Derived categories and rationality of conic bundles

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Classical setting and the main results

Let $\pi : X \to S$ be a smooth standard conic bundle over a smooth projective surface, and $\tilde{C} \to C$ the associated double cover of the discriminant curve given by connected components of singular conics. A classical question in algebraic geometry is to determine the rationality of X.

Necessary conditions: S is rational, C is connected and the intermediate Jacobian $J(X)$ is isomorphic to the direct sum of Jacobians of smooth projective curves.

Consequence [Clemens-Griffiths]. Any smooth cubic threefold in $\mathbb{P}^4$ is not rational.

[Beauville]. $J(X)$ is isomorphic to the Prym variety $P(\tilde{C}/C)$.

Consequence [Beauville, Shokurov]. If S is minimal X is rational if and only if $J(X)$ splits as the sum of Jacobians of curves. The only possible cases are: $S = \mathbb{P}^2$ and C is a smooth cubic, $S = \mathbb{P}^2$ and C is a quartic, $S = \mathbb{P}^2$ and C is a quintic and $\tilde{C} \to C$ is given by an even θ-characteristic, $S \to \mathbb{P}^1$ is ruled and C is either trigonal or hyperelliptic and the g^1_r is induced by the ruling.

Question. Can we relate the derived category $\mathrm{D^b}(X)$ and the rationality of X? The most promising way is looking at semiorthogonal decompositions of $\mathrm{D^b}(X)$.

Example [BMMS]. If X is a smooth cubic threefold

$$\mathrm{D^b}(X) = \langle \mathbf{T}, \mathcal{O}_X, \mathcal{O}_X(1)\rangle,$$

and the equivalence class of the category $\mathbf{T}$ corresponds to the isomorphism class of $J(X)$.

Main Results. Let $\pi : X \to S$ be a standard conic bundle over a smooth rational surface and $\mathrm{D^b}(S, \mathcal{B}_0)$ the Kuznetsov component of its derived category.

Theorem 1 *If there are smooth projective curves Γ_i with fully faithful functors $\Psi_i : \mathrm{D^b}(\Gamma_i) \to \mathrm{D^b}(S, \mathcal{B}_0)$, exceptional objects E_j in $\mathrm{D^b}(S, \mathcal{B}_0)$ and a semiorthogonal decomposition*

$$\mathrm{D^b}(S, \mathcal{B}_0) = \langle \Psi_1 \mathrm{D^b}(\Gamma_1), \ldots \Psi_k \mathrm{D^b}(\Gamma_k), E_1, \ldots, E_l\rangle, \tag{1}$$

then $J(X) \cong \oplus J(\Gamma_i)$.

Theorem 2 *If S is minimal, then X is rational and $J(X) \cong \oplus J(\Gamma_i)$ if and only if $\mathrm{D^b}(S, \mathcal{B}_0)$ decomposes like (1).*

Remark. We work exclusively with varieties defined over $\mathbb{C}$.

Motives and derived categories of conic bundles

The motive of a conic bundle. We consider here the category of Chow motives with rational coefficients.

Recall that if Γ is a smooth projective curve

$$h(\Gamma) = h^0(\Gamma) \oplus h^1(\Gamma) \oplus h^2(\Gamma),$$

where $h^0(\Gamma) = \mathbb{Q}$, $h^2(\Gamma) = \mathbb{Q}(-1)$ and $h^1(\Gamma)$ corresponds to $J(\Gamma)$ up to isogenies, in the sense that $\mathrm{Hom}(h^1(\Gamma), h^1(\Gamma')) = \mathrm{Hom}(J(\Gamma), J(\Gamma')) \otimes \mathbb{Q}$. Finally, no nontrivial map $h^1(\Gamma) \to h^1(\Gamma)$ factors through $\mathbb{Q}(-j)$ for any j.

[Nagel-Saito]: If $\pi : X \to S$ is a standard conic bundle, there is a submotive $Prym \subset h^1(\tilde{C})$, corresponding to the Prym variety, and $Prym(-1) \subset h^3(X)(-1)$. If S is rational, the motive $h(X)$ is the direct sum of $Prym(-1)$ and a finite number of copies of $\mathbb{Q}(-j)$ (with different twists).

