal{M}_1$, condition (i) gets stronger. Indeed, for $n = 0$ the kernel of α is allowed to have degree 1, which is no more possible for $n = 1$, in which case the maximum degree allowed to the kernel is 0.

If we want to spell out for which sheaves V the pair (V, α) belongs to $\mathcal{M}_0$ and not to $\mathcal{M}_{i+1}$, we just have to restrict the value of the degree of $\ker\alpha$ step by step. We then get that if a pair (V, α) belongs both to $\mathcal{M}_0$ and to $\mathcal{M}_i$, V is given by an extension

$$0 \longrightarrow \mathcal{O}_S((1-s)f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma + (-1+s-t)f) \otimes I_Z \longrightarrow 0,$$

where $0 \le s \le t$ and its kernel has to be $\ker\alpha = \mathcal{O}_S((1-i)f) \subset \mathcal{O}_S((1-s)f)$. Then any such extension with $s \ge i$ has a good kernel. Recall indeed Proposition 2.21 which states that any map $\mathcal{O}_S(af) \to V$ factors through $\mathcal{O}_S((1-s)f) \to V$. This analysis should not lead to think that any extension with $i < s$ does not appear in a pair belonging to $\mathcal{M}_i$. An important issue to be calculated is then the vector space $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ or at least its dimension.

Let us now consider condition (ii), which gets weaker when n gets bigger. This will tell us for which sheaves V the pair (V, α) belongs to $\mathcal{M}_{i+1}$ but not to $\mathcal{M}_i$. First consider $\mathcal{M}_0$. If a pair (V, α) belongs to $\mathcal{M}_0$, V is stable as a sheaf. Condition (ii) tells us that for $t = 0$ we allow subline bundles of V to have maximal degree 1. If (V, α) belongs to the moduli space $\mathcal{M}_1$ the maximal degree of a subline bundle of V goes up to 2. Hence when passing from $\mathcal{M}_1$ to $\mathcal{M}_0$ we lose those pairs (V, α) such that V has a subline bundle of degree 2. By Proposition 2.23, such a subline bundle is of the form $\mathcal{O}_S(\sigma + af)$ and then a straight calculation gives $a = -t - 1$. Then passing from $\mathcal{M}_1$ to $\mathcal{M}_0$ we lose pairs (V, α) where V admits $\mathcal{O}_S(\sigma - (t+1)f)$ as a subline bundle.

A similar argument applies if we want to spell out for which sheaves V the pair (V, α) belongs to $\mathcal{M}_t$ and not to $\mathcal{M}_i$ for $0 < i < t$, increasing the value of a step by step.

Remark that all such considerations are still valid for semistable sheaves, just recall Theorem 2.22.

Let $\tilde{\mathcal{M}}_t \subset \mathcal{M}_t$ be the subscheme whose elements are those pairs (V, α) with locally free kernel. In this case, V is given by an extension

$$0 \longrightarrow \mathcal{O}_S((1-t)f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - f) \otimes I_Z \longrightarrow 0, \tag{13}$$

with Z in $\mathrm{Hilb}^t(S)$. This subscheme $\tilde{\mathcal{M}}_t$ fibers over $\mathrm{Hilb}^t(S)$ with fiber $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_S(\sigma - f) \otimes I_Z, \mathcal{O}_S((1-t)f))$ over the point Z.

PROPOSITION 2.27. *Let us fix the determinant of V to be $\mathcal{O}_S(\sigma - tf)$. The set of all pairs (V, α) in $\mathcal{M}_t$ with $\ker(\alpha)$ locally free forms a projective scheme over $\mathrm{Hilb}^t(S)$ with fiber isomorphic to $\mathbb{P}\mathrm{Ext}^1(\mathcal{O}_S(\sigma - f) \otimes I_Z, \mathcal{O}_S((1-t)f))$ over Z.*

PROOF. This is Corollary 2.14 in [**HL95b**]. Indeed if $(\mathcal{V}, A)$ is a universal family over $\mathcal{M}_t \times S$, then the set of points $p \in \mathcal{M}_t$ with $l((\mathrm{coker} A)_p)$ maximal is closed. We have $(\mathrm{coker} A)_p \simeq \mathrm{coker}(A_p)$ and $l((\mathrm{coker} A)_p)$ is maximal if it is equal to t and then if $\ker(A_p)$ is locally free. Therefore the set of all pairs with locally free kernel is closed and the quotient $\mathcal{O}_S(\sigma - f)/\mathrm{Im}(\alpha)$ gives the morphism to $\mathrm{Hilb}^t(S)$. $\square$

5. Stable pairs in the case $t = 2$

We are going to describe explicitly the behavior of the moduli spaces $\mathcal{M}_i$ in the case we consider pairs (V, α) with $\det(V)$ fixed to be $\mathcal{O}_S(\sigma - 2f)$. If we consider such stable sheaves, the dimension of $M(\sigma - 2f, 1)$ is 4 and Friedman ([**Fri95**], Section 4 of part III) explicitly constructed an isomorphism between $\mathrm{Hilb}^2(S)$ and this moduli space. In this Section, we use such explicit construction to describe birational maps between the different moduli spaces of pairs. This will allow us to recover the isomorphism $M(\sigma - 2f, 1) \simeq \mathrm{Hilb}^2(S)$ through these birational maps (see Corollary 2.37).

In this case, we have three different moduli spaces: $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$. Consider V an extension

$$0 \longrightarrow \mathcal{O}_S(-f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - f) \otimes I_Z \longrightarrow 0,$$

with $l(Z) = 2$. Let D_σ be the effective divisor of $\mathrm{Hilb}^2(S)$ which is the closure of the locus of pairs $\{p, q\}$ where $p \in \sigma$. Let D be the irreducible smooth divisor in $\mathrm{Hilb}^2(S)$ given by

$$D = \{Z \in \mathrm{Hilb}^2(S) \mid h^0(\mathcal{O}_S(f) \otimes I_Z) = 1\}.$$

Remark that, for Z in D_σ, the extension V is not locally free since the Cayley-Bacharach property fails.

5.1. The subscheme $\tilde{\mathcal{M}}_2$. Consider $\tilde{\mathcal{M}}_2$ and its fiber structure over $\mathrm{Hilb}^2(S)$ as described in Proposition 2.27. We know by Proposition 2.24 that for Z not in $\mathrm{Sym}^2\sigma$ the dimension of $\mathrm{Ext}^1(\mathcal{O}_S(\sigma - f) \otimes I_Z, \mathcal{O}_S(-f))$ is one and hence there is an isomorphism between a dense open subscheme of $\tilde{\mathcal{M}}_2$ and the open subscheme $\mathrm{Hilb}^2(S) \setminus \mathrm{Sym}^2\sigma$. We can actually show that $\tilde{\mathcal{M}}_2$ is the blow up of $\mathrm{Hilb}^2(S)$ along $\mathrm{Sym}^2\sigma$.

