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Algebraic Geometry

Stable pairs on elliptic K3 surfaces

Couples stables sur une surface K3 elliptique

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ABSTRACT

We study semistable pairs on elliptic K3 surfaces with a section: we construct a family of moduli spaces of pairs, related by wall crossing phenomena, which can be studied to describe the birational correspondence between moduli spaces of sheaves of rank 2 and Hilbert schemes on the surface. In the 4-dimensional case, this can be used to get the isomorphism between the moduli space and the Hilbert scheme described by Friedman.

RÉSUMÉ

On considère sur une surface K3 elliptique avec une section une notion de stabilité pour un couple. On obtient une famille d'espaces de modules reliés par wall crossing, dont l'étude permet de décrire les correspondances birationnelles entre les espaces de modules des faisceaux stables de rang 2 et les schémas de Hilbert sur la surface. En particulier, en dimension 4, ceci permet de décrire l'isomorphisme entre l'espace des module et le schéma de Hilbert demontré par Friedman.

Version française abrégée

Soit $\pi : S \to \mathbb{P}^1$ une surface elliptique K3 lisse sur $\mathbb{C}$ dont toute fibre singulière est au plus nodale. Si on considère sur S un faisceau V de rang 2 et de classes de Chern $c_1(V)$ et $c_2(V)$, il existe une polarisation, dite $(c_1(V), c_2(V))$-convenable, pour laquelle V est stable si et seulement si sa restriction à la fibre générique de π est stable. Si on demande aussi que le degré de V sur la fibre soit impair, Friedman montre que l'espace des modules des fibrés avec telles classes de Chern semistable par rapport à une polarisation convenable est, lorsque non vide, une variété lisse projective de dimension paire $2t$ birationnelle à $\mathrm{Sym}^t J^{e+1}(S)$, où $2e+1$ est le degré sur la fibre et $J^d(S)$ dénote la surface elliptique dont la fibre générale est isomorphe á l'espace des fibrés en droites de degré d sur la fibre générale de S.

Si $\pi : S \to \mathbb{P}^1$ admet une section σ et on dénote par f la fibre de π, on peut, sans perte de généralité, étudier les cas où $c_2 = 1$ et $c_2 = \sigma - tf$ pour t entier positif. On dénotera dans la suite par $M(t)$ l'espace des modules des tels fibés stables. Friedman obtient donc une application birationnelle ε entre $\mathrm{Hilb}^t(S)$ et $M(t)$, qui est un isomorphisme pour $t \leqslant 2$, et conjecture que ceci soit vrai pour tout t [2, Conj. III, 4.13].

Pour étudier dans le détail la correspondance birationnelle ε, on étudie dans ce papier des espaces de modules de couples stables sur S. Huybrechts et Lehn [3,4] définissent une notion de stabilité pour un couple (V, α) où V est un faisceau cohérent sur une variété lisse projective et $\alpha : V \to E_0$ un morphisme vers un faisceau E_0 fixé. Cette notion dépend

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d'un polynôme à coefficients rationnels et, pour un choix convenable (d'ailleurs, générique), il existe un espace des modules fin projectif des couples stables.

Nous définissons une condition de stabilité, dépendant d'un paramètre rationnel positif δ, pour un couple (V,α), où V est un faisceau de rang 2, $c_1(V)=\sigma - tf$ et $c_2(V)=1$ et $\alpha : V \to \mathcal{O}_S(\sigma - f)$. Pour $\delta > t+1/2$ cette condition devient trop stricte et il n'y a donc pas de couple stable. Pour tout entier n, la condition de stabilité ne change pas si δ est compris entre $\max\{0, n-1/2\}$ et $n+1/2$ et, dans ce cas, tout couple semistable est stable. On peut donc se ramener à l'étude d'une famille finie d'espaces de modules $\mathcal{M}_n$ projectifs pour n entier compris entre 0 et t.

La première propriété qu'on observe est que le premier espace $\mathcal{M}_0$ de la famille admet une fibration en espaces projectifs au dessus de l'espace des modules $M(t)$. En effet, un couple (V,α) est 0-stable si et seulement si le faisceau V est stable. Pour tout V stable la fibre est donc donnée par l'espace projectif $\mathbb{P}\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$. Dans le cas en question, un tel espace n'est jamais vide et est réduit à un point pour un V générique. On a donc un morphisme birationnel $\mathcal{M}_0 \to M(t)$. Comme on peut décrire toujours un tel V comme extension, il est facile de vérifier que le couple générique (V,α) dans $\mathcal{M}_0$ est n-stable pour tout $n=1,\dots,t$ et de determiner les couples (V,α) qui sont 0-stables mais pas n-stables pour $n \geqslant 1$.

