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Homological projective duality for determinantal varieties

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ABSTRACT

In this paper we prove Homological Projective Duality for categorical resolutions of several classes of linear determinantal varieties. By this we mean varieties that are cut out by the minors of a given rank of a $m \times n$ matrix of linear forms on a given projective space. As applications, we obtain pairs of derived-equivalent Calabi–Yau manifolds, and address a question by A. Bondal asking whether the derived category of any smooth projective variety can be fully faithfully embedded in the derived category of a smooth Fano variety. Moreover we discuss the relation between rationality and categorical representability in codimension two for determinantal varieties.

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1. Introduction

Homological Projective Duality (HPD) is one of the most exciting recent breakthroughs in homological algebra and algebraic geometry. It was introduced by A. Kuznetsov in [24] and its goal is to generalize classical projective duality to a homological framework. One of the important features of HPD is that it offers a very important tool to study the bounded derived category of a projective variety together with its linear sections, providing interesting semiorthogonal decompositions as well as derived equivalences, cf. [22,28,26,2,30].

Roughly speaking, two (smooth) varieties X and Y are HP-dual if X has an ample line bundle $\mathscr{O}_X(1)$ giving a map $X \to \mathbb{P}W$, Y has an ample line bundle $\mathscr{O}_Y(1)$ giving a map $Y \to \mathbb{P}W^\vee$, and X and Y have dual semiorthogonal decompositions (called *Lefschetz* decompositions) compatible with the projective embedding. In this case, given a generic linear subspace $L \subset W$ and its orthogonal $L^\perp \subset W^\vee$, one can consider the linear sections X_L and Y_L of X and Y respectively. Kuznetsov shows the existence of a category $\mathbf{C}_L$ which is admissible both in $\mathrm{D^b}(X_L)$ and in $\mathrm{D^b}(Y_L)$, and whose orthogonal complement is given by some of the components of the Lefschetz decompositions of $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$ respectively. That is, both $\mathrm{D^b}(X_L)$ and $\mathrm{D^b}(Y_L)$ admit a semiorthogonal decomposition by a "Lefschetz" component, obtained via iterated hyperplane sections, and a common "nontrivial" or "primitive" component.

HPD is closely related to classical projective duality: [24, Theorem 7.9] states that the critical locus of the map $Y \to \mathbb{P}W^\vee$ coincides with the classical projective dual of X. The main technical issue of this fact is that one has to take into account singular varieties, since the projective dual of a smooth variety is seldom smooth – *e.g.* the dual of certain Grassmannians are singular Pfaffian varieties [12]. On the other hand, derived (dg-enhanced) categories should provide a so-called *categorical* or *noncommutative* resolution of singularities [25,36]. Roughly speaking, one needs to find a sheaf of $\mathscr{O}_Y$-(dg)-algebras $\mathscr{R}$ such that the category $\mathrm{D^b}(Y, \mathscr{R})$ of bounded complexes of coherent $\mathscr{R}$-modules is proper, smooth and $\mathscr{R}$ is locally Morita-equivalent to some matrix algebra over $\mathscr{O}_Y$ (this latter condition translates the fact that the resolution is birational). In the case where Y is singular, one of the most difficult tasks in proving HPD is to provide such a resolution with the required Lefschetz decomposition (for example, see [30, §4.7]). On the other hand, given a non-smooth variety, it is a very interesting question to provide such resolutions and study their properties such as crepancy, minimality and so forth.

The main application of HPD is that it is a direct method to produce semiorthogonal decompositions for projective varieties with non-trivial canonical sheaf, and derived equivalences for Calabi–Yau varieties. The importance of this application is due to the fact that determining whether a given variety admits or not a semiorthogonal decomposition is a very hard problem in general. Notice that there are cases where it is known that the answer to this question is negative, for example if X has trivial canonical bundle [13, Ex. 3.2], or if X is a curve of positive genus [33]. On the other hand, if X is Fano, then any line bundle is exceptional and gives then a semiorthogonal decomposition. Almost all the known cases of semiorthogonal decompositions of Fano varieties described in the literature (see, *e.g.*, [22,27,28,6,2]) can be obtained via HPD or its relative version.

Derived equivalences of Calabi–Yau (CY for short) varieties have deep geometrical insight. First of all, it was shown by Bridgeland that birational CY-threefolds are derived equivalent [14]. The converse is not true: the first example – that has been shown to be also a consequence of HPD in [29] – was displayed by Borisov and Caldararu in [12].