LEMMA 2.28. *The subscheme $\tilde{\mathcal{M}}_2$ is the blow-up of* $\mathrm{Hilb}^2(S)$ *along* $\mathrm{Sym}^2\sigma$. *This also implies that $\tilde{\mathcal{M}}_2$ is smooth.*

PROOF. Let us consider the sheaf ζ on $\mathrm{Hilb}^2(S)$ whose stalk over a point Z is given by

$$\zeta_Z := \mathrm{Ext}^1(\mathcal{O}_S(\sigma - f) \otimes I_Z, \mathcal{O}_S(-f))^\vee \simeq H^1(S, I_Z(\sigma)).$$

This datum gives actually a sheaf, since $H^2(S, I_Z(\sigma)) = 0$, and shows that $\tilde{\mathcal{M}}_2$ is $\mathbb{P}(\zeta)$. Moreover, if we define π_i to be the projection from $S \times \mathrm{Hilb}^2(S)$ to i-th factor, we have $\zeta = R^1\pi_{2*}(I_{\mathcal{Z}} \otimes \pi_1^*\mathcal{O}_S(\sigma))$, where $\mathcal{Z}$ denote the universal subscheme of $S \times \mathrm{Hilb}^2(S)$. Our aim is then to show that ζ is isomorphic to $I_{\mathrm{Sym}^2\sigma}$ up to tensoring with a line bundle.

To this end, consider the exact sequence

$$0 \to I_{\mathcal{Z}} \otimes \pi_1^*\mathcal{O}_S(\sigma) \to \pi_1^*\mathcal{O}_S(\sigma) \to \mathcal{O}_{\mathcal{Z}} \otimes \pi_1^*\mathcal{O}_S(\sigma) \to 0$$

and apply the functor $R\pi_{2*}$. We then get the exact sequence

$$R^0\pi_{2*}(\pi_1^*\mathcal{O}_S(\sigma)) \xrightarrow{\delta} R^0\pi_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes \pi_1^*\mathcal{O}_S(\sigma)) \to \zeta \to 0,$$

since $H^1(S, \mathcal{O}_S(\sigma)) = 0$. Then ζ is the cokernel of δ. Consider the diagram

$$\begin{tikzcd} S \times \mathrm{Hilb}^2(S) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathrm{Hilb}^2(S) \arrow[d, "p_0"] \\ S \arrow[r, "q_0"] & \mathrm{spec}(k). \end{tikzcd}$$

Flat base change gives $R^0\pi_{2*}(\pi_1^*\mathcal{O}_S(\sigma)) = R^0p_{0*}(q_0^*\mathcal{O}_S(\sigma)) = \mathcal{O}_{\mathrm{Hilb}^2(S)}$. If we denote by $E := R^0\pi_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes \pi_1^*\mathcal{O}_S(\sigma))$, we get ζ as the cokernel of the map $\mathcal{O}_{\mathrm{Hilb}^2(S)} \xrightarrow{\gamma} E$, where E is a rank two vector bundle on $\mathrm{Hilb}^2(S)$. The map γ is then a section of E. If we let $L = \Lambda^2 E$ to be the determinant of E and $Z(\gamma)$ be the zero locus of the section, we get Koszul resolution

$$\mathcal{O}_{\mathrm{Hilb}^2(S)} \longrightarrow E \longrightarrow L \otimes I_{Z(\gamma)} \longrightarrow 0.$$

The zero locus of a section of E is then, up to a twist with L, the locus of points Z of the Hilbert scheme for which Z is contained in σ. Then, set-theoretically, we have $\mathrm{Sym}^2\sigma$ up to a twist with L. Moreover, such zero locus is reduced. We know indeed that it is locally a complete intersection, then it is enough to show it is generically reduced. Consider the generic point Z in the Hilbert scheme which is contained in

σ. Then Z consists of two distinct points and each one of these points admits a neighborhood which does not contain the other one. Then the zero locus of such a section is generically reduced. Then ζ is $I_{\mathrm{Sym}^2\sigma} \otimes L$, which gives us the proof. □

Let $\tilde{G}$ be the exceptional divisor of the blow up, that is the subscheme of $\tilde{\mathcal{M}}_2$ of pairs (V, α) corresponding to points in $\mathrm{Sym}^2\sigma$. Then $\tilde{G}$ fibers with $\mathbb{P}^1$-fibers over $\mathrm{Sym}^2\sigma$.

Let us call $\tilde{G}_Z$ the fiber over a point Z. If Z consists of two distinct points on σ, there are exactly two points in $\tilde{G}_Z$ corresponding to extensions which are not locally free. Indeed consider the local to global extensions spectral sequence. Recalling that $H^1(S, \mathcal{O}_S(-\sigma)) = 0$ by Theorem 2.38, we get the following sequence

$$0 \to \mathrm{Ext}^1(I_Z(\sigma), \mathcal{O}_S) \xhookrightarrow{i} H^0(\underline{\mathrm{Ext}}^1(I_Z(\sigma), \mathcal{O}_S)) \simeq H^0(S, \mathcal{O}_Z). \tag{14}$$

Moreover since the dimension of both $\mathrm{Ext}^1(I_Z(\sigma), \mathcal{O}_S)$ and $H^0(S, \mathcal{O}_Z)$ is two, the injective morphism i turns out to be an isomorphism. A local extension of $\mathcal{O}_S$ by $I_Z(\sigma)$ is not locally free if and only if it splits. This is the case if its image in $H^0(S, \mathcal{O}_Z) \simeq \mathbb{C}^2$ is of the form $(k, 0)$ or $(0, k)$, that is for the stalk at the two points of Z. We then have exactly two split sequences and then two non locally free extensions.

If Z is a point with a tangent vector along σ, then we have exactly one non locally free extension in the fiber G_Z. Indeed, in this case $H^0(S, \mathcal{O}_Z)$ is $\mathbb{C}[\epsilon]$ and then there is only one nonsplit and hence non locally free extension, by the same arguments as before.

We can then make the structure of $\tilde{\mathcal{M}}_2$ clearer: we have $\tilde{G}$, the exceptional divisor of the blow-up. Let us call $\tilde{D}_\sigma$ (resp. $\tilde{D}$) the strict transform of D_σ (resp. of D). Since local freeness is an open condition, there is an open irreducible subscheme consisting all pairs (V, α) with V locally free. Then $\tilde{D}_\sigma$ turns out to be the closed complement of this open irreducible subscheme. Then $\tilde{D}_\sigma$ intersects a fiber $\tilde{G}_Z$ exactly in the points described above.

Finally, the exceptional divisor $\tilde{G}$ is a $\mathbb{P}^1$-fibration over $\mathrm{Sym}^2\sigma$. A dense open subset of any of such fiber is contained in the open subset of locally free extensions, the complement is given by the two non locally free extensions contained in $\tilde{D}_\sigma$.

