Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems

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Abstract

Let $X \to Y$ be a fibration whose fibers are complete intersections of r quadrics. We develop new categorical and algebraic tools—a theory of relative homological projective duality and the Morita invariance of the even Clifford algebra under quadric reduction by hyperbolic splitting—to study semiorthogonal decompositions of the bounded derived category $\mathsf{D}^{\mathsf{b}}(X)$. Together with results in the theory of quadratic forms, we apply these tools in the case where $r = 2$ and $X \to Y$ has relative dimension 1, 2, or 3, in which case the fibers are curves of genus one, Del Pezzo surfaces of degree 4, or Fano threefolds, respectively. In the latter two cases, if $Y = \mathbb{P}^1$ over an algebraically closed field of characteristic zero, we relate rationality questions to categorical representability of X.
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Résumé

Soit $X \to Y$ une fibration en intersections complètes de r quadriques. Nous étudions une décomposition semiorthogonale de la catégorie dérivée $\mathsf{D}^{\mathsf{b}}(X)$ grâce à des nouveaux outils catégoriques et algébriques : une théorie relative de la dualité homologique projective et l'invariance par équivalence de Morita de l'algèbre de Clifford paire sous un scindage hyperbolique. Avec des résultats sur les formes quadratiques, ceci nous permet d'étudier en détail les cas $r = 2$ et $X \to Y$ de dimension relative 1, 2, ou 3. Dans ces cas, les fibres sont respectivement des courbes de genre 1, des surfaces de Del Pezzo de degré 4, ou des variétés de Fano de dimension 3. Dans les deux derniers cas, si $Y = \mathbb{P}^1$ et si le corps de base est algébriquement clos de caractéristique zéro, les résultats developpés nous permettent de mettre en relation la rationalité de X et sa représentabilité catégorique.
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0. Introduction

One of the numerous applications of the study of triangulated categories in algebraic geometry is understanding how to extract, from the bounded derived category of coherent sheaves $\mathsf{D}^b(X)$, information about the birational geometry of a given smooth projective variety X.

Since the seminal work of Bondal and Orlov [21], it has become understood that such information should be encoded in semiorthogonal decompositions

$$\mathsf{D}^b(X) = \langle \mathsf{A}_1, \ldots, \mathsf{A}_n \rangle$$

by admissible triangulated subcategories: purely homological properties of the components of such a decomposition often reflect geometric properties of X. For example, if for each $i > 1$, the component A_i is "zero-dimensional" (i.e., equivalent to the bounded derived category of the base field), then A_1 should contain nontrivial information about the birational geometry of X. When X is a Fano threefold, many examples support this idea [12,13,21,59,60].

In particular, in the case that $X \to S$ is a conic bundle over a rational complex surface, a semiorthogonal decomposition by derived categories of points and smooth projective curves allows one to reconstruct the intermediate Jacobian $J(X)$ as the sum of the Jacobians of the curves. This can determine the rationality of X when S is minimal [13]. More generally, this works if X is a complex threefold with negative Kodaira dimension (e.g., a Fano threefold) whose codimension 2 cycles are universally described by a principally polarized abelian variety [14, Section 3.2]. In such cases, homological properties of semiorthogonal decompositions are related to classical notions of representability of cycles on X.

Attempting to trace the link between derived categories and algebraic cycles, the second and third named authors defined in [14] the notion of *categorical representability* in a given dimension m (or codimension $\dim(X) - m$) of a smooth projective variety X, by requiring the existence of a semiorthogonal decomposition whose components can be fully and faithfully embedded in derived categories of smooth projective varieties of dimension at most m. Categorical representability in dimension one is equivalent to the existence of a semiorthogonal decomposition by copies of the derived category of a point and derived categories of smooth projective curves. One might wonder if categorical representability in codimension 2 is a necessary condition for rationality. The work of Kuznetsov on cubic fourfolds [64], developed before the definition of categorical representability, shows how this philosophy persists as a tool to conjecturally understand rationality problems in dimension larger than three, where one cannot appeal to more classical methods, such as the study of the intermediate Jacobian.

In this paper, we provide two new instances where categorical representability is strictly related to birational properties. These arise as fibrations $X \to \mathbb{P}^1$ whose fibers are complete intersections of two quadrics. We impose a genericity hypothesis on such fibrations (see Definition 1.2.4) so that the associated pencil of quadrics has simple degeneration along a smooth divisor.

In Section 4, we consider fibrations $X \to \mathbb{P}^1$ whose fibers are Del Pezzo surfaces of degree four. Such threefolds have negative Kodaira dimension and their rationality (over the complex numbers) is completely classified [1,83]. We provide a purely categorical criterion for rationality of X based on [13].

Theorem 1. *(See Section 4.) Let $X \to \mathbb{P}^1$ be a generic Del Pezzo fibration of degree four over the complex numbers. Then X is rational if and only if it is categorically representable in codimension 2. Moreover, there is a semiorthogonal decomposition*

$$\mathsf{D}^b(X) = \langle \mathsf{D}^b(\Gamma_1), \ldots, \mathsf{D}^b(\Gamma_k), E_1, \ldots, E_l \rangle,$$

with Γ_i smooth projective curves and E_i exceptional objects if and only if $J(X) = \bigoplus J(\Gamma_i)$ as principally polarized abelian varieties.

In Section 5, we consider fibrations $X \to \mathbb{P}^1$ whose fibers are complete intersections of two four-dimensional quadrics. Such fourfolds have a semiorthogonal decomposition

$$\mathsf{D}^b(X) = \langle \mathsf{A}_X, E_1, \ldots, E_4 \rangle,$$

where E_i are exceptional objects. Moreover, we construct a fibration $T \to \mathbb{P}^1$ in hyperelliptic curves and a Brauer class $\beta \in \mathrm{Br}(T)$ such that $\mathsf{A}_X \simeq \mathsf{D}^b(T, \beta)$. We state a conjecture in the same spirit as Kuznetsov's conjecture for cubic fourfolds [64, Conj. 1.1].

Conjecture 1. *Let* $X \to \mathbb{P}^1$ *be a fibration whose fibers are intersections of two four-dimensional quadrics over the complex numbers. Then* X *is rational if and only if it is categorically representable in codimension* 2.

The main evidence for this conjecture is provided in two cases of rational fibrations that are categorically representable in codimension 2. Recall from [31, Thm. 2.2] that if X contains a surface generically ruled over $\mathbb{P}^1$, then X is rational.

Theorem 2. *(See Section 5.) Let* $X \to \mathbb{P}^1$ *be a generic fibration whose fibers are intersections of two four-dimensional quadrics over a field* k. *Let* $T \to \mathbb{P}^1$ *and* $\beta \in \mathrm{Br}(T)$ *be the associated fibration in hyperelliptic curves and Brauer class. If* $\beta = 0$, *then* X *is rational and categorically representable in codimension* 2. *In particular, this is the case if* X *contains a surface generically ruled over* $\mathbb{P}^1$.

These results are obtained from new general constructions involving the derived category of a smooth projective variety X admitting a fibration $X \to Y$ in complete intersection of quadrics. Over an algebraically closed field, the derived category of an intersection of two quadrics was studied by Kapranov [51] and Bondal and Orlov [21, Section 2]. Along with results in the algebraic theory of quadratic forms, we utilize three main tools: homological projective duality [62] extended to a relative context for quadric fibrations (see Section 2.3), Kuznetsov's [63] description of the derived category of a quadric fibration via the even Clifford algebra, and a Morita invariance result for the even Clifford algebra under "quadric reduction" by hyperbolic splitting (see Section 1.3). We now give an overview of these tools.

Let $Q \to S$ be a flat quadric fibration of relative dimension $n - 2$ over a scheme with associated even Clifford algebra $\mathscr{C}_0$. Kuznetsov [63], extending the seminal work of Kapranov [50], [52, Section 4] on derived categories of quadrics, exhibits a semiorthogonal decomposition

$$\mathsf{D}^b(Q) = \langle \mathsf{D}^b(S, \mathscr{C}_0), \mathsf{D}^b(S)_1, \ldots, \mathsf{D}^b(S)_{n-2} \rangle, \tag{0.0.1}$$

where $\mathsf{D}^b(S)_i \simeq \mathsf{D}^b(S)$ for all $1 \leqslant i \leqslant n - 2$. The category $\mathsf{D}^b(S, \mathscr{C}_0)$ is the component of $\mathsf{D}^b(Q)$ encoding nontrivial information about the quadric. One of the main results of this paper is the derived invariance of this category under the geometric process of "quadric reduction" by hyperbolic splitting, see Section 1.3. This allows us to pass to smaller-dimensional quadric fibrations.

"Quadric reduction" by hyperbolic splitting is a classical construction which, starting from a quadric $Q \subset \mathbb{P}^{n+1}$ with a smooth rational point x, describes a quadric $Q' \subset \mathbb{P}^{n-1}$ with the same degeneration type. Roughly, Q' is the base of the cone obtained by intersecting $T_x Q$ and Q. This is the analogue, in the language of quadratic forms, of "splitting off a hyperbolic plane". This construction can be performed relatively over a base scheme S, given quadric fibration $Q \to S$ with possibly singular fibers and a smooth section (i.e., a section $S \to Q$ avoiding singular points of fibers), see Section 1.4.

Theorem 3. *(See Corollary 1.8.8.) Let* S *be a regular integral scheme,* $Q \to S$ *a quadric fibration with simple degeneration along a regular divisor,* $Q' \to S$ *the quadric fibration obtained by quadric reduction along a smooth section, and* $\mathscr{C}_0$ *and* $\mathscr{C}_0'$ *the respective even Clifford algebras. If the relative dimension is odd, assume that* 2 *is invertible on* S. *Then there is an equivalence* $\mathsf{D}^b(S, \mathscr{C}_0) \simeq \mathsf{D}^b(S, \mathscr{C}_0')$.

This result is classical if S is the spectrum of a field and the quadric Q is smooth, see [33, Lemma 14.2]. Along the way to proving Theorem 3, we provide new generalizations, to a degenerate setting, of fundamental results in the theory of quadratic forms (see Section 1.3). These include an isotropic splitting principle (Theorem 1.3.8) and an orthogonal sum formula for the even Clifford algebra (Lemma 1.8.1). In general, algebraic results of this type are quite limited in the literature (see e.g., [10] and [56, IV.4.8]). Some stack theoretic considerations are required in the case of odd rank. In Appendices A and B, we prove the equality of differing constructions and interpretations of the even Clifford algebra in the literature.

Such results on degenerate forms may prove useful in their own right. There has been recent focus on such forms from various number theoretic perspectives. An approach to Bhargava's [18] construction of moduli spaces of "rings of low rank" over arbitrary base schemes is developed by Wood [87], who must deal with degenerate forms (of higher degree). In related developments, building on the work of Delone and Faddeev [32] over $\mathbb{Z}$ and Gross and

Lucianovic [38] over local rings, V. Balaji [85], and independently Voight [86], used Clifford algebras of degenerate quadratic forms of rank 3 to classify degenerations of quaternion algebras over arbitrary bases.

Homological projective duality was introduced by Kuznetsov [62] to study semiorthogonal decompositions of hyperplane sections (see also [61]). For example, consider a finite set of quadric hypersurfaces $\{Q_i\}_{i=0}^r$ in a projective space $\mathbb{P}^{n-1} = \mathbb{P}(V)$ and their complete intersection X. Let Q be the linear system of quadrics generated by the Q_i, which is a quadric fibration $Q \to \mathbb{P}^r$ of relative dimension $n-2$, and let $\mathscr{C}_0$ be its associated even Clifford algebra. The derived categories $\mathsf{D}^b(X)$ and $\mathsf{D}^b(\mathbb{P}^r, \mathscr{C}_0)$ are strongly related [63]. In particular, if X is Fano or Calabi–Yau, then there is a fully faithful functor $\mathsf{D}^b(\mathbb{P}^r, \mathscr{C}_0) \to \mathsf{D}^b(X)$. In Section 2, we describe a relative version of this construction, replacing V by a vector bundle E over a smooth variety Y, the quadrics by flat quadric fibrations $Q_i \to Y$ contained in $\mathbb{P}(E)$, the intersection by the relative complete intersection $X \to Y$ of the quadric fibrations, the linear system of quadrics by the *linear span quadric fibration* $Q \to S$ (which is a flat quadric fibration over a projective bundle $S \to Y$, see Definition 1.2.3), and $\mathscr{C}_0$ by the even Clifford algebra of $Q \to S$. Then $\mathsf{D}^b(X)$ and $\mathsf{D}^b(S, \mathscr{C}_0)$ are similarly related: if the generic fiber of $X \to Y$ is Fano or Calabi–Yau, then there is a fully faithful functor $\mathsf{D}^b(S, \mathscr{C}_0) \to \mathsf{D}^b(X)$.

Considering the relationship between intersections of quadrics and linear spans also has arithmetic roots. The Amer–Brumer theorem (Theorem 1.9.1), which is indispensable for our work, states that the intersection of two quadrics has a rational point if and only if the linear system of quadrics (or pencil of quadrics) has a rational section over $\mathbb{P}^1$. Versions of this theorem also hold for 0-cycles of degree 1 on intersections of more quadrics [30]. The general subject is also considered in [31].

Using the above described categorical and algebraic tools, we proceed in the relative setting described above. Assuming X is a smooth projective variety and the generic fiber of $X \to Y$ is Fano (resp. Calabi–Yau)—a condition that is satisfied when $2r + 2 < n$ (resp. $2r + 2 = n$)—we obtain a semiorthogonal decomposition

$$\mathsf{D}^b(X) = \langle \mathsf{D}^b(S, \mathscr{C}_0), \mathsf{D}^b(Y)_1, \ldots, \mathsf{D}^b(Y)_{n-2r} \rangle,$$

where $\mathsf{D}^b(Y)_i \simeq \mathsf{D}^b(Y)$ for $1 \leqslant i \leqslant n - 2r$ (resp. an equivalence $\mathsf{D}^b(S, \mathscr{C}_0) \simeq \mathsf{D}^b(X)$). The category $\mathsf{D}^b(S, \mathscr{C}_0)$ is then the important component A_X of $\mathsf{D}^b(X)$ remarked on earlier. Moreover, if $Q \to S$ admits a smooth section, we can more closely examine the relationship between $\mathsf{D}^b(X)$ and $\mathsf{D}^b(Q)$ by performing quadric reduction. In the cases we consider, such sections are guaranteed by the work of Alekseev [1] and by the theory of quadratic forms (see Lemma 1.9.3), or by the general results of [27] and [37] on fibrations over curves with rationally connected fibers.

This article is organized as follows. In Section 1, we adapt many standard results in the theory of regular quadratic forms and even Clifford algebras to possibly degenerate forms with values in line bundles, eventually leading to a proof of the Morita invariance of the even Clifford algebra under quadric reduction by hyperbolic splitting. In Section 2, we introduce the basic notions and relevant results in derived category theory and then detail the relative version of homological projective duality for quadric fibrations. Finally, we study the explicit case of varieties $X \to Y$ which are intersection of two quadric fibrations: in Section 3, we consider the case of genus 1 fibrations, where our results are analogous to (but different from) well-known ones; in Section 4, we consider the case of quartic Del Pezzo fibrations over curves; and in Section 5, we consider fourfolds $X \to \mathbb{P}^1$ fibered in complete intersections of two quadrics. The last two applications require working over an algebraically closed field (assumed to be of characteristic zero for certain statements). In Appendix A, we compare various constructions of the even Clifford algebra in the literature and recall some of its functorial properties. In Appendix B, we provide a proof that for quadric fibrations of relative dimension 2 and 4, the Brauer class of the even Clifford algebra over the discriminant cover coincides with that arising from the Stein factorization of the Lagrangian Grassmannian considered in [43] and [64].

Notations

In general, we work over an arbitrary base field k. As we indicate in Section 2.1, the basic results in derived categories and semiorthogonal decompositions that we appeal to hold over any field. Any triangulated category is assumed to be essentially small and support a k-linear structure. Whenever a functor between derived categories is given, it will be denoted as if underived. For example, if $f : X \to Y$ is a morphism, then denote by f^* and f_* the derived pull-back and push-forward, respectively. By a scheme, we mean a Noetherian separated scheme. By a variety, we mean a reduced scheme of finite type over a field. If X is a scheme, $\mathscr{O}_X(1)$ a line bundle, and $\mathsf{A} \subset \mathsf{D}^b(X)$ a subcategory, we denote by $\mathsf{A}(i)$ the image of A under the autoequivalence $\otimes \mathscr{O}_X(i)$ of $\mathsf{D}^b(X)$. In the relative setting $X \to Y$, we use the same notation for a given line bundle $\mathscr{O}_{X/Y}(1)$ on X. If $\mathscr{A}$ is an $\mathscr{O}_X$-algebra, which is coherent as an $\mathscr{O}_S$-module and right (or left) Noetherian, then we denote by $\mathsf{Coh}(X, \mathscr{A})$ the abelian category of finitely presented

right (or left) $\mathscr{A}$-modules and by $\mathsf{D}^{\mathrm{b}}(X, \mathscr{A})$ its derived category. By a vector bundle on X we mean a locally free $\mathscr{O}_X$-module of finite constant rank. By a projective bundle on Y we mean $\mathbb{P}(E) = \mathbf{Proj}\, S^{\bullet}(E^{\vee}) \to Y$ for a vector bundle E on X.

1. Quadratic forms and even Clifford algebras

This section is mostly devoted to generalizing some well-known results in the theory of regular quadratic forms (or "quadratic spaces") to quadratic forms with degeneration over schemes. While our treatment proceeds in the "algebraic theory of quadratic forms" style, we nevertheless keep the geometric perspective of quadric fibrations in mind.

1.1. Quadratic forms

Let Y be a scheme. Fix vector bundles E and L on Y. Denote by $T^2 E$, $S^2 E$, and $S_2 E$ the tensor square, symmetric square, and submodule of symmetric tensor squares of E, respectively. We first collect together various notions of quadratic form with values in a L, and show that they are all equivalent.

Lemma 1.1.1. *The following sets of objects are in natural bijection.*

1. *Morphisms of sheaves $q : E \to L$ satisfying*: $q(av) = a^2 q(v)$ *on sections a of $\mathscr{O}_Y$ and v of E; and that the morphism of sheaves $b_q : E \times E \to L$, defined by $b_q(v, w) = q(v + w) - q(v) - q(w)$ on sections v and w of E, is $\mathscr{O}_Y$-bilinear.*
2. *Morphisms of $\mathscr{O}_Y$-modules $S_2 E \to L$, i.e., global sections $\Gamma(Y, \mathscr{H}om(S_2 E, L))$.*
3. *Morphisms of $\mathscr{O}_Y$-modules $L^{\vee} \to S^2(E^{\vee})$, i.e., global sections $\Gamma(Y, \mathscr{H}om(L^{\vee}, S^2(E^{\vee})))$.*
4. *Global sections $\Gamma(Y, S^2(E^{\vee}) \otimes L)$.*
5. *Global sections $s_q \in \Gamma(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)/Y}(2) \otimes p^* L)$, where $p : \mathbb{P}(E) \to Y$ is the projection.*

Proof. Let $\mathscr{Q}uad(E, L)$ be the Zariski presheaf (which is actually a sheaf, and moreover, naturally an $\mathscr{O}_Y$-module), whose sections over U are the morphisms of sheaves $q : E|_U \to L|_U$ on U, satisfying the conditions in (1). Let $\mathscr{B}il(E, L)$ be the Zariski presheaf (which is an $\mathscr{O}_Y$-module), whose sections over U are the $\mathscr{O}_U$-bilinear morphisms $b : E|_U \times E|_U \to L|_U$. Then there is a commutative diagram of morphisms of $\mathscr{O}_Y$-modules:

$$
\begin{array}{ccc}
T^2(E^{\vee}) \otimes L & \longrightarrow & \mathscr{H}om(S_2 E, L) \\
& S^2(E^{\vee}) \otimes L & \\
\wr \Big\downarrow & \overset{\sim}{\nearrow} \quad \overset{\sim}{\searrow} & \Big\updownarrow \wr \\
\mathscr{B}il(E, L) & \longrightarrow & \mathscr{Q}uad(E, L)
\end{array}
\tag{1.1.1}
$$

The left vertical map is the canonical isomorphism $f \otimes g \otimes l \mapsto ((v, w) \mapsto f(v)g(w)l)$. The bottom horizontal map is $b \mapsto (v \mapsto b(v, v))$. The composition of these maps clearly factors through $S^2(E^{\vee}) \otimes L$, hence the diagonal maps from upper left to lower right. The vertical downward map at right is $\varphi \mapsto (v \mapsto \varphi(v \otimes v))$, while the upward map is $q \mapsto (v \otimes v \mapsto q(v))$. The top horizontal map is $f \otimes g \otimes l \mapsto (v \otimes v \mapsto f(v)g(v)l)$, this clearly factor through $S^2(E^{\vee}) \otimes L$, hence the diagonal map from the center to upper right. (In particular, this gives a canonical isomorphism $S^2(E^{\vee}) \to S_2(E)^{\vee}$.) In defining maps to $\mathscr{H}om(S_2 E, L)$, we observe that $S_2 E$ is the sheafification of the $\mathscr{O}_Y$-submodule of $E \otimes E$ generated by all sections of the form $v \otimes v$, thus by the universal property of sheafification, to define an $\mathscr{O}_Y$-module morphism $S_2 E \to L$ it suffices to specify it on sections of the form $v \otimes v$.

The fact that the triangle at right consists of isomorphisms is checked locally, see [84, Lemma 2.1] or [87, Prop. 6.1]. Taking global sections in this triangle establishes the bijection between (1), (2), and (4). The evident $\mathscr{O}_Y$-isomorphisms $S^2(E^{\vee}) \otimes L \to (L^{\vee})^{\vee} \otimes S^2(E^{\vee}) \to \mathscr{H}om(L^{\vee}, S^2(E^{\vee}))$ prove the bijection between (4) and (3). Finally, the isomorphism $\Gamma(Y, S^2(E^{\vee}) \otimes L) \cong \Gamma(\mathbb{P}(E), \mathscr{O}(2) \otimes p^* L)$ establishes the bijection between (4) and (5). $\quad\square$

By a (*line bundle-valued*) *quadratic form* on Y, we mean a triple (E, q, L) as in Lemma 1.1.1(1), with L a line bundle on Y. We will mostly dispense with the title "line bundle-valued." The *rank* of (E, q, L) is the rank of E.

A quadratic form (E, q, L) is *primitive* if the associated $\mathscr{O}_Y$-module morphism $S_2 E \to L$ is an epimorphism. This is equivalent to q being nonzero over the residue field of any point of Y or to the associated morphism $L^\vee \to S^2(E^\vee)$ being a monomorphism with locally free cokernel.

There is an evident *polar* $\mathscr{O}_Y$-module morphism $\psi_q : E \to \mathscr{H}om(E, L)$ associated to b_q. A quadratic form (E, q, L) is *regular* if ψ_q is an isomorphism. A quadratic form of odd rank can never be regular over a characteristic 2 point, and the notion of *semiregularity* is needed, see [56, IV.3]. We say that a quadratic form is *generically (semi)regular* if it is (semi)regular at every maximal generic point of Y (this is called *non-degenerate* in [56]).

If (E, q, L) has rank n, then the signed determinant $(-1)^{n(n-1)/2} \det \psi_q : \det E \to \det E^\vee \otimes L^{\otimes n}$ gives rise to a global section $\mathrm{disc}(q) \in \Gamma(Y, (\det E^\vee)^{\otimes 2} \otimes L^{\otimes n})$ called the *discriminant* (in odd rank, one must take the *half-discriminant*, cf. [56, IV.3.1]). The zero scheme $D \subset Y$ of the (half-)discriminant is the *discriminant divisor*. Its reduced subscheme is the locus of points of Y where q is not (semi)regular. If q is generically (semi)regular, then the discriminant divisor is the subscheme associated to an effective Cartier divisor.

Assume that (E, q, L) is generically (semi)regular, let $j : D \to S$ denote the discriminant divisor and $i : U \to S$ its open complement. The restriction of the polar morphism $\psi_q|_U : E|_U \to \mathscr{H}om(E, L)|_U$ is then an $\mathscr{O}_U$-module isomorphism and we have an exact sequence of $\mathscr{O}_D$-modules

$$0 \to R \to j^* E \xrightarrow{j^* \psi_q} j^* \mathscr{H}om(E, L) \to C \to 1$$

where we define R and C to be the sheaf kernel and cokernel of $j^* \psi_q$, respectively. We call R the *bilinear radical* of $j^* q$. The *quadratic radical* is the $\mathscr{O}_D$-submodule $R_q \subset R$ where $q|_D$ vanishes. The bilinear and quadratic radical coincide over any point where 2 is invertible. Note that as E is locally free and q is generically (semi)regular, ψ_q is injective and R and R_q are torsion-free $\mathscr{O}_D$-modules.

A quadratic form (E, q, L) of rank n has (*at most*) *simple degeneration* on Y if it has a (semi)regular quadratic subform of rank at least $n - 1$ over the local ring of every point of Y. Equivalently, the quadratic radical R_q has rank $\leqslant 1$. For an (nonempty) effective Cartier divisor $D \subset S$, we say that q has simple degeneration *along* D if (E, q, L) has simple degeneration and is generically (semi)regular with discriminant divisor D.

A *similarity* between quadratic forms (E, q, L) and (E', q', L') is a pair (φ, λ) consisting of $\mathscr{O}_Y$-module isomorphisms $\varphi : E \to E'$ and $\lambda : L \to L'$ such that either of the following equivalent diagrams,

$$\begin{array}{ccc} E & \xrightarrow{q} & L \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \lambda} \\ E' & \xrightarrow{q'} & L' \end{array} \qquad \begin{array}{ccc} S_2 E & \longrightarrow & L \\ {\scriptstyle S_2 \varphi}\downarrow & & \downarrow{\scriptstyle \lambda} \\ S_2 E' & \longrightarrow & L' \end{array} \qquad (1.1.2)$$

commute, which happens if and only if we have $q'(\varphi(v)) = \lambda(q(v))$ on sections.

Given quadratic forms (E_1, q_1, L) and (E_2, q_2, L) with values in the same line bundle, their *orthogonal sum* $(E_1, q_1, L) \perp (E_2, q_2, L) = (E_1 \oplus E_2, q_1 \perp q_2, L)$ is defined by $(q_1 \perp q_2)(v) = q_1(v) + q_2(v)$.