De l'autre côté, il existe un sous-schéma fermé $\tilde{\mathcal{M}}_t$ dans $\mathcal{M}_t$ qui admet une fibration en espaces projectifs au dessus du schéma de Hilbert $\mathrm{Hilb}^t(S)$. En fait lorsque on considère un couple (V,α) dans $\mathcal{M}_t$ tel que le noyau de α est localement libre, on peut décrire V comme extension de $\mathcal{O}_S(\sigma - f)\otimes I_Z$ par $\mathcal{O}_S((1-t)f)$, où Z est le conoyau de α, et donc un sous-schéma localement intersection complète dans S de codimension 2 et longueur t. Dans le cas en question, pour un Z générique, une telle extension existe et est unique, ce qui donne donc un morphisme birationnel $\tilde{\mathcal{M}}_t \to \mathrm{Hilb}^t(S)$. Le couple générique (V,α) de $\tilde{\mathcal{M}}_t$ est 0-stable et on peut décrire dans $\tilde{\mathcal{M}}_t$ les lieux des couples (V,α) non n-stables pour $n \leqslant t$ par l'étude du fibré déstabilisant maximal du faisceau V.

On dispose finalement d'une suite d'espaces de modules $\mathcal{M}_n$ pour n entier compris entre 0 et t et tels que le premier et le dernier des espaces de la suite admettent un morphisme birationnel respectivement sur l'espace des modules des fibrés stables de rang deux et sur le schéma de Hilbert. La description des couples 0-stables et t-stables comme extensions permet de décrire les lieux d'indétermination de l'application birationnelle ε à travers les correspondances birationnelles entre les espaces de modules des couples induites par les wall crossing.

Lorsque on fixe $t=2$, ces transformations birationnelles peuvent être décrites explicitement en appliquant des tranformations élémentaires au couple universel $(\mathcal{V}, A)$ de l'espace des modules $\tilde{\mathcal{M}}_2$, qui, dans ce cas, est lisse et projectif. Ceci nous permet de montrer l'existence d'un morphisme injectif $\tilde{\mathcal{M}}_2 \hookrightarrow \mathcal{M}_1$ et d'un morphisme birationnel $\phi_0 : \tilde{\mathcal{M}}_2 \to \mathcal{M}_0$. On obtient comme corollaire le résultat suivant [2, Thm. 4.9].

Théorème 0.1 *(Corollary 3.6). Le morphisme ϕ_0 induit un isomorphisme* $\mathrm{Hilb}^2(S) \xrightarrow{\cong} M(2)$.

Le but de cette Note est donc de donner un nouveau regard sur la correspondance birationnelle ε, en explicitant dans le cas $t=2$ comment retracer dans ce langage l'isomorphisme déjà connu.

1. Introduction

Let $\pi : S \to \mathbb{P}^1$ be a complex elliptic K3 surface whose singular fibers have at most nodal singularities. Given a rank 2 torsion free sheaf V with Chern classes $c_1(V)$ and $c_2(V)$, there exists a polarization, called $(c_1(V), c_2(V))$-suitable, with respect to which V is stable if and only if its restriction to the generic fiber is stable. This allows Friedman [2] to show that, if nonempty, the moduli space of such stable sheaves with odd fiber degree $2e+1$ is smooth, of even dimension $2t$ and birational to $\mathrm{Sym}^t J^{e+1}(S)$, where $J^d(S)$ denotes the elliptic surface whose general fiber is the set of line bundles of degree d on the general fiber of S.

If $\pi : S \to \mathbb{P}^1$ admits a section σ and f denotes the generic fiber, we can restrict to the cases $c_2 = 1$ and $c_1 = \sigma - tf$ for a nonnegative integer t and denote by $M(t)$ the moduli space of rank 2 stable sheaves with such Chern classes. Friedman's result gives in this case a birational map ε between $\mathrm{Hilb}^t(S)$ and $M(t)$, which he shows to be an isomorphism for $t \leqslant 2$. This leads to conjecture that this map is an isomorphism for all t [2, Conj. III, 4.13].