Moreover, we can say for which Z in $\mathrm{Hilb}^2(S)$ an extension like (13) is actually stable and which is the maximal destabilizing subline bundle if not. This allows us to say whether an extension like (13) appears in a pair belonging to $\mathcal{M}_i$ for $i \leq 2$.

LEMMA 2.29. *Let (V, α) be a pair in $\tilde{\mathcal{M}}_2$. Then V is not stable if and only if either $(V, \alpha) \in \tilde{G}$ or $(V, \alpha) \in \tilde{D}$ or $(V, \alpha) \in \tilde{D}_\sigma$. If (V, α) belongs to $\tilde{D}$ or to $\tilde{D}_\sigma$, then it belongs to $\mathcal{M}_i$ if and only if $i = 2$. If (V, α) belongs to $\tilde{G} \setminus (\tilde{D} \cup \tilde{D}_\sigma)$, then it belongs to $\mathcal{M}_i$ if and only if $i \geq 1$.*

Proof. Proposition 2.24 tells us whether the sheaf V is stable and which is the maximal destabilizing subline bundle if not (see also [**Fri95**], III, Lemma 4.11). We know that the maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma - 2f)$ for Z in D and $\mathcal{O}_S(\sigma - 3f)$ if V is a locally free extension and $Z \in \mathrm{Sym}^2\sigma$. Recall moreover that D does not intersect $\mathrm{Sym}^2\sigma$. Then pairs in $\tilde{D}$ do not belong neither to $\mathcal{M}_1$ nor to $\mathcal{M}_0$, and pairs in $\tilde{G} \setminus \tilde{D}_\sigma$ belong to $\mathcal{M}_1$, but not to $\mathcal{M}_0$.

If (V, α) is in $\tilde{D}_\sigma$, we have two possibilities. If V is a non locally free extension corresponding to a Z in D_σ, Z has not the Cayley Bacharach property with respect to $\mathcal{O}_S(\sigma)$. If $Z = \{p, q\}$, where $p = \sigma \cap Z$, then q does not lie on σ. This, combined with a Chern class argument, gives rise to the exact sequence

$$0 \longrightarrow \mathcal{O}_S(\sigma - 2f) \otimes \mathfrak{m}_q \longrightarrow V \longrightarrow \mathcal{O}_S \longrightarrow 0. \tag{15}$$

The other possibility is when the pair (V, α) lies in $\tilde{D}_\sigma \cap \tilde{G}$. In this case, for any Z in $\mathrm{Sym}^2\sigma$, there are two points of the fiber $\tilde{G}_Z$ contained in $\tilde{D}_\sigma$ and for each of them we can repeat the same argument as before, since the choice of the non locally free sheaf in the fiber $\tilde{G}_Z$ corresponds to the choice of one of the two points of Z. Then for (V, α) such a pair, V fits the sequence (15) as well. This allows us to conclude that pairs in $\tilde{D}_\sigma$ do not belong to $\mathcal{M}_1$, since in any case $\mathcal{O}_S(\sigma - 2f)$ is a subline bundle. □

5.2. The structure of $\mathcal{M}_0$. The moduli space $\mathcal{M}_0$ fibers over $M(\sigma - 2f, 1)$ with fibers given by $\mathbb{P}\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$. Indeed, if (V, α) is a pair in $\mathcal{M}_0$, then V is stable as a sheaf on S with respect to the fixed suitable polarization, and any $\alpha : V \to \mathcal{O}_S(\sigma - f)$ fits the pair.

If V is a stable sheaf in $M(\sigma - 2f, 1)$, then Proposition 2.21 gives us four possible types of extensions for V. In the generic case it exists a length 2 codimension 2 l.c.i. subscheme $Z \subset S$ and V is given by

$$\text{(type 1)}\quad 0 \longrightarrow \mathcal{O}_S(-f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - f) \otimes I_Z \longrightarrow 0.$$

Other possibilities are

$$\text{(type 2)}\quad 0 \longrightarrow \mathcal{O}_S \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - 2f) \otimes \mathfrak{m}_q \longrightarrow 0,$$

$$\text{(type 3)}\quad 0 \longrightarrow \mathcal{O}_S(f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - 3f) \longrightarrow 0,$$

$$\text{(type 4)}\quad 0 \longrightarrow \mathfrak{m}_q \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - 2f) \longrightarrow 0.$$

In the last case, V is not locally free. For any of these extension types, we can study the dimension of the extension space, the stability of V and give its maximal destabilizing bundle if it exists. In order to keep a clear exposition, we are stating and proving such results in Section 6.

In what follows, we will prove that passing from $\mathcal{M}_0$ to $\mathcal{M}_1$ we lose all type 3 extensions and passing from $\mathcal{M}_1$ to $\mathcal{M}_2$ we lose all type 2 and type 4 extensions.

LEMMA 2.30. *Let V be a stable bundle given by a type 3 extension. Then the dimension of* $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ *is 3 and all pairs (V, α) with such a V do not belong neither to $\mathcal{M}_1$ nor to $\mathcal{M}_2$.*

PROOF. Let us calculate the dimension of $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$. Applying the functor $\mathrm{Hom}(-, \mathcal{O}_S(\sigma - f))$ to the exact sequence (type 3), we get the following long exact sequence

$$0 \to H^0(S, \mathcal{O}_S(2f)) \to \mathrm{Hom}(V, \mathcal{O}_S(\sigma - f)) \to H^0(S, \mathcal{O}_S(\sigma - 2f)) = 0.$$

Since $\pi_*(\mathcal{O}_S(2f)) = \mathcal{O}_{\mathbb{P}^1}(2)$, we can conclude that the dimension of $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ is $h^0(\mathcal{O}_{\mathbb{P}^1}(2)) = 3$.

This result tells us that any map $\alpha : V \to \mathcal{O}_S(\sigma - f)$ factors through the map $V \to \mathcal{O}_S(\sigma - 3f)$ given in the (type 3) sequence. Indeed, $\mathrm{Hom}(\mathcal{O}_S(\sigma - 3f), \mathcal{O}_S(\sigma - f)) = H^0(S, \mathcal{O}_S(2f))$. We can then say that all pairs (V, α) with V given by a type 3 extension have $\ker(\alpha) = \mathcal{O}_S(f)$ and then they do not belong to $\mathcal{M}_i$ for $i \geq 1$. $\square$

Let us call Σ^2 the subscheme of $\mathcal{M}_0$ of pairs (V, α) whose sheaf component V is a type 3 stable extension. The scheme Σ^2 is then a $\mathbb{P}^2$-fibration over a subscheme of $M(\sigma - 2f, 1)$ isomorphic to $\mathrm{Sym}^2\sigma$ by Lemma 2.41.