1.2. Quadric fibrations

The *quadric fibration* $\pi : Q \to Y$ associated to a nonzero quadratic form (E, q, L) of rank $n \geqslant 2$ is the restriction of the projection $p : \mathbb{P}(E) \to Y$ via the closed embedding $j : Q \to \mathbb{P}(E)$ defined by the vanishing of the global section $s_q \in \Gamma(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)/Y}(2) \otimes p^* L)$. Write $\mathscr{O}_{Q/Y}(1) = j^* \mathscr{O}_{\mathbb{P}(E)/Y}(1)$. The form (E, q, L) is primitive if and only if $\pi : Q \to Y$ is flat of relative dimension $n - 2$, see [71, 8 Thm. 22.5]. The fiber Q_y is a smooth projective quadric (resp. a quadric cone with isolated singularity) over any point y where (E, q, L) is (semi)regular (resp. has simple degeneration).

Define the *projective similarity* class of a quadratic form (E, q, L) to be the set of similarity classes of quadratic forms $(N, q_N, N^{\otimes 2}) \otimes (E, q, L) = (N \otimes E, q_N \otimes q, N^{\otimes 2} \otimes L)$ ranging over all line bundles N, where $q_N : N \to N^{\otimes 2}$ is the squaring form. In [9], this is referred to as a *lax-similarity* class. Though the following should be well known, we could not find a proof in the literature.

Proposition 1.2.1. *Let Y be an integral locally factorial scheme. Let $\pi : Q \to Y$ and $\pi' : Q' \to Y$ be quadric fibrations associated to primitive generically (semi)regular quadratic forms (E, q, L) and (E', q', L'). Then (E, q, L) and (E', q', L') are in the same projective similarity class if and only if Q and Q' are Y-isomorphic.*

Proof. Let η be the generic point of Y and $\pi : Q \to Y$ a flat quadric bundle of relative dimension $\geqslant 1$ (the case of relative dimension 0 is easy). Restriction to the generic fiber of π gives rise to a complex

$$0 \to \mathrm{Pic}(Y) \xrightarrow{\pi^*} \mathrm{Pic}(Q) \to \mathrm{Pic}(Q_\eta) \to 0. \tag{1.2.1}$$

We claim that since Y is locally factorial, (1.2.1) is exact in the middle. First, note that flat pullback and restriction to the generic fiber give rise to an exact sequence of Weil divisor groups

$$0 \to Z^1(Y) \xrightarrow{\pi^*} Z^1(Q) \to Z^1(Q_\eta) \to 0. \tag{1.2.2}$$

Indeed, since $Z^1(Q_\eta) = \varinjlim Z^1(Q_U)$ (where the limit is taken over all dense open sets $U \subset Y$ and we write $Q_U = Q \times_Y U$) the exactness at right and center of (1.2.2) then follows from the exactness of the excision sequence

$$Z^0\big(\pi^{-1}(Y \setminus U)\big) \to Z^1(Q) \to Z^1(Q_U) \to 0$$

associated to the closed subscheme $\pi^{-1}(Y \setminus U) \subset Q$. Note that the exactness at right of the excision sequence follows by taking the closure of a prime divisor. The exactness at left of (1.2.2) follows since π is surjective on codimension 1 points.

Since π is dominant and the function fields of Q and Q_η coincide, the sequence (1.2.2) of Weil divisor groups induces a sequence

$$\mathrm{Cl}(Y) \xrightarrow{\pi^*} \mathrm{Cl}(Q) \to \mathrm{Cl}(Q_\eta) \to 0$$

of Weil divisor class groups, which is exact by a diagram chase. We then have the following commutative diagram

$$
\begin{array}{ccccccc}
\mathrm{Pic}(Y) & \xrightarrow{\pi^*} & \mathrm{Pic}(Q) & \longrightarrow & \mathrm{Pic}(Q_\eta) & \longrightarrow & 0 \\
\cap & & \cap & & \cap & & \\
\mathrm{Cl}(Y) & \xrightarrow{\pi^*} & \mathrm{Cl}(Q) & \longrightarrow & \mathrm{Cl}(Q_\eta) & \longrightarrow & 0
\end{array}
$$

of abelian groups. The vertical inclusions are equalities since Y is locally factorial, cf. [39, Tome 4 Cor. 21.6.10]. Finally, a diagram chase shows that (1.2.1) is exact in the middle.

Let (E, q, L) and (E', q', L') be projectively similar with respect to a line bundle N and $\mathscr{O}_Y$-module isomorphisms $\varphi : E' \to N \otimes E$ and $\lambda : L' \to N^{\otimes 2} \otimes L$ preserving the quadratic forms. Write $h = \mathbf{Proj}(\varphi^\vee) : \mathbb{P}(E') \to \mathbb{P}(N \otimes E)$ for the associated Y-isomorphism. There is a natural Y-isomorphism $g : \mathbb{P}(N \otimes E) \to \mathbb{P}(E)$ satisfying $g^*\mathscr{O}_{\mathbb{P}(E)/Y}(1) \cong \mathscr{O}_{\mathbb{P}(E\otimes N)/Y}(1) \otimes g^*p^*N$, see [41, II Lemma 7.9]. Denote by $f = g \circ h : \mathbb{P}(E') \to \mathbb{P}(E)$ the composition. Then via the isomorphism

$$\Gamma\big(\mathbb{P}(E'), f^*(\mathscr{O}_{\mathbb{P}(E)/Y}(2) \otimes p^*L)\big) \to \Gamma\big(\mathbb{P}(E'), \mathscr{O}_{\mathbb{P}(E')/Y}(2) \otimes p'^*L'\big)$$

induced by $f^*\mathscr{O}_{\mathbb{P}(E)/Y}(2) \cong \mathscr{O}_{\mathbb{P}(E')}(2) \otimes (p'^*N)^{\otimes 2}$ and $p'^*\lambda^{-1} : (p'^*N)^{\otimes 2} \otimes p'^*L \to p'^*L'$, the global section f^*s_q is taken to $s_{q'}$, hence f restricts to a Y-isomorphism $Q' \to Q$.

Conversely, let $f : Q' \to Q$ be a Y-isomorphism. First, we will prove that f can be extended to a Y-isomorphism $\tilde{f} : \mathbb{P}(E') \to \mathbb{P}(E)$ satisfying $\tilde{f} \circ j' = j \circ f$. To this end, considering the long exact sequence associated to applying p_* to the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}(E)/Y}(-1) \otimes p^*L^\vee \xrightarrow{s_q} \mathscr{O}_{\mathbb{P}(E)/Y}(1) \to j_*\mathscr{O}_{Q/Y}(1) \to 0 \tag{1.2.3}$$

and keeping in mind that $R^i p_*\mathscr{O}_{\mathbb{P}(E)/Y}(-1) = 0$ for $i = 0, 1$ (as $p : \mathbb{P}(E) \to Y$ is a projective bundle of positive relative dimension by assumption), we arrive at a canonical isomorphism $p_*\mathscr{O}_{\mathbb{P}(E)/Y}(1) \cong \pi_*\mathscr{O}_{Q/Y}(1)$. In particular, we have a canonical identification $E^\vee = \pi_*\mathscr{O}_{Q/Y}(1)$. We have a similar identification $E'^\vee = \pi'_*\mathscr{O}_{Q'/Y}(1)$.

We claim that $f^*\mathscr{O}_{Q/Y}(1) \cong \mathscr{O}_{Q'/Y}(1) \otimes \pi'^*N$ for some line bundle N on Y. Indeed, over the generic fiber, we have $f^*\mathscr{O}_{Q/Y}(1)_\eta = f_\eta^*\mathscr{O}_{Q_\eta}(1) \cong \mathscr{O}_{Q'_\eta}(1)$ by the case of smooth quadrics (as q is generically (semi)regular) over a field, cf. [33, Lemma 69.2]. Then the exactness of (1.2.1) in the middle finishes the proof of the claim.

Finally, by the projection formula and our assumption that $\pi' : Q' \to Y$ is of positive relative dimension, we have that f induces an $\mathscr{O}_Y$-module isomorphism

$$E^\vee \otimes N^\vee \cong \pi_*\mathscr{O}_{Q/Y}(1) \otimes \pi'_*\pi'^*N^\vee \cong \pi_*f_*\big(f^*\mathscr{O}_{Q/Y}(1) \otimes \pi'^*N^\vee\big) \cong \pi'_*\mathscr{O}_{Q'/Y}(1) = E'^\vee$$

with induced dual isomorphism $\varphi : E' \to N \otimes E$. Now define $\tilde{f} : \mathbb{P}(E') \to \mathbb{P}(E)$ to be the composition of **Proj**$(\varphi^\vee) : \mathbb{P}(E') \to \mathbb{P}(N \otimes E)$ with the natural Y-isomorphism $\mathbb{P}(N \otimes E) \to \mathbb{P}(E)$, as earlier in this proof. Then by the construction of $\tilde{f}$, we have that $\tilde{f}^* \mathscr{O}_{\mathbb{P}(E)/Y}(1) \cong \mathscr{O}_{\mathbb{P}(E')/Y}(1) \otimes p'^* N$ and that $j \circ f = \tilde{f} \circ j'$ (an equality that can be checked on fibers using [33, Thm. 69.3]). Equivalently, there exists an isomorphism $\tilde{f}^*(\mathscr{O}_{\mathbb{P}(E)(2)/Y} \otimes p^* L) \cong \mathscr{O}_{\mathbb{P}(E')/Y}(2) \otimes p'^* L'$ taking $f^* s_q$ to $s_{q'}$. However, as $\tilde{f}^*(\mathscr{O}_{\mathbb{P}(E)(2)/Y} \otimes p^* L) \cong \mathscr{O}_{\mathbb{P}(E')/Y}(2) \otimes p'^*(N^{\otimes 2} \otimes L')$, we have an isomorphism $p'^* L' \cong p'^*(N^{\otimes 2} \otimes L)$. Upon taking pushforward, we arrive at an isomorphism $\lambda : L' \to N^{\otimes 2} \otimes L$. By the construction of φ and λ, it follows that (φ, λ) is a similarity $(E, q, L) \to (E', q', L')$, proving the converse. $\quad\square$

We now recall a moduli space theoretic characterization of quadric fibrations. We first recall Grothendieck's moduli characterization of projective bundles, see [41, II Prop. 7.12] or [72, Section 5.1.5(1)]. If E is a vector bundle, the projective bundle $p : \mathbb{P}(E) \to Y$ represents the moduli functor of *line subbundles* of E on the category of Y-schemes:

$$u : U \to Y \quad \mapsto \quad \left\{ N \xrightarrow{\varphi} u^* E \right\}$$

of equivalence classes of invertible $\mathscr{O}_U$-modules N and $\mathscr{O}_U$-monomorphisms $\varphi : N \to u^* E$ with locally free quotient, where $\varphi : N \to u^* E$ is equivalent to $\varphi' : N' \to u^* E$ if and only if there exists an $\mathscr{O}_U$-module isomorphism $\mu : N' \to N$ such that $\varphi' = \varphi \circ \mu$. Given a Y-morphism $g : U \to \mathbb{P}(E)$, pulling back the universal line subbundle $\mathscr{O}_{\mathbb{P}(E)/Y}(-1) \to p^* E$ via g gives the a line subbundle of $u^* E$.

Let (E, q, L) be a quadratic form and $\pi : Q \to Y$ the associated quadric fibration. Then Q represents the moduli functor of *isotropic* line subbundles of E, i.e., line subbundles on which the quadratic form $q : E \to L$ vanishes.

Theorem 1.2.2. *The Y-scheme Q represents the moduli functor*

$$u : U \to Y \quad \mapsto \quad \left\{ N \xrightarrow{\varphi} u^* E :\ u^* q \big|_N = 0 \right\}$$

of line subbundles $\varphi : N \to u^* E$ such that the restriction of the quadratic form $u^* q : u^* E \to u^* L$ to N is identically zero.

Proof. Composing a Y-morphism $g : U \to Q$ with the closed embedding $j : Q \to \mathbb{P}(E)$, we obtain a line subbundle $\varphi : N \to \pi^* E$, where $N = (j \circ g)^* \mathscr{O}_{\mathbb{P}(E)/Y}(-1) = g^* \mathscr{O}_{Q/Y}(-1)$. Thus is suffices to observe that the universal line subbundle $\mathscr{O}_{Q/Y}(-1) \to \pi^* E$ is isotropic for the pull-back quadratic form $\pi^* q : \pi^* E \to \pi^* L$ on Q. $\quad\square$

Finally, we need the following generalization of the notions of pencils, nets, and webs of quadrics.

Definition 1.2.3. Let E and L be vector bundles on S and $r : S = \mathbb{P}(L^\vee) \to Y$. To any vector bundle-valued quadratic form $(E, \boldsymbol{q}, L)$, define the *linear span quadratic form* $(r^* E, q, \mathscr{O}_{S/Y}(1))$ on S by the sheaf morphism $q : r^* E \to \mathscr{O}_{S/Y}(1)$ over S associated, via adjunction, to the sheaf morphism $\boldsymbol{q} : E \to L = r_* \mathscr{O}_{S/Y}(1)$ over Y. One immediately checks that the linear span is a line bundle-valued quadratic form on S. The associated quadric fibration $Q \to S$ is called the *linear span quadric fibration*.

We apply this construction in the following situation. Let (E, q_i, L_i) for $1 \leqslant i \leqslant m$ be a finite set of line bundle-valued quadratic forms on the same vector bundle E. Letting $L = L_1 \oplus \cdots \oplus L_m$, then

$$\boldsymbol{q} = q_1 \oplus \cdots \oplus q_m : E \to L_1 \oplus \cdots \oplus L_m = L$$

is a vector bundle-valued quadratic form, to which we can form the linear span quadratic form.

As we are mostly interested in linear span quadratic forms in the sequel, we will now denote by S our base space. (We keep in mind the case where $S \to Y$ is a projective bundle.)

In most geometric applications, we will consider quadric fibrations with "good" properties, which we summarize in the following definition.

Definition 1.2.4. We say that a finite set of generically (semi)regular primitive quadratic forms (E, q_i, L_i) (or quadric fibrations $Q_i \to Y$) for $1 \leqslant i \leqslant m$ is *generic* if the following properties hold:

1. The images of $L_i^\vee \to S^2(E^\vee)$ (associated to q_i via Lemma 1.1.1(3)) span an $\mathscr{O}_Y$-submodule $L^\vee \subset S^2(E^\vee)$ of rank m,
2. The associated linear span quadric fibration $Q \to S$ has simple degeneration with regular discriminant divisor,
3. The associated relative intersection $X \to Y$ of the quadric fibrations $Q_i \subset \mathbb{P}(E)$ is a relative complete intersection.

By a *generic relative intersection of quadrics* we mean any relative intersection $X \to Y$ of a generic set of quadric fibrations.

We now mention how the regularity of the discriminant divisor is related to the simple degeneration hypothesis.

Proposition 1.2.5. *Let S be a scheme smooth over an arbitrary field k of characteristic $\neq 2$. Let $\pi : Q \to S$ be a flat generically regular quadric fibration with discriminant divisor D. Then D is smooth over k if and only if Q is smooth over k and $Q \to S$ has simple degeneration.*

Proof. As Q is locally of finite presentation (by construction) and flat (by hypothesis) over S, and S is smooth (hence locally of finite presentation and flat, see [39, Tome 4 Thm. 17.5.1]) over k, we have that Q is locally of finite presentation and flat over k, being a composition of such morphisms. Thus Q is smooth over k if and only if Q is geometrically regular, see [39, Tome 4 Cor. 17.5.2]. To prove the smoothness of Q over k, it would be enough to verify that for each point $x \in S$, the local quadric $Q \times_S \mathscr{O}_{S,x}$ is a smooth scheme over k, hence geometrically regular. Indeed, for any point y of Q, the local ring $\mathscr{O}_{Q,y}$ would then be geometrically regular, as it resides on the local quadric $Q \times_S \mathscr{O}_{S,\pi(y)}$. If $x \in S \setminus D$, then $Q \times_S \mathscr{O}_{S,x}$ is actually smooth over $\mathscr{O}_{S,x}$. So we can assume that $x \in D$.

Let $Q \to S$ be defined by a quadratic form (E, q, L) of rank n on S. Over the residue field of the local ring $\mathscr{O}_{S,x}$ at x, the regular subform $\bar{q}_0$ of q_x can be diagonalized (being a quadratic form over a field of characteristic $\neq 2$) as $\sum_{i=1}^r \bar{a}_i x_i^2$ for some $0 < r < n$ (since q_x is nonzero by the flatness assumption) and $\bar{a}_i \neq 0$. By [8, Cor. 3.4], this lifts to an orthogonal decomposition $q_{\mathscr{O}_{S,x}} = q_0 + q_1$, where $q_0 = \sum_{i=1}^r a_i x_i^2$ is a regular quadratic form of rank $r < n$ over $\mathscr{O}_{S,x}$ lifting $\bar{q}_0$, and q_1 is a quadratic form of rank $n - r$ over $\mathscr{O}_{S,x}$ that vanishes modulo the maximal ideal $\mathfrak{m}_{S,x}$. As a consequence, we find that the discriminant of $q_{\mathscr{O}_{S,x}}$, which defines D at x, is in $\mathfrak{m}_{S,x}^{n-r}$. Hence, if D is smooth at x then the discriminant of $q_{\mathscr{O}_{S,x}}$ must be contained in $\mathfrak{m}_{S,x} \setminus \mathfrak{m}_{S,x}^2$, and hence the regular subform q_0 of $q_{\mathscr{O}_{S,x}}$ has rank $n - 1$, i.e., (E, q, L) and $Q \to S$ have simple degeneration at x.

Thus we are left to prove that if $Q \to S$ has simple degeneration, then the smoothness of Q over k is equivalent to the smoothness of D over k. To this end, we can assume that k is algebraically closed and pass the complete local ring $\widehat{\mathscr{O}}_{S,x} \cong k[\![u_1, \ldots, u_d]\!]$, where d is the dimension of S at x, see [39, Tome 4 Prop. 17.5.3]. If $\delta \in \mathfrak{m}_{S,x}$ is a local equation of D at x, then up to a unit in $\widehat{\mathscr{O}}_{S,x}$, the discriminant of q_1 (which has rank 1 by hypothesis) is δ. Then we can write (noting that over $\widehat{\mathscr{O}}_{S,x}$ all units are squares)

$$Q \times_S \operatorname{Spec} \widehat{\mathscr{O}}_{S,x} \cong \operatorname{Spec} k[\![u_1, \ldots, u_d]\!][x_1, \ldots, x_n] / (x_1^2 + \cdots + x_{n-1}^2 + \delta x_n^2).$$

An application of the Jacobian criterion shows that $Q \times_S \operatorname{Spec} \widehat{\mathscr{O}}_{S,x}$ is smooth over k if and only if $\delta \in k[\![u_1, \ldots, u_n]\!]$ contains a nonzero linear term in the u_i (i.e., D is smooth at x) and c is a unit. $\square$

Note that special cases of Proposition 1.2.5 are proved in [11, I Prop. 1.2(iii)] and [43, Lemma 5.2].

We remark that D can have mild singularities and yet the total space of the quadric bundle Q can still be smooth, but then the quadric bundle must have worse degeneration over the singular points of D, see [11, I Prop. 1.2] for example.

1.3. Smooth sections and hyperbolic splitting

In this section, on the way to developing a splitting principle for isotropic subbundles, we first establish some local algebraic results for isotropic subbundles of possibly degenerate quadratic forms.

Definition 1.3.1. A section $s : S \to Q$ of a quadric fibration $Q \to S$ is called *smooth* if the image of s only consists of smooth points of the fibers of $Q \to S$. An isotropic line subbundle $N \subset E$ of a quadratic form (E, q, L) is called *smooth* if the associated section of its quadric fibration is smooth.

We relate smoothness of the base and total space with smoothness of a section.

Lemma 1.3.2. *Let $\pi : Q \to S$ be a flat morphism of schemes with a section s. If $y \in S$ is a regular point such that $x = s(y) \in Q$ is regular, then $x \in Q_y$ is a regular point of the fiber. In particular, if Q and S are smooth schemes over a field k, then any section of π is smooth.*

Proof. Denoting by $j : Q_y \to Q$ the closed embedding of a fiber, we have local homomorphisms $\pi^\# : \mathscr{O}_{S,y} \to \mathscr{O}_{Q,x}$ (which is a splitting of $s^\# : \mathscr{O}_{Q,x} \to \mathscr{O}_{S,y}$) and $j^\# : \mathscr{O}_{Q,x} \to \mathscr{O}_{Q_y,x}$ (which is surjective), whose composition factors through the residue field $\kappa := \kappa(y) = \kappa(x)$. Thus the induced sequence of κ-vector spaces

$$0 \to \mathfrak{m}_{S,y}/\mathfrak{m}_{S,y}^2 \xrightarrow{\pi^\#} \mathfrak{m}_{Q,x}/\mathfrak{m}_{Q,x}^2 \xrightarrow{j^\#} \mathfrak{m}_{Q_y,x}/\mathfrak{m}_{Q_y,x}^2 \to 0$$

is split exact. If $\dim_\kappa \mathfrak{m}_{S,y}/\mathfrak{m}_{S,y}^2 = \dim \mathscr{O}_{S,y}$ and $\dim_\kappa \mathfrak{m}_{Q,x}/\mathfrak{m}_{Q,x}^2 = \dim \mathscr{O}_{Q,x}$, then $\dim_\kappa \mathfrak{m}_{Q_y,x}/\mathfrak{m}_{Q_y,x}^2 = \dim \mathscr{O}_{Q,x} - \dim \mathscr{O}_{S,y} = \dim \mathscr{O}_{Q_y,x}$ by the local fiber dimension theorem using the flatness of π, see [71, Thm. 15.1]. Hence x is regular in the fiber Q_y. For the final claim, the geometric points $\bar{y} \in S$ and $\bar{x} = s(\bar{y}) \in Q$ are regular by hypothesis, hence $\bar{x} \in Q_{\bar{y}}$ is regular. Thus $x \in Q_y$ is a smooth point. $\square$

Remark 1.3.3. Assume that S is a smooth scheme over a field k of characteristic $\neq 2$ and that $\pi : Q \to S$ is a flat quadric fibration with smooth discriminant divisor $D \subset S$. Then by Proposition 1.2.5, Q is smooth over k, and hence by Lemma 1.3.2, any section of π is smooth.

We have the following algebraic reinterpretation of the smoothness of an isotropic line subbundle in terms of the bilinear radical defined in Section 1.1.

Lemma 1.3.4. *An isotropic line subbundle N of (E, q, L) is smooth if and only if $R_q \cap j^*N = 0$.*

This motivates the following definition. An isotropic subbundle $N \subset E$ (of any rank) of a generically (semi)regular quadratic form (E, q, L) is called *smooth* if $R_q \cap j^*N = 0$.

Let $g : N \to E$ be the inclusion of an isotropic subbundle. The *orthogonal complement* $g^\perp : N^\perp \to E$ is defined by the exact sequence

$$0 \to N^\perp \xrightarrow{g^\perp} E \xrightarrow{\psi_{q,N}} \mathscr{H}om(N, L),$$

where $\psi_{q,N}$ is the composition of $\psi_q : E \to \mathscr{H}om(E, L)$ with the projection $\mathscr{H}om(E, L) \to \mathscr{H}om(N, L)$. Note that if N is isotropic then $N \subset N^\perp$.

Given a vector bundle N and a line bundle L on S, the *hyperbolic quadratic form* $H_L(N)$ on S is defined by the evaluation pairing $N \oplus \mathscr{H}om(N, L) \to L$. The isotropic subbundle $N \subset H_L(N)$ is smooth and is maximal amongst isotropic subbundles. Such an isotropic subbundle is called a *Lagrangian*.

Lemma 1.3.5 (*Local hyperbolic splitting*). *Let $(\mathscr{O}, \mathfrak{m})$ be a local ring. Let N be an isotropic direct summand of a generically (semi)regular quadratic form (E, q, L) over $\mathscr{O}$. If N is smooth then we have an orthogonal decomposition $(E, q, L) \cong H_L(N) \perp (N^\perp/N, q|_{N^\perp/N}, L)$ such that the inclusion of the first factor fits into the following commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & H_L(N) & \longrightarrow & \mathscr{H}om(N, L) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N^\perp & \xrightarrow{g^\perp} & E & \xrightarrow{\psi_{q,N}} & \mathscr{H}om(N, L) & &
\end{array}$$
(1.3.1)

of free $\mathscr{O}$-modules, where the top sequence arises from the inclusion of the Lagrangian $N \subset H_L(N)$.

Proof. First note that since N is smooth, by Lemma 1.3.4 we have a decomposition $\bar{E} \cong \bar{N} \oplus R_q \oplus \bar{M}$ over the residue field $k(\mathfrak{m})$ of $\mathscr{O}$, where $\bar{M} = \bar{E}/(\bar{N} + R_q)$. Fixing a generator $L = l\mathscr{O}$, we identify $L = \mathscr{O}$ throughout the proof.

First, suppose $N = v\mathcal{O}$ has rank 1. Since $\bar{v} \notin R_q$ there exists $\overline{w} \in \overline{M}$ such that $b_q(\bar{v}, \overline{w}) = a \in k(\mathfrak{m})^\times$. Then $\bar{v}k(\mathfrak{m}) \oplus \overline{w}k(\mathfrak{m})$ is a regular orthogonal summand of $(\overline{E}, \bar{q})$, which can be lifted (by [8, I Cor. 3.4, Prop. 3.2ii]) to a regular orthogonal summand $H = v\mathcal{O} \oplus w\mathcal{O}$ of (E, q), for some lift $w \in E$ of $\overline{w}$ over $\mathcal{O}$. Hence there is an orthogonal decomposition $(E, q) \cong (H, q|_H) \perp (H^\perp, q|_{H^\perp})$. We use the following classical trick to modify this orthogonal decomposition. Choosing a lift $c \in \mathcal{O}^\times$ of a^{-1}, then $\{cv, w - cq(w)v\}$ forms a hyperbolic basis of H with Lagrangian N, hence the isometric inclusion $H_L(N) \cong (H, q|_H) \subset (E, q)$ defining the middle vertical arrow in the diagram. The commutativity of the diagram is then immediate. Finally, we see that $E \cong w\mathcal{O} \oplus N^\perp$, hence that $N^\perp \cong N \oplus H^\perp$ and thus $H^\perp \cong N^\perp/N$. We thus arrive at the desired decomposition $(E, q, L) \cong H_L(N) \perp (N^\perp/N, q|_{N^\perp/N}, L)$.

Now suppose that N has rank n and write $N \cong N_1 \oplus N_2$ with $\mathrm{rk}\, N_1 = 1$. Then $N \cap N_1^\perp / N_1$ has rank $n - 1$ and its reduction to $k(\mathfrak{m})$ does not intersect R_q. Hence we can apply induction on $\mathrm{rk}\, N$. $\quad\square$

Proposition 1.3.6. *Let (E, q, L) be a generically (semi)regular quadratic form over a scheme S and $g : N \to E$ an isotropic subbundle. If N is smooth then the sequence*

$$0 \to N^\perp \xrightarrow{g^\perp} E \xrightarrow{\psi_{q,N}} \mathscr{H}om(N, L) \to 0$$

is exact. In particular, $N^\perp$ is locally free.