Besides their geometric relevance, derived equivalences between CY varieties play an important role in theoretical physics. First of all, Kontsevich's homological mirror symmetry conjectures an equivalence between the bounded derived category of a CY-threefold X and the Fukaya category of its mirror. More recently, it has been conjectured that homological projective duality should be realized physically as phases of abelian *gauged linear sigma models* (GLSM) (see [17] and [3]).

As an example, denote by X and Y the pair of equivalent CY-threefolds considered by Borisov and Caldararu. Rødland [35] argued that the families of X's and Y's (letting the linear section move in the ambient space) seem to have the same mirror variety Z (a more string theoretical argument has been given recently by Hori and Tong [18]). The equivalence between X and Y would then fit Kontsevich's Homological Mirror Symmetry conjecture via the Fukaya category of Z. It is thus fair to say that HPD plays an important role in understanding these questions and potentially providing new examples. Notice in particular that some determinantal cases were considered in [20].

In this paper, we describe new families of HP Dual varieties. We consider two vector spaces U and V of dimension m and n respectively with $m \leq n$. Let $\mathbb{G} = \mathbb{G}(U, r)$ denote the Grassmannian of r-dimensional quotients of U, set $\mathscr{Q}$ and $\mathscr{U}$ for the universal quotient and sub-bundle respectively. Let $\mathscr{X} := \mathbb{P}(V \otimes \mathscr{Q})$ and $\mathscr{Y} := \mathbb{P}(V^\vee \otimes \mathscr{U}^\vee)$, for any $0 < r < m$. Let $p : \mathscr{X} \to \mathbb{G}$ and $q : \mathscr{Y} \to \mathbb{G}$ be the natural projections. Set H_X and H_Y for the relatively ample tautological divisors on $\mathscr{X}$ and $\mathscr{Y}$. Orlov's result [34] provides semiorthogonal decompositions

$$\begin{aligned} \mathrm{D^b}(\mathscr{X}) &= \langle p^*\mathrm{D^b}(\mathbb{G}), \dots, p^*\mathrm{D^b}(\mathbb{G}) \otimes \mathscr{O}_{\mathscr{X}}((rn-1)H_X)\rangle, \\ \mathrm{D^b}(\mathscr{Y}) &= \langle q^*\mathrm{D^b}(\mathbb{G}) \otimes \mathscr{O}_{\mathscr{Y}}(((r-m)n+1)H_Y), \dots, q^*\mathrm{D^b}(\mathbb{G})\rangle. \end{aligned} \tag{1.1}$$

Theorem 3.5. *In the previous notation, $\mathscr{X}$ and $\mathscr{Y}$ with Lefschetz decompositions (1.1) are HP-dual.*

The proof of the previous result is a consequence of Kuznetsov's HPD for projective bundles generated by global sections (see [24, §8]). Here, the spaces of global sections of $\mathscr{O}_{\mathscr{X}}(H_X)$ and $\mathscr{O}_{\mathscr{Y}}(H_Y)$ sheaves are, respectively, $W = V \otimes U$ and $W^\vee = V^\vee \otimes U^\vee$.

The main interest of Theorem 3.5 is that $\mathscr{X}$ is known to be the resolution of the variety $\mathscr{Z}^r$ of $m \times n$ matrices of rank at most r. Write such a matrix as $M : U \to V^\vee$. Then $\mathscr{Z}^r$ is naturally a subvariety of $\mathbb{P}W$, which is singular in general, with resolution

$f : \mathscr{X} \to \mathscr{Z}^r$. Dually, $g : \mathscr{Y} \to \mathscr{Z}^{m-r}$ is a desingularization of the variety of $m \times n$ matrices of corank at least r. Theorem 3.5 provides the categorical framework to describe HPD between the classical projectively dual varieties $\mathscr{Z}^r$ and $\mathscr{Z}^{m-r}$ (see, *e.g.*, [38]).

In the affine case, categorical resolutions for determinantal varieties have been constructed by Buchweitz, Leuschke and van den Bergh [15,16]. Such resolution is crepant if $m = n$ (that is, in the case where $\mathscr{Z}^r$ has Gorenstein singularities). The starting point is Kapranov's construction of a full strong exceptional collection on Grassmannians [21]. One can use the decompositions in exceptional objects (1.1) to produce a sheaf of algebras $\mathscr{R}'$ and a categorical resolution of singularities $\mathrm{D^b}(\mathscr{Z}^r, \mathscr{R}') \simeq \mathrm{D^b}(\mathscr{X})$. For simplicity, we will denote by $\mathscr{R}'$ the algebra on any of the determinantal varieties $\mathscr{Z}^r$ (forgetting about the dependence of $\mathscr{R}'$ on the rank r). This gives a geometrically deeper version of Theorem 3.5.