LEMMA 2.31. *Let V be given by a type 2 extension. Then the dimension of* $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ *is 2 and all pairs (V, α) with such a V belong to $\mathcal{M}_1$ but not to $\mathcal{M}_2$.*

PROOF. Let us calculate the dimension of $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$. Applying the functor $\mathrm{Hom}(-, \mathcal{O}_S(\sigma - f))$ to the exact sequence (type 2), we get the following long exact sequence

$$0 \to \mathrm{Hom}(\mathfrak{m}_q, \mathcal{O}_S(f)) \to \mathrm{Hom}(V, \mathcal{O}_S(\sigma - f)) \to 0$$

We then have $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f)) \simeq \mathrm{Hom}(\mathfrak{m}_q, \mathcal{O}_S(f))$. In order to study this vector space, consider the exact sequence

$$0 \longrightarrow \mathfrak{m}_q \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_q \longrightarrow 0$$

and apply the functor $\mathrm{Hom}(-, \mathcal{O}_S(f))$. This gives us

$$0 \to H^0(S, \mathcal{O}_S(f)) \to \mathrm{Hom}(\mathfrak{m}_q, \mathcal{O}_S(f)) \to 0,$$

since $\mathrm{Hom}(\mathcal{O}_q, \mathcal{O}_S(f)) = \mathrm{Ext}^1(\mathcal{O}_q, \mathcal{O}_S(f)) = 0$. Since $\pi_*(\mathcal{O}_S(f)) = \mathcal{O}_{\mathbb{P}^1}(1)$, we can conclude that the dimension of $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ is $h^0(\mathcal{O}_{\mathbb{P}^1}(1)) = 2$.

This result tells us that any map $\alpha : V \to \mathcal{O}_S(\sigma - f)$ factors through the map $V \to \mathcal{O}_S(\sigma - 2f)$ given in the (type 2) sequence. Indeed, $\mathrm{Hom}(\mathcal{O}_S(\sigma - 2f), \mathcal{O}_S(\sigma - f)) = H^0(S, \mathcal{O}_S(f))$. Moreover, Proposition 2.23 tells us that if V is unstable, then the maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma - 3f)$. We can then say that all pairs (V, α) with V given by a type 2 have $\ker(\alpha) = \mathcal{O}_S$ and then they do not belong to $\mathcal{M}_i$ for $i \geq 2$. $\square$

Lemma 2.32. *Let V be given by a type 4 extension. Then the dimension of* $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ *is 2 and pairs (V, α) with such a V belong to $\mathcal{M}_1$ and $\mathcal{M}_0$ if and only if V is stable, and they never belong to $\mathcal{M}_2$.*

Proof. The dimension calculation is the same performed in Lemma 2.31. If V is a type 4 extension, this tells also that the kernel of any α in a pair (V, α) is $\mathcal{O}_S$, and then condition (i) is satisfied for $\delta < 2$. If V is unstable the maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma - 2f)$ by Lemma 2.40 and then no pair (V, α) with V unstable belong to $\mathcal{M}_i$ for any i. If V is stable, then any pair (V, α) belongs to $\mathcal{M}_i$ for $i = 0, 1$. $\square$

The last problem to tackle in order to detail the structure of $\mathcal{M}_0$ is the dimension of $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ for V a (stable) type 1 extension.

Lemma 2.33. *Let V be a stable type 1 extension, then* $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ *has dimension one. Moreover, if we let V be any type 1 extension, the dimension jumps if and only if Z lies in* $\mathrm{Sym}^2\sigma$.

Proof. Recall that $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f)) \simeq V^\vee(\sigma - f)$. Since $\det(V) = \sigma - 2f$, we have $V^\vee \simeq V(-\sigma + 2f)$ and then $V^\vee(\sigma - f) \simeq V(f)$.

Dualizing the exact sequence (type 1) and tensoring with $\mathcal{O}_S(\sigma - f)$, we get the following long exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_S \to V^\vee(\sigma - f) \to \mathcal{O}_S(\sigma) \xrightarrow{\gamma} \\ \xrightarrow{\gamma} \underline{\mathrm{Ext}}^1(I_Z, \mathcal{O}_S) \to 0, \end{aligned} \tag{16}$$

since V is locally free. Lemma 2.38 together with Serre duality, tells us that $h^0(\mathcal{O}_S(\sigma)) = h^2(\mathcal{O}_S(-\sigma)) = 1$.

We have $\mathrm{Ext}^1(I_Z, \mathcal{O}_S) = H^1(S, I_Z)^\vee$. Then the long cohomology sequence associated to

$$0 \longrightarrow I_Z \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

gives us $h^1(I_Z) = 1$.

Then if we come back to the exact sequence (16) the vector space $H^0(S, V^\vee(\sigma - f))$ has dimension one if and only if the map γ is an isomorphism and this is true if and only if γ is not the zero map.

To see this, recall $V^\vee \simeq V(-\sigma + 2f)$ and then we get the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_S \arrow[r] \arrow[d, equal] & V^\vee(\sigma - f) \arrow[r] \arrow[d, equal] & \mathcal{O}_S(\sigma) \arrow[r, "\gamma"] & \underline{\mathrm{Ext}}^1(I_Z, \mathcal{O}_S) \\ 0 \arrow[r] & \mathcal{O}_S \arrow[r] & V(f) \arrow[r] & I_Z(\sigma) \arrow[r] \arrow[u, hook] & 0. \end{tikzcd}$$

Then the map γ is the zero map if and only if Z is contained in σ, that is the case if and only if Z is in $\mathrm{Sym}^2(\sigma)$. $\square$

These Lemmas allow us to detail the structure of the projective variety $\mathcal{M}_0$. Indeed, there is a natural fibration $\mathcal{M}_0 \to M(\sigma - 2f, 1)$ with projective fibers given by $\mathbb{P}\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ over any V in $M(\sigma - 2f, 1)$. This fibration is then an isomorphism over the dense open subset of $M(\sigma - 2f, 1)$ consisting of sheaves which are stable type 1 extensions, the fiber is $\mathbb{P}^1$ over stable type 2 and stable type 4 extensions, and it is $\mathbb{P}^2$ over stable type 3 extensions.

5.3. Birational transformations between the $\mathcal{M}_i$. Let us detail how we can perform birational transformations between the moduli spaces $\mathcal{M}_i$. The idea is to consider the universal pair $(\mathcal{V}, A)$ on $S \times \tilde{\mathcal{M}}_2$ and to perform elementary transformations along the divisors of $\tilde{\mathcal{M}}_2$ for which the pair (V, α) does not belong to $\mathcal{M}_1$, to get a universal pair $(\mathcal{U}, B)$ and then to perform again an elementary transformation to finally get a universal pair $(\mathcal{W}, \Gamma)$ on $S \times \mathcal{M}_0$. We have there to be careful, since Theorem 2.16, which is needed to assure the good properties of the elementary transformation of the family, requires the smoothness of the varieties involved. We will then perform such a transformation only over $\tilde{\mathcal{M}}_2$, which is smooth by Lemma 2.28. This will give first a birational map $\phi_1 : \tilde{\mathcal{M}}_2 \dashrightarrow \mathcal{M}_1$ and then a birational map $\phi_0 : \tilde{\mathcal{M}}_2 \dashrightarrow \mathcal{M}_0$. Indeed, at every step, the sheaf given by the elementary transformation is unique but this is not the case for the corresponding framing map. However, ϕ_0 induces a bijection between $\mathrm{Hilb}^2(S)$ and $M(\sigma - 2f, 1)$, giving rise to the isomorphism of Theorem 4.9 of [**Fri95**], part III.