Proof. For an isotropic $\mathcal{O}_S$-submodule $N \subset E$, consider the commutative diagram of $\mathcal{O}_S$-modules

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N^\perp & \xrightarrow{g^\perp} & E & \xrightarrow{\psi_{q,N}} & \mathscr{H}om(N, L) \\
& & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & N^\perp/N & \longrightarrow & E/N & \xrightarrow{\psi_q/N} & \mathscr{H}om(N, L)
\end{array}
$$

with exact rows. Here ψ_q/N is defined by $v + N \mapsto (n \mapsto b_q(v, n))$. Hence the surjectivity of $\psi_{q,N}$ is equivalent to the surjectivity of ψ_q/N. Both of these are local questions.

Let $\mathcal{O}$ be the local ring of a point x of S. Since N is smooth, we can apply Lemma 1.3.5 to N over $\mathcal{O}$ and we have an orthogonal decomposition $(E, q, L)_\mathcal{O} \cong H_L(N)_\mathcal{O} \perp (N^\perp/N, q|_{N^\perp/N}, L)_\mathcal{O}$ of quadratic forms over $\mathcal{O}$. But then we have the following commutative diagram of $\mathcal{O}$-modules

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (N^\perp/N)_\mathcal{O} & \longrightarrow & E_\mathcal{O} & \longrightarrow & H_L(N)_\mathcal{O} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (N^\perp/N)_\mathcal{O} & \longrightarrow & (E/N)_\mathcal{O} & \xrightarrow{\psi_q/N} & \mathscr{H}om(N, L)_\mathcal{O} & &
\end{array}
$$

whose top row defines a splitting of the inclusion $H_L(N)_\mathcal{O} \to E_\mathcal{O}$ from the middle column of (1.3.1). In particular, ψ_q/N is surjective over each $\mathcal{O}$, hence is surjective over S. $\quad\square$

Definition 1.3.7. Let (E, q, L) be a quadratic form over a scheme and $N \to E$ be a smooth isotropic subbundle. Then $q|_{N^\perp} : N^\perp \to L$ vanishes on N, hence defines a quadratic form $q' : N^\perp/N \to L$ on $E' = N^\perp/N$. We call (E', q', L) the *reduced quadratic form* associated to N.

The following is a generalization of the hyperbolic splitting principle for regular quadratic forms (see [8, I Thm. 3.6]).

Theorem 1.3.8 (Isotropic splitting principle). *Let S be a scheme and (E, q, L) be a generically (semi)regular quadratic form over S. Let (E', q', L) be the reduced quadratic form associated to a smooth isotropic subbundle $g : N \to E$. Then there exists a (Zariski) locally trivial affine bundle $p : V \to S$ such that $p^*(E, q, L) = H_{p^*L}(p^*N) \perp p^*(E', q', L)$. In particular, then there exists a Zariski open covering of S such that $(E, q, L)|_U = H_{L|_U}(N|_U) \perp (E', q', L)|_U$ for each element U of the covering.*

Proof. Following Fulton [34, Section 2], let $V' \subset \mathscr{H}om(E, N)$ be the sheaf of local retractions of $g : N \to E$, i.e., whose sections over $U \to S$ are $\mathcal{O}_U$-module morphisms $\varphi : E|_U \to N|_U$ such that $\varphi \circ g|_U = \mathrm{id}_N|_U$. Then V' has

a (left) translation action by the $\mathscr{O}_S$-module $\mathscr{H}om(\mathrm{coker}(g), N) \subset \mathscr{H}om(E, N)$, making V' into a (left) torsor. By abuse of notation, denote by $p' : V' \to S$ the corresponding Zariski locally trivial affine bundle, which has a section if and only if g is split over S. The pullback p'^*E has a tautological map to p'^*N, which is a retraction of p'^*g, hence p'^*N is a direct summand of p'^*E.

By Proposition 1.3.6, $g^\perp : N^\perp \to E$ is locally free with locally free quotient. Let $V \subset \mathscr{H}om(p'^*E, p'^*N^\perp)$ be the sheaf of local retractions of $g^\perp$ over V'. Denote by $p : V \to S$ the corresponding composition of Zariski locally trivial affine bundles. Then $p^*N^\perp$ is a direct summand of p^*E. Considering the Nine Lemma applied to the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & N^\perp & \longrightarrow & N^\perp/N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \stackrel{g}{\longrightarrow} & E & \longrightarrow & E/N & \longrightarrow & 0
\end{array}
$$

of $\mathscr{O}_S$-modules, we have (using Proposition 1.3.6) an exact sequence

$$0 \to N^\perp/N \to E/N \to \mathscr{H}om(N, L) \to 0,$$

which becomes split upon application of p^*. Note that $p^*N^\perp = (p^*N)^\perp$ (which can be checked locally). Finally, the direct sum decomposition $p^*E \cong p^*(N \oplus \mathscr{H}om(N, L)) \oplus p^*(N^\perp/N)$, induces the required isometry of quadratic forms, as in the affine case [8, I Thm. 3.6] or [56, I Prop. 3.7.1]. Indeed, the associated bilinear form induces a pairing $b_q : N \times E/N^\perp \to L$, which is locally checked to be a perfect pairing. By Proposition 1.3.6, this is a perfect pairing $N \times \mathscr{H}om(N, L) \to L$, which remains perfect after pulling back to V. If $p^*\mathscr{H}om(N, L)$ is isotropic in p^*E, then the pairing pulls back to the associated bilinear form of the hyperbolic quadratic form $p^*H_L(N)$ and we are done. Otherwise, we employ the same trick appearing in the proof of Lemma 1.3.5 to modify the orthogonal decomposition of p^*E, in order to make the summand $p^*(N \oplus \mathscr{H}om(N, L))$ hyperbolic. The final claim follows since p has a section Zariski locally. $\square$

Corollary 1.3.9. *The quadratic forms* (E, q, L) *and* (E', q', L) *have the same degeneration divisor. In particular,* (E, q, L) *has simple degeneration if and only if* (E', q', L) *does.*

Proof. The degeneration divisors D and D' of (E, q, L) and (E', q', L) agree over a Zariski open covering of S by Theorem 1.3.8 since hyperbolic spaces have trivial discriminant. Hence $D = D'$. The final statement is local and follows from Lemma 1.3.5. $\square$

1.4. Quadric reduction by hyperbolic splitting

We now describe a geometric interpretation, in terms of quadratic fibrations, of the reduced quadratic form associated to a smooth isotropic line subbundle. Given a quadric hypersurface $Q \subset \mathbb{P}^{n-1}$ over an arbitrary field k and a smooth k-point x, consider the embedded tangent space T_xQ. The intersection $T_xQ \cap Q$ is a quadric cone with vertex x, over a quadric $Q' \subset \mathbb{P}^{n-2}$. In particular, projecting $T_xQ \cap Q$ away from x, we find Q' as a quadric hypersurface in the target $\mathbb{P}^{n-3}$, which we call the *quadric reduction by hyperbolic splitting* associated to Q and x. Recently, this construction was considered [47] in the study of nets of quadrics over $\mathbb{P}^2$ and associated K3 surfaces.

Conversely, consider a quadric $Q' \subset \mathbb{P}^{n-3}$ and an embedding of $\mathbb{P}^{n-3}$ as a hyperplane of $\mathbb{P}^{n-2}$. Let $\varepsilon : \widetilde{\mathbb{P}}^{n-2} \to \mathbb{P}^{n-2}$ be the blow-up of $\mathbb{P}^{n-2}$ along Q'. Then the quadric Q is obtained as the image of $\widetilde{\mathbb{P}}^{n-2}$ when contracting the proper transform of $\mathbb{P}^{n-3}$.

Proposition 1.4.1. *Let N be a smooth isotropic line subbundle of a primitive quadratic form (E, q, L) over a scheme S. Let Q and Q' be the quadric fibrations associated to (E, q, L) and the reduced quadratic form (E', q', L) associated to N. Let $s : S \to Q$ be the smooth section associated to N. Then the fiber Q'_x is the quadric reduction of Q_x associated to the smooth point $s(x)$.*

Proof. Since $s(x)$ is a smooth point of Q_x, the projective space $\mathbb{P}(N_x^\perp)$ is canonically identified with the embedded tangent space of Q_x at $s(x)$, see [33, Prop. 22.1]. Moreover, the quotient map $N_x^\perp \to N_x^\perp/N_x$ corresponds to the

projection away from $s(x)$. Hence it is immediate that the reduced quadratic form q'_x on $N_x^\perp/N_x$, induced by q_x and N_x, defines the quadric Q'_x on $\mathbb{P}(N_x^\perp/N_x)$. $\square$

1.5. Even Clifford algebra and Clifford bimodule

In this section, we give a new tensorial construction of the even Clifford algebra of a line bundle-valued quadratic form. In Appendix A, we show that it coincides with the one in [63, Section 3.3]. The usefulness of this construction lies in its universal property, cf. [5, Section 1].

Let (E, q, L) be a (line bundle-valued) quadratic form of rank n on a scheme S. Write $n = 2m$ or $n = 2m + 1$. Inspired by [57, II Lemma 8.1], we define ideals J_1 and J_2 of the tensor algebra $T(E \otimes E \otimes L^\vee)$, generated by

$$v \otimes v \otimes f - f(q(v)), \quad \text{and} \quad u \otimes v \otimes f \otimes v \otimes w \otimes g - f(q(v))u \otimes w \otimes g,$$

respectively, for sections u, v, w of E and f, g of $L^\vee$. We define the *even Clifford algebra* of (E, q, L) as the quotient algebra

$$\mathscr{C}_0(E, q, L) = T(E \otimes E \otimes L^\vee)/(J_1 + J_2).$$

Then $\mathscr{C}_0(E, q, L)$ has an $\mathscr{O}_S$-module filtration

$$\mathscr{O}_S = F_0 \subset F_2 \subset \cdots \subset F_{2m} = \mathscr{C}_0(E, q, L),$$

where F_{2i} is the image of the truncation of the tensor algebra $T^{\leqslant i}(E \otimes E \otimes L^\vee)$ in $\mathscr{C}_0(E, q, L)$, for each $0 \leqslant i \leqslant m$. The fact that $F_{2m} = \mathscr{C}_0(E, q, L)$ is a consequence of the Poincaré–Birkhoff–Witt theorem, see [17, Prop. 3.5]. This filtration has associated graded pieces $F_{2i}/F_{2(i-1)} \cong \bigwedge^{2i} E \otimes (L^\vee)^{\otimes i}$. In particular, $\mathscr{C}_0(E, q, L)$ is a locally free $\mathscr{O}_S$-algebra of rank 2^{n-1}. By its tensorial construction, the even Clifford algebra has the following universal property: given an $\mathscr{O}_S$-algebra $\mathscr{A}$ and an $\mathscr{O}_S$-module morphism $\psi : E \otimes E \otimes L^\vee \to \mathscr{A}$ satisfying

$$\psi(v \otimes v \otimes f) = f(q(v)) \cdot 1_\mathscr{A} \quad \text{and} \quad \psi(u \otimes v \otimes f) \cdot \psi(v \otimes w \otimes g) = f(q(v)) \psi(u \otimes w \otimes g), \quad (1.5.1)$$

there exists a unique $\mathscr{O}_S$-algebra homomorphism $\Psi : \mathscr{C}_0(E, q, L) \to \mathscr{A}$ satisfying $\psi = \Psi \circ i$, where $i : E \otimes E \otimes L^\vee \to \mathscr{C}_0(E, q, L)$ is the canonical morphism.

When $L = \mathscr{O}_S$, and (E, q) is a classical ($\mathscr{O}_S$-valued) quadratic form, then we can define the "full" *Clifford algebra* as

$$\mathscr{C}(E, q) = T(E)/J \qquad (1.5.2)$$

where J is the ideal generated by $v \otimes v - q(v)$ for sections v of E. As J is generated by relations in even degree, the full Clifford algebra inherits a $\mathbb{Z}/2\mathbb{Z}$-graded structure $\mathscr{C}(E, q) = \mathscr{C}_0(E, q) \oplus \mathscr{C}_1(E, q)$ into components of even and odd degree. The canonical morphism $E \otimes E \otimes \mathscr{O}_S = E \otimes E \to \mathscr{C}_0(E, q)$ satisfies the universal property (1.5.1). The induced $\mathscr{O}_S$-algebra morphism $\mathscr{C}_0(E, q, \mathscr{O}_S) \to \mathscr{C}_0(E, q)$ is an isomorphism. If $L \neq \mathscr{O}_S$, then quadratic forms with values in L do not generally enjoy a "full" Clifford algebra, of which the even Clifford algebra is the even degree part. In this case, the Clifford bimodule is a replacement for the odd degree part.

Now we define the Clifford bimodule. Inspired by [57, II Section 9], we proceed as follows. The $\mathscr{O}_S$-module $E \otimes T(E \otimes E \otimes L^\vee)$ has a natural right $T(E \otimes E \otimes L^\vee)$-module structure denoted by $\otimes$; we now define a commuting left $T(E \otimes E \otimes L^\vee)$-module structure denoted by $*$ and defined by

$$(v_1 \otimes u_1 \otimes f_1 \otimes \cdots \otimes v_r \otimes u_r \otimes f_r) * w = v_1 \otimes (u_1 \otimes v_2 \otimes f_1 \otimes u_2 \otimes v_3 \otimes f_2 \otimes \cdots \otimes u_r \otimes w \otimes f_r),$$

for sections w, u_i, v_i of E and f_i of $L^\vee$. We define the *Clifford bimodule* of (E, q, L) as the quotient module

$$\mathscr{C}_1(E, q, L) = E \otimes T(E \otimes E \otimes L^\vee)/(E \otimes J_1 + J_1 * E).$$

One immediately checks that $E \otimes J_2 \subset J_1 * E$ and $J_2 * E \subset E \otimes J_1$, hence $\mathscr{C}_1(E, q, L)$ inherits a $\mathscr{C}_0(E, q, L)$-bimodule structure. Also, $\mathscr{C}_1(E, q, L)$ has an $\mathscr{O}_S$-module filtration

$$E = F_1 \subset F_3 \subset \cdots \subset F_{2m+1} = \mathscr{C}_1(E, q, L),$$

where F_{2i+1} is the image of the truncation $E \otimes T^{\leqslant i}(E \otimes E \otimes L^\vee)$ in $\mathscr{C}_1(E, q, L)$, for each $0 \leqslant i \leqslant m$. This filtration has associated graded pieces $F_{2i+1}/F_{2i-1} \cong \bigwedge^{2i+1} E \otimes (L^\vee)^{\otimes i}$. In particular, $\mathscr{C}_1(E, q, L)$ is a locally free $\mathscr{O}_S$-module of rank 2^{n-1}.

For a list of basic functorial properties of the even Clifford algebra and Clifford bimodule, see Appendix A. In particular, Proposition A.2(4) shows that the $\mathscr{O}_S$-algebra isomorphism class $\mathscr{C}_0(E, q, L)$ is an invariant of the projective similarity class of (E, q, L), hence by Proposition 1.2.1, of the associated quadric fibration $\pi : Q \to S$. We can thus speak of *the* even Clifford algebra (defined up to isomorphism) of a quadric fibration.

1.6. The discriminant cover: even rank case

Let (E, q, L) be a generically regular quadratic form of even rank on a scheme S and $\mathscr{Z} = \mathscr{Z}(E, q, L)$ the center of $\mathscr{C}_0 = \mathscr{C}_0(E, q, L)$. We call the associated affine morphism $f : T = \mathbf{Spec}\,\mathscr{Z} \to S$ the *discriminant cover*.

Lemma 1.6.1. *Let (E, q, L) be a generically regular quadratic form of even rank over a locally factorial integral scheme. Then $\mathscr{Z}$ is a locally free $\mathscr{O}_S$-algebra of rank 2, hence $f : T \to S$ is a finite flat morphism of degree 2.*

Proof. As pointed out before [56, IV Prop. 4.8.3], the proof of the statement in the affine case [56, IV Prop. 4.8.1] goes through over any factorial domain. □

We denote by $\mathscr{B}_0 = \mathscr{B}_0(E, q, L)$ the $\mathscr{O}_T$-algebra associated to the $\mathscr{Z}$-algebra $\mathscr{C}_0$. By Proposition A.2(1), $\mathscr{Z}$ is étale over every point of S where (E, q, L) is regular. By Proposition A.2(2), $\mathscr{B}_0$ is Azumaya over every point of T laying over a point of S where (E, q, L) is regular. The following generalization will be extremely important in the sequel (cf. [63, Section 3.5]).

Proposition 1.6.2. *Let (E, q, L) be a generically regular quadratic form of even rank on a locally factorial integral scheme S. Let $f : T \to S$ be the discriminant cover. Then $\mathscr{B}_0$ is an Azumaya algebra over every point of T lying over a point of S where (E, q, L) has simple degeneration. In particular, if (E, q, L) has simple degeneration on S, then $\mathscr{B}_0$ is an Azumaya algebra on T.*

Proof. Let $(\mathscr{O}, \mathfrak{m})$ be the local ring of a point of S where (E, q, L) has simple degeneration. The center $\mathscr{Z}_\mathscr{O}$ of $\mathscr{C}_0$ restricted to $\mathscr{O}$, is a free $\mathscr{O}$-algebra of rank 2 by Lemma 1.6.1. Suppressing the choice of a trivialization of L over $\mathscr{O}$, we write $(E, q)_\mathscr{O} = (E_1, q_1) \perp (\mathscr{O}, \langle \pi \rangle)$ for q_1 a semiregular quadratic form of odd rank over $\mathscr{O}$ and $\pi \in \mathscr{O}$. We claim that the map $\psi : \mathscr{C}_0(E_1, q_1) \otimes_\mathscr{O} \mathscr{Z}_\mathscr{O} \to \mathscr{C}_0$, defined by the canonical monomorphism $\mathscr{C}_0(E_1, q_1) \to \mathscr{C}_0$ (and multiplication in $\mathscr{C}_0$), is a $\mathscr{Z}_\mathscr{O}$-algebra isomorphism. As (E_1, q_1) is semiregular of odd rank, $\mathscr{C}_0(E_1, q_1)$ is an Azumaya algebra on S (cf. Proposition A.2(1)), hence remains Azumaya upon tensoring by $\mathscr{Z}_\mathscr{O}$. In particular, ψ is a monomorphism. By [56, IV Prop. 7.3.1], ψ is surjective upon identifying $\mathscr{Z} \cong \mathscr{C}(\mathscr{Z}_1(E_1, q_1) \otimes \langle -\pi \rangle)$, where $\mathscr{Z}_1(E_1, q_1) = \mathscr{Z}(E_1, q_1) \cap \mathscr{C}_1(E_1, q_1)$ is an invertible $\mathscr{O}$-module. This shows that $\mathscr{B}$ is a locally free $\mathscr{O}_T$-algebra of constant rank that is Zariski locally Azumaya, hence is Azumaya. A different proof can be found in [63, Prop. 3.13], cf. [6, Prop. 1.13]. □

1.7. The discriminant stack: odd rank case

Let (E, q, L) be a generically regular quadratic form of odd rank $n = 2m + 1$ on S. We further assume that 2 is invertible on S. Recall that the discriminant divisor $D \subset S$ is the zero locus of the global section $\mathrm{disc}(q) \in \Gamma(Y, (\det E^\vee)^{\otimes 2} \otimes L^{\otimes n})$. We define the *discriminant stack* to be the square root stack $\mathcal{T}$ associated to the pair $((\det E^\vee)^{\otimes 2} \otimes L^{\otimes n}, \mathrm{disc}(q))$ over S, cf. [25, Section 2.2]. As 2 is invertible, $\mathcal{T}$ is a Deligne–Mumford stack (see [25, Thm. 2.3.3]) and has coarse moduli space $f : \mathcal{T} \to S$ (see [25, Cor. 2.3.7]). Furthermore, if S and D are smooth over a field k, then $\mathcal{T}$ is smooth (see [25, Ex. 2.4.5]).

We have the following results analogous to Lemma 1.6.1 and Proposition 1.6.2 (cf. [63, Section 3.6]). Recall the construction of the "full" Clifford algebra (1.5.2) in the case where $L = \mathscr{O}_S$.

Lemma 1.7.1. *Let (E, q) be a generically regular quadratic form (with values in $\mathscr{O}_S$) of odd rank over a locally factorial integral scheme. Then the center $\mathscr{Z}$ of the "full" Clifford algebra is a locally free $\mathscr{O}_S$-algebra of rank 2.*

Proof. See the proof of Lemma 1.6.1. □

Proposition 1.7.2. *Let (E, q, L) be a generically regular quadratic form of odd rank $n = 2m + 1$ on a locally factorial integral scheme S with 2 invertible. Let $f : \mathcal{T} \to S$ be the discriminant stack. Then there exists a locally free algebra $\mathscr{B}_0$ on $\mathcal{T}$ such that $f_* \mathscr{B}_0 = \mathscr{C}_0$, and which is Azumaya over every point of $\mathcal{T}$ lying over a point of S where (E, q, L) has simple degeneration. In particular, if (E, q, L) has simple degeneration on S, then $\mathscr{B}_0$ is an Azumaya algebra on $\mathcal{T}$.*

Proof. We first give a local description of the discriminant stack (cf. [63, Section 3.6]). Let $U \subset S$ be a Zariski affine open trivializing E and also L via $l : \mathscr{O}_U \to L|_U$. Then there is an $\mathscr{O}_U$-valued quadratic form $(E|_U, l^{-1}q|_U)$ on U. Hence we can consider the classical "full" Clifford algebra $\mathscr{C}_U = \mathscr{C}(E|_U, l^{-1}q|_U)$, and $\mathscr{Z}_U$ its center. By Lemma 1.7.1, $\mathscr{Z}_U$ is a locally free $\mathscr{O}_U$-algebra of rank 2. As S is integral, we have $\mathscr{Z}_U \cong \mathscr{O}_U[z]/(z^2 - d)$, where $d = l^{-n} \operatorname{disc}(q)|_U$ by [56, IV Props. 4.8.5(1), 4.8.9(1)]. Then $T_U = \operatorname{Spec} \mathscr{Z}_U \to U$ is a finite flat double cover branched along $D \cap U$. By the local description of the root stack (cf. [25, Ex. 2.4.1]), we have that $\mathcal{T}|_U \cong [T_U / \mu_2]$, where μ_2 acts $\mathscr{O}_U$-linearly on T_U via the trivial character on $1_{\mathscr{Z}_U}$ and the standard character on z. As 2 is invertible, this coincides with the action of the Galois group $\mathbb{Z}/2\mathbb{Z} \cong \mu_2$ of T_U over U.

We now construct the $\mathscr{O}_{\mathcal{T}}$-algebra $\mathscr{B}_0$. For a Zariski affine open $U \subset S$, let $T_U \to U$ and $\mathscr{C}_U$ as above, and denote by $\mathscr{B}_U$ the $\mathscr{O}_{T_U}$-algebra associated to $\mathscr{C}$. Negation on E, when restricted to U, induces an $\mathscr{O}_U$-automorphism of $\mathscr{C}_U$ of order 2, which restricts to the Galois automorphism of $\mathscr{Z}_U$ over $\mathscr{O}_U$ (since $\mathscr{Z}_U$ is generated by an element z in odd degree). Hence $\mathscr{B}_U$ is naturally a $\mathbb{Z}/2\mathbb{Z} \cong \mu_2$-equivariant $\mathscr{O}_{T_U}$-algebra for the Galois action on T_U over U, hence defines an $\mathscr{O}_{[T_U/\mu_2]}$-algebra. Since the equivariant structure over U is induced by a global automorphism of E over S, this construction glues over a Zariski affine open cover to yield an $\mathscr{O}_{\mathcal{T}}$-algebra $\mathscr{B}_0$.

To check where $\mathscr{B}_0$ is Azumaya, we work locally. Let $(\mathscr{O}, \mathfrak{m})$ be the local ring of a point of S where (E, q, L) has simple degeneration and let $l : \mathscr{O} \to L_{\mathscr{O}}$ be a trivialization (if (E, q, L) is regular at $\mathscr{O}$, then the argument simplifies). We can write $(E_{\mathscr{O}}, l^{-1}q_{\mathscr{O}}) = (E_1, q_1) \perp (\mathscr{O}, \langle \pi \rangle)$ for q_1 a regular quadratic form of even rank over $\mathscr{O}$ and $\pi \in \mathscr{O}$. The center $\mathscr{Z}_{\mathscr{O}}$ of the full Clifford algebra $\mathscr{C}_{\mathscr{O}} = \mathscr{C}(E_{\mathscr{O}}, l^{-1}q_{\mathscr{O}})$ is a free $\mathscr{O}$-algebra of rank 2 by Lemma 1.7.1. We claim that the map $\psi : \mathscr{C}(E_1, q_1) \otimes_{\mathscr{O}} \mathscr{Z}_{\mathscr{O}} \to \mathscr{C}_{\mathscr{O}}$, defined by the canonical monomorphism $\mathscr{C}(E_1, q_1) \to \mathscr{C}_{\mathscr{O}}$ and multiplication in $\mathscr{C}_{\mathscr{O}}$, is a $\mathscr{Z}_{\mathscr{O}}$-algebra isomorphism. As (E_1, q_1) is regular of even rank, $\mathscr{C}(E_1, q_1)$ is an Azumaya $\mathscr{O}$-algebra (cf. Proposition A.2(1)), hence remains Azumaya upon tensoring with $\mathscr{Z}_{\mathscr{O}}$. In particular, ψ is a monomorphism. By [56, IV Prop. 7.2.1], ψ is surjective upon identifying $\mathscr{Z}_{\mathscr{O}} \cong \mathscr{C}(\mathscr{Z}_{\mathscr{O}}^0 \otimes \langle \pi \rangle)$, where $\mathscr{Z}_{\mathscr{O}}^0 = \ker(\operatorname{tr} : \mathscr{Z}(E_1, q_1) \to \mathscr{O})$ coincides with the submodule generated by z as above. This shows that $\mathscr{B}_0$ is glued from μ_2-equivariant Azumaya algebras of constant rank, hence defines an Azumaya $\mathscr{O}_{\mathcal{T}}$-algebra. A different proof can be found in [63, Prop. 1.15]. □

1.8. A Morita equivalence

The main result of this section is Theorem 3. Leading up to its proof, we provide some useful formulas for the even Clifford algebra of a (line bundle-valued) quadratic form. See [56, IV Thm. 1.3.1] for analogous formulas when the value line bundle is trivial.