In this paper, in order to understand closely the birational correspondence ε between $\mathrm{Hilb}^t(S)$ and $M(t)$, we consider stable pairs and their moduli spaces as defined and studied in [3,4]. We give a definition of a δ-stable pair depending on a rational parameter δ, which gives rise to a finite family of moduli spaces related by wall crossing phenomena. The first and the last moduli spaces are birational respectively to $M(t)$ and $\mathrm{Hilb}^t(S)$ and the wall crossing phenomena accurately describe the locus of indeterminacy of ε. Indeed, if one consider an element Z of $\mathrm{Hilb}^t(S)$, the Serre construction defines an extension V which is only generically stable, but for such an extension, any pair (V,α) is t-stable. On the other side, any semistable sheaf V in $M(t)$ can be described as an extension, depending on two codimension 2 subschemes of S. For such a V, any pair (V,α) is 0-stable. The birational correspondences between the moduli spaces can be then described basing upon the codimension 2 subschemes appearing in the extensions. In the case $t=2$, a detailed description of such correspondences, based on [2], allows to define a birational morphism from an irreducible component of the moduli spaces of 2-stable pairs (which turns out to be a blow up of $\mathrm{Hilb}^t(S)$) and the moduli space of 0-stable pairs inducing an isomorphism between $\mathrm{Hilb}^2(S)$ and $M(2)$.

2. Stable pairs on elliptic K3 surfaces

If t is a nonnegative integer, we fix $L = \mathcal{O}_S(\sigma + (t+5)f)$ as a $(\sigma - tf, 1)$-suitable polarization. We say that a sheaf is (semi)stable if it is μ-(semi)stable.

Definition 2.1. Let V be a rank 2 coherent sheaf over S with $c_1(V) = \sigma - tf$ and $c_2(V) = 1$, $\alpha : V \to \mathcal{O}_S(\sigma - f)$ a morphism and $\delta \in \mathbb{Q}_{>0}$. The pair (V, α) is δ-semistable if

(i) $\deg G \leqslant 3/2 - \delta$ for all nontrivial submodules $G \subset \ker(\alpha)$,
(ii) $\deg G \leqslant 3/2 + \delta$ for all nontrivial submodules $G \subset V$.

Such a pair is δ-stable if both inequalities hold strictly.

This definition is just a special case of Definition 1.1 in [3]. Then for any positive δ the fine moduli space of stable pairs with respect to δ exists and is projective [3,4]. We denote it by $\mathcal{M}_\delta$.

For any integer n, if δ varies in $(\max\{0, n-1/2\}, n+1/2)$, the moduli spaces $\mathcal{M}_\delta$ are all isomorphic and all semistable pairs are stable. By condition (i) there is no semistable pair with respect to $\delta > t + 1/2$. There is then a family $\{\mathcal{M}_n\}_{0 \leqslant n \leqslant t}$ of nonempty projective moduli spaces related by wall crossing phenomena which give rise to birational maps.

A pair (V, α) is 0-stable if and only if V is stable. By [2], any V in $M(t)$ fits a sequence

$$0 \longrightarrow \mathcal{O}_S\big((1-s)f\big) \otimes I_{Z_1} \longrightarrow V \longrightarrow \mathcal{O}_S\big(\sigma + (1+s-t)f\big) \otimes I_{Z_2} \longrightarrow 0, \tag{1}$$

where $0 \leqslant s \leqslant t$ and $l(Z_1) + l(Z_2) = s$. Then V admits at least one nonzero map to $\mathcal{O}_S(\sigma - f)$ and $\mathcal{M}_0$ fibers over $M(t)$ with fibers given by $\mathbb{P}\operatorname{Hom}(V, \mathcal{O}_S(\sigma - f))$. Since such fiber is never empty and generically one-dimensional, $\mathcal{M}_0 \to M(t)$ is birational.

Condition (i) gets stronger as δ grows. Pairs (V, α) in $\mathcal{M}_0$ which do not belong to $\mathcal{M}_n$ are given by extensions (1) with $s \geqslant n$.

On the other side, let $\tilde{\mathcal{M}}_t \subset \mathcal{M}_t$ be the subscheme whose elements are those pairs (V, α) with $\ker(\alpha)$ locally free. In this case, V is given by an extension

$$0 \longrightarrow \mathcal{O}_S\big((1-t)f\big) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - f) \otimes I_Z \longrightarrow 0, \tag{2}$$

with Z in $\operatorname{Hilb}^t(S)$. Moreover such an extension is unique for Z generic, which, together with the following proposition, tells us that $\tilde{\mathcal{M}}_t$ is projective and birational to $\operatorname{Hilb}^t(S)$.