First recall that in part III of [**Fri95**], Friedman constructed a universal sheaf $\mathcal{V}'$ over $S \times (\mathrm{Hilb}^2(S) \setminus D_\sigma)$ which gives the stable type 1 extension for Z not in D and a stable uniquely determined type 4 extension for Z in D. We briefly recall this construction in Lemma 2.42, performing it on the whole $S \times \tilde{\mathcal{M}}_2$. Then we can start from such a construction to get a birational map between $\tilde{\mathcal{M}}_2$ and $\mathcal{M}_1$ and detail its properties.

Theorem 2.34. *There is a birational map $\phi_1 : \tilde{\mathcal{M}}_2 \dashrightarrow \mathcal{M}_1$. The map ϕ_1 is an isomorphism on the open complement of $\tilde{D} \cup \tilde{D}_\sigma$.*

Proof. Let us consider the universal pair $(\mathcal{V}, A)$ on $S \times \mathcal{M}_2$ restricted to $\tilde{\mathcal{M}}_2$, and consider just the sheaf datum, namely $\mathcal{V}$. Recall that if (V, α) lies in $\tilde{D}$, then it does not belong to $\mathcal{M}_1$. We start by elementarily transform the universal sheaf $\mathcal{V}$ along this divisor. This construction has already been performed by Friedman and it is straightforward generalized to $\mathcal{V}$, see Lemma 2.42, to get a flat reflexive sheaf $\mathcal{V}'$ over $S \times \tilde{\mathcal{M}}_2$ such that

- if (V, α) belongs to $\tilde{D}$, then $\mathcal{V}'_{(V,\alpha)}$ is the unique type 4 extension as described by Friedman,
- otherwise $\mathcal{V}'_{(V,\alpha)}$ is V.

Using Theorem 2.16, Friedman can show that for (V, α) in $\tilde{D} \cap \tilde{D}_\sigma$ the unique type 4 extension given by $\mathcal{V}'$ is not stable, since its double dual splits. Recall Theorem 2.32, which tells that pairs with an unstable type 4 sheaf component do not belong to $\mathcal{M}_1$. Moreover, no pair (V, α) in $\tilde{D}_\sigma \setminus \tilde{D}$ belongs to $\mathcal{M}_1$. We then have to perform an elementary transformation of $\mathcal{V}'$ over $\tilde{D}_\sigma$ to get a universal sheaf for $\mathcal{M}_1$.

Let (V, α) be a pair in $\tilde{D}_\sigma$ and let $V' := \mathcal{V}'_{(V,\alpha)}$. We have two possibilities for V'.

In the case (V, α) does not lie on $\tilde{D}$, then $V' = V$ and in any case, as we have seen in the proof of Lemma 2.29, V' fits the sequence

$$0 \longrightarrow \mathcal{O}_S(\sigma - 2f) \otimes \mathfrak{m}_q \longrightarrow V' \longrightarrow \mathcal{O}_S \longrightarrow 0, \tag{17}$$

with q suitable and unique.

In the other case, (V, α) belongs to $\tilde{D} \cap \tilde{D}_\sigma$ and then V' is an unstable type 4 extension, which means that its double dual $V'^{\vee\vee}$ is the split extension

$$0 \longrightarrow \mathcal{O}_S \longrightarrow V'^{\vee\vee} \longrightarrow \mathcal{O}_S(\sigma - 2f) \longrightarrow 0.$$

This implies that the maximal destabilizing subline bundle of V' is $\mathcal{O}_S(\sigma - 2f)$ and then that V' fits the sequence (17) by a Chern class argument.

In order to perform an elementary transformation along $\tilde{D}_\sigma$, let us observe that since $\dim\mathrm{Hom}(\mathcal{O}_S(\sigma - 2f) \otimes \mathfrak{m}_q, V') = 1$ for any such V', we have a line bundle $\mathcal{L}_2$ on S and a surjective morphism

$$\mathcal{V}'_{|S \times \tilde{D}_\sigma} \to \pi_1^* \mathcal{O}_S \otimes \pi_2^* \mathcal{L}_2 \to 0,$$

over $S \times \tilde{D}_\sigma$. We can then perform the elementary transformation

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V}' \longrightarrow i_*(\pi_1^* \mathcal{O}_S \otimes \pi_2^* \mathcal{L}_2) \longrightarrow 0,$$

where i is the embedding of $S \times \tilde{D}_\sigma$ in $S \times \tilde{\mathcal{M}}_2$. By Theorem 2.16, $\mathcal{U}$ is flat and reflexive and, if (V, α) belongs to $\tilde{D}_\sigma$, then $U := \mathcal{U}_{(V,\alpha)}$ is a type 2 extension. By Lemma 2.39, U is unstable if and only if q lies in σ. This is the case for $U = \mathcal{U}_{(V,\alpha)}$ when (V, α) belongs to $\tilde{D}_\sigma \cap \tilde{G}$. In this case, the maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma - 3f)$ by Proposition 2.23, and then any pair (U, β) with such a U belongs to $\mathcal{M}_1$.

We finally get a universal sheaf $\mathcal{U}$ over $S \times \tilde{\mathcal{M}}_2$, such that

- if (V, α) belongs to $\tilde{D} \setminus \tilde{D}_\sigma$, then $\mathcal{U}_{(V,\alpha)}$ is the unique type 4 stable extension as described by Friedman,
- if (V, α) belongs to $\tilde{D}_\sigma$, then $\mathcal{U}_{(V,\alpha)}$ is a type 2 extension whose sheaf component is unstable if and only if (V, α) belongs to $\tilde{D}_\sigma \cap \tilde{G}$,

- otherwise $\mathcal{U}_{(V,\alpha)}$ is V.

In any case, $\mathcal{U}_{(V,\alpha)}$ will belong to a pair in $\mathcal{M}_1$. If (V,α) is a pair in $\tilde{D}_\sigma \cup \tilde{D}$, then the sheaf $U := \mathcal{U}_{(V,\alpha)}$ is uniquely determined, but the choice of the map β to give a pair (U,β) is not unique. Indeed, since we have a unique map $\beta' : U \to \mathcal{O}_S(\sigma - 2f)$, all maps $\mathcal{O}_S(\sigma - 2f) \to \mathcal{O}_S(\sigma - f)$ give rise by composition to a map β as required.We then have a $\mathbb{P}^1$ parametrizing the possible choices of the map for the elementary transformation of any (V,α) in $\tilde{D}_\sigma \cup \tilde{D}$.