The derived category of a conic bundle. If $\mathbf{T}$ is a linear triangulated category a semiorthogonal decomposition

$$\mathbf{T} = \langle \mathbf{T}_1, \ldots, \mathbf{T}_n\rangle,$$

is an ordered collection of orthogonal (from right to left) subcategories generating the whole category.

An object E of $\mathbf{T}$ is exceptional if $\hom(E, E[i]) = 1$ for $i = 0$ and 0 otherwise. It generates a triangulated subcategory of $\mathbf{T}$ which is equivalent to the derived category of a point.

Orlov's Formula for blow-ups. If Z is smooth projective and $\chi : Y \to Z$ is the blow-up along a smooth r-codimensional subvariety W, then

$$\mathrm{D^b}(Y) = \langle \Psi_1 \mathrm{D^b}(W), \ldots, \Psi_{r-1}\mathrm{D^b}(W), \chi^* \mathrm{D^b}(Z)\rangle,$$

where Ψ_i and χ^* are fully faithful.

[Kuznetsov]. If $\pi : X \to S$ is a conic bundle, let $\mathcal{B}_0$ be the sheaf of even parts of the Clifford algebra associated to it and $\mathrm{D^b}(S, \mathcal{B}_0)$ the derived category of $\mathcal{B}_0$-algebras.

$$\mathrm{D^b}(X) = \langle \Phi \mathrm{D^b}(S, \mathcal{B}_0), \pi^* \mathrm{D^b}(S)\rangle,$$

where Φ and π^* are fully faithful.

If S is rational, $\mathrm{D^b}(S)$ is generated by exceptional objects and then the only nontrivial component in the semiorthogonal decomposition of $\mathrm{D^b}(X)$ is $\mathrm{D^b}(S, \mathcal{B}_0)$ (the *Kuznetsov component*).

From semiorthogonal decomposition to rationality

The key of the proof of Theorem 1 is the study of the map induced by a fully faithful functor $\Psi : \mathrm{D^b}(\Gamma) \to \mathrm{D^b}(X)$ on the motive $h^1(\Gamma)$, where Γ is a smooth projective curve and $g(\Gamma) > 0$.

If $\Psi : \mathrm{D^b}(\Gamma) \to \mathrm{D^b}(X)$ is fully faithful, then it is a Fourier–Mukai functor. Moreover, it admits a right adjoint Ψ_R, also a FM. Let $\mathcal{E}$ and $\mathcal{F}$ in $\mathrm{D^b}(\Gamma \times X)$ be the kernels of Ψ and Ψ_R respectively. Then $\Psi \circ \Psi_R = Id_{\mathrm{D^b}(\Gamma)}$.

Define $e := ch(\mathcal{E}).Td(X)$ and $f := ch(\mathcal{F}).Td(\Gamma)$, mixed cycles in $CH^*_{\mathbb{Q}}(X \times \Gamma)$. By Grothendieck–Riemann–Roch the composition $f \cdot e = Id_{h(\Gamma)}$.

By the decomposition of $h(X)$, $(f_i \cdot e_{4-i})_{|h^1(\Gamma)}$ is zero unless $i = 2$. Then $h^1(\Gamma)$ is a direct summand $h^3(X)(-1) = Prym(\tilde{C}/C)(-1)$ and we have an isogeny $\psi_{\mathbb{Q}}$ between $J(\Gamma)$ and a subvariety of $J(X)$. This isogeny is the algebraic morphism $\psi : J(\Gamma) \to J(X)$ given by the cycle $ch_2(\mathcal{E})$. The cycle $-ch_2(\mathcal{E})$ gives its inverse.

The Prym variety $P(\tilde{C}/C)$ is the algebraic representative of the algebraically trivial part $A^2(X)$ of the Chow group. The polarization θ_P is the incidence polarization with respect to X. In particular $\psi^* \theta_{J(X)} = \theta_{J(\Gamma)}$ and then ψ is an isomorphism between $J(\Gamma)$ and a principally polarized abelian subvariety of $J(X)$.