Proposition 2.1. *(See [3], Corollary 2.14.) The set $\tilde{\mathcal{M}}_t$ is a projective scheme over $\operatorname{Hilb}^t(S)$ with fiber over Z isomorphic to $\mathbb{P}\operatorname{Ext}^1(\mathcal{O}_S(\sigma - f) \otimes I_Z, \mathcal{O}_S((1-t)f))$.*

Condition (ii) gets stronger as δ decreases. Indeed, for $\delta = 0$, this condition implies that V has no destabilizing subline bundles, while for $\delta \geqslant 1$, the sheaf V can have destabilizing subline bundles. By [2, III, Prop. 4.4], the maximal destabilizing subline bundle is of the form $\mathcal{O}_S(\sigma - af)$ for some integer a, then it is not contained in $\ker(\alpha)$. Pairs (V, α) belonging to $\tilde{\mathcal{M}}_t$ but not to $\mathcal{M}_n$ are then unstable extensions (2) such that the maximal destabilizing subline bundle of V is $\mathcal{O}_S(\sigma - af)$ with $a > 1 + t - n$.

3. Stable pairs in the case $t = 2$

Consider pairs (V, α) with $c_1(V) = \mathcal{O}_S(\sigma - 2f)$. We show that the scheme $\tilde{\mathcal{M}}_2$ is smooth and there is an injective morphism $\tilde{\mathcal{M}}_2 \hookrightarrow \mathcal{M}_1$, and that there is a birational morphism $\tilde{\mathcal{M}}_2 \to \mathcal{M}_0$, inducing an isomorphism $\operatorname{Hilb}^2(S) \simeq M(2)$.

If (V, α) is a pair in $\tilde{\mathcal{M}}_2$, then V is given by

$$0 \longrightarrow \mathcal{O}_S(-f) \longrightarrow V \longrightarrow \mathcal{O}_S(\sigma - f) \otimes I_Z \longrightarrow 0, \tag{3}$$

for Z in $\operatorname{Hilb}^2(S)$. The dimension of $\operatorname{Ext}^1(\mathcal{O}_S(\sigma - f) \otimes I_Z, \mathcal{O}_S(-f))$ is 2 if Z is in $\operatorname{Sym}^2\sigma$ and 1 otherwise (see [2]). Consider the ideal sheaf $I := I_{\operatorname{Sym}^2\sigma}$. It can be shown [1] that the projectivization $\mathbb{P}(I)$ is isomorphic to $\tilde{\mathcal{M}}_2$. Indeed, up to a twist with a line bundle on $\operatorname{Hilb}^2(S)$, the sheaf I is isomorphic to the sheaf whose stalks are given by extensions (3).

Lemma 3.1. *(See [1], Lemma 2.28.) The subscheme $\tilde{\mathcal{M}}_2$ is the blow-up of $\operatorname{Hilb}^2(S)$ along $\operatorname{Sym}^2\sigma$.*

Let D_σ be the effective divisor of $\operatorname{Hilb}^2(S)$ which is the closure of the locus of pairs $\{p, q \mid p \in \sigma\}$. Let D be the irreducible smooth divisor in $\operatorname{Hilb}^2(S)$ given by

$$D = \big\{Z \in \operatorname{Hilb}^2(S) \mid h^0\big(\mathcal{O}_S(f) \otimes I_Z\big) = 1\big\}.$$

An argument for the smoothness of D can be found in [2]. We denote by $\tilde{D}_\sigma$ (resp. $\tilde{D}$) the strict transform of D_σ (resp. of D) and by $\tilde{G}$ the exceptional divisor of the blow-up.

Studying destabilizing subline bundles for the extension (3) as Z varies in $\mathrm{Hilb}^2(S)$ allows us to say whether V appears in a pair belonging to $\mathcal{M}_n$ for $n < 2$.