Lemma 1.8.1. *Let (E_1, q_1, L) and (E_2, q_2, L) be quadratic forms over a scheme S. Then there is an isomorphism of $\mathscr{O}_S$-algebras*

$$\mathscr{C}_0(q_1 \perp q_2) \cong \mathscr{C}_0(q_1) \otimes \mathscr{C}_0(q_2) \oplus \mathscr{C}_1(q_1) \otimes \mathscr{C}_1(q_2) \otimes L^{\vee}$$

where the algebra structure on the right is given by multiplication in $\mathscr{C}_0$, by the module action of $\mathscr{C}_0$ on $\mathscr{C}_1$, and by the multiplication map in Proposition A.3(1).

Proof. See [5, Eq. (9)] for a proof making use of the universal property (1.5.1). □

Proposition 1.8.2. *Let $(E, q, L) \cong H_L(N) \perp (E', q', L)$ be an orthogonal decomposition of quadratic forms on a scheme S, with N a line bundle and (E', q', L') primitive. Then we have an $\mathscr{O}_S$-algebra isomorphism*

$$\mathscr{C}_0(q) \cong \mathscr{E}nd_{\mathscr{C}_0(q')}(P),$$

where $P = \mathscr{C}_0(q') \oplus N^\vee \otimes \mathscr{C}_1(q')$ is a locally free right $\mathscr{C}_0(q')$-module of rank 2. In particular, $\mathscr{C}_0(q)$ is Morita S-equivalent to $\mathscr{C}_0(q')$. Furthermore, assuming that S is integral and locally factorial and that E has even rank, then there is an $\mathscr{O}_S$-algebra isomorphism $\mathscr{Z}(q) \cong \mathscr{Z}(q')$ of centers, with respect to which the induced left and right actions of the centers on P commute.

Proof. By Lemma 1.8.1, we have

$$\mathscr{C}_0(q) \cong (\mathscr{O}_S \oplus \mathscr{O}_S) \otimes \mathscr{C}_0(q') \oplus (N \oplus \mathscr{H}om(N, L)) \otimes \mathscr{C}_1(q') \otimes L^\vee.$$

Then we have to trace through the multiplication to show that we can write this algebra in a block matrix decomposition

$$\begin{pmatrix} \mathscr{C}_0(q') & N \otimes \mathscr{C}_1(q') \otimes L^\vee \\ N^\vee \otimes \mathscr{C}_1(q') & \mathscr{C}_0(q') \end{pmatrix}$$

using the fact that

$$\mathscr{H}om_{\mathscr{C}_0(q')}(N^\vee \otimes \mathscr{C}_1(q'), \mathscr{C}_0(q')) \cong N \otimes \mathscr{H}om_{\mathscr{C}_0(q')}(\mathscr{C}_1(q'), \mathscr{C}_0(q')) \cong N \otimes \mathscr{C}_1(q') \otimes L^\vee.$$

In this last step, we needed the fact that the multiplication map from Proposition A.3(1) yields an isomorphism

$$\mathscr{C}_1(q') \otimes_{\mathscr{C}_0(q')} \mathscr{C}_1(q') \to \mathscr{C}_0(q') \otimes L$$

of $\mathscr{C}_0(q')$-bimodules (since q' is assumed primitive). For the final statement, restricting the isomorphism of Lemma 1.8.1 to the center yields an $\mathscr{O}_S$-algebra monomorphism $\psi : \mathscr{Z}(q) \to \mathscr{Z}(H_L(N)) \otimes \mathscr{Z}(q')$. Note that $\mathscr{Z}(H_L(N)) \cong \mathscr{O}_S \times \mathscr{O}_S$. Considering the Galois automorphisms ι of $\mathscr{Z}(H_L(N))$ and ι' of $\mathscr{Z}(q')$ over $\mathscr{O}_S$, then as in the proof of [5, Prop. 2.3], we compute that $(1 - \iota \otimes \iota')$ annihilates the image of ψ. Hence we arrive at an $\mathscr{O}_S$-monomorphism $\mathscr{Z}(q) \to \mathscr{Z}(q')$, which is an isomorphism over $S \setminus D$ by [56, IV Section 4.4] and over D by Corollary 1.3.9, where D is the discriminant divisor of q. $\square$

Let S be a locally ringed topos. We say that $\mathscr{O}_S$-algebras $\mathscr{A}$ and $\mathscr{A}'$ are *Morita S-equivalent* if the fibered categories of finitely presented (right) modules $\mathsf{COH}(S, \mathscr{A})$ and $\mathsf{COH}(S, \mathscr{A}')$ are S-equivalent, i.e., inducing $\mathscr{O}_S$-module isomorphisms on $\mathscr{H}om$ sheaves, see Lieblich [68, Section 2.1.4] or Kashiwara and Schapira [54, Section 19.5]. First recall "Morita theory" in the context of locally ringed topoi, which is proved in [68, Prop. 2.1.4.4] and [54, Thm. 19.5.4].

Theorem 1.8.3 (*Morita theory*). *Let S be a locally ringed topos and $\mathscr{A}$ and $\mathscr{A}'$ be $\mathscr{O}_S$-algebras. Then*:

I. *Any invertible $\mathscr{A}' \otimes_{\mathscr{O}_S} \mathscr{A}^{\mathrm{op}}$-module $\mathscr{P}$ gives rise to Morita S-equivalences*

$$\mathscr{P} \otimes_{\mathscr{A}} - : \mathsf{COH}(S, \mathscr{A}) \to \mathsf{COH}(S, \mathscr{A}'), \qquad \mathscr{H}om_{\mathscr{A}'}(\mathscr{P}, -) : \mathsf{COH}(S, \mathscr{A}') \to \mathsf{COH}(S, \mathscr{A}).$$

II. *Any Morita S-equivalence $F : \mathsf{COH}(S, \mathscr{A}) \to \mathsf{COH}(S, \mathscr{A}')$ is isomorphic to $\mathscr{P} \otimes_{\mathscr{A}} -$ for some invertible $\mathscr{A}' \otimes_{\mathscr{O}_S} \mathscr{A}^{\mathrm{op}}$-module $\mathscr{P}$.*

We also recall that if $\mathscr{A}$ and $\mathscr{A}'$ are Azumaya algebras on S then Morita S-equivalence is equivalent to Brauer equivalence, cf. [68, Prop. 2.1.5.6].

Let $T \to S$ be a morphism of locally ringed topoi and fix a $\mathbf{G}_\mathrm{m}$-gerbe $\mathcal{Y} \to T$ (e.g., the gerbe of splittings of an fixed Azumaya algebra). Given a sheaf $\mathscr{V}$ on $\mathcal{Y}$, denote by $\mathscr{V} \times \mathbf{G}_\mathrm{m} \to \mathscr{V}$ the natural inertial action. We recall that $\mathcal{Y}$ defines a class in $H^2(T, \mathbf{G}_\mathrm{m})$, see [36, IV Section 3]. The Brauer group of Azumaya algebras $\mathrm{Br}(T)$ naturally injects into the torsion subgroup of $H^2(T, \mathbf{G}_\mathrm{m})$, see [40, I Section 2].

Definition 1.8.4. A $\mathcal{Y}$-*twisted sheaf* on T is an $\mathscr{O}_\mathcal{Y}$-module $\mathscr{V}$ such that the inertial action $\mathscr{V} \times \mathbf{G}_\mathrm{m} \to \mathscr{V}$ is equal to the right action associated to the left module action $\mathbf{G}_\mathrm{m} \times \mathscr{V} \to \mathscr{V}$.

Proposition 1.8.5. *Let T be an integral Noetherian Artin stack with generic point η. Let $\mathcal{Y} \to T$ be a $\mathbf{G}_m$-gerbe. Then*:

1. *Any quasi-coherent $\mathscr{O}_{\mathcal{Y}}$-module is the colimit of its coherent $\mathscr{O}_{\mathcal{Y}}$-submodules.*
2. *Any coherent $\mathcal{Y}$-twisted sheaf on η extends to a coherent $\mathcal{Y}$-twisted sheaf on T.*

Proof. The first statement follows from [66, Prop. 15.4], as any $\mathbf{G}_m$-gerbe over an Artin stack is itself an Artin stack, cf. [69, 2.2.1.5]. The second statement immediately follows from the first. $\square$

We need the following generalization of [7] or [40, II Cor. 1.10] to the setting of stacks.

Proposition 1.8.6. *Let T be an regular integral Noetherian Artin stack with generic point η. Then the restriction map $\mathrm{Br}(T) \to \mathrm{Br}(\eta)$ is injective.*

Proof. The proof in [70, Prop. 3.1.3.3] (with the hypothesis that T is an algebraic space) immediately generalizes to any Noetherian Artin stack by appealing to Proposition 1.8.5. $\square$

This can be interpreted as saying that two Azumaya algebras on a Noetherian Artin stack that are Zariski locally Brauer equivalent are actually Brauer equivalent. Finally, we can state the main result of this section, of which Theorem 3 is a special case.

Theorem 1.8.7. *Let (E, q, L) be a quadratic form over a regular integral scheme S, with simple degeneration along a regular divisor D. In the case of odd rank, assume that 2 is invertible on S. Let (E', q', L') be the reduced quadratic form associated to a smooth isotropic subbundle $N \to E$. Then the even Clifford algebras $\mathscr{C}_0(E, q, L)$ and $\mathscr{C}_0(E', q', L)$ are Morita S-equivalent. Furthermore, the associated Azumaya algebras $\mathscr{B}_0$ and $\mathscr{B}_0'$ on the discriminant cover or stack are Brauer equivalent.*

As an immediate consequence, we get the following in terms of derived categories. Denote by β the Brauer class of the Azumaya algebra $\mathscr{B}_0$ on the discriminant cover $T \to S$ or stack $\mathcal{T} \to S$, see Sections 1.6–1.7.

Corollary 1.8.8. *Let $Q \to S$ be a quadric fibration over a regular integral scheme with simple degeneration along a regular divisor and $Q' \to S$ obtained by quadric reduction along a smooth section. In odd relative dimension, assume that 2 is invertible on S. Let $\mathscr{C}_0$ and $\mathscr{C}_0'$ be the respective even Clifford algebras. Then there are equivalences $\mathsf{D}^b(S, \mathscr{C}_0) \simeq \mathsf{D}^b(S, \mathscr{C}_0')$ and $\mathsf{D}^b(T, \beta) \simeq \mathsf{D}^b(T, \beta')$ or $\mathsf{D}^b(\mathcal{T}, \beta) \simeq \mathsf{D}^b(\mathcal{T}, \beta')$ depending on the parity of the relative dimension.*

Proof of Theorem 1.8.7. By induction on the rank of N using Theorem 1.3.8, we can assume that $N \subset E$ is an isotropic line subbundle. By Corollary 1.3.9, (E', q', L) also has simple degeneration along D. We can assume that D is nonempty, otherwise the statement of the theorem is classical, see [56, IV Prop. 8.1.1]. Write $\mathscr{C}_0 = \mathscr{C}_0(E, q, L)$ and $\mathscr{C}_0' = \mathscr{C}_0(E', q', L)$.

First we handle the case of even rank. Let $f : T \to S$ be the discriminant cover as in Section 1.6 and $\mathscr{B}_0$ and $\mathscr{B}_0'$ the Azumaya algebras on T associated to $\mathscr{C}_0$ and $\mathscr{C}_0'$ by Proposition 1.6.2. Since S and D are regular and $f : T \to S$ is a finite flat morphism branched along D, we have that T is regular. By Theorem 1.3.8 and Propositions 1.8.2, we have that $\mathscr{B}_0$ and $\mathscr{B}_0'$ are Zariski locally Brauer equivalent, hence they are Brauer equivalent Proposition 1.8.6. In particular, $\mathscr{C}_0 = f_*\mathscr{B}_0$ and $\mathscr{C}_0' = f_*\mathscr{B}_0'$ are Morita S-equivalent.

Now we deal with the case of odd rank. Let $f : \mathcal{T} \to S$ be the discriminant stack as in Section 1.7 and $\mathscr{B}_0$ and $\mathscr{B}_0'$ the Azumaya algebras on $\mathcal{T}$ associated to $\mathscr{C}_0$ and $\mathscr{C}_0'$ by Proposition 1.7.2. Since S and D are regular, the local discriminant covers are regular, hence $\mathcal{T}$ is regular, being covered by quotient stacks of regular schemes by finite étale group schemes (we use again our hypothesis that 2 is invertible). Since S is integral and D is nonempty, $\mathcal{T}$ is integral. By Theorem 1.3.8 and Propositions 1.8.2, we have that $\mathscr{C}_0$ and $\mathscr{C}_0'$ are Zariski locally Morita equivalent on S, hence $\mathscr{B}_0$ and $\mathscr{B}_0'$ are Zariski locally Brauer equivalent on $\mathcal{T}$ (by the uniqueness of the coarse moduli space), hence they are Brauer equivalent by Proposition 1.8.6 (hence Morita $\mathcal{T}$-equivalent). In turn, this implies that $\mathscr{C}_0 = f_*\mathscr{B}_0$ and $\mathscr{C}_0' = f_*\mathscr{B}_0'$ are Morita S-equivalent. $\square$

Remark 1.8.9. The equivalence in the proof of Corollary 1.8.8 is actually a Fourier–Mukai functor whose kernel is an object $P \in \mathsf{D}^{\mathsf{b}}(S \times S)$ whose local structure over S (see Theorem 1.3.8) is described by Proposition 1.8.2.

1.9. Two useful results from the algebraic theory of quadratic forms

We use two results from the classical algebraic theory of quadratic forms: the Amer–Brumer theorem and a Clifford algebra condition for isotropy of quadratic forms of rank 4. To fit the context, we will state these results in the geometric language of quadric fibrations.

Theorem 1.9.1 (*Amer–Brumer Theorem*). *Let $X \to Y$ be the relative complete intersection of two quadric fibrations over an integral scheme Y and let $Q \to S$ be the associated linear span quadric fibration. Then $X \to Y$ has a rational section if and only if $Q \to S$ has a rational section. Furthermore, if $X \to Y$ has a section* (*resp. smooth section*)*, then so does $Q \to S$.*

Proof. The first assertion immediately reduces to the classical Amer–Brumer theorem for fields (see [33, Thm. 17.14]) by going to the generic point. Any section of $X \to Y$ immediately yields a section of $Q \to S$ and the final assertion follows since the regular locus of a fiber of $X \to Y$ is contained in the intersection of the regular loci of the corresponding fibers of the two quadric fibrations. □

Remark 1.9.2. There is an amplification of the Amer–Brumer Theorem (cf. Leep [67, Thm. 2.2], see also [2] or [77]), stating that $X \to Y$ rationally contains a linear subspace of a given dimension if and only if the linear span $Q \to S$ does.

We deduce one corollary which will be useful in the sequel. While (at least in characteristic zero) we could appeal to [27] or [37], here we give a direct argument.

Lemma 1.9.3. *Let $X \to Y$ be a relative complete intersection of two quadric fibrations over a smooth complete curve Y over an algebraically closed field. If $X \to Y$ has relative dimension > 2 then it has a section.*

Proof. Let $Q \to S$ be the associated linear span quadric fibration represented by a quadratic form q over S of rank $\geqslant 5$. Since Y is an integral curve, S is an integral surface, whose function field K (over an algebraically closed field) is thus a C_2-field (cf. [33, Thm. 97.7]). Hence q has a nontrivial zero over K, i.e., $Q \to S$ has a rational section. Then by the Amer–Brumer Theorem 1.9.1, $X \to Y$ has a rational section, hence a section (since Y is a smooth curve and X is proper). □

Now we come to a geometric rephrasing of a classical fact about the isotropy of quadratic forms of rank 4.

Theorem 1.9.4. *Let $\pi : Q \to S$ be a quadric surface fibration with simple degeneration along a smooth divisor over a regular integral scheme S and let $T \to S$ be its discriminant cover. Then $\mathscr{B}_0 \in \mathrm{Br}(T)$ is trivial if and only if π has a rational section.*

Proof. Let (E, q, L) be a quadratic form of rank 4 over S whose associated quadric surface fibration is $Q \to S$. Let K be the function field of S and L the function field of T. Then on the generic fiber, $\mathscr{C}_{0,K}$ is the even Clifford algebra of the quadratic form $(E, q, L)_K$ of rank 4 over K. By [58, Thm. 6.3] (also see [81, 2 Thm. 14.1, Lemma 14.2] in characteristic $\neq 2$ and [8, II Prop. 5.3] in characteristic 2), the regular quadratic form q_K is isotropic (i.e., $\pi : Q \to S$ has a rational section) if and only if q_L is isotropic if and only if $\mathscr{B}_{0,L} \in \mathrm{Br}(L)$ is trivial. Since T is regular (since S and the discriminant divisor are), $\mathrm{Br}(T) \to \mathrm{Br}(L)$ is injective (see Proposition 1.8.6). In particular, $\mathscr{B}_0 \in \mathrm{Br}(T)$ is trivial if and only if $\mathscr{B}_{0,L} \in \mathrm{Br}(L)$ is trivial. □

We shall state an often (albeit implicitly) used corollary of this after recalling some obvious facts about rationality of relative schemes. Let k be an arbitrary field and S an integral k-scheme with function field $k(S)$. To any morphism $\pi : Z \to S$ denote by $Z_{k(S)} = Z \times_S \mathrm{Spec}\, k(S) \to \mathrm{Spec}\, k(S)$ its generic fiber.

Lemma 1.9.5. *Let Z and Z' be schemes over an integral k-scheme S.*

1. *If the $k(S)$-schemes $Z_{k(S)}$ and $Z'_{k(S)}$ are $k(S)$-birational then the k-schemes Z and Z' are k-birational.*
2. *Assume that S is k-rational. If the $k(S)$-scheme $Z_{k(S)}$ is $k(S)$-rational then the k-scheme Z is k-rational.*

Proof. For (1), note that we have a canonical isomorphism of function fields $k(Z') \cong k(S)(Z'_{k(S)})$. Hence if $Z_{k(S)}$ and $Z'_{k(S)}$ are $k(S)$-birational, then the total ring of fractions $k(Z) \cong k(S)(Z_{k(S)})$ and $k(Z') \cong k(S)(Z'_{k(S)})$ are $k(S)$-isomorphic. In particular, they are k-isomorphic, hence Z and Z' are k-birational.

For (2), we apply (1) with $Z' = \mathbb{P}^n_S$. First note that since S is k-rational, we have that $\mathbb{P}^n_S$ is k-rational. Now by (1), $Z_{k(S)}$ is $k(S)$-rational implies $Z_{k(S)}$ is $k(S)$-birational to $\mathbb{P}^n_{k(S)}$ implies Z is k-birational to $\mathbb{P}^n_S$ implies that Z is k-rational. $\square$

Corollary 1.9.6. *With the hypotheses of* Theorem 1.9.4*, assume that S is a rational k-scheme. If the class of $\mathscr{B}_0$ in* $\mathrm{Br}(T)$ *is trivial then Q is k-rational.*

Proof. Let $Q_{k(S)}$ be the generic fiber of $Q \to S$, which is a smooth quadric over the function field $k(S)$. Note that $Q_{k(S)}$ is $k(S)$-rational if and only if (see [33, Prop. 22.9]) $Q_{k(S)}$ has a $k(S)$-rational point (i.e., $\pi : Q \to S$ has a rational section) if and only if (by Theorem 1.9.4) $\mathscr{B}_0 \in \mathrm{Br}(T)$ is trivial. Now since S is rational, if $Q_{k(S)}$ is $k(S)$-rational then Q is k-rational, by Lemma 1.9.5. $\square$

As a simple application, Corollary 1.9.6 gives a proof, in the spirit of quadratic forms, of one of the main results of [42]: if Z is a smooth cubic fourfold containing a k-plane such that the associated quadric surface bundle $Z' \to \mathbb{P}^2$ has simple degeneration, then Z is k-rational if the associated Brauer class of $\mathscr{B}_0$ on the discriminant cover (which is a K3 surface of degree 2) is trivial.

2. Categorical tools

In this section, we introduce the main categorical tools that we will apply to geometric examples. The main new technical tool that we introduce (in Section 2.3) is relative version of homological projective duality, which we apply in the case of quadric fibrations. This is a generalization of Kuznetsov's original theory [62]. We note that most definitions in this section were originally stated over an algebraically closed field of characteristic zero. Where necessary, we will show that this hypothesis is often not needed to prove the basic results we'll utilize.

2.1. Semiorthogonal decompositions over a base

Let k be a field and T a k-linear triangulated category. Any unadorned product of k-schemes will denote a fiber product over k. In their seminal work [20], Bondal and Kapranov define semiorthogonal decompositions for k-linear triangulated categories, in the case where k is algebraically closed of characteristic zero. We will briefly recall the definitions and main results from [20, Sections 2–3], arguing that they are still valid over any field.

Given a full triangulated subcategory A of T, we denote by $\mathsf{A}^\perp$ its *right orthogonal*, that is the full subcategory of T whose objects are all the B satisfying $\mathrm{Hom}_\mathsf{T}(A, B) = 0$. Similarly we define the *left orthogonal* $^\perp\mathsf{A}$. A full triangulated subcategory A of T is called *right* (resp. *left*) *admissible* if the embedding functor admits a right (resp. left) adjoint. The subcategory A is *admissible*, if it is both right and left admissible. Notice that the original definition of admissibility is different, but equivalent to this one [20, Def. 1.2, Prop. 1.5].

Proposition 2.1.1. *(See [20, Prop. 1.5].) Let A be a right (resp. left) admissible subcategory of T. Then T is generated by A and $\mathsf{A}^\perp$ (resp. A and $^\perp\mathsf{A}$) as a triangulated category. In particular, if A is right admissible, $\mathsf{A}^\perp$ is left admissible, and if A is left admissible, $^\perp\mathsf{A}$ is right admissible.*

Proof. The proof of [20, Prop. 1.5] works over any field: the first statement is the implication $(b) \Rightarrow (c)$ in the notation of [20]; the second and third statements follow from the implication $(c) \Rightarrow (b)$. $\square$

Definition 2.1.2. (See [21, Def. 2.4].) A *semiorthogonal decomposition* of T is an ordered sequence of admissible subcategories $\mathsf{A}_1, \ldots, \mathsf{A}_n$ of T such that:

- For all objects A_i of A_i and A_j of A_j, $\mathrm{Hom}_\mathsf{T}(A_i, A_j) = 0$ for all $i > j$, and
- For every object T of T, there is a chain of morphisms $0 = T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 = T$ such that the cone of $T_k \to T_{k-1}$ is an object of A_k for all $1 \leqslant k \leqslant n$.

Such a decomposition will be written

$$\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_n \rangle.$$

Now fix a scheme Y smooth over k and of finite Krull dimension.

Definition 2.1.3. Let $p : X \to Y$ be a Y-scheme. A strictly full subcategory A of $\mathsf{D}^b(X)$ is *Y-linear* if for every E in A and G in $\mathsf{D}^b(Y)$ we have that $p^*G \otimes E$ is in A. A semiorthogonal decomposition of $\mathsf{D}^b(X)$ is Y-linear if all the components are Y-linear. If $p' : X' \to Y$ is another Y-scheme, then a functor $F : \mathsf{D}^b(X) \to \mathsf{D}^b(X')$ is Y-linear if for every E in $\mathsf{D}^b(X)$ and G in $\mathsf{D}^b(Y)$, there is a bifunctorial isomorphism $F(p^*G \otimes E) \cong p'^*G \otimes F(E)$.

Lemma 2.1.4. *Let X be a Y-scheme. If A in $\mathsf{D}^b(X)$ is a Y-linear admissible subcategory, then both $^\perp\mathsf{A}$ and $\mathsf{A}^\perp$ are Y-linear. If X' is another Y-scheme and $F : \mathsf{D}^b(X) \to \mathsf{D}^b(X')$ is a Y-linear functor admitting a right or left adjoint, then this adjoint is also Y-linear.*

Proof. The first claim is [61, Lemma 2.36]. The second claim is [61, Lemma 2.33] for the right adjoint; for the left adjoint, the proof is the same, but uses the contravariant version of Yoneda's lemma.   □

Fourier–Mukai functors provide a geometric way of producing admissible subcategories. Let X and X' be smooth schemes over k and E an object of $\mathsf{D}^b(X' \times X)$. The *Fourier–Mukai functor* with kernel E is the functor $\Phi_E : \mathsf{D}^b(X') \to \mathsf{D}^b(X)$ defined by

$$\Phi_E(-) = q_*\big(p^*(-) \otimes E\big),$$

where p and q are the projections from $X' \times X$ to X' and X, respectively (recall the we denote derived functors as if they were underived). If X and X' are Y-schemes, $i : X' \times_Y X \to X' \times X$ is the natural embedding, and E is an object in $i_*\mathsf{D}^b(X' \times_Y X) \subset \mathsf{D}^b(X' \times X)$, then the Fourier–Mukai functor $\Phi_E : \mathsf{D}^b(X') \to \mathsf{D}^b(X)$ is Y-linear, see [61, Lemma 2.35].

Proposition 2.1.5. *Let X and X' be smooth schemes over k and $\Phi_E : \mathsf{D}^b(X') \to \mathsf{D}^b(X)$ a Fourier–Mukai functor. If $\mathrm{Supp}(E)$ is proper over X' and X then Φ_E has a right and left adjoint of Fourier–Mukai type. If in addition, Φ_E is fully faithful, then there is a semiorthogonal decomposition*

$$\mathsf{D}^b(X) = \big\langle \mathsf{A}, \Phi_E\big(\mathsf{D}^b(X')\big) \big\rangle,$$

where A is the left orthogonal of $F(\mathsf{D}^b(X'))$.

Proof. This is [61, Lemmas 2.4–2.5], noting that since X and X' are smooth, the various conditions on the finiteness of cohomological amplitude are satisfied, ensuring that the adjoints land in the derived category of bounded complexes and are of Fourier–Mukai type. Also, the projectivity hypothesis can be relaxed to properness, which is all that is required to invoke Grothendieck–Verdier duality.   □

We will mostly use Proposition 2.1.4 in the following relative situation. Let X and X' be schemes proper over Y and smooth over k. Then any Y-linear Fourier–Mukai functor $\Phi_E : \mathsf{D}^b(X) \to \mathsf{D}^b(X')$ has right and left adjoints. In fact, there is a stronger result about the existence of adjoints.

Proposition 2.1.6. *Let X and X' be smooth schemes over k. If X' is proper, then any exact functor $F : \mathsf{D}^b(X') \to \mathsf{D}^b(X)$ has a left and right adjoint. In particular, if F is fully faithful, then there is a semiorthogonal decomposition*

$$\mathsf{D}^b(X) = \langle \mathsf{A}, F\big(\mathsf{D}^b(X')\big) \rangle,$$

where A is the left orthogonal of $F(\mathsf{D}^b(X'))$.