Consider a semiorthogonal decomposition like (1). Since S is rational, we get

$$\mathrm{D^b}(X) = \langle \Psi_1 \mathrm{D^b}(\Gamma_1), \ldots \Psi_k \mathrm{D^b}(\Gamma_k), E_1, \ldots, E_r\rangle.$$

Each Ψ_i gives a morphism ψ_i. Let $\psi = \oplus \psi_i$. Moreover

$$CH^*_{\mathbb{Q}}(X) = \bigoplus_{i=1}^{k} CH^*_{\mathbb{Q}}(\Gamma^i) \oplus \mathbb{Q}^r = \bigoplus_{i=1}^{k} Pic_{\mathbb{Q}}(\Gamma_i) \oplus \mathbb{Q}^{r+k}.$$

The cokernel of $\psi_{\mathbb{Q}}$ is a finite $\mathbb{Q}$-vector space. Since $\psi : \oplus J(\Gamma_i) \to J(X)$ is a morphism of abelian varieties, such cokernel is trivial. Then ψ is an isomorphism of principally polarized abelian varieties.

Corollary 3 *If S is minimal and $\mathrm{D^b}(S, \mathcal{B}_0)$ admits a decomposition like (1), then X is rational and $J(X) \cong \oplus J(\Gamma_i)$.*

From rationality to semiorthogonal decomposition

Let $\pi : X \to S$ be a rational standard conic bundle over a minimal rational surface.

$\mathcal{B}_0$ is isomorphic over the generic point to a quaternion algebra. Since $Br(S) = 0$ the double cover $\tilde{C} \to C$ determines a unique quaternion algebra in $Br(K(S))$ [Artin-Mumford]. Then the category $\mathrm{D^b}(S, \mathcal{B}_0)$ is fixed by $\tilde{C} \to C$. Theorem 2 is proved providing an example for each possible case.

In each case we provide an explicit construction as follows:

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\chi"] & \\ S & & Z. \end{tikzcd}$$

Z is a smooth projective rational threefold with known semiorthogonal decomposition,
$\pi : X \to S$ is induced by an explicit linear system on Z, and
χ is the blow up of the smooth curve Γ in the base locus.

The decompositions are obtained comparing, via mutations, the decompositions induced respectively by the blow-up and by the conic bundle structure:

$$(A) \qquad \mathrm{D^b}(X) = \langle \Psi \mathrm{D^b}(\Gamma), \chi^* \mathrm{D^b}(Z)\rangle,$$

$$(B) \qquad \mathrm{D^b}(X) = \langle \Phi \mathrm{D^b}(S, \mathcal{B}_0), \pi^* \mathrm{D^b}(S)\rangle.$$

Here is a table summarizing the four different cases

$C \subset S$	$\mathrm{D^b}(S)$	Z	Γ	$\mathrm{D^b}(Z)$	$\mathrm{D^b}(S, \mathcal{B}_0)$
quintic in $\mathbb{P}^2$	3 exc.	$\mathbb{P}^3$	genus 5	4 exc.	$\mathrm{D^b}(\Gamma)$, 1 exc.
quartic in $\mathbb{P}^2$	3 exc.	Quadric	genus 2	4 exc.	$\mathrm{D^b}(\Gamma)$, 1 exc.
sm. cubic in $\mathbb{P}^2$	3 exc.	$\mathbb{P}^1$-bd. over $\mathbb{P}^2$	$\emptyset$	6 exc.	3 exc.
trigonal in $\mathbb{F}_n$	4 exc.	$\mathbb{P}^2$-bd. over $\mathbb{P}^1$	tetragonal	6 exc.	$\mathrm{D^b}(\Gamma)$, 2 exc.
hyperell. in $\mathbb{F}_n$	4 exc.	Quadr. bd. over $\mathbb{P}^1$	hyperell.	$\mathrm{D^b}(\Gamma')$, 4 exc.	$\mathrm{D^b}(\Gamma), \mathrm{D^b}(\Gamma')$

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