Lemma 3.2. *(See [1], Lemma 2.29.) Let (V, α) be a pair in $\tilde{\mathcal{M}}_2$. If (V, α) belongs to $\tilde{D} \cup \tilde{D}_\sigma$, then it is n-stable if and only if $n = 2$. If (V, α) belongs to $\tilde{G} \setminus (\tilde{D} \cup \tilde{D}_\sigma)$, then it is n-stable if and only if $n = 1, 2$. In any other case, (V, α) is n-stable for $n = 0, 1, 2$.*

On the other side, a pair (V, α) belongs to $\mathcal{M}_0$ if and only if V is stable and α is any morphism $V \to \mathcal{O}_S(\sigma - f)$. The sheaf V is then an extension (1) with $l(Z_1) + l(Z_2) \leqslant 2$. There are four possible extension types (see [2]), which we call type a if $l(Z_2) = 2$, type b if $l(Z_2) = 1$ and $l(Z_1) = 0$, type c if $l(Z_2) = l(Z_1) = 0$ and type d if $l(Z_2) = 0$ and $l(Z_1) = 1$.

The generic stable sheaf is given by a type a (which is indeed of the form (3)) extension. In this case, there is a unique choice for α and the pair (V, α) belongs to $\tilde{\mathcal{M}}_2$. In particular, such pairs form the open complementary of $\tilde{D} \cup \tilde{D}_\sigma \cup \tilde{G}$ in $\tilde{\mathcal{M}}_2$. In the nongeneric case, the dimension of $\mathrm{Hom}(V, \mathcal{O}_S(\sigma - f))$ is greater then 1, but any morphism $V \to \mathcal{O}_S(\sigma - f)$ factors through the extension map [1] and we can say for which n extensions of type b, c and d are n-stable.

Lemma 3.3. *(See [1], Lemmas 3.3, 3.4 and 3.5.) If V is a stable type c extension, then any pair (V, α) is n-stable if and only if $n = 0$. Moreover, such extensions form in $M(2)$ a subscheme isomorphic to $\mathrm{Sym}^2\sigma$.*

If V is a type b extension, then any pair (V, α) is n-stable if and only if $n = 0, 1$.

If V is a stable type d extension, then any pair (V, α) is n-stable if and only if $n = 0, 1$. If V is unstable, then for any n no pair (V, α) is n-stable.

We are now ready to describe birational morphisms between the spaces of pairs. The main tool is given by elementary transformations of the universal pair $(\mathcal{V}, A)$ on $S \times \tilde{\mathcal{M}}_2$.

If (V, α) lies in $\tilde{D} \cup \tilde{D}$, then it does not belong to $\mathcal{M}_1$. We can perform first an elementary transformation of $\mathcal{V}$ along $S \times \tilde{D}$ by straightforward generalizing a construction by Friedman [2, III, Prop. 4.12]. We get a flat reflexive sheaf $\mathcal{V}'$ over $S \times \tilde{\mathcal{M}}_2$ such that if (V, α) belongs to $\tilde{D}$, then $\mathcal{V}'_{(V,\alpha)}$ is a type d extension, and in that case it is not stable if and only if (V, α) is in $\tilde{D} \cap \tilde{D}_\sigma$.

We now perform an elementary transformation of $\mathcal{V}'$ along $S \times \tilde{D}_\sigma$ to get a family of sheaves for pairs in $\mathcal{M}_1$. For any (V, α) in $\tilde{D}_\sigma$, there is a unique morphism $\mathcal{V}'_{(V,\alpha)} \to \mathcal{O}_S$ (see [1]). We then have a line bundle $\mathcal{L}$ on S and a surjective morphism

$$\mathcal{V}'_{|S \times \tilde{D}_\sigma} \to \pi_1^* \mathcal{O}_S \otimes \pi_2^* \mathcal{L} \to 0,$$

over $S \times \tilde{D}_\sigma$. Define $\mathcal{U}$ as the elementary transformation

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V}' \longrightarrow i_*\big(\pi_1^* \mathcal{O}_S \otimes \pi_2^* \mathcal{L}\big) \longrightarrow 0,$$

where i is the embedding of $S \times \tilde{D}_\sigma$ in $S \times \tilde{\mathcal{M}}_2$. By [2, Prop. A2], the sheaf $\mathcal{U}$ is flat and reflexive. If (V, α) belongs to $\tilde{D}_\sigma$, then $\mathcal{U}_{(V,\alpha)}$ is a type b extension, which is unstable if and only if $Z_2 = q \in \sigma$ (recall that a type b extension is an extension (1) with $l(Z_2) = 1$ and $l(Z_1) = 0$). Summarizing (see [1]):

- if (V, α) belongs to $\tilde{D} \setminus \tilde{D}_\sigma$, then $\mathcal{U}_{(V,\alpha)}$ is a stable type d extension,
- if (V, α) belongs to $\tilde{D}_\sigma$, then $\mathcal{U}_{(V,\alpha)}$ is a type b extension and it is unstable if and only if $(V, \alpha) \in \tilde{D}_\sigma \cap \tilde{G}$.