In order to define the birational morphism $\phi_1 : \tilde{\mathcal{M}}_2 \dashrightarrow \mathcal{M}_1$ we have to check that for any pair (V,α) not in $\tilde{D} \cup \tilde{D}_\sigma$, there exist a unique pair (U,β) in $\mathcal{M}_1$ such that $\mathcal{U}_{(V,\alpha)} = U$. It is clear that ϕ_1 is defined on such pairs, and it is actually the identity. □

Anyway, it is important to remark that the indeterminacy of ϕ_1 is just given by the choice of the framing map. If U is a type 2 extension, then any pair (U,β) belongs to $\mathcal{M}_1$. Indeed in this case, if U is unstable, the maximally destabilizing subline bundle is $\mathcal{O}_S(\sigma - 3f)$. If we have a type 2 extension given by a q not in σ, then it arises as the elementary transformation of a type 1 extension with Z in D_σ of the form $Z = \{q,p\}$ with p in σ. Moreover, if we fix the point q, the set of such extensions is parametrized by a $\mathbb{P}^1 \simeq \sigma$, by Lemma 2.39. Then take U to be the elementary transformation of a type 1 (possibly through a type 4) extension associated to $Z = \{q,p\}$ with p in σ. Then there is a bijection between the pairs on $\tilde{D}_\sigma$ and type 2 extensions fitting a pair in $\mathcal{M}_1$. If U is a type 4 stable extension, it is clear from the construction by Friedman recalled in Lemma 2.42, that there is a unique type 1 extension whose elementary transformation is U. Then there is a bijection between the pairs on $\tilde{D} \setminus \tilde{D}_\sigma$ and type 4 extensions fitting a pair in $\mathcal{M}_1$.

THEOREM 2.35. *There is a birational map $\phi_0 : \tilde{\mathcal{M}}_2 \dashrightarrow \mathcal{M}_0$. The map ϕ_0 is an isomorphism over the open complement of $\tilde{D} \cup \tilde{D}_\sigma \cup \tilde{G}$.*

PROOF. Let us consider the universal sheaf $\mathcal{U}$ over $S \times \tilde{\mathcal{M}}_2$ constructed in the proof of Theorem 2.34. Remark that $\mathcal{U}_{(V,\alpha)}$ is not stable if and only if (V,α) lies in $\tilde{G}$. Indeed, this is clear for (V,α) in $\tilde{G} \setminus (\tilde{D}_\sigma \cup \tilde{D})$, since in this case $\mathcal{U}_{(V,\alpha)}$ is just V. If (V,α) lies in $\tilde{G} \cap (\tilde{D}_\sigma \cup \tilde{D})$, then we have seen in the proof of Theorem 2.34 that $\mathcal{U}_{(V,\alpha)}$ is an unstable type 2 extension. In any case, the maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma - 3f)$ and then $U := \mathcal{U}_{(V,\alpha)}$ fits the exact sequence

$$0 \longrightarrow \mathcal{O}_S(\sigma - 3f) \longrightarrow U \longrightarrow \mathcal{O}_S(f) \longrightarrow 0.$$

We can then perform, as done over $\tilde{D}$ and $\tilde{D}_\sigma$, an elementary transformation of $\mathcal{U}$ along $\tilde{G}$ to get a sheaf $\mathcal{W}$ flat and reflexive. Repeating the same arguments as in Theorem 2.34, we can say that if (V,α) is in $\tilde{G}$, then $\mathcal{W}_{(V,\alpha)}$ is a stable type 3 extension. To summarize, we get a flat reflexive sheaf $\mathcal{W}$ over $S \times \tilde{\mathcal{M}}_2$ such that

- If (V, α) belongs to $\tilde{G}$, then $\mathcal{W}_{(V,\alpha)}$ is a stable type 3 extension,
- If (V, α) belongs to $\tilde{D}_\sigma \setminus \tilde{G}$, then $\mathcal{W}_{(V,\alpha)}$ is a stable type 2 extension,
- If (V, α) belongs to $\tilde{D} \setminus (\tilde{G} \cup \tilde{D}_\sigma)$, then $\mathcal{W}_{(V,\alpha)}$ is a stable type 4 extension,
- otherwise $\mathcal{W}_{(V,\alpha)}$ is V, which in this case is always a stable type 1 extension.

If (V, α) belongs to $\tilde{G}$, then the elementary transformation we performed gives us a sheaf $W := \mathcal{W}_{(V,\alpha)}$ which is a type 3 extension. To give a pair (W, β) in $\mathcal{M}_0$, the choice of the map β is not unique. Indeed, since we have a unique map $\beta' : W \to \mathcal{O}_S(\sigma - 3f)$, all maps $\mathcal{O}_S(\sigma - 3f) \to \mathcal{O}_S(\sigma - f)$ give rise by composition to such a β. We then have a $\mathbb{P}^2$ parametrizing the possible choices of the map for the elementary transformation of any such (V, α). □

Remark now that (V, α) is in $\mathcal{M}_0$ if and only if V is stable and elementary transformations have given all such vector bundles. Moreover, the indeterminacy of ϕ_0 is just given by the choice of the framing map on $(\tilde{D} \cup \tilde{D}_\sigma) \setminus \tilde{G}$. Nevertheless, if we consider (V_1, α_1) and (V_2, α_2) in the same fiber $\tilde{G}_Z$, we have $\mathcal{W}_{(V_1,\alpha_1)} = \mathcal{W}_{(V_2,\alpha_2)}$, then we do not have any more a bijection between the set of pairs of $\tilde{\mathcal{M}}_2$ and the set of sheaves fitting a pair in $\mathcal{M}_0$, that is the underling set of the moduli space $M(\sigma - 2f, 1)$.

Let us consider the moduli space $M(\sigma - 2f, 1)$ and its subscheme Σ of type 3 stable extensions. Lemma 2.41 shows that Σ is isomorphic to $\mathrm{Sym}^2\sigma$. Consider $\tilde{M}$, the blow-up of $M(\sigma - 2f, 1)$ along Σ.

COROLLARY 2.36. *There is an isomorphism* $\tilde{M} \simeq \tilde{\mathcal{M}}_2$.