Proof. By [20, Prop. 2.14] and [23, Cor. 3.1.5], $\mathsf{D}^b(Y)$ is saturated, i.e., every functor (covariant or contravariant) of finite type, from $\mathsf{D}^b(Y)$ to the category of k-vector spaces, is representable. By [20, Props. 2.6, 2.14], any full triangulated subcategory is admissible if it is saturated. □

In particular, this shows that any fully faithful functor $\mathsf{D}^b(\operatorname{Spec} k) \to \mathsf{D}^b(X)$ has admissible image.

Corollary 2.1.7. *Let X be smooth and $X'_1, \ldots, X'_n$ be smooth proper schemes over k, and $F_i : \mathsf{D}^b(X'_i) \to \mathsf{D}^b(X)$ fully faithful functors, such that $F_i(\mathsf{D}^b(X'_i)) \subset F_j(\mathsf{D}^b(X'_j))^\perp$ whenever $i > j$. Then there is a semiorthogonal decomposition*:

$$\mathsf{D}^b(X) = \langle \mathsf{A}, F_1\big(\mathsf{D}^b(X'_1)\big), \ldots, F_n\big(\mathsf{D}^b(X'_n)\big) \rangle,$$

where A is the left orthogonal of the category generated by the $F_i(\mathsf{D}^b(X'_i))$.

So far, we have seen that many basic definitions and formal results concerning semiorthogonal decompositions can be given and hold over any field k. The main question is, given a smooth projective variety X over k, how to produce a semiorthogonal decomposition of $\mathsf{D}^b(X)$. In the literature, almost all constructions are described over an algebraically closed field of characteristic zero, even when this restrictive assumption is not necessary.

We will now present some simple descent results for semiorthogonal decompositions. Given a smooth scheme X over k, consider its scalar extension $\overline{X} := X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\overline{k})$. We will also denote $\overline{E}$ the pullback of a coherent sheaf E on X, under the projection morphism $\overline{X} \to X$. Denote by $\overline{\mathsf{A}} \subset \mathsf{D}^b(\overline{X})$ the base change of a triangulated subcategory $\mathsf{A} \subset \mathsf{D}^b(X)$ to $\overline{k}$. A theory of base change for triangulated category has been developed in [65]. In order to study how semiorthogonal decompositions behave under base change, the following is very useful.

Lemma 2.1.8. *(See [75, Lemma 2.12].) Let X and X' be smooth schemes over k. The Fourier–Mukai functor $\Phi : \mathsf{D}^b(X') \to \mathsf{D}^b(X)$ is an equivalence (resp. fully faithful) if and only if the Fourier–Mukai functor $\overline{\Phi} : \mathsf{D}^b(\overline{X}') \to \mathsf{D}^b(\overline{X})$ is an equivalence (resp. fully faithful).*

Lemma 2.1.9. *Let X be a smooth scheme over k. Suppose that there exist admissible triangulated categories A_i in $\mathsf{D}^b(X)$ such that $\mathsf{D}^b(\overline{X}) = \langle \overline{\mathsf{A}}_1, \ldots, \overline{\mathsf{A}}_n \rangle$. Then $\mathsf{D}^b(X) = \langle \mathsf{A}_1, \ldots, \mathsf{A}_n \rangle$.*

Proof. The ordered sequence $\mathsf{A}_1, \ldots, \mathsf{A}_n$ of subcategories is semiorthogonal by flat base change. Consider $\langle \mathsf{A}_1, \ldots, \mathsf{A}_n \rangle$ and its orthogonal complement A, which are both admissible in $\mathsf{D}^b(X)$ by Proposition 2.1.1. By hypothesis, we have $\overline{\mathsf{A}} = 0$. This means that for each object A of A, we have that $\overline{A} = 0$, and hence $A = 0$, since otherwise, A would have a nonzero homology group, which would remain nonzero over $\overline{k}$ by flat base change. Hence $\mathsf{A} = 0$ and the ordered sequence of subcategories generate $\mathsf{D}^b(X)$. A similar argument appears in the proof of [3, Prop. 2.1]. □

We observe that Lemma 2.1.9 holds if we replace $\overline{k}$ with any field extension of k. A special case of semiorthogonal decompositions is provided by exceptional collections.

Definition 2.1.10. (See [19, Section 2].) An object E of T is *exceptional* if $\operatorname{Hom}_\mathsf{T}(E, E) = k$ and $\operatorname{Hom}_\mathsf{T}(E, E[i]) = 0$ for all $i \neq 0$. An ordered sequence $(E_1, \ldots, E_l)$ of exceptional objects is an *exceptional collection* if $\operatorname{Hom}_\mathsf{T}(E_j, E_k[i]) = 0$ for all $j > k$ and for all $i \in \mathbb{Z}$.

If E in T is an exceptional object, the triangulated subcategory generated by E (that is, the smallest full triangulated subcategory of T containing E) is equivalent to the derived category of $\operatorname{Spec} k$, see [19, Section 6].

By Proposition 2.1.6, this subcategory is admissible. Hence, given an exceptional collection $(E_1, \ldots, E_l)$ in the derived category $\mathsf{D}^b(X)$ of a smooth scheme, Corollary 2.1.7 provides a semiorthogonal decomposition (see also [21, Section 2])

$$\mathsf{D}^b(X) = \langle \mathsf{A}, E_1, \ldots, E_l \rangle,$$

where E_i denotes, by abuse of notation, the category generated by E_i and A is the full triangulated subcategory consisting of objects A satisfying $\mathrm{Hom}_\mathsf{T}(E_i, A) = 0$ for all $1 \leqslant i \leqslant l$.

2.2. Semiorthogonal decomposition for quadric fibrations

We now describe the semiorthogonal decomposition of the derived category of a quadric fibration given by Kuznetsov [63], generalizing the work of Kapranov [50,52]. See Section 1 for precise definitions of the notions of (line bundle-valued) quadratic forms, quadric fibration, and even Clifford algebras. Let Y be a smooth scheme and $\pi : Q \to Y$ a flat quadric fibration of relative dimension $n - 2$ associated to a quadratic form $q : E \to L$ on a locally free $\mathscr{O}_Y$-module E of rank $n \geqslant 2$ (see Section 1.2). Denote by $\mathscr{O}_{Q/Y}(1)$ the restriction to $Q \to Y$ of the relative ample line bundle $\mathscr{O}_{\mathbb{P}(E)/Y}(1)$ and by $\mathscr{C}_0 = \mathscr{C}_0(E, q, L)$ the even Clifford algebra, which is a locally free $\mathscr{O}_Y$-algebra whose isomorphism class is an invariant of $\pi : Q \to Y$ (see Section 1.5).

Theorem 2.2.1. *(See [63, Thm. 4.2].) Let $\pi : Q \to Y$ be a quadric fibration of relative dimension $n - 2$ over a scheme Y smooth over a field. There is a fully faithful Fourier–Mukai functor $\Phi : \mathsf{D}^b(Y, \mathscr{C}_0) \to \mathsf{D}^b(Q)$ and a semiorthogonal decomposition*

$$\mathsf{D}^b(Q) = \langle \Phi \mathsf{D}^b(Y, \mathscr{C}_0), \pi^* \mathsf{D}^b(Y)(1), \ldots, \pi^* \mathsf{D}^b(Y)(n - 2) \rangle.$$

Proof. We observe that this result has been stated in [63, Thm. 4.2] over an algebraically closed field of characteristic 0. We will briefly explain why the main technical results needed in [63] hold over a general field, the rest of the proof being very formal.

First, in Proposition A.1, we prove that the even Clifford algebra $\mathscr{C}_0$ and Clifford bimodule $\mathscr{C}_1$ defined in Section 1.5 are isomorphic to the ones defined in [63, Section 3] (which are only correct in characteristic $\neq 2$). Second, the fact that the coordinate algebra and homogeneous Clifford algebra (see Appendix A) of a flat quadric fibration are Koszul dual algebras (as stated in [63, Lemma 3.1]) holds over a general base scheme by sheafifying the construction in [78, Ch. 2, Section 6, Ex. 4]. Third, the construction in [63, Lemma 4.5] of a Fourier–Mukai kernel inducing a fully faithful functor $\mathsf{D}^b(Y, \mathscr{C}_0) \to \mathsf{D}^b(Q)$ is explicit and carries over to our notion of Clifford algebra and bimodule. The proof, in [63, Lemma 4.4], of the semiorthogonality of the other components is a formal consequence of Grothendieck–Verdier duality for π and the cohomology of the exact sequence (1.2.3). Finally, and most importantly, the resolution of the diagonal of a quadric fibration stated in [63, Thm. 2.4] holds over a general base scheme by sheafifying the construction in [53, Section 4.4]. This allows the proof, in [63, p. 1365], that the subcategories generate $\mathsf{D}^b(Q)$, to carry over. $\square$

More generally, if $\pi : Z \to Y$ is flat morphism of smooth schemes, which is a relative hypersurface in a projective bundle $\mathbb{P}(E) \to Y$ in such a way that $\omega_{Z/Y} = \mathscr{O}_{\mathbb{P}(E)/Y}(-l)|_Z$, then the following well-known result carries over to the relative setting.

Proposition 2.2.2. *For all $i \in \mathbb{Z}$, the functors $\pi^*(-) \otimes \mathscr{O}_{Z/Y}(i)$ are fully faithful. For all $j \in \mathbb{Z}$, there is a semiorthogonal decomposition*:

$$\mathsf{D}^b(Z) = \langle \mathsf{A}^j, \pi^* \mathsf{D}^b(Y)(j), \ldots, \pi^* \mathsf{D}^b(Y)(j + l - 1) \rangle,$$

where A^j is the orthogonal complement.

Proof. The proof of fully faithfulness and orthogonality goes exactly as the one for projective bundles in [74, Section 2]. In fact, one constructs a functor from $\mathsf{D}^b(Y)$ to $\mathsf{D}^b(Z)$ as the derived functor associated to the inverse image functor for coherent sheaves. If $\mathrm{rk}\, E = 2$, then π is finite flat, hence affine and locally free. If $\mathrm{rk}\, E \geqslant 3$,

then $\pi_* \mathscr{O}_Z = \mathscr{O}_Y$, and then we have that π is flat hence π^* is fully faithful (see Lemma 2.1 of [74]). Tensoring with a line bundle is an equivalence of $\mathsf{D}^b(Z)$, so composing π^* with $- \otimes \mathscr{O}_{Z/Y(i)}$ is a fully faithful functor for all i integer. Finally, the subcategories $\pi^*\mathsf{D}^b(Y)(j), \ldots, \pi^*\mathsf{D}^b(Y)(j + l - 1)$ are admissible and the projection formula (Lemma 2.5, [74]) shows that they are semiorthogonal. Then use Corollary 2.1.7 to conclude. $\square$

2.3. Relative homological projective duality for quadric fibrations

Homological projective duality was introduced by Kuznetsov [62] in order to study derived categories of hyperplane sections (see also [61]). In particular, it can be applied to relate the derived category of a complete intersection of quadrics inside the projective space to the derived category of the associated linear system of quadrics [63, Section 5]. In what follows, we spell out this latter construction over a smooth base scheme Y, providing a straightforward generalization of Kuznetsov's construction to quadric fibrations and their intersections.

Remark 2.3.1. Homological projective duality involves *noncommutative schemes*, by which we mean (following Kuznetsov [63, Section 2.1]) a scheme N together with an $\mathscr{O}_N$-algebra $\mathscr{A}$, coherent as an $\mathscr{O}_N$-module. Morphisms are defined accordingly. By definition, a noncommutative scheme $(N, \mathscr{A})$ has $\mathsf{Coh}(N, \mathscr{A})$ as category of coherent sheaves and $\mathsf{D}^b(N, \mathscr{A})$ as bounded derived category. Following Bondal and Orlov [22, Section 5], a non-commutative resolution of singularities $(N, \mathscr{A})$ of a possibly singular scheme N is a torsion-free $\mathscr{O}_N$-algebra $\mathscr{A}$ of finite rank such that $\mathsf{Coh}(N, \mathscr{A})$ has finite homological dimension (i.e., is smooth in the noncommutative sense). For Y any scheme, a flat Y-noncommutative scheme (or a noncommutative scheme over a base Y) is a pair $(N, \mathscr{A})$ with $\pi : N \to Y$ a Y-scheme and $\pi_* \mathscr{A}$ a flat $\mathscr{O}_Y$-module. Also, $(N, \mathscr{A})$ is projective if N is. A Fourier–Mukai functor $\Phi : \mathsf{D}^b(N, \mathscr{A}) \to \mathsf{D}^b(N', \mathscr{A}')$ is an integral transform whose kernel is an object E in $\mathsf{D}^b(N \times N', \mathscr{A}^{\mathrm{op}} \boxtimes \mathscr{A}')$.

Remark. In the case when $\mathscr{A}$ and $\mathscr{A}'$ are Azumaya algebras, this coincides with the notion of "twisted" Fourier–Mukai functors developed in [29].

We now recast the basic notions of homological projective duality from [62, Def. 4.1] in a relative setting. Let $M \to Y$ be a flat projective morphism of schemes smooth over a field k and $\mathscr{O}_{M/Y}(1)$ a relatively very ample line bundle on M.

Definition 2.3.2. A *Lefschetz decomposition* of $\mathsf{D}^b(M)$ with respect to $\mathscr{O}_{M/Y}(1)$ is a semiorthogonal decomposition

$$\mathsf{D}^b(M) = \langle \mathsf{A}_0, \mathsf{A}_1(1), \ldots, \mathsf{A}_{i-1}(i-1) \rangle, \tag{2.3.1}$$

with

$$0 \subset \mathsf{A}_{i-1} \subset \cdots \subset \mathsf{A}_0.$$

Let $\mathbb{P}(V) \to Y$ be a projective bundle and $f : M \to \mathbb{P}(V)$ be a Y-morphism such that $f^* \mathscr{O}_{\mathbb{P}(V)/Y}(1) \cong \mathscr{O}_{M/Y}(1)$. Let $\mathcal{M} \subset M \times_Y \mathbb{P}(V^\vee)$ be the universal hyperplane section

$$\mathcal{M} := \left\{ (p, H) \in M \times_Y \mathbb{P}(V^\vee) : p \text{ belongs to } H \right\}.$$

We denote by $\mathrm{pr}_1 : \mathcal{M} \to M$ the restriction of the projection onto the first factor.

Definition 2.3.3. (See [62, Def 6.1].) Let V be a vector bundle over a smooth scheme Y over a field k. By a *homological projective duality pair* (or *HP dual pair*) over Y we mean the data of scheme M smooth over k and flat over Y, a noncommutative scheme $(N, \mathscr{A})$ flat over Y, a morphism $f : M \to \mathbb{P}(V)$, a morphism $g : N \to \mathbb{P}(V^\vee)$, a Lefschetz decomposition of $\mathsf{D}^b(M)$ with respect to $\mathscr{O}_{M/Y} = f^* \mathscr{O}_{\mathbb{P}(V)/Y}(1)$, and an object E in $\mathsf{D}^b(\mathcal{M} \times_{\mathbb{P}(V^\vee)} N, \mathscr{O}_\mathcal{M} \boxtimes \mathscr{A}^{\mathrm{op}})$, such that the Fourier–Mukai functor $\Phi_E : \mathsf{D}^b(N, \mathscr{A}) \to \mathsf{D}^b(\mathcal{M})$ is fully faithful and gives the semiorthogonal decomposition

$$\mathsf{D}^b(\mathcal{M}) = \langle \Phi_E(\mathsf{D}^b(N, \mathscr{A})), \mathsf{A}_1(1) \boxtimes \mathsf{D}^b(\mathbb{P}(V^\vee)), \ldots, \mathsf{A}_{i-1}(i-1) \boxtimes \mathsf{D}^b(\mathbb{P}(V^\vee)) \rangle. \tag{2.3.2}$$

Let the rank of V be n. For a vector subbundle $L \subset V^\vee$ with locally free quotient, denote its orthogonal by $L^\perp \subset V$, and consider the following relative linear sections

$$M_L = M \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp), \qquad N_L = N \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L)$$

of M and N. If $\mathscr{A}$ is an $\mathscr{O}_N$-algebra, then denote by $\mathscr{A}_L = \mathscr{A} \boxtimes_{\mathscr{O}_{\mathbb{P}(V^\vee)/Y}} \mathscr{O}_{\mathbb{P}(L)/Y}$.

We now prove that the main result of the theory of homological projective duality holds in the relative setting.

Theorem 2.3.4. *Let V be a vector bundle on a smooth scheme Y over k. Let M and $(N, \mathscr{A})$ be an HP dual pair over Y. If the Lefschetz decomposition is Y-linear then*:

(i) *N is smooth over k and admits a* dual *Lefschetz decomposition*

$$\mathsf{D}^{\mathsf{b}}(N, \mathscr{A}) = \langle \mathsf{B}_{j-1}(1-j), \ldots, \mathsf{B}_1(-1), \mathsf{B}_0 \rangle, \quad 0 \subset \mathsf{B}_{j-1} \subset \cdots \subset \mathsf{B}_1 \subset \mathsf{B}_0;$$

(ii) *for any vector subbundle $L \subset V^\vee$ of rank r with locally free quotient such that*

$$\dim M_L = \dim M - r, \quad and \quad \dim N_L = \dim N + r - n,$$

there exists a triangulated category C_L and semiorthogonal decompositions:

$$\mathsf{D}^{\mathsf{b}}(M_L) = \langle \mathsf{C}_L, \mathsf{A}_r(1), \ldots, \mathsf{A}_{i-1}(i-r) \rangle,$$
$$\mathsf{D}^{\mathsf{b}}(N_L, \mathscr{A}_L) = \langle \mathsf{B}_{j-1}(N-r-j), \ldots, \mathsf{B}_{N-r}(-1), \mathsf{C}_L \rangle.$$

Proof. The proof goes along the lines of [62, Sections 5–6]. Notably [62, Rem. 6.4], gives three conditions (which we will denote by (1), (2) and (3) in this proof) that are sufficient for homological projective duality to hold for a given HP dual pair. The description in [62] is very precise, so for the sake of readability, we refrain from giving too many details here and we address the interested reader to the original paper. Roughly, the first of these conditions requires the existence of a fully faithful Fourier–Mukai functor $\Phi_E : \mathsf{D}^{\mathsf{b}}(N, \mathscr{A}) \to \mathsf{D}^{\mathsf{b}}(\mathcal{M})$ satisfying the orthogonality condition as in (2.3.2). Hence, Condition (1) is part of the definition of HP dual pair.

Conditions (2) and (3) require roughly that the functor $\Phi_E^* \circ \mathrm{pr}_1^* : \mathsf{D}^{\mathsf{b}}(M) \to \mathsf{D}^{\mathsf{b}}(N, \mathscr{A})$ is well-behaved on all the *primitive* components of the Lefschetz decomposition. Here Φ_E^* denotes the left adjoint to Φ_E, while the i-th primitive component of a Lefschetz decomposition is the orthogonal complement of A_i in A_{i+1}. A full and accurate treatment is given in [62, Section 4], which requires an immense amount of notation that we refrain from introducing here.

Conditions (2) and (3) are proved in the case where Y is a point in [62, Prop. 5.7, Cor. 5.8] and [62, Lemma 5.9, Prop. 5.10] respectively. As remarked upon in [62, Rem. 5.13], these statements from [62, Section 5] hold true also under the wider hypothesis assumed in our statement (even more generally, without projectivity assumptions on Y). Indeed, Y-linearity allows us to work relatively over the scheme Y, replacing Hom_X by $p_* \underline{\mathrm{Hom}}_X$ in all the homological arguments presented in [62, Section 5]. As the three conditions of [62, Rem. 6.4] are satisfied, the theorem is proved. $\square$

Next, we will apply relative homological projective duality to the case of flat quadric fibrations. Let E be a vector bundle of rank n on a smooth scheme Y. Consider the projective bundle $p : M = \mathbb{P}(E) \to Y$, the relative ample line bundle $\mathscr{O}_{M/Y}(1)$, and the semiorthogonal decomposition (see [74, Thm. 2.6])

$$\mathsf{D}^{\mathsf{b}}(M) = \langle p^* \mathsf{D}^{\mathsf{b}}(Y)(-1), p^* \mathsf{D}^{\mathsf{b}}(Y), \ldots, p^* \mathsf{D}^{\mathsf{b}}(Y)(n-2) \rangle. \tag{2.3.3}$$

Let us denote by $m = \lfloor n/2 \rfloor$ and put

$$\mathsf{A}_0 = \mathsf{A}_1 = \cdots = \mathsf{A}_{m-1} = \langle p^* \mathsf{D}^{\mathsf{b}}(Y)(-1), p^* \mathsf{D}^{\mathsf{b}}(Y) \rangle,$$

$$\mathsf{A}_m = \begin{cases} \langle p^* \mathsf{D}^{\mathsf{b}}(Y)(-1), p^* \mathsf{D}^{\mathsf{b}}(Y) \rangle & \text{if } n \text{ is even,} \\ \langle p^* \mathsf{D}^{\mathsf{b}}(Y)(-1) \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Then the decomposition (2.3.3) is a Lefschetz decomposition

$$\mathsf{D}^{\mathrm{b}}(M) = \langle \mathsf{A}_0, \mathsf{A}_1(2), \dots, \mathsf{A}_m(2m) \rangle, \tag{2.3.4}$$

with respect to the relative double Veronese embedding $f : M = \mathbb{P}(E) \to \mathbb{P}(S_2 E) =: \mathbb{P}(V)$, as $f^* \mathscr{O}_{\mathbb{P}(S_2 E)/Y}(1) \cong \mathscr{O}_{\mathbb{P}(E)/Y}(2)$. Here, we use the submodule $S_2 E$ of symmetric tensors for defining the relative Veronese embedding (working even in characteristic 2), given our convention for defining projective bundles, see Section 1 for more details. Recall the canonical isomorphism $S^2 E^\vee \cong (S_2 E)^\vee$.

Definition 2.3.5. Let $\mathcal{Q} \subset \mathbb{P}(E) \times_Y \mathbb{P}((S_2 E)^\vee) = \mathbb{P}(E) \times_Y \mathbb{P}(S^2 E^\vee)$ be the universal hyperplane section with respect to f, then we will refer to the projection

$$\pi : \mathcal{Q} \to \mathbb{P}(S^2 E^\vee),$$

as the *universal relative quadric fibration* in $\mathbb{P}(E)$.

Roughly speaking, this is equivalent to the fact that the double Veronese embedding f is defined by the full linear systems of quadrics on $\mathbb{P}(E)$, hence the universal hyperplane section carries the universal family of relative quadrics in $\mathbb{P}(E)$. Indeed, if a section $s : Y \to \mathbb{P}(S^2 E^\vee)$ of π corresponds to a line subbundle $L^\vee \subset S^2 E^\vee \cong (S_2 E)^\vee$, then the pullback $s^* \mathcal{Q} \to Y$ is the quadric fibration associated to the quadratic form defined by Lemma 1.1.1. Thus $\pi : \mathcal{Q} \to \mathbb{P}(S^2 E^\vee)$ is a flat quadric fibration. Let $\mathscr{C}_0$ be its even Clifford algebra.

Theorem 2.3.6. *Let Y be a smooth projective scheme and E a vector bundle. Then $\mathbb{P}(E)$ and the noncommutative variety $(\mathbb{P}(S^2 E^\vee), \mathscr{C}_0)$ form an HP dual pair over Y with respect to the Lefschetz decomposition (2.3.4).*

Proof. The pair $\mathbb{P}(E)$, $(\mathbb{P}(S^2 E^\vee), \mathscr{C}_0)$ forms a HP dual pair thanks to the semiorthogonal decomposition of the universal quadric fibration. $\square$

Now we describe some consequences of Theorem 2.3.4 in the case of flat quadric fibrations. Let (E_i, q_i, L) be a finite set of primitive generically (semi)regular quadratic forms. Denote by $L^\vee \to S^2 E^\vee$ the $\mathscr{O}_Y$-submodule generated by the line subbundles $L_i^\vee$. Then the linear section $M_{L^\vee} = \mathbb{P}(E) \times_{\mathbb{P}(S_2 E)} \mathbb{P}(L^{\vee \perp})$ (which we denote by X) is a relative intersection of the quadric fibrations $Q_i \to Y$ in $\mathbb{P}(E)$. Indeed, the projection map $\pi : X \to Y$ has fibers the intersection of the fibers of $Q_i \to Y$ in the projective space given by the fibers of $\mathbb{P}(E)$. On the other hand, the linear section $N_{L^\vee} = \mathbb{P}(S^2 E^\vee) \times_{\mathbb{P}(S^2 E^\vee)} \mathbb{P}(L^\vee)$ (which we denote by S) is precisely $\mathbb{P}(L^\vee) \subset \mathbb{P}(S^2 E^\vee)$. Then the restriction $\mathscr{C}_0|_{\mathbb{P}(L^\vee)} = \mathscr{C}_0 \boxtimes_{\mathscr{O}_{\mathbb{P}(S^2 E^\vee)}} \mathscr{O}_{\mathbb{P}(L^\vee)}$ (which we shamelessly denote by $\mathscr{C}_0$) of the even Clifford algebra of $\mathcal{Q} \to \mathbb{P}(S^2 E^\vee)$ to $\mathbb{P}(L^\vee)$ is then isomorphic to the even Clifford algebra of the corresponding linear span (see Definition 1.2.3) quadric fibration $Q \to \mathbb{P}(L^\vee)$ associated to the $Q_i \to Y$. We assume that $L^\vee \cong \bigoplus_i L_i^\vee$ and that this relative intersection is complete. Notice that we have $\omega_{X/Y} = \mathscr{O}_{X/Y}(2m - n)$. We will record the following application of Theorems 2.3.4 and 2.3.6 for future use.

Theorem 2.3.7 *(HP duality for quadric fibration intersections). Let Y be a smooth scheme, $Q \to S$ a linear span of m quadric fibrations of relative dimension $n - 2$ over Y, and $X \to Y$ their relative complete intersection. Let $\mathscr{C}_0$ be the even Clifford algebra of $Q \to S$.*

1. *If $2m < n$, then the fibers of $X \to Y$ are Fano and relative homological projective duality yields*

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{D}^{\mathrm{b}}(S, \mathscr{C}_0), \pi^* \mathsf{D}^{\mathrm{b}}(Y)(1) \dots \pi^* \mathsf{D}^{\mathrm{b}}(Y)(n - 2m) \rangle.$$

2. *If $2m = n$ then the fibers of $X \to Y$ are generically Calabi–Yau and relative homological projective duality yields*

$$\mathsf{D}^{\mathrm{b}}(X) \simeq \mathsf{D}^{\mathrm{b}}(S, \mathscr{C}_0).$$

3. *If $2m > n$, then the fibers of $X \to Y$ are generically of general type and there exists a fully faithful functor $\mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(S, \mathscr{C}_0)$ with explicitly describable orthogonal complement.*

Remark. We note that the claims on the Kodaira dimension of the fibers contained in (2) and (3) hold only for the generic fibers since it can change over a closed subscheme of the base. Just think about a family of plane cubics with a singular central fiber.