In any case, $\mathcal{U}_{(V,\alpha)}$ belongs to some pair in $\mathcal{M}_1$. Moreover, if (V, α) is in $\tilde{D} \cup \tilde{D}_\sigma$, the sheaf $U := \mathcal{U}_{(V,\alpha)}$ is uniquely determined and we have a natural choice for a framing map $\beta : U \to \mathcal{O}_S(\sigma - f)$. Indeed, if (V, α) lies in $\tilde{D}$, then the elementary transformation of $\mathcal{V}$ at that point is induced by the destabilizing exact sequence

$$0 \longrightarrow \mathcal{O}_S(\sigma - 2f) \xrightarrow{\iota} V \longrightarrow \mathfrak{m}_q \longrightarrow 0.$$

Such extension class uniquely determines the extension class (see [2, Prop. A2])

$$0 \longrightarrow \mathfrak{m}_q \longrightarrow \mathcal{V}'_{(V,\alpha)} \xrightarrow{\gamma} \mathcal{O}_S(\sigma - 2f) \longrightarrow 0.$$

The map $\alpha' := \iota \circ \alpha : \mathcal{O}(\sigma - 2f) \to \mathcal{O}(\sigma - f)$ cannot be zero, because $\mathcal{O}(\sigma - 2f)$ is not in $\ker(\alpha)$. There is then a natural choice of a nontrivial framing for U, namely $\beta := \gamma \circ \alpha'$. A similar argument works also for the second elementary transformation.

Theorem 3.4. *There is an injective morphism $\phi_1 : \tilde{\mathcal{M}}_2 \hookrightarrow \mathcal{M}_1$.*

Proof. The universal sheaf $\mathcal{U}$ over $S \times \tilde{\mathcal{M}}_2$ defines a morphism because for any (V,α) there is a unique 1-stable pair $(\mathcal{U}_{(V,\alpha)}, \beta)$. Injectivity is not straightforward only in $\tilde{D} \cap \tilde{D}_\sigma$. Fix a point $q \in S$ and let U be a corresponding type b extension. Such extensions are parametrized by σ (see [1, Lemma 2.39]). Let V be the type a extension with $Z = (q, p)$ such that p in σ corresponds to the extension class of U. Then (V,α) is the unique pair such that $U = \mathcal{U}_{(V,\alpha)}$. $\square$

If (V,α) is not in $\tilde{G}$, then $\mathcal{U}_{(V,\alpha)}$ is stable. If (V,α) is in $\tilde{G}$ then $\mathcal{U}_{(V,\alpha)}$ is unstable and the maximal destabilizing subline bundle is $\mathcal{O}_S(\sigma - 3f)$, which gives, for all (V,α) in $\tilde{G}$,

$$0 \longrightarrow \mathcal{O}_S(\sigma - 3f) \longrightarrow \mathcal{U}_{(V,\alpha)} \longrightarrow \mathcal{O}_S(f) \longrightarrow 0.$$

We then perform an elementary transformation of $\mathcal{U}$ along $\tilde{G}$ to get a flat and reflexive sheaf $\mathcal{W}$ over $S \times \tilde{\mathcal{M}}_2$ such that if (V,α) is in $\tilde{G}$, then $\mathcal{W}_{(V,\alpha)}$ is a stable type c extension. Arguing as before, we get a morphism $\tilde{\mathcal{M}}_2 \to \mathcal{M}_0$.

Theorem 3.5. *There is a birational morphism $\phi_0 : \tilde{\mathcal{M}}_2 \to \mathcal{M}_0$ which is an isomorphism over the open complement of $\tilde{G}$.*

Corollary 3.6. *There is an isomorphism* $\mathrm{Hilb}^2(S) \simeq M(2)$.

Proof. Recall that the locus Σ of stable type c extensions in $M(2)$ is isomorphic to $\mathrm{Sym}^2\sigma$. The map ϕ_0 induces an isomorphism between $\tilde{\mathcal{M}}_2$ and the blow up of $M(2)$ along Σ. This is obtained just by forgetting the framing map of the image of ϕ_0. If we take a point Z in $\mathrm{Sym}^2\sigma$, the fiber $\tilde{G}_Z$ over it corresponds, under this isomorphism, to a fiber over a single point of Σ. We then have a birational map which is a bijection between smooth varieties. $\square$

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