PROOF. The birational map ϕ_0 induces a birational map $\phi : \tilde{\mathcal{M}}_2 \dashrightarrow M(\sigma - 2f, 1)$, which is not defined over Σ. This obtained indeed by ϕ_0 forgetting the framing data. If we blow up the indeterminacy locus, the sheaf $\mathcal{W}$ constructed in the proof of Theorem 2.35 allows us to define a morphism $\tilde{\phi} : \tilde{\mathcal{M}}_2 \to \tilde{M}$ which is a bijection. Remark that $\tilde{M}$ and $\tilde{\mathcal{M}}_2$ are smooth, then $\tilde{\phi}$ is an isomorphism between $\tilde{M}$ and $\tilde{\mathcal{M}}_2$. □

COROLLARY 2.37. *There is an isomorphism* $\mathrm{Hilb}^2(S) \simeq M(\sigma - 2f, 1)$

PROOF. This is a straightforward consequence of Corollary 2.36. Indeed we have an isomorphism between $\mathrm{Hilb}^2(S) \setminus \mathrm{Sym}^2\sigma$ and $M(\sigma - 2f, 1) \setminus \Sigma$, a subscheme isomorphic to $\mathrm{Sym}^2\sigma$. Moreover, if we take a point Z in Sym^2, the fiber $\tilde{G}_Z$ over it corresponds under the isomorphism $\tilde{\phi}$ to a fiber over a point of Σ. We then have a bijection between $\mathrm{Hilb}^2(S)$ and $M(\sigma - 2f, 1)$ which is a birational map and then an isomorphism. $\square$

6. Some dimension calculations

In this Section we show how to get some result we used previously and we recall some result taken from part III of [**Fri95**] which we frequently use. We are focusing firstly on the dimension of the extension spaces parametrizing type 2, 3 and 4 extensions and on stability of such sheaves. But let us first recall a Lemma by Friedman.

LEMMA 2.38. *We have the following results.*

(i) *For all integers a, $h^0(-\sigma + af) = 0$.*

(ii) *For all integers a,*

$$h^1(-\sigma + (2+a)f) = \begin{cases} 0, & a < 0 \\ a+1 & a \geq 0. \end{cases}$$

(iii) *For all integers a,*

$$h^2(-\sigma + (2-a)f) = \begin{cases} a-1, & a \geq 2 \\ 0 & a \leq 1. \end{cases}$$

PROOF. This is Lemma 4.1 of part III in [**Fri95**] in the K3 case. $\square$

LEMMA 2.39. **(Type 2 extensions).** *Let q be any point of S, then the dimension of $\mathrm{Ext}^1(\mathcal{O}_S(\sigma - 2f) \otimes \mathfrak{m}_q, \mathcal{O}_S)$ is two. Then, if we fix any point q, type 2 extensions are parametrized by a $\mathbb{P}^1$. Such extensions are unstable if and only if q lies on σ. Moreover, if V is an unstable type 2 extension, then the maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma - 3f)$.*

PROOF. Consider V a type 2 extension

$$0 \to \mathcal{O}_S \to V \to \mathcal{O}_S(\sigma - 2f) \otimes \mathfrak{m}_q \to 0.$$

The set of such extensions is parametrized by $\mathrm{Ext}^1(\mathcal{O}_S(\sigma - 2f) \otimes \mathfrak{m}_q, \mathcal{O}_S) \cong \mathrm{Ext}^1(\mathfrak{m}_q, \mathcal{O}_S(-\sigma + 2f))$. In order to study this vector space, consider the exact sequence

$$0 \to \mathfrak{m}_q \to \mathcal{O}_S \to \mathcal{O}_q \to 0$$

and apply to it the functor $\mathrm{Hom}(-, \mathcal{O}_S(-\sigma + 2f))$ to get the following exact sequence

$$0 \to H^1(S, \mathcal{O}_S(-\sigma + 2f)) \to \mathrm{Ext}^1(\mathfrak{m}_q, \mathcal{O}_S(-\sigma + 2f)) \to$$

$$\to \mathrm{Ext}^2(\mathcal{O}_q, \mathcal{O}_S(-\sigma + 2f)) \to 0.$$

Indeed $\mathrm{Ext}^1(\mathcal{O}_q, \mathcal{O}_S(-\sigma + 2f)) = 0$ and $\mathrm{Ext}^2(\mathcal{O}_S, \mathcal{O}_S(-\sigma + 2f)) = H^2(S, \mathcal{O}_S(-\sigma + 2f)) = 0$ by Lemma 2.38.

The same Theorem gives us $h^1(S, \mathcal{O}_S(-\sigma + 2f)) = 1$. The proof is completed once we observed that $\mathrm{Ext}^2(\mathcal{O}_q, \mathcal{O}_S(-\sigma + 2f)) = 1$.

In order to show that a type 2 extension is unstable if and only if q belongs to σ, we follow the argument following the proof of Proposition 4.6 in part III of [**Fri95**]. Indeed if V has a maximal destabilizing subline bundle, it has to be of the form $\mathcal{O}_S(\sigma - 3f)$ by Proposition 2.23. Applying $\underline{\mathrm{Hom}}(\mathcal{O}_S(\sigma - 3f), -)$ to the type 2 exact sequence, we get the exact sequence

$$(18)\quad 0 \to \mathcal{O}_S(-\sigma + 3f) \to \underline{\mathrm{Hom}}(\mathcal{O}_S(\sigma - 3f), V) \to \mathcal{O}_S(f) \otimes \mathfrak{m}_q \to 0,$$

and V is unstable if and only if a nonzero section of $\mathcal{O}_S(f) \otimes \mathfrak{m}_q$ lifts to a morphism $\mathcal{O}_S(\sigma - 3f) \to V$. Such a nonzero section defines an exact sequence

$$(19)\qquad 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(f) \otimes \mathfrak{m}_q \longrightarrow \mathcal{O}_{f_q}(-q) \longrightarrow 0,$$

where f_q is the fiber containing the point q. Consider the coboundary map of (18)

$$H^0(\mathcal{O}_S(f) \otimes \mathfrak{m}_q) \to H^1(\mathcal{O}_S(-\sigma + 3f)),$$

which is given by taking the cup product of a nonzero section with the extension class ξ in $\mathrm{Ext}^1(\mathcal{O}_S(f) \otimes \mathfrak{m}_q, \mathcal{O}_S(-\sigma + 3f))$ corresponding to V. We can see that this is the same as taking the image of ξ under the map

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}_S(f) \otimes \mathfrak{m}_q, \mathcal{O}_S(-\sigma + 3f)) \quad &\to \mathrm{Ext}^1(\mathcal{O}_S, \mathcal{O}_S(-\sigma + 3f)) \\ &= \\ &H^1(S, \mathcal{O}_S(-\sigma + 3f)). \end{aligned}$$

Consider the long exact sequence obtained applying $\mathrm{Hom}(-, \mathcal{O}(-\sigma + 3f))$ to (19) and recall that $H^0(-\sigma+3f) = 0$ by Lemma 2.38. We then get the following exact sequence

$$\begin{aligned} 0 \to \quad & \mathrm{Ext}^1(\mathcal{O}_{f_q}(-q), \mathcal{O}_S(-\sigma + 3f)) \to \\ \to \quad & \mathrm{Ext}^1(\mathcal{O}_S(f) \otimes \mathfrak{m}_q, \mathcal{O}_S(-\sigma + 3f)) \to \quad H^1(-\sigma + 3f). \end{aligned}$$

By Lemma 2.38 and by the previous discussion, the two latter spaces have both dimension 2. Then ξ is sent to 0 if and only if

$$\mathrm{Ext}^1(\mathcal{O}_{f_q}(-q), \mathcal{O}_S(-\sigma + 3f)) \neq 0.$$

Our aim is then to show that this space is non trivial if and only if q lies in σ. By Lemma VII, 1.27 of [**FM94**], we have

$$\mathrm{Ext}^1(\mathcal{O}_{f_q}(-q), \mathcal{O}_S(-\sigma + 3f)) = H^0(f_q, \mathcal{O}_{f_q}(q - p)),$$

where $p = \sigma \cap f_q$. The latter space is zero unless $q = p$, that is unless q lies on σ.