2.4. Semiorthogonal decompositions, representability, and rationality

Since the pioneering work of Bondal and Orlov [21], semiorthogonal decompositions have proved themselves to be a very useful tool in studying Fano varieties and their birational properties, see [59] and [60] for example. In particular, semiorthogonal decompositions should encode, in a categorical way, the geometric information contained in the intermediate Jacobian of a Fano threefold, see [14]. Unlike the theory of intermediate Jacobians, this categorical approach can be extended to higher dimensions. We now introduce the notion of categorical representability, which places our results in a general conjectural framework for the study of rationality via semiorthogonal decompositions.

Definition 2.4.1. (See [14, Def. 2.3].) A triangulated category T is *representable* in dimension m if it admits a semiorthogonal decomposition

$$\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_l \rangle,$$

such that for each $1 \leqslant i \leqslant l$, there exists a smooth projective variety M_i with $\dim M_i \leqslant m$, such that A_i is equivalent to an admissible subcategory of $\mathsf{D}^{\mathrm{b}}(M_i)$.

Remark 2.4.2. Notice that in the definition, we can assume the categories A_i to be indecomposable and also the varieties M_i to be connected. Indeed, the derived category $\mathsf{D}^{\mathrm{b}}(M)$ of a scheme M is indecomposable if and only if M is connected (see [24, Example 3.2]).

Definition 2.4.3. (See [14, Def. 2.4].) Let X be a smooth projective variety of dimension n. We say that X is *categorically representable* in dimension m (or equivalently in codimension $n - m$) if $\mathsf{D}^{\mathrm{b}}(X)$ is representable in dimension m.

In the sequel, we develop general classes of threefolds and fourfolds with a fibration in intersection of quadrics over $\mathbb{P}^1$, where categorical representability is intimately related to rationality.

3. Genus 1 fibrations

In this section, we introduce a warm-up example: a genus 1 fibration $X \to Y$ (by *genus* 1 *fibration* we mean a proper flat surjective map whose generic fibers are smooth genus 1 curves) over a smooth projective variety Y, obtained as the generic (see Definition 1.2.4) relative complete intersection of two quadric surface fibrations.

3.1. Clifford algebras of genus 1 fibrations

Let $Q_1 \to Y$ and $Q_2 \to Y$ be a generic pair of quadric surface fibrations defined by quadratic forms (E, q_1, L_1) and (E, q_2, L_2) over a scheme Y. Consider the $\mathbb{P}^1$-bundle $S = \mathbb{P}(L_1^\vee \oplus L_2^\vee) \to Y$, the linear span quadric fibration $Q \to S$ (see Definition 1.2.3) and its associated even Clifford algebra $\mathscr{C}_0$. Then the generic relative complete intersection $X \to Y$, of $Q_1 \to Y$ and $Q_2 \to Y$ in $\mathbb{P}(E)$, is a genus 1 fibration and $\mathscr{C}_0$ gives rise to an Azumaya algebra $\mathscr{B}_0$ on the discriminant cover $T \to S$ (see Proposition 1.6.2), whose Brauer class we denote by $\beta \in \mathrm{Br}(T)$.

By the genericity hypothesis, the discriminant divisor D of $Q \to S$ is smooth. It intersects each fiber of the ruled surface $S \to Y$ in a closed point of degree 4. We note that the composite map $T \to S \to Y$ is thus also a genus 1 fibration, since every generic fiber is a double cover of the projective line branched at 4 points. We point out that for a geometric point y of Y of characteristic zero, the fiber X_y is a principal homogeneous space under the Jacobian $J(T_y)$, see [79, Thm. 4.8].

Theorem 3.1.1. *Let $X \to Y$ be a genus* 1 *fibration arising as the generic relative complete intersection of two quadric surface fibrations. Then there is an equivalence $\mathsf{D}^{\mathrm{b}}(X) \simeq \mathsf{D}^{\mathrm{b}}(T, \beta)$. Moreover, $\beta \in \mathrm{Br}(T)$ is trivial if and only if the genus* 1 *fibration $X \to Y$ admits a section.*

Proof. Relative homological projective duality (Theorem 2.3.7(2)) yields an equivalence $\mathsf{D}^{\mathsf{b}}(X) \simeq \mathsf{D}^{\mathsf{b}}(S, \mathscr{C}_0)$. Since $T \to S$ is affine, we have an equivalence $\mathsf{D}^{\mathsf{b}}(S, \mathscr{C}_0) \simeq \mathsf{D}^{\mathsf{b}}(T, \beta)$, since β is the class in the Brauer group corresponding to the Azumaya algebra $\mathscr{B}_0$.

As for the second assertion, the genus 1 fibration $X \to Y$ admits a section if and only if it admits a rational section if and only if (by the Amer–Brumer Theorem 1.9.1) $Q \to S$ has a rational section if and only if (by Lemma 1.9.4) $\beta \in \mathrm{Br}(T)$ is trivial. $\square$

On the other hand—as we shall also detail for a higher-dimensional case in Section 5—the genus 1 fibration $T \to Y$ is the relative moduli space of spinor bundles on $X \to Y$ (actually, restrictions to X of spinor bundles on the quadrics of the pencil, see [21, Ch. 2] for details). The obstruction to the existence of a universal class on $X \times_Y T$ is given by a Brauer class ω. Combining results from [21] and [26], one gets an equivalence $\mathsf{D}^{\mathsf{b}}(T, \omega) \simeq \mathsf{D}^{\mathsf{b}}(X)$ by the Fourier–Mukai functor whose kernel is the ω-twisted universal sheaf on $X \times_Y T$.

Moreover, we note that a similar result was already proved by Căldăraru [26, Section 4]. Indeed, suppose that $X \to Y$ is a smooth elliptic fibration and let $J \to Y$ be the relative Jacobian. There is an element α in $\mathrm{Br}(J)$ giving the obstruction to $J \to Y$ to be a fine moduli space for degree zero relative line bundles on $X \to Y$. By considering the α-twisted Poincaré line bundle on $X \times_Y J$, one gets an equivalence $\mathsf{D}^{\mathsf{b}}(X) \simeq \mathsf{D}^{\mathsf{b}}(J, \alpha)$. That the triviality of α is equivalent to the existence of a section of $X \to Y$ is detailed in [26, Section 4] and arises from the fact that the image of α under the canonical map $\mathrm{Br}(J) \to \text{III}(J/Y)$ coincides with the class of $X \to Y$.

Question 3.1.2. (See Question 5.3.2.) Let $X \to Y$ be a smooth genus 1 fibration arising as the generic relative complete intersection of two quadric surface fibrations. Let $J \to Y$ be the Jacobian fibration and $T \to S \to Y$ the discriminant cover of the linear span. Are $J \to Y$ and $T \to Y$ related (or at least their Brauer groups)? If so, are the classes α, β, and ω related (or possibly coincide)?

4. Del Pezzo fibrations of degree four

Del Pezzo fibrations over $\mathbb{P}^1$ form a very important class of varieties in the classification of smooth projective threefolds with negative Kodaira dimension. Notably, they form one of the three classes to which any such threefold can be geometrically birationally equivalent. In this section, whenever we use the results from [1], we need to work over $\mathbb{C}$. We will state when a broader generality for the field of definition is possible.

4.1. Minimal Del Pezzo fibrations

Definition 4.1.1. A 3-fold X is a minimal Del Pezzo fibration if there exists a proper flat morphism $\delta : X \to C$ to a smooth projective curve C whose generic fiber is a Del Pezzo surface and $\mathrm{Pic}(X) = \delta^* \mathrm{Pic}(C) \oplus \mathbb{Z}$.

In this section, we employ our techniques to study quartic Del Pezzo fibrations $X \to C$ over a curve. Over an algebraically closed field, such fibrations can always be realized as relative complete intersections of two three-dimensional quadric fibrations over C. Our study will be mainly devoted to the problem of rationality in the case where $C = \mathbb{P}^1$. In fact, the rationality of such a Del Pezzo fibration only depends on the Euler characteristic, as shown by Alekseev [1] and Shramov [83]. Starting from the classical point of view of [1] and the categorical tools developed in [13] and [14], we lead up to a proof of Theorem 1, establishing a purely categorical criterion of rationality of quartic Del Pezzo fibrations over $\mathbb{P}^1$.

Let Y be a scheme over a field k and $\pi : X \to Y$ be a quartic Del Pezzo fibration with X be smooth over k. Then there exists a vector bundle E of rank 5 such that $X \subset \mathbb{P}(E)$ is the complete intersection of two quadric fibrations $Q_1 \to Y$ and $Q_2 \to Y$ of relative dimension 3 (see [83]). We consider here the generic case, according to Definition 1.2.4. We get a $\mathbb{P}^1$-bundle over $S \to Y$, and a fibration $Q \to Y$ of relative dimension 3. Moreover, $\omega_{X/Y} = \mathscr{O}_{X/Y}(-1)$ and thus, by Proposition 2.2.2, for all integers j, we have a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(X) = \langle \mathsf{A}_X^j, \mathsf{D}^{\mathsf{b}}(Y)(j) \rangle. \tag{4.1.1}$$

We observe moreover that A_X^j does not contain $\mathsf{D}^b(Y)(l)$ for any $l \neq j$. To simplify notation let us denote $\mathsf{A}_X = \mathsf{A}_X^1$. Letting $\mathscr{C}_0$ be the even Clifford algebra of $Q \to S$, then relative homological projective duality (Theorem 2.3.7(1)) provides an equivalence $\mathsf{A}_X \simeq \mathsf{D}^b(S, \mathscr{C}_0)$.

4.2. Reduction by hyperbolic splitting vs. Alekseev's construction

In the case where $Y = C$ is a smooth projective complex curve, Alekseev [1] has shown that X is birational to a conic bundle Q'' over a ruled surface over $S' \to C$. Part of this birationality result is based on the construction of a particular smooth section of $X \to C$. Hence, by the (easy direction of the) Amer–Brumer theorem (Theorem 1.9.1), there exists also a smooth section of $Q \to S$, along which we can perform quadric reduction to obtain a conic bundle $Q' \to S$. The mere existence of a smooth section is guaranteed by Lemma 1.9.3. Even though Alekseev's construction is not given in terms of quadric reduction, but by explicit birational transformations, we will show that $S' = S$, that the conic bundle $Q'' \to S$ is birational to the $Q' \to S$, and that their associated Clifford algebras are Morita equivalent.

Definition 4.2.1. A *standard conic bundle* $\pi : Z \to T$ over a surface T is a proper flat surjective morphism whose geometric fibers are isomorphic to plane conics, such that for any irreducible curve $B \subset T$ the surface $\pi^{-1}(B)$ is irreducible (this second condition is also called relative minimality).

The discriminant locus of a standard conic bundle is a curve $D \subset T$, which can be possibly empty, with at most double points [11, Prop. 1.2]. The fiber of π over a smooth point of D is the union of two lines intersecting in a single point, while the fiber over a node is a double line. Recall that any conic bundle is birationally equivalent to a standard one via elementary transformations [80], and that, if T is a rational surface, then the discriminant double cover $\tilde{D} \to D$ identifies the isomorphism class of the associated Clifford algebra $\mathscr{C}_0$ [13, Lemma 3.2].

Recall that $\pi : X \to C$ is the complete intersection of two three-dimensional quadric fibrations $Q_1 \to C$ and $Q_2 \to C$ given by line bundle valued quadratic forms on a rank 5 vector bundle E over C. Moreover, X is embedded in $\mathbb{P}(E)$ via its relative anticanonical system $-\omega_{X/C}$, that is $\omega_{X/C} = \mathscr{O}_{X/C}(-1) = \mathscr{O}_{\mathbb{P}(E)/C}(-1)|_X$. Hence, the anticanonical bundle corresponds to the relative hyperplane section of X over C.

One of the main ideas of [1] is summarized in the following:

Proposition 4.2.2. *There exists a standard conic bundle Q'', over a ruled surface $S' \to C$, which is birational to X.*

Proof. We will only give a sketch of the construction; for complete proofs see [1]. Let s be a smooth section of π, and consider the blow-up $\eta : \widetilde{X} \to X$ along s, and the composition $\tilde{\pi} := \pi \circ \eta : \widetilde{X} \to C$, which is a flat two-dimensional fibration. The exceptional divisor of the blow-up is a ruled surface $\widetilde{F} \to C$. The linear system $|-\omega_{X/C} - s|$ gives the relative projection of X off s over C. This gives a rational map $X \dashrightarrow \mathbb{P}(\mathscr{F})$, where $\mathbb{P}(\mathscr{F}) \to C$ is a projective bundle of relative dimension 3. Resolving this rational map by blowing up s gives a closed embedding $\widetilde{X} \hookrightarrow \mathbb{P}(\mathscr{F})$, corresponding to the linear system $|-\omega_{X/C} - \widetilde{F}| = |-\omega_{\widetilde{X}/C}|$, so that $\widetilde{X}$ is relatively anticanonically embedded in $\mathbb{P}(\mathscr{F})$.

For any point c of C, considering the projective anticanonical embedding $X_c \subset \mathbb{P}(E_c) \simeq \mathbb{P}^4$, we can think of X_c as the blow up of a projective plane along 5 points in general position. Alekseev shows that the smooth section s can be chosen such that, over all but a finite number of points of C, the point cut out by S on the fiber X_c is in general position with respect to those 5 points [1]. This means that the blow-up of X_c along that point is a degree 3 del Pezzo surface. It follows that all but a finite number of the fibers of $\widetilde{X} \to C$ are cubic surfaces anticanonically embedded in the corresponding fiber of $\mathbb{P}(\mathscr{F}) \to C$.

Consider the linear system $|L_2| = |-\omega_{\widetilde{X}/C} - \widetilde{F}| = |\eta^*(-\omega_{X/C}) - 2\widetilde{F}|$ on $\widetilde{X}$. On each fiber, $\widetilde{F}$ is a line, while for all but a finite number of points of C, the anticanonical class corresponds to the hyperplane section. So the linear system $|L_2|$ forms, fiber-wise in all but a finite number of fibers, a pencil of residual conics. Working by birational transformations along the fibers which are not cubic surfaces, Alekseev finally describes a standard conic bundle Q'', over a ruled surface $S' \to C$. Denote by $|L_1| := |-\omega_{X/C} - 2s|$, and notice that $\eta^* L_1 = L_2$, so that $S' = \mathbb{P}(\pi_*(-\omega_{X/C} - 2s))$. The conic bundle $Q'' \to S'$ is standard and birational to X [1].

On the other hand, consider the quadric fibration $Q \to S$, the linear span of the quadric pencils $Q_1 \to C$ and $Q_2 \to C$. In particular, $S \to C$ is a $\mathbb{P}^1$-bundle. Since s is a smooth section of X, it provides a smooth section of Q

and hence we get, by hyperbolic splitting, a conic bundle $Q' \to S$. By [31, Thm. 3.2], X is birational to Q', so Q' is birational to Q''. $\quad\square$

Theorem 4.2.3. *The two ruled surfaces S and S' are isomorphic, and the two conic bundles Q' and Q'' are isomorphic over the complement of a finite number of fibers.*

Proof. Consider once again the linear system $|L_2| = |-\omega_{X/C} - 2s|$ on X. Since $\omega_{X/C} = \mathcal{O}_{X/C}(-1)$, this linear system is the pencil of vertical hyperplanes in $\mathbb{P}(E)$ which are tangent to X at s. Let us denote by $Q_\lambda \to C$ a quadric fibration in the quadric pencil $Q \to S$. That is, we are fixing a section λ of the $\mathbb{P}^1$-bundle $S \to C$ and we are considering the corresponding 3-dimensional quadric fibration. In particular, for each λ, s is contained in Q_λ. If we let λ vary in the pencil and take the relative tangent hyperplanes to s in each Q_λ in Q, these give all the relative tangent hyperplanes to X at s. Hence they cut out on X the full linear system $|L_2|$.

In order to prove our claim, let us first explain the relation between the relative pencil of quadrics $Q \to S$ in $\mathbb{P}(E) \times \mathbb{P}^1$ and relative conics, residual to $\widetilde{F}$, that compose the linear system $|L_1|$ on $\widetilde{X}$. Take again a fixed 3-dimensional quadric fibration $Q_\lambda \to C$ and the tangent hyperplane T_λ to it at s. The surface $T_\lambda \cap X$ (which is an element of $|L_2|$) has a fibration in quartic curves $G_\lambda \to C$, induced by restriction under the inclusion $G_\lambda \subset Q_\lambda$. Now, s is contained in G_λ and all the quartic curves are singular at the section s. On the other hand, T_λ cuts out a family of quadric cone surfaces $\widetilde{Q}_\lambda \to C$ from $Q_\lambda \to C$ that contains G_λ. We denote $\widetilde{G}_\lambda \to C$ the family of bases of the cone.

Consider now the rational projection $\mathbb{P}(E) \dashrightarrow \mathbb{P}(\mathscr{F})$ from the center s. We recognize the rational map that sends X onto $\widetilde{X}$, the rational inverse of η. It sends T_λ to a family of 2-planes $\mathbb{P}^2_\lambda \to C$ containing $\widetilde{F}$. It sends Q_λ to its base conic $\widetilde{G}_\lambda \subset \mathbb{P}^2_\lambda$. It sends G_λ birationally onto $\widetilde{G}_\lambda$ (note in fact that we are projecting fiber-wise from the double point of quartic curve, hence the image has fiber-wise degree two). We note that $\widetilde{G}_\lambda$ is contained in $\widetilde{X}$ and that it is a divisor of the linear system of conics $|L_1|$. If $\widetilde{X}_c$ is a cubic surface in $\mathbb{P}(\mathscr{F}_c) = \mathbb{P}^3$, we get the residual conics with respect to the exceptional line. Furthermore, by construction, $\widetilde{G}_\lambda$ is the quadric reduction of Q_λ with respect to the section s.

Finally, we get that $S = \mathbb{P}(\pi_*(L_1)) = \mathbb{P}(\pi_*(-\omega_{X/C} - 2s)) = S'$. On the fibers where s cuts out a general point, the conic pencils $Q' \to S$ and $Q'' \to S$ coincide. $\quad\square$

4.3. Clifford algebras associated to quartic Del Pezzo fibrations and rationality

The previous constructions give three quadric fibrations $Q \to S$, $Q' \to S$, and $Q'' \to S$. Now we will show that their even Clifford algebras are Morita equivalent.

Let $\mathscr{C}_0$ (resp. $\mathscr{C}_0'$) be the even Clifford algebra associated of $Q \to S$ (resp. $Q' \to S$). The following is then an immediate consequence of the Morita invariant of the Clifford algebra under hyperbolic splitting (Corollary 1.8.8) and of relative homological projective duality (Theorem 2.3.7(1)).

Corollary 4.3.1. *Let $\pi : X \to C$ be a generic quartic Del Pezzo fibration over a smooth proper curve over a field of characteristic $\neq 2$. Then the categories A_X, $\mathsf{D}^b(S, \mathscr{C}_0)$, and $\mathsf{D}^b(S, \mathscr{C}_0')$ are equivalent.*

Now recall that the even Clifford algebra of a conic bundle gives rise to a (generally ramified) 2-torsion element of $\mathrm{Br}(k(S))$. Notably, if two conic bundles coincide on an open subset of S then their even Clifford algebra define the same Brauer class. Let $\mathscr{C}_0''$ be the even Clifford algebra of $Q'' \to S$. Considering Theorem 4.2.3, the following is now immediate.

Corollary 4.3.2. *The category A_X is equivalent to $\mathsf{D}^b(S, \mathscr{C}_0'')$.*

An interesting case of quartic Del Pezzo fibrations arises when $C = \mathbb{P}^1$ and it makes sense to wonder about the rationality of X. By [31, Thm. 3.2], X is $k(t)$-birational to the conic bundle $Q' \to S$ and Lemma 1.9.5 implies that the rationality of X is equivalent to that of Q'. Alekseev [1] and Shramov [83], by reducing to the study of Q'', proved that the rationality of X depends only on the topological Euler characteristic of X.

On the other hand, if $C = \mathbb{P}^1$, then S is a Hirzebruch surface. For conic bundles on Hirzebruch surfaces there exists a criterion of rationality in terms of derived categories.

Theorem 4.3.3. *(See [13, Thm. 1.2].) If S is a Hirzebruch surface or $\mathbb{P}^2$ over $\mathbb{C}$, then Q'' is rational if and only if it representable in codimension* 2. *In particular, there is a semiorthogonal decomposition*

$$\mathsf{D}^b\big(S, \mathscr{C}_0''\big) = \big\langle \mathsf{D}^b(\varGamma_1), \dots, \mathsf{D}^b(\varGamma_k), E_1, \dots, E_l \big\rangle,$$

where $\varGamma_i$ are smooth projective curves and E_i exceptional objects if and only if we have $J(Y) = \bigoplus J(\varGamma_i)$.

Proof. While [13, Thm. 1.2] is stated only for *minimal* rational surfaces, the same arguments can be used for the non-minimal Hirzebruch surface $S := \mathbb{F}_1$ as well. One implication is a corollary of [13, Thm. 1.1]: from the semiorthogonal decomposition it is possible to split the intermediate Jacobian of X into Jacobian of curves as principally polarized abelian varieties. This is sufficient to get rationality, thanks to Shokurov [82, Thm. 10.1]. Actually, Shokurov states the theorem for minimal surfaces, but he actually proves it for $\mathbb{F}_1$ as well [82, Section 10], [48, Rem. 1]. The other implication follows by the case-by-case analysis from [13, Section 6]. Indeed (see [48, Conj. I, Rem. 1]), the discriminant divisor of a rational conic bundle over S has either a trigonal or a hyperelliptic structure induced by the natural fibration $S \to \mathbb{P}^1$. □

Finally, we can give a proof of Theorem 1, obtaining a categorical criterion for rationality of the threefold X.

Proof of Theorem 1. For a quartic Del Pezzo fibration $X \to \mathbb{P}^1$, the semiorthogonal decomposition takes the form

$$\mathsf{D}^b(X) = \big\langle \mathsf{A}_X, \mathscr{O}_{X/\mathbb{P}^1}(1), \pi^*\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathscr{O}_{X/\mathbb{P}^1}(1) \big\rangle,$$

where $\mathscr{O}_{X/\mathbb{P}^1}(1)$ and $\pi^*\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathscr{O}_{X/\mathbb{P}^1}(1)$ are exceptional objects. The result then follows as a corollary of Corollary 4.3.2 and Theorem 4.3.3. □

5. Relative intersections of two 4-dimensional quadrics

In this section, we use our techniques to study the derived category of a fourfold X admitting a fibration $X \to \mathbb{P}^1$ in intersections of two four-dimensional quadrics. Moreover we propose a conjecture—in the spirit of Kuznetsov's on cubic fourfolds [64]—relating the rationality of such varieties to their categorical representability. Our analysis is based on the study of the semiorthogonal decompositions of the quadric fibrations involved and the even Clifford algebras associated to the relative surface fibration defined by homological projective duality. We will also present a moduli space-theoretic way to attack this problem, which will lead us to consider twisted universal sheaves. In this section we will eventually need to work over an algebraically closed field of characteristic zero.

5.1. Intersections of 4-dimensional quadrics and Clifford algebras

Let Y be a scheme and $\pi : X \to Y$ a fibration in complete intersections of two four-dimensional quadrics. More precisely, we suppose that there exists a vector bundle E of rank 6 such that $X \subset \mathbb{P}(E)$ is the relative complete intersection of two quadric fibrations $Q_1 \to Y$ and $Q_2 \to Y$ of relative dimension 4. In this case, we have $\omega_{X/Y} = \mathscr{O}_{X/Y}(-2)$ and thus, by Proposition 2.2.2, for every integer j, we have a semiorthogonal decomposition

$$\mathsf{D}^b(X) = \big\langle \mathsf{A}_X^j, \mathsf{D}^b(Y)(j), \mathsf{D}^b(Y)(j+1) \big\rangle. \tag{5.1.1}$$

To simplify the notation, let us denote $\mathsf{A}_X = \mathsf{A}_X^1$.

The two quadric bundles $Q_1 \to Y$ and $Q_2 \to Y$ span a quadric bundle $Q \to S$ over a $\mathbb{P}^1$-bundle $S \to Y$. Let $\mathscr{C}_0$ be the even Clifford algebra of $Q \to S$. Recall that by relative homological projective duality (Theorem 2.3.7(1)) we have $\mathsf{A}_X \simeq \mathsf{D}^b(S, \mathscr{C}_0)$. If $Q \to S$ has a regular section, then by quadric reduction we obtain a quadric surface fibration $Q' \to S$, with associated even Clifford algebra $\mathscr{C}_0'$. Assuming we are in the generic situation (see Definition 1.2.4), then $Q \to S$, and therefore $Q' \to S$ (by Corollary 1.3.9), has simple degeneration with smooth discriminant divisor. By Theorem 1.8.8, we have $\mathsf{D}^b(S, \mathscr{C}_0) \simeq \mathsf{D}^b(S, \mathscr{C}_0')$ and $\mathscr{C}_0$ and $\mathscr{C}_0'$ define Azumaya algebras $\mathscr{B}_0$ and $\mathscr{B}_0'$ on the discriminant double cover $T \to S$, with the same Brauer class $\beta \in \mathrm{Br}(T)$.

Note that if Y is a curve over an algebraically closed field k, then $X \to Y$ always has a generically smooth section by Lemma 1.9.3. Then, by the Amer–Brumer Theorem 1.9.1, $Q \to S$ always has a generically smooth section. In order

to perform quadric reduction we need a smooth section: we will explicitly underline when it is needed. When X is smooth over k (e.g., if Y is smooth over k of characteristic $\neq 2$ and $X \to Y$ is generic, by Proposition 1.2.5), then any section of $X \to Y$ is smooth by Lemma 1.3.2.

As previously recalled, relative homological projective duality (Theorem 2.3.7(1)) gives the first description of the category A_X. If we suppose that $Q \to S$ has a globally smooth section (which is the case, by Theorem 1.9.1, if $X \to Y$ has a smooth section), then this can be extended further thanks to Theorem 1.8.8, and the fact that $Q \to S$ has even relative dimension.

Corollary 5.1.1. *Let $\pi : X \to Y$ be a generic relative complete intersection of two four-dimensional quadric fibrations over an integral scheme Y smooth over k. If the linear span quadric fibration $Q \to S$ has a smooth section (e.g., if π has a smooth section) then the categories A_X, $\mathsf{D}^b(S, \mathscr{C}_0)$, $\mathsf{D}^b(S, \mathscr{C}_0')$, and $\mathsf{D}^b(T, \beta)$ are equivalent.*

Proof. This is a corollary of relative homological projective duality (Theorem 2.3.7(1)) and the Morita invariance of the even Clifford algebra under quadric reduction (Corollary 1.8.8). In fact, the equivalence $\mathsf{D}^b(S, \mathscr{C}_0) \simeq \mathsf{D}^b(T, \beta)$ does not require the existence of a smooth section. $\quad\square$

Consider $Y = \mathbb{P}^1$. The rationality of X is in general unknown. The techniques developed in this paper and the language of categorical representability (see Definition 2.4.3) allow us to state a conjecture relating the rationality and the derived category of X, in the same spirit as Kuznetsov's conjecture on cubic fourfolds [64].

Conjecture 5.1.2. *Let $X \to \mathbb{P}^1$ be a fibration in complete intersections of two four-dimensional quadrics over an algebraically closed field of characteristic zero.*

- **Weak version.** *The fourfold X is rational if and only if it is categorically representable in dimension at most two.*
- **Strong version.** *The fourfold X is rational if and only if A_X is representable in dimension at most two.*

We say that a morphism $X \to Y$ *contains a line* over Y if X contains a surface generically ruled over Y via the restriction of the morphism. We will prove the strong version of Conjecture 5.1.2 in two cases: if the Brauer class β on T is trivial then we prove that $\mathsf{A}_X \simeq \mathsf{D}^b(T)$ and X is rational; if X contains a line over $\mathbb{P}^1$ then X is rational and we establish an equivalence $\mathsf{A}_X \simeq \mathsf{D}^b(T)$.

Even though, as Lemma 5.2.2 shows, this second case is contained in the first one, it still deserves its own treatment. Indeed, relative quadric intersections containing lines were already considered in the literature (see [31]), where the associated rational parameterizations can be explicitly described. Also, in this case, we provide an independent moduli space-theoretic interpretation of the equivalence $\mathsf{A}_X \simeq \mathsf{D}^b(T)$ that generalizes to other contexts.

Proposition 5.1.3. *Let $\pi : X \to Y$ be a generic relative complete intersection of two four-dimensional quadric fibrations over an integral scheme Y smooth over k. Then X and Q' are k-birationally equivalent. If furthermore, $\beta \in \mathrm{Br}(T)$ is trivial, and Y is k-rational, then X is k-rational and there is an equivalence $\mathsf{A}_X \simeq \mathsf{D}^b(T)$.*

Proof. For the first claim, by [31, Thm. 3.2], X and Q' are $k(Y)$-birational, hence k-birational by Lemma 1.9.5. For the second claim, we have that β is trivial in $\mathrm{Br}(T)$ if and only if Q' has a rational section over S, by Theorem 1.9.4. In this case, since Y is k-rational (hence S is also k-rational), Q' is thus $k(S)$-rational, hence k-rational. But by the first claim Q' is k-birational to X. The last assertion is a straightforward consequence of Corollary 5.1.1. $\quad\square$

5.2. Quadric intersection fibrations containing a line

The case of complete intersections of two quadrics of dimension $n \geqslant 4$ over a field that contain a line was studied in [31]. In particular, an explicit geometric proof of the rationality of $X \to \mathbb{P}^1$ can be given as a special case (see also [31, Prop. 2.2]).

Proposition 5.2.1. *Let k be a field, Y be an integral k-rational variety, and $W \to Y$ be a generic relative complete intersection of two quadric fibrations of dimension $n \geqslant 4$. If W contains a line over Y, then W is k-rational.*

Proof. Consider the generic fiber $W_\eta \subset \mathbb{P}^{n+1}_{k(Y)}$, and l the line contained in W_η. The fibers of the projection $p_l :$ $\mathbb{P}^{n+1}_{k(Y)} \dashrightarrow \mathbb{P}^{n-1}_{k(Y)}$ off the line l are the 2-dimensional planes containing the line l. Consider now the two quadrics $Q_{1,\eta}$ and $Q_{2,\eta}$, cutting out W_η. A generic fiber of the projection p_l intersects $Q_{i,\eta}$ along a pair of lines: l and a second line l_i, for $i = 1, 2$. The two lines l_1 and l_2 meet in just one point in each fiber of p_l. Then by restricting p_l to W_η we get the desired birational map $W_\eta \dashrightarrow \mathbb{P}^{n-1}_{k(Y)}$. Then W is birational to a projective bundle over the k-rational scheme Y, and hence W is k-rational. $\square$

In our case of a generic relative complete intersection $X \to Y$ of two quadrics of dimension 4, we can say more. We prove that containing a line implies the vanishing of the associated Brauer class. Then rationality, and indeed Conjecture 5.1.2 for this case, is then a consequence of Proposition 5.1.3.

Lemma 5.2.2. *Let $\pi : X \to Y$ be a generic relative complete intersection of two four-dimensional quadric fibrations over a regular integral scheme Y. If X contains a line over Y then β is trivial.*

Proof. If $X \to Y$ contains a line then $Q \to S$ contains a line l. A local computation proves that, if $Q \to S$ has simple degeneration, then there exists a smooth section s whose image is contained in the line l. Let Q' be the quadric reduction along s. In the process, l is contracted to a section of Q' over S. By Theorem 1.9.4, the Brauer class β' associated to $Q' \to S$ is trivial. Finally, by Corollary 1.8.8, we have $\beta = \beta'$ in $\text{Br}(T)$, hence β is trivial as well. $\square$

If the base scheme $Y = \mathbb{P}^1$ over an algebraically closed field then we can even say more.

Lemma 5.2.3. *Let k be algebraically closed and $\pi : X \to \mathbb{P}^1$ be a generic relative complete intersection of two four-dimensional quadric fibrations. Then X contains a line over $\mathbb{P}^1$ if and only if β is trivial.*

Proof. One direction is Lemma 5.2.2. For the other, by Lemma 1.9.3, π has a rational section (which is smooth by Remark 1.3.3), hence by the Amer–Brumer Theorem 1.9.1, so does the associated quadric fibration $Q \to S$. Then we can perform quadric reduction and obtain a quadric surface fibration $Q' \to S$. The Brauer class associated to Q' is again β, hence by Theorem 1.9.4, Q' has a rational section. The existence of a rational section of Q' implies the existence of a rational line over $\mathbb{P}^1$ in Q. Finally, we use the Amer Theorem (cf. Remark 1.9.2) to obtain a rational line over $\mathbb{P}^1$ inside X. $\square$

Corollary 5.2.4. *Let k be algebraically closed and $X \to \mathbb{P}^1$ be a generic relative complete intersection of two four-dimensional quadrics. If X contains a line over $\mathbb{P}^1$ then X is rational and A_X is equivalent to $\mathsf{D}^b(T)$, hence is representable in dimension 2.*

We recall that if M is a K3 surface and $\beta \in \text{Br}(M)$ is nontrivial, then $\mathsf{D}^b(M, \beta)$ is never equivalent to $\mathsf{D}^b(M)$ [46, Rem. 7.10]. We do not know if the same result holds for the class of surfaces T that carry a fibration $T \to \mathbb{P}^1$ in hyperelliptic curves, and the Brauer classes $\beta \in \text{Br}(T)$, considered above. Such a result would strengthen Corollary 5.2.4.

5.3. A moduli space-theoretic interpretation

On the other hand, the rich geometry of the fibration $X \to \mathbb{P}^1$ allows one to give another description of A_X as a derived category of twisted sheaves on T over the complex numbers.

More generally, let $W \to Y$ be the intersection of two $2m$-dimensional quadric fibrations over a smooth projective variety, and $Q \to S$ the associated quadric span. For any point y of Y, there is a hyperelliptic curve T_y parameterizing maximal isotropic subspaces of the fiber Q_y. In fact, T_y is also the fine moduli space for spinor bundles over W_y (see [21] and [73] for definitions and constructions). One can in general consider the same moduli problem in the relative case, see [45, Thm. 4.3.7]. The most substantial difference concerns the existence of a universal family on the product $W \times T$. While this can be constructed explicitly on $W_y \times T_y$ [21, Section 2], the obstruction to the existence of a universal family over T is given by an element ω of the Brauer group $\text{Br}(T)$. That is, we have an ω-twisted universal sheaf E on $W \times T$ [26]. The existence of a linear space of dimension $(m - 1)$ in W over Y implies the existence of a universal object and then the vanishing of ω.

Combining this description of T as a non-fine moduli space and the semiorthogonal decomposition in the absolute case given in [21], we get an equivalence between A_W and a category of twisted sheaves on T.

Proposition 5.3.1. *Let Y be a smooth projective variety over $\mathbb{C}$ and $W \to Y$ be a generic relative complete intersection of two $2m$-dimensional quadric fibrations. Then there is an equivalence $A_W \simeq D^b(T, \omega)$. In particular, if W contains an $(m-1)$-plane over Y, then $\omega = 0$ and W is categorically representable in dimension 2. Moreover, in this case, if Y is rational then so is W.*

Proof. By an ω-twisted skyscraper sheaf $\mathscr{F}_t$ of T we mean a simple ω-twisted sheaves supported on the closed point t of T (such a sheaf exists for every closed point, e.g., by [26, Cor. 1.2.6]). Then one can check, as in the untwisted case (e.g., [44, Prop. 3.17]), that the collection of all ω-twisted skyscraper sheaves forms a spanning class (as defined in [44, Def. 1.47]) for $D^b(T, \omega)$.

The ω-twisted universal family E over $W \times T$ is a $pr_2^*\omega$-twisted sheaf such that for each closed point y of Y, the sheaf E_y is the universal spinor bundle on the product $W \times_{k(y)} T = W_y \times T_y$. Hence, the restricted Fourier–Mukai functor $\Phi_{E_y} : D^b(T_y) \to D^b(W_y)$ is fully faithful by [21, Thm. 2.7] (note that the Brauer class ω_y is trivial on T_y by Tsen's theorem). We observe that, for each structure sheaf $\mathscr{F}_t$ of a closed point t of T on the fiber T_y, we have that $\Phi_E(\mathscr{F}_t) = j_*\Phi_{E_y}(\mathscr{F}_t)$, where $j : W_y \to W$ is the closed embedding of the fiber. As j is an affine morphism, hence cohomologically acyclic, we have that

$$\mathrm{Hom}_{D^b(T,\omega)}(\mathscr{F}_{t_1}, \mathscr{F}_{t_2}) = \mathrm{Hom}_{D^b(W)}\big(\Phi_E(\mathscr{F}_{t_1}), \Phi_E(\mathscr{F}_{t_2})\big),$$

for any closed point t_1 and t_2 of T. Hence we can apply [44, Prop. 1.49] to prove that Φ_E is fully faithful.

The existence of an $(m-1)$-plane over Y allows one to construct the universal family, so that $\omega = 0$, and the categorical representability in dimension 2 follows. Recalling Theorem 5.2.1, we get that if W contains a linear space of dimension $(m-1)$ over Y (and hence a line), it is rational. □

Proposition 5.3.1 generalizes (and gives a different proof of) Lemma 5.2.2.

Recall that in [21], it is shown that if Z is a smooth intersection of quadric fourfolds and contains a line (which is always the case over an algebraically closed field of characteristic zero), then there is a direct way to construct a universal family of spinor bundles on Z. In the relative case this translates into another proof of Proposition 5.3.1.

Consider now $X \to \mathbb{P}^1$ the intersection of two generic quadric fibrations of relative dimension 4 (that is, $m = 2$ in the above discussion). The family $T \to \mathbb{P}^1$ factors through $T \to S \to \mathbb{P}^1$, where $T \to S$ is the discriminant double cover of the associated linear span quadric fibration. We obtained an equivalence $D^b(T, \omega) \simeq A_X$ providing another description of A_X. Some natural questions remain.

Question 5.3.2. Let $\pi : X \to \mathbb{P}^1$ be a generic relative complete intersection of two four-dimensional quadric fibrations over an algebraically closed field and $T \to S$ the associated discriminant cover.

1. Are $\beta, \omega \in \mathrm{Br}(T)$ the same Brauer class?
2. Does X contain a line over $\mathbb{P}^1$ if and only if ω is trivial?
3. Is X rational if and only if it contains a line over $\mathbb{P}^1$?

We conjecture that the answer to the first (hence also the second) question is positive, whereas it is expected that the third question has a negative answer.

5.4. An explicit fourfold example

The aim of this section is to give an explicit example of a rational fourfold fibered over $\mathbb{P}^1$ in intersections of two four-dimensional quadrics. As before it will be embedded in a $\mathbb{P}^5$ bundle $\mathbb{P}(E)$ over $\mathbb{P}^1$.

Let us first develop the geometric construction over a point p. Let X_p be the intersection of two quadrics in $\mathbb{P}(E)_p \simeq \mathbb{P}^5$. First recall that, if X_p contains a line L, then the projection off L is a rational morphism from X_p into $\mathbb{P}^5$, inducing a birational map between X_p and a $\mathbb{P}^3$ (see 5.2.1). The corresponding birational map decomposes as the blow-up of L followed by the contraction of a surface swept by lines intersecting L onto a curve of genus 2 and degree 5 in $\mathbb{P}^3$ (see [49, Prop. 3.4.1(ii)]). Notably the exceptional divisor is sent onto a quadric surface in $\mathbb{P}^3$ containing the genus 2 curve.

On the other hand, suppose that we have a curve B of genus 2 and degree 5 in $\mathbb{P}^3$, for example embedded by the linear system $|2K_B + D|$, where K_B is the canonical divisor and D a degree one divisor. Then standard computations, that we omit, show that the birational inverse map from $\mathbb{P}^3$ onto X_x is given by the ideal of cubics in $\mathbb{P}^3$ containing the curve B. By easy Riemann–Roch computations (see for instance [15, Section 3]) B is contained in a quadric surface and it comes:

(a) Either as a divisor of type $(2, 3)$ on a smooth quadric surface (the quadric surface is the trisecant scroll of B),
(b) Or as the following construction: the intersection of a quadric cone of corank 1 with a cubic surface passing through the singularity is a sextic curve that decomposes in two components. The first is a line, a ruling of the cone. The second, residual to the line, is our curve B. Also in this case, the quadric cone is the trisecant scroll of B.

The two different kinds of genus 2 degree 5 curves in $\mathbb{P}^3$ correspond to the different values of the normal sheaf N_{L/X_p}. That is: in the generic case, the normal sheaf is $\mathscr{O}_L \oplus \mathscr{O}_L$. In this case the exceptional divisor of L is $\mathbb{P}^1 \times \mathbb{P}^1$ and it is isomorphically sent onto the smooth quadric surface of case (a) by the projection off L. On the other hand, we can also have a normal sheaf equal to $\mathscr{O}_L(1) \oplus \mathscr{O}_L(-1)$. In this case the exceptional divisor is isomorphic to $\mathbb{F}_2$, that is the desingularization of the quadric cone that contains B in case (b). There are no other possible normal bundles (see for example [49, Lemma 3.3.4]). In both cases the map given by cubics contracts the trisecant scroll (or its desingularization) onto the line $L \subset X_x$. Note that if we work over complex numbers, X always contains a line.

A semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(X) = \big\langle \mathsf{D}^{\mathsf{b}}(B), \mathscr{O}_X, \mathscr{O}_X(1) \big\rangle$$

of the derived category of the intersection of two even-dimensional quadrics was described in [21] over $\mathbb{C}$ by an explicit Fourier–Mukai functor. Homological projective duality gives the decomposition over any field [63, Cor. 5.7]. In case (a), we can give an alternative proof based on the explicit description and mutations.

Suppose we are in case (a) and consider the diagram:
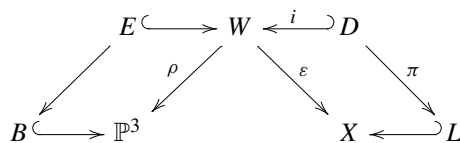
where $\rho: W \to \mathbb{P}^3$ is the blow-up of the hyperelliptic curve with exceptional divisor E and $\varepsilon: W \to X$ is the blow-up of the line L with exceptional divisor $i: D \hookrightarrow W$. Let $\pi: D \to L$ be the $\mathbb{P}^1$-bundle map. Let us denote by $H = \rho^* \mathscr{O}_{\mathbb{P}^3}(1)$ and by $h = \varepsilon^* \mathscr{O}_X(1)$ the two generators of $\mathrm{Pic}(W)$. Then by the previous description we have $h = 3H - E$ and $H = h - D$, which give $E = 3H - h$, and $D = h - H$. Finally, we have $\omega_W = -4H + E = -H - h$.

Now consider the semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^3) = \langle \mathscr{O}_{\mathbb{P}^3}(-2), \ldots, \mathscr{O}_{\mathbb{P}^3}(1) \rangle$. By the blow-up formula we get:

$$\mathsf{D}^{\mathsf{b}}(W) = \big\langle \Phi \mathsf{D}^{\mathsf{b}}(B), -2H, -H, \mathscr{O}_W, H \big\rangle,$$

where Φ is a fully faithful functor. Mutating H to the left with respect to its orthogonal complement, we get

$$\mathsf{D}^{\mathsf{b}}(W) = \big\langle -h, \Phi \mathsf{D}^{\mathsf{b}}(B), -2H, -H, \mathscr{O}_W \big\rangle,$$

using [20, Prop. 3.6] and $\omega_W = -H - h$. Now mutating $\Phi \mathsf{D}^{\mathsf{b}}(B)$ to the right with respect to $\langle -2H, -H \rangle$, we get a fully faithful functor $\Phi' = \Phi \circ R_{\langle -2H, -H \rangle}: \mathsf{D}^{\mathsf{b}}(B) \to \mathsf{D}^{\mathsf{b}}(W)$ and a semiorthogonal decomposition:

$$\mathsf{D}^{\mathsf{b}}(W) = \big\langle -h, -2H, -H, \Phi' \mathsf{D}^{\mathsf{b}}(B), \mathscr{O}_W \big\rangle. \tag{5.4.1}$$

On the other hand, consider the decompositions $\mathsf{D}^{\mathsf{b}}(X) = \langle \mathsf{A}_X, \mathscr{O}_X, \mathscr{O}_X(1) \rangle$ and $\mathsf{D}^{\mathsf{b}}(L) = \langle \mathscr{O}_L, \mathscr{O}_L(1) \rangle$. By the blow-up formula [74, Thm. 4.3], we get the following semiorthogonal decomposition:

$$\mathsf{D}^{\mathsf{b}}(W) = \big\langle i_*\big(\pi^* \mathscr{O}_L \otimes \mathscr{O}_\pi(-1)\big), i_*\big(\pi^* \mathscr{O}_L(1) \otimes \mathscr{O}_\pi(-1)\big), \mathsf{A}_X, \mathscr{O}_W, h \big\rangle,$$

where we identified, up to the equivalence ε^* the category A_X with its pull-back to $\mathsf{D}^{\mathsf{b}}(W)$.

Now notice that $\mathcal{O}_\pi(1) = H_D$. Then we can rewrite the semiorthogonal decomposition as

$$\mathsf{D}^b(W) = \langle \mathcal{O}_D(-H), \mathcal{O}_D(h - H), \mathsf{A}_X, \mathcal{O}_W, h \rangle.$$

First, mutate A_X to the left with respect to $\mathcal{O}_D(h - H)$ to get

$$\mathsf{D}^b(W) = \langle \mathcal{O}_D(-H), \mathsf{A}_X, \mathcal{O}_D(h - H), \mathcal{O}_W, h \rangle,$$

where we identify, up to equivalence, A_X with the image of the mutation. Notice that $\mathcal{O}_D(h - H) = \mathcal{O}_D(D)$. It is standard to check, by definition of mutation and using the evaluation sequence of the smooth divisor $D = h - H$, that the mutation of the pair $\langle \mathcal{O}_D(D), \mathcal{O}_W \rangle$ gives the pair $\langle \mathcal{O}_W, D \rangle = \langle \mathcal{O}_W, h - H \rangle$. Performing this mutation we get then:

$$\mathsf{D}^b(W) = \langle \mathcal{O}_D(-H), \mathsf{A}_X, \mathcal{O}_W, h - H, h \rangle.$$

Now mutate $\langle h - H, h \rangle$, recalling $\omega_W = -H - h$, to the left with respect to its orthogonal complement to get

$$\mathsf{D}^b(W) = \langle -2H, -H, \mathcal{O}_D(-H), \mathsf{A}_X, \mathcal{O}_W \rangle.$$

Once again, by definition of mutation and the evaluation sequence, up to a twist with $-H$, the mutation of the pair $\langle -H, \mathcal{O}_D(-H) \rangle$ gives the pair $\langle -H - D, -H \rangle$, and then the decomposition

$$\mathsf{D}^b(W) = \langle -2H, -h, -H, \mathsf{A}_X, \mathcal{O}_W \rangle, \tag{5.4.2}$$

since $D = h - H$. Comparing the decompositions (5.4.1) and (5.4.2), we get that the pair $\langle -h, -H \rangle$ is completely orthogonal and then we get the required equivalence between A_X and $\mathsf{D}^b(B)$.

Let us come to the relative setting. Suppose that we have a rational family in the universal degree 1 Picard variety $\mathscr{P}ic^1$ over the moduli space $\mathcal{M}_2$ of genus two curves. This is just a rational family of couples $(B_t, D_t)_{t \in \mathbb{P}^1}$, where D_t is a degree one divisor on the genus two curve B_t. Call $\mathbb{B}$ the total space of the family B_t and $\tau : \mathbb{B} \to \mathbb{P}^1$ the projection. Then B_t is embedded as a family of degree 5 curves in the rank 3 projective bundle $\mathbb{P}(U) = \mathbb{P}(\tau_*(2K_{B_t} + D_t))$ over $\mathbb{P}^1$. We can suppose that we have chosen (B_t, D_t) so that each curve B_t has a smooth trisecant scroll in $\mathbb{P}(U_t)$ as in case (a).

By our preceding remarks, if we apply to $\mathbb{P}(U)$ the birational transformation given by relative cubics in $\mathbb{P}(U)$ vanishing on $\mathbb{B}$, we get a fourfold $\mathcal{X}$, fibered over $\mathbb{P}^1$ in intersections of quadrics. Moreover, by construction, X is clearly rational and $\mathbb{B}$ is a fibration in hyperelliptic curves. Now we use Orlov's formula [74, Thm. 4.3] for the derived category of the blow-up of a smooth projective variety in order to get a semiorthogonal decomposition of the derived category of $\mathcal{X}$. Moreover, by mutations, we can prove the strong version of Conjecture 5.1.2. We note that the trisecant lines of the curves in the family B_t make up a quadric surface fibration $\mathcal{Q}$ contained in $\mathbb{P}(U)$. Let φ the rational map defined on $\mathbb{P}(U)$ by the ideal of relative cubics containing $\mathbb{B}$.

Theorem 5.4.1. *Let $\mathcal{X}$ be the rational fourfold obtained as the image of φ. Then $\mathcal{X}$ is categorically representable in dimension* 2 *as follows*:

$$\mathsf{D}^b(\mathcal{X}) = \langle \mathsf{D}^b(\mathbb{B}), E_1, E_2, E_3, E_4 \rangle,$$

where E_i are exceptional objects. Moreover, there is an equivalence

$$\mathsf{A}_\mathcal{X} \simeq \mathsf{D}^b(\mathbb{B}).$$

Proof. The fourfold $\mathcal{X}$ is obtained by first blowing-up $\mathbb{P}(U)$ along $\mathbb{B}$ and then contracting $\mathcal{Q}$ onto a ruled surface $\mathbb{F}$. The quadric fibration $\mathcal{Q}$ is the exceptional divisor over $\mathbb{F}$ when one blows-up $\mathcal{X}$ along $\mathbb{F}$. Let us denote by $\mathcal{W}$ the blow-up of $\mathbb{P}(U)$ along $\mathbb{B}$, which is naturally isomorphic to the blow-up of $\mathcal{X}$ along $\mathbb{F}$.

The two assertions are proved by comparing the two semiorthogonal decompositions of $\mathsf{D}^b(\mathcal{W})$ induced by the two descriptions of $\mathcal{W}$, as the blow-up on one hand of $\mathbb{P}(U)$ and on the other of $\mathcal{X}$. To this end, recall that the derived category of $\mathbb{P}^1$ admits a semiorthogonal decomposition by two categories generated by exceptional line bundles. we get two semiorthogonal decompositions of $\mathsf{D}^b(\mathcal{W})$ given by exceptional objects with orthogonal complements respectively $\mathsf{D}^b(\mathbb{B})$ and $\mathsf{A}_\mathcal{X}$.

In order to get the required equivalence explicitly, one has to perform the same mutations as before in the relative setting. Indeed one has a diagram
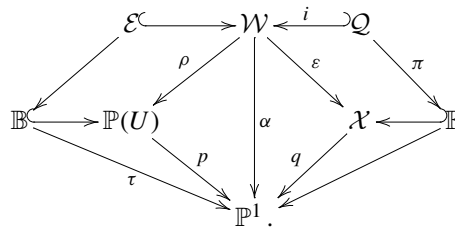
We will keep the same notations for the relative line bundles in $\mathrm{Pic}(\mathcal{X}/\mathbb{P}(U))$ that we used for $\mathrm{Pic}(X)$ in the previous mutations. Moreover, notice that $\rho \circ p = \alpha = \varepsilon \circ q$. Let us moreover introduce, for L in $\mathrm{Pic}(\mathcal{X}/\mathbb{P}^1)$, the notation $\alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)(L)$ for the admissible category whose objects are of the form $\alpha^* A \otimes L$, where A is in $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)$. So we end up with the following semiorthogonal decomposition (a relative version of (5.4.1)):

$$\mathsf{D}^{\mathsf{b}}(\mathcal{W}) = \langle \Phi \mathsf{D}^{\mathsf{b}}(\mathbb{B}), \alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)(-2H), \alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)(-H), \alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1), \alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)(H) \rangle, \qquad (5.4.3)$$

by blowing up $\mathbb{P}(U)$ along $\mathbb{B}$, where Φ is a fully faithful functor. On the other hand, recall that $\mathsf{D}^{\mathsf{b}}(\mathbb{F})$ decomposes into two copies of $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)$, and consider the blow-up formula for $\mathcal{W} \to \mathcal{X}$. In order to complete the proof, notice that we can perform the same mutations as before in the relative setting in the following sense: any occurrence of $\alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)$ can be replaced by two exceptional objects (actually, two line bundles in $\alpha^* \mathrm{Pic}(\mathbb{P}^1)$). The previous mutations carry on these exceptional sets: whenever we act by a mutation inside $\alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)$, this is just changing the choice of the pair of line bundle decomposing it. The mutations involving relative line bundles carry on just as before. We end up hence with the following semiorthogonal decomposition (a relative version of (5.4.2)):

$$\mathsf{D}^{\mathsf{b}}(\mathcal{W}) = \langle \alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)(-2H), \alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)(-h), \alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)(-H), \mathsf{A}_{\mathcal{X}}, \alpha^* \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1) \rangle. \qquad (5.4.4)$$

Comparing via mutations the decompositions (5.4.3) and (5.4.4) we get the proof. $\quad\square$

Remark 5.4.2. Notice that the proof of Theorem 5.4.1 applies whenever we consider the base to be a rational smooth projective variety whose derived category admits a full exceptional sequence of vector bundles, as for example any $\mathbb{P}^n$ or any smooth quadric hypersurface.