The calculation of the maximal destabilizing subline bundle is a straight application of Theorem 2.23.

$\square$

LEMMA 2.40. **(Type 4 extensions).** *Let q be any point of S, then the dimension of* $\mathrm{Ext}^1(\mathcal{O}_S(\sigma-2f),\mathfrak{m}_q)$ *is two. Then, if we fix any point q, type 4 extensions are parametrized by a $\mathbb{P}^1$. Such an extension is unstable if and only if its double dual split. Moreover, if V is an unstable type 4 extension, the maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma-2f)$.*

PROOF. The calculation of the dimension of the extension space is the same as in the proof of Lemma 2.39.

Let V be a type 4 extension. Then it is unstable if and only if its double dual splits. Indeed, the double dual $V^{\vee\vee}$ fits

$$o\longrightarrow \mathcal{O}_S\longrightarrow V^{\vee\vee}\longrightarrow \mathcal{O}_S(\sigma-2f)\longrightarrow 0$$

and then it is, up to a shift with a line bundle, the bundle V_0 which is stable if and only if it is nonsplit.

If V is an unstable type 4 extension, its maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma-2f)$ because of the splitting of the sequence. □

LEMMA 2.41. **(Type 3 extensions).** *The dimension of the extension space* $\mathrm{Ext}^1(\mathcal{O}_S(\sigma-3f),\mathcal{O}_S(f))$ *is three. Stable type 3 extensions are parametrized by* $\mathrm{Sym}^2\sigma$.

PROOF. This is Corollary 4.5 of [**Fri95**]. Indeed the first assertion is easily deduced by $\mathrm{Ext}^1(\mathcal{O}_S(\sigma-3f),\mathcal{O}_S(f))\cong H^1(S,\mathcal{O}_S(-\sigma+4f))$ and Lemma 2.38.

If such an extension is unstable, then Proposition 2.24 tells us that the maximal destabilizing subline bundle has to be $\mathcal{O}_S(\sigma-3f)$, which splits the sequence. Moreover $H^1(S,\mathcal{O}_S(-\sigma+4f))\cong H^0(R^1\pi_*\mathcal{O}_S(-\sigma+4f))=H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2))$. Now use the isomorphism $\mathbb{P}H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2))=\mathrm{Sym}^2\sigma$ given by associating to a section the two points where it vanishes. □

Let us now recall the construction of the universal sheaf we called $\mathcal{V}'$ in section 5.3.

LEMMA 2.42. *There is a sheaf $\mathcal{V}'$ reflexive and flat over $S\times\tilde{\mathcal{M}}_2$. The restriction of $\mathcal{V}'$ to a slice $S\times\{(V,\alpha)\}$ is V if (V,α) does not belong to $\tilde{D}$ and is a torsion free type 4 extension if (V,α) belong to $\tilde{D}$ which is stable unless (V,α) lies in $\tilde{D}\cap\tilde{D}_\sigma$.*

PROOF. This is fundamentally Proposition 4.12 of [**Fri95**], III. Let us briefly recall the construction of such a sheaf. Consider the universal pair $(\mathcal{V},A)$ and forget the map datum. If (V,α) belongs to $\tilde{D}$, then there is a unique point q such that $\mathcal{V}_{(V,\alpha)}$ maps surjectively to $\mathfrak{m}_q$. This is straightforward generalization of Friedman's construction to the pair case. Then we can define $\mathcal{V}'$ as the elementary transformation

$$0\to\mathcal{V}'\to\mathcal{V}\to i_*\pi_1^*\mathcal{O}_S\otimes\pi_2^*\mathcal{L}_2\otimes I_{\mathcal{Y}}\to 0,$$

where i is the inclusion of $S \times \tilde{D}$ in $S \times \tilde{\mathcal{M}}_2$, π_1 and π_2 the projection from $S \times \tilde{\mathcal{M}}_2$ on the first and the second factor respectively, $\mathcal{L}_2$ is a line bundle on $\tilde{\mathcal{M}}_2$ and $\mathcal{Y}$ is a smooth divisor isomorphic to $S \times \tilde{D}$. Then by Theorem 2.16 $\mathcal{V}'$ is flat and reflexive over $S \times \tilde{\mathcal{M}}_2$ and for each (V, α) in $\tilde{D}$ the restriction of $\mathcal{V}'$ is a type 4 extension

$$0 \to \mathfrak{m}_q \to \mathcal{V}'_{(V,\alpha)} \to \mathcal{O}_S(\sigma - 2f) \to 0.$$

Such an extension is unstable if and only if its double dual splits. Nontrivial calculations show that this is the case if and only if (V, α) belongs to $\tilde{D} \cap \tilde{D}_\sigma$ [**Fri95**] . $\square$

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Résumé. Dans la première partie de la thèse, on rappelle la définition et les propriétés des catégories dérivées bornées des faisceaux cohérents sur une variété lisse projective. On généralise ici un théorème d'Orlov en donnant une décomposition semiorthogonale de la catégorie dans le cas d'une variété de Brauer-Severi. On montre en plus comment, lorsque deux courbes lisses ont la même catégorie dérivée, l'isomorphisme induit est exactement celui qui nous est donné par le théorème de Torelli.

Dans la deuxième partie, on étudie les espaces de modules de couples sur une surface elliptique $K3$, en utilisant une construction de Friedman. Ceci permet de reécrire l'isomorphisme entre un espace de modules et un schéma de Hilbert en termes de transformations birationnelles entre les espaces des couples semistables.

Abstract. In the first part of the thesis, we recall the definition and the main properties of bounded derived categories of coherent sheaves on smooth projective varieties. We are generalizing a Theorem by Orlov by giving a semiorthogonal decomposition of the derived category of a Brauer-Severi scheme. We also show that, if two smooth projecticve curves have equivalent derived categories, the induced isomorphism between the curves is exactly the one induced by the Torelli Theorem.

In the second part, we study moduli spaces of pairs on elliptic $K3$ surfaces, using a construction by Friedman. This allows to describe the isomorphism between a moduli space and a Hilbert scheme by the composition of birational maps between the moduli spaces of semistable couples.