Remark 5.4.3. Theorem 1.9.4 has already been (implicitly) used by some authors to describe the birational geometry of high-dimensional varieties. Namely, if a cubic fourfold contains a plane, then it is birational to a quadric surface bundle over $\mathbb{P}^2$. The study of the Brauer class associated to this quadric bundle has been developed in [42] (implicitly) and [64]. In Appendix B, we verify that the Brauer class considered in [42] is the same as the one considered here and in [64].

Acknowledgements

Appendix A. A comparison of even Clifford algebras

In his thesis, Bichsel [16] constructed an even Clifford algebra of a line bundle-valued quadratic form on an affine scheme using faithfully flat descent. Parimala and Sridharan [76, Section 4] generalized this construction to any scheme. Bichsel and Knus [17] and Caenepeel and van Oystaeyen [28] gave constructions of a *generalized* or $\mathbb{Z}$-graded

Clifford algebra, of which the even Clifford algebra is the degree zero piece. These constructions are detailed in [4, Section 1.8].

Independently, Kapranov [52, Section 4.1] introduced a *homogeneous* Clifford algebra, which was further developed by Kuznetsov [63, Section 3], to study the derived category of projective quadrics and quadric fibrations. In this context, the even Clifford algebra is defined as a certain limit of the graded pieces of the homogeneous Clifford algebra. In this appendix, we will show that our construction of the even Clifford algebra coincides with that of Kuznetsov.

First, we would like to point out some differences between our conventions in Section 1 and those in [63, Section 3]. The value line bundle L differs by a dual and all quadratic forms are assumed to be primitive in [63, Section 3]. Also, certain noncanonical isomorphisms involving symmetric squares and dualization are used (thereby requiring the standing hypothesis of working in characteristic zero), and the definition of the Clifford algebra of a quadratic form over a field given in [63, Section 2.4] is incorrect if 2 is not invertible.

Let S be a scheme and (E, q, L) be a quadratic form over S (see Section 1.1). Kuznetsov defines the *homogeneous Clifford algebra* as the graded quotient

$$\mathfrak{B} = T(E)/I = \bigoplus_{n \geq 0} T^n(E)/I_n = \bigoplus_{n \geq 0} \mathfrak{B}_n$$

of the tensor algebra of E by the homogeneous ideal $I = \bigoplus_{n \geq 0} I_n$ generated by

$$I_2 = \ker(q : S_2 E \to L) \subset T^2(E).$$

We point out that $\mathfrak{B}_0 = \mathscr{O}_S$, $\mathfrak{B}_1 = E$, and that $\mathfrak{B}_2$ fits into a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S_2 E & \longrightarrow & T^2(E) & \longrightarrow & \bigwedge^2 E & \longrightarrow & 0 \\
 & & \downarrow{q} & & \downarrow{\varphi_2} & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & \mathfrak{B}_2 & \longrightarrow & \bigwedge^2 E & \longrightarrow & 0
\end{array}
\tag{A.1}
$$

of locally free $\mathscr{O}_S$-modules, where $\varphi = \bigoplus_{n \geq 0} \varphi_n$ is the quotient map $T(E) \to \mathfrak{B}$. Via the subbundle $L \to \mathfrak{B}_2$ there is a canonical morphism $\mathfrak{B}_n \otimes L \to \mathfrak{B}_{n+2}$ for every $n \geq 0$. By induction, there is a canonical monomorphism $L^{\otimes m} \to \mathfrak{B}_{2m}$ for all $m \geq 0$. Thus there is an monomorphism $S(L) = \bigoplus_{n \geq 0} L^{\otimes n} \to \mathfrak{B}$ of degree 2, whose image is actually central, see [63, Lemma 3.6]. Also, there is a canonical morphism $\mathfrak{B}_{2m} \otimes L^{\vee \otimes m} \to \mathfrak{B}_{2(m+1)} \otimes L^{\vee \otimes (m+1)}$ for every $m \geq 0$. Kuznetsov [63, Section 3.3] defines the even Clifford algebra as the colimit

$$\mathcal{B}_0 = \varinjlim \mathfrak{B}_{2m} \otimes L^{\vee \otimes m}$$

of this directed system of $\mathscr{O}_S$-module morphisms.

Proposition A.1. *Let (E, q, L) be a quadratic form of rank n on a scheme S. The even Clifford algebras, $\mathscr{C}_0$ as defined in Section 1.5 and $\mathcal{B}_0$ as defined in [63, Section 3.3], are isomorphic $\mathscr{O}_S$-algebras.*

Proof. Tensoring the quotient morphism $\varphi_2 : T^2(E) \to \mathfrak{B}_2$ by $L^\vee$, there is an induced morphism $\psi : T^2(E) \otimes L^\vee \to \mathcal{B}_0$. We will verify that ψ satisfies the universal property of $\mathscr{C}_0$, deducing the existence of an $\mathscr{O}_S$-algebra morphism $\Psi : \mathscr{C}_0 \to \mathcal{B}_0$.

First, we observe that by diagram (A.1), $\psi(v \otimes v \otimes f) = q(v) \otimes f$ in $L \otimes L^\vee \to \mathfrak{B}_2 \otimes L^\vee$, hence $\psi(v \otimes v \otimes f) = f(q(v))$ under the identification $L \otimes L^\vee = \mathscr{O}_S \subset \mathcal{B}_0$ with the identity in the limit.

Second, we calculate that

$$
\begin{aligned}
\psi(u \otimes v \otimes f)\psi(v \otimes w \otimes g) &= \varphi_2(u \otimes v) \otimes f \otimes \varphi_2(v \otimes w) \otimes g = f \otimes \varphi_4(u \otimes v \otimes v \otimes w) \otimes g \\
&= f \otimes q(v) \otimes \varphi_2(u \otimes w) \otimes g = f(q(v)) \psi(u \otimes w \otimes g)
\end{aligned}
$$

using the facts that $L \subset \mathfrak{B}_2$ is central in $\mathfrak{B}$, the quotient map $\varphi = \bigoplus \varphi_n$ is a graded morphism, and that φ_4 evaluated on the submodule $E \otimes S_2 E \otimes E$ factors through $L \otimes \mathfrak{B}_2$.

Hence ψ satisfies the universal property of $\mathscr{C}_0$, so there exists an $\mathscr{O}_S$-algebra morphism $\Psi : \mathscr{C}_0 \to \mathcal{B}_0$. Since these algebras are locally free of the same rank, it suffices to check that Ψ is an isomorphism on fibers. Hence we are

reduced to the case where S is the spectrum of a field k. Choosing a generator $L = lk$ and a k-basis $e_1, \ldots, e_n$ of E, then by its construction, $\mathscr{C}_0$ has a k-basis consisting of $e_{i_1} \cdots e_{i_{2m}} l^{-m}$ where $1 \leqslant i_1 < \cdots < i_{2m} \leqslant n$. For $2m \geqslant n$, the limit stabilizes and $\mathcal{B}_0 = \mathfrak{B}_{2m} \otimes L^{\vee \otimes m}$, showing that $\mathcal{B}_0$ has a k-basis of the same shape. Since ψ preserves basis elements of the shape $e_{i_1} e_{i_2} l^{-1}$ by its very definition, Ψ will preserve the shape of the entire basis. Hence Ψ is an isomorphism. $\quad\square$

Kuznetsov [63, Section 3.3] also defines the Clifford bimodule ("the odd part of the Clifford algebra") as the colimit

$$\mathcal{B}_1 = \varinjlim \mathfrak{B}_{2m+1} \otimes L^{\vee \otimes m}$$

of the directed system of $\mathscr{O}_S$-module morphisms $\mathfrak{B}_{2m+1} \otimes L^{\vee \otimes m} \to \mathfrak{B}_{2(m+1)+1} \otimes L^{\vee \otimes (m+1)}$ defined similarly as above. Arguing similarly as in the proof of Proposition A.1, there is an isomorphism of Clifford bimodules $\mathscr{C}_1$ and $\mathcal{B}_1$ equivariant for the above isomorphism of even Clifford algebras.

As constructed, $\mathfrak{B}$ is a quadratic $\mathscr{O}_S$-algebra in the sense of [78, Ch. 1, Section 2]. There is another quadratic $\mathscr{O}_S$-algebra associated to (E, q, L): the coordinate algebra $\mathfrak{A} = \bigoplus_{l \geqslant 0} \pi_* \mathscr{O}_{Q/S}(l)$. The cohomology of the exact sequence (1.2.3) tensored by $\mathscr{O}_{Q/S}(l-1)$, implies, noting that $R^1 \pi_* \mathscr{O}_{Q/S}(l) = 0$, that $\pi_* \mathscr{O}_{Q/S}(l) \cong S^l(E^\vee)/(S^{l-2}(E^\vee) \otimes s_q)$. Dualizing the diagram (A.1), we see that $\mathfrak{B}_2^\vee \subset T^2(E^\vee)$ is the ideal of relations of degree 2 in $\mathfrak{A}$. In fact, the quotient $T(E^\vee) \to \mathfrak{A}$ has ideal of relations generated in by $\mathfrak{B}_2^\vee$. Hence $\mathfrak{A}$ is the quadratic dual $\mathscr{O}_S$-algebra to $\mathfrak{B}$ in the terminology of [78, Ch. 1, Section 2, Def. 1]. In fact, both $\mathfrak{A}$ and $\mathfrak{B}$ are Koszul algebras (cf. [78, Ch. 2, Section 1, Def. 1]), which can be proved as in [78, Section 6].

The following list of properties of the even Clifford algebra are all proved by descent, using the corresponding classical properties of even Clifford algebra of $\mathscr{O}_S$-valued forms, see [4, Section 1.8].

Proposition A.2. *Let (E, q, L) be a quadratic form of rank n on S.*

1. *Assume that (E, q, L) is (semi)regular. If n is odd, then $\mathscr{C}_0(E, q, L)$ is a central $\mathscr{O}_S$-algebra. If n is even, then the center $\mathscr{Z}(E, q, L)$ of $\mathscr{C}_0(E, q, L)$ is an étale quadratic $\mathscr{O}_S$-algebra.*
2. *If (E, q, L) is (semi)regular and n is odd, then $\mathscr{C}_0(E, q, L)$ is an Azumaya $\mathscr{O}_S$-algebra. If (E, q, L) is regular and n is even, then $\mathscr{C}_0(E, q, L)$ is an Azumaya algebra over its center.*
3. *Any similarity $(\varphi, \lambda) : (E, q, L) \to (E', q', L')$ induces an $\mathscr{O}_S$-algebra isomorphism*

$$\mathscr{C}_0(\varphi, \lambda) : \mathscr{C}_0(E, q, L) \to \mathscr{C}_0(E', q', L').$$

4. *Any $\mathscr{O}_S$-module isomorphism $\phi : N^{\otimes 2} \otimes L \to L'$ induces an $\mathscr{O}_S$-algebra isomorphism*

$$\mathscr{C}_0(N \otimes E, \phi \circ (q_N \otimes q), L') \to \mathscr{C}_0(E, q, L),$$

where $q_N : N \to N^{\otimes 2}$ is the canonical quadratic form.
5. *For any morphism of schemes $g : S' \to S$, there is a canonical $\mathscr{O}_S$-module isomorphism*

$$g^* \mathscr{C}_0(E, q, L) \to \mathscr{C}_0(g^*(E, q, L)).$$

Proof. For (1) and (2), see [56, IV Thm. 2.2.3, Prop. 3.2.4] or [17, Thm. 3.7]. For (3) and (4), see [56, IV Props. 7.1.1, 7.1.2]. The tensorial nature of the construction immediately implies (5). $\quad\square$

The following list of properties of the Clifford bimodule are all proved by descent, using the corresponding classical properties of odd Clifford algebra of $\mathscr{O}_S$-valued forms, see [4, Section 4.1].

Proposition A.3. *Let (E, q, L) be a quadratic form of rank n on S.*

1. *Via multiplication in the tensor algebra, there is a canonical morphism*

$$m : \mathscr{C}_1(E, q, L) \otimes_{\mathscr{C}_0(E, q, L)} \mathscr{C}_1(E, q, L) \to \mathscr{C}_0(E, q, L) \otimes_{\mathscr{O}_S} L$$

of $\mathscr{C}_0(E, q, L)$-bimodules (with trivial action on L). If (E, q, L) is primitive then $\mathscr{C}_1(E, q, L)$ is an invertible $\mathscr{C}_0(E, q, L)$-bimodule and the map m is an isomorphism.

2. *Any similarity transformation* $(\varphi, \lambda): (E, q, L) \to (E', q', L')$ *induces an* $\mathscr{O}_S$-*module isomorphism*

$$\mathscr{C}_1(\varphi, \lambda): \mathscr{C}_1(E, q, L) \to \mathscr{C}_1(E', q', L')$$

that is $\mathscr{C}_0(\varphi, \lambda)$-*semilinear according to the diagram*

$$
\begin{array}{ccc}
\mathscr{C}_1(E, q, L) \otimes_{\mathscr{C}_0(E, q, L)} \mathscr{C}_1(E, q, L) & \xrightarrow{\ m\ } & \mathscr{C}_0(E, q, L) \otimes_{\mathscr{O}_S} L \\
\downarrow{\scriptstyle \mathscr{C}_1(\varphi, \lambda) \otimes \mathscr{C}_1(\varphi, \lambda)} & & \downarrow{\scriptstyle \mathscr{C}_0(\varphi, \lambda) \otimes \lambda} \\
\mathscr{C}_1(E', q', L') \otimes_{\mathscr{C}_0(E', q', L')} \mathscr{C}_1(E', q', L') & \xrightarrow{\ m\ } & \mathscr{C}_0(E', q', L') \otimes_{\mathscr{O}_S} L'
\end{array}
$$

3. *Any* $\mathscr{O}_S$-*module isomorphism* $\phi: N^{\otimes 2} \otimes L \to L'$ *induces an* $\mathscr{O}_S$-*module isomorphism*

$$\mathscr{C}_1\big(N \otimes E, \phi \circ (q_N \otimes q), L'\big) \to N \otimes \mathscr{C}_1(E, q, L).$$

4. *For any morphism of schemes* $g: S' \to S$, *there is a canonical* $\mathscr{O}_S$-*module isomorphism*

$$g^* \mathscr{C}_1(E, q, L) \to \mathscr{C}_1\big(g^*(E, q, L)\big).$$

Proof. For (1), see [16, Section 2] or [17, Lemma 3.1]. For a proof of the final statement in (1), we have the following local calculation: if (E, q, L) is a primitive quadratic form over a local ring then there exists a line subbundle $N \subset E$ such that $q|_N$ is regular, and in this case, N generates $\mathscr{C}_1$ over $\mathscr{C}_0$. For (2), see [16, Prop. 2.6]. For (3), we can appeal to [17, Lemma 3.1]. The tensorial nature of the construction immediately implies (4). $\quad\square$

Appendix B. The equality of some Brauer classes related to quadric fibrations

Our perspective has been to use the even Clifford algebra as an algebraic way of getting at the Brauer class associated to a quadric fibration. Other authors, notably in [42,43,35], prefer to use geometric manifestations of Brauer classes. One of these geometric manifestations involves the Stein factorization of the relative Lagrangian Grassmannian. Over a field (more generally, when the quadric fibration is regular), it's a classical fact that in dimensions 2 and 4, the Brauer classes arising from these two methods coincide, see [33, XVI Ex. 85.4]. This is proved by appealing to the exceptional isomorphisms of projective homogeneous varieties associated to algebraic groups of low rank, as in [57, Section 15]. The aim of this section is provide a proof for quadric fibrations with simple degeneration. For quadric surface bundles, this fact has been noted in [64, Lemma 4.2] and [47, Thm. 4.5].

Let (E, q, L) be a quadratic form of rank n on a scheme S. We recall the standard moduli theoretic description of the *isotropic Grassmanian fibration* $\Lambda\mathrm{G}_r(q) \to S$.

Theorem B.4. *For each integer* $0 \leqslant r \leqslant \lceil n/2 \rceil$, *the* S-*scheme* $\Lambda\mathrm{G}(q)$ *represents the the moduli subfunctor*

$$u: U \to S \quad \mapsto \quad \big\{ W \xrightarrow{v} u^* E: W \text{ has rank } r + 1 \text{ and } u^* q|_W = 0 \big\}$$

of the Grassmannian of rank $r + 1$ *sub vector bundles of* E.

Note that $\Lambda\mathrm{G}_0(q) = Q$ recovers Theorem 1.2.2. When n is even, the $\Lambda\mathrm{G}(q) = \Lambda\mathrm{G}_{n/2}(q)$ is called the *Lagrangian Grassmannian fibration* associated to (E, q, L).

We recall that the *Stein factorization* of a proper morphism $p: X \to S$ of schemes is a decomposition $X \xrightarrow{r} Z \xrightarrow{f} S$ such that $r_* \mathscr{O}_X \cong \mathscr{O}_Z$ and f is affine, and which satisfies the following universal property: for any other factorization $X \to Z' \xrightarrow{f'} S$ with f' affine, there exists a unique morphism $\alpha: Z \to Z'$ such that $f' \circ \alpha = f$. The Stein factorization may be constructed by taking $Z = \mathbf{Spec}\, p_* \mathscr{O}_X$. Note that the following functoriality property holds: given a proper morphism $p': X' \to S$ and an S-morphism $j: X' \to X$, there exists a commutative diagram

$$
\begin{array}{ccccc}
X' & \xrightarrow{\ r'\ } & Z' & \xrightarrow{\ f'\ } & S \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} & & \| \\
X & \xrightarrow{\ r\ } & Z & \xrightarrow{\ f\ } & S
\end{array}
$$

of Stein factorizations.

Let $f : T \to S$ be the discriminant cover, and $Q \to S$ be the quadric fibration, associated to (E, q, L). Let

$$\Lambda\mathrm{G}(q) \xrightarrow{r'} T' \xrightarrow{f'} S \tag{B.1}$$

be the Stein factorization of the morphism $\Lambda\mathrm{G}(q) \to S$. In [43, Section 3.2], it's proved that if $Q \to S$ has simple degeneration, then r' is smooth and f' is finite flat of degree 2. By the classical theory, if $n = 4$ or $n = 6$, then r' is an étale locally trivial projective space bundle of relative dimensions 1 and 2, respectively.

Our aim is to show that in these cases, the projective space bundle r' and the Azumaya algebra $\mathscr{B}_0$ (the even Clifford algebra considered over the discriminant cover, see Section 1.6) define the same Brauer class. We must first verify that $f' : T' \to S$ is indeed isomorphic to the discriminant cover, a result that seems to be often used in the literature, though no complete proof could be readily found.

Proposition B.5. *Let S be a smooth scheme over a field and (E, q, L) a quadratic form of even rank n on S. Assume that (E, q, L) has simple degeneration along a smooth divisor D of S. Then there is a canonical S-isomorphism $T' \to T$.*

Proof. We first construct the morphism $T' \to T$, following [33, Section 85]. For each $u : U \to S$ and each Lagrangian submodule $v : W \to u^*E$, there is a canonical $\mathscr{O}_U$-module morphism $\psi : \det W \otimes L^{\vee \otimes n/4} \to \mathscr{C}_0(E, q, L)$ (if $n \equiv 0 \bmod 4$) or $\psi : \det W \otimes L^{\vee \otimes (n-2)/4} \to \mathscr{C}_1(E, q, L)$ (if $n \equiv 2 \bmod 4$). But the action of the center $u^*\mathscr{Z}$ stabilizes the image of ψ, hence there is an $\mathscr{O}_U$-algebra morphism $u^*\mathscr{Z} \to \mathscr{E}nd_{\mathscr{O}_U}(\det W) \cong \mathscr{O}_U$, hence an $\mathscr{O}_S$-algebra morphism $\mathscr{Z} \to u_*\mathscr{O}_U$, hence a morphism $U \to T$. By the functoriality of the even Clifford algebra and Clifford bimodule (cf. Propositions A.2(5) and A.3(4)), we've just defined a morphism of moduli functors $\Lambda\mathrm{G}(q) \to T$ over S. Since $T \to S$ is affine, the universal property of Stein factorization provides an S-morphism $t : T' \to T$. Note that under our hypotheses, both T and T' are finite flat of degree 2 over S.

Since T and T' are affine over S, to argue that $t : T' \to T$ is an isomorphism we may assume that S is the spectrum of a local ring $(R, \mathfrak{m})$. Note that t is equivariant for the Galois action since any reflection of (E, q, L) induces the nontrivial S-isomorphism of both T and T'.

First note that if (E, q, L) is regular over S, then both T and T' are étale of degree 2 and hence, as both correspond to torsors for the group $\mathbb{Z}/2\mathbb{Z}$, any equivariant morphism between them is an isomorphism. Thus we may assume that (E, q, L) has simple degeneration along a regular divisor D of S generated by a nonsquare nonzero divisor $\pi \in \mathfrak{m}$ and (modifying up to units) that π is the discriminant of q.

Now we proceed by induction on the rank. If (E, q, L) has rank 2, then (multiplying by units, which does not change the quadric) it can be diagonalized as $q = \langle 1, -\pi \rangle$. Now $\Lambda\mathrm{G}(q) = \operatorname{Proj} R[x, y]/(x^2 - \pi y^2)$ and note that the standard affine patch U_y (where y is invertible) is canonically isomorphic to $\operatorname{Spec} R[x]/(x^2 - \pi)$ which is precisely T. Finally, note that $U_x \subset U_y$ since π is a nonzero divisor, so that $\Lambda\mathrm{G}(q) \cong T$.

Now assume that the proposition holds in rank $n \geqslant 2$. Let (E, q, L) have rank $n + 2$. We first reduce to the case that q is isotropic. In general, there exists an finite étale extension $S' \to S$ over which q becomes isotropic since q has simple degeneration and rank > 2, it has a (semi)regular subform of rank > 1, which acquires a zero over a finite étale extension. Then if $t : T' \to T$ becomes an isomorphism over S', it was an isomorphism.

Now assume that (E, q, L) is isotropic. Then since D and S are smooth, and the associated quadric fibration has simple degeneration, then any isotropic line $N \subset E$ is smooth (see Lemma 1.3.2). Hence we have a decomposition $(E, q, L) = H_L(N) \perp (E', q', L)$ corresponding to quadric reduction (see Lemma 1.3.5), with q' having the same degeneration as q by Corollary 1.3.9. There is a closed embedding $j : \Lambda\mathrm{G}(q') \to \Lambda\mathrm{G}(q)$ defined by $W \mapsto l^{-1}(W)$ where $l : N^\perp \to E'$ is the quotient map. Thus there is a commutative diagram

$$
\begin{array}{ccccc}
\Lambda\mathrm{G}(q') & \xrightarrow{\ r''\ } & T'' & \xrightarrow{\ f''\ } & S \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} & & \| \\
\Lambda\mathrm{G}(q) & \xrightarrow{\ r'\ } & T' & \xrightarrow{\ f'\ } & S
\end{array}
$$

of Stein factorizations.

We claim that $g : T'' \to T'$ is an isomorphism. Pushing forward the ideal sheaf $\mathscr{I}$ of the embedding j, we arrive at an exact sequence

$$0 \to r'_* \mathscr{I} \to \mathscr{O}_{T'} \to g_* \mathscr{O}_{T''} \to R^1 r'_* \mathscr{I} \to 0$$

which is exact at right since $R^1 r'_* \mathscr{O}_{AG(q)}$ vanishes (equivalently, $f'_* R^1 r'_* \mathscr{O}_{AG(q)} \cong R^1 p'_* \mathscr{O}_{AG(q)}$ vanishes, since f' is affine). Indeed, the fibers of r' are projective homogeneous varieties whose structure sheaves have no higher cohomology by Kempf's vanishing theorem [55] (cf. [43, Proof of Prop. 3.3]). As the generic fiber of $R^i r'_* \mathscr{I}$ is trivial for $i = 0, 1$ (again, since the generic fiber of $T'' \to T'$ is an equivariant morphism of étale quadratic covers, it is an isomorphism), these are torsion sheaves. In particular, $r'_* \mathscr{I} = 0$ since it is torsion-free. By étale localization, we are reduced to the following: if q is a quadratic form with simple degeneration (i.e., radical of rank 1) over a separably closed field k, then the affine part of the Stein factorization of $AG(q)$ is isomorphic to $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. This follows by a geometric argument: considering a flat family with smooth generic fiber and special fiber q, the two connected components of the Lagrangian Grassmannian over the generic fiber come together in the special fiber. Then, since any injective algebra endomorphism between rings of dual numbers is an isomorphism, the special fiber of $\mathscr{O}_{T'} \to g_* \mathscr{O}_{T''}$ is an isomorphism. Thus $R^1 r'_* \mathscr{I} = 0$ and we have proved the claim.

By Proposition 1.8.2, $\mathscr{C}_0(E, q, L)$ and $\mathscr{C}_0(E', q', L)$ have isomorphic centers, hence the discriminant cover of q' is isomorphic to $T \to S$. But now by the induction hypothesis, the induced morphism $T'' \to T$ is an isomorphism, hence by the above, $T' \to T$ is an isomorphism. $\quad\square$

Proposition B.6. *Let S be a smooth connected scheme over a field. Let $Q \to S$ be a quadric fibration of relative dimension 2 or 4 with simple degeneration along a regular divisor and discriminant double cover $T \to S$. Then there is an equality classes in $\operatorname{Br}(T) \cong \operatorname{Br}(T')$:*

- *The class λ of the Severi–Brauer scheme appearing as the connected part of the Stein factorization $AG(Q) \to T' \to S$.*
- *The class β arising from the even Clifford algebra $\mathscr{B}_0$ on the discriminant cover $T \to S$.*

Proof. Since S and D are smooth, T will be smooth. Since S is connected and D is nonempty, T will be connected. Denote by L the function field of T. Thus $\operatorname{Br}(T) \to \operatorname{Br}(L)$ is injective (see [7] or [40, II Cor. 1.10]). Identifying $\operatorname{Br}(T') \cong \operatorname{Br}(T)$ by Proposition B.5, we need only compare the two Brauer classes λ and β at the generic point, where this statement is classical (see [33, Exer. 85.4] for example). $\quad\square$

It would be interesting to give a direct explicit isomorphism between the Severi–Brauer scheme of $\mathscr{B}_0$ and the connected part of the Stein factorization of $AG(Q)$. See [47, Thm. 4.5] for an approach in the case of quadric surface bundles. This would seem most naturally accomplished via the moduli space interpretations of the two objects.

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