Theorem 3.6. *In the previous notations, $\mathscr{Z}^r$ admits a categorical resolution of singularities $\mathrm{D^b}(\mathscr{Z}^r, \mathscr{R}')$, which is crepant if $m = n$. Moreover, $\mathrm{D^b}(\mathscr{Z}^r, \mathscr{R}')$ and $\mathrm{D^b}(\mathscr{Z}^{m-r}, \mathscr{R}')$ are HP-dual.*

Once the equivalence $\mathrm{D^b}(\mathscr{Z}^r, \mathscr{R}') \simeq \mathrm{D^b}(\mathscr{X})$ constructed, Theorem 3.6 is proved by applying directly Theorem 3.5. However, the geometric relevance of Theorem 3.6, and its difference with Theorem 3.5, is that it shows HPD directly on noncommutative structures over the determinantal varieties $\mathscr{Z}^r$ and $\mathscr{Z}^{m-r}$ with respect to their natural embedding in $\mathbb{P}W$ and $\mathbb{P}W^\vee$ respectively. That is, these natural smooth and proper noncommutative scheme structures are well-behaved with respect to projective duality and hyperplane sections. Finally, notice that whenever we pick a smooth linear section Z_L of $\mathscr{Z}^r$ (or a smooth section Z^L of $\mathscr{Z}^{m-r}$), the restriction to Z_L of the sheaf $\mathscr{R}'$ is Morita-equivalent to $\mathscr{O}_{Z_L}$, so that we get the derived category $\mathrm{D^b}(Z_L)$ of the section itself.

As a consequence, given a matrix of linear forms on some projective space, one can see the locus Z where the matrix has rank at most r as a linear section of $\mathscr{Z}^r$. Assuming Z to have expected dimension, Theorem 3.6 gives a categorical resolution of singularities $\mathrm{D^b}(Z, \mathscr{R}')$ of Z and a semiorthogonal decomposition of this category involving the dual linear section of $\mathscr{Z}^{m-r}$.

Our construction of Homological Projective Duality allows us to recover some Calabi–Yau equivalences appeared in [20] and many more (see Corollary 3.7).

A special case is obtained by setting $r = 1$. In this case $\mathscr{X}$ is a Segre variety and $\mathscr{Y}$ is the variety of degenerate matrices rank.

As an application of this new instance of Homological Projective Duality, we try to address a fascinating question, asked by A. Bondal in Tokyo in 2011. Since any Fano variety admits semiorthogonal decompositions, it is natural to ask whether the derived category of any variety can be realized as a component of a semiorthogonal decomposition of a Fano variety. Under this perspective, considering Fano varieties will be enough to study all "geometric" triangulated categories.

Bondal's Question 1.1. Let X be a smooth and projective variety. Is there any smooth Fano variety Y together with a full and faithful functor $\mathrm{D^b}(X) \to \mathrm{D^b}(Y)$?

We will say that X is *Fano-visitor* if Question 1.1 has a positive answer (see Definition 2.9).

On the other hand, an interesting geometrical insight of semiorthogonal decompositions is to provide a conjectural obstruction to rationality of a given variety X. In [5], the first and second named authors introduced, based on existence of semiorthogonal decompositions, the notion of *categorical representability* of a variety X (see Definition 2.8). This notion allows to formulate a natural question about categorical obstructions to rationality.

Question 1.2. Is a rational projective variety always categorically representable in codimension at least 2?

The motivating ideas of Question 1.2 can be traced back to the work of Bondal and Orlov, and to their address at the 2002 ICM [10], and to Kuznetsov's remarkable contributions (*e.g.* [28] or [31]). Notice that a projective space is representable in dimension 0. Roughly speaking, the idea supporting Question 1.2 is based on a motivic argument which let us suppose that birational transformations should not add components representable codimension 1 or less (see also [5]).

Several examples seem to suggest that Question 1.2 may have a positive answer. Let us mention conic bundles over minimal surfaces [6], fibrations in intersections of quadrics [2], or some classes of cubic fourfolds [28]. Moreover, Question 1.2 is equivalent to one implication of Kuznetsov Conjecture on the rationality of a cubic fourfold [28], which was proved to coincide with Hodge theoretical expectations for a general cubic fourfold by Addington and Thomas [1].

As consequences of Theorems 3.5 and 3.6, we can show that (the categorical resolution of singularities of) any determinantal hypersurface of general type is Fano visitor (§5), and that (the categorical resolution of singularities of) a rational determinantal variety is categorically representable in codimension at least two (§6). Hence we provide a large family of varieties for which Questions 1.1 and 1.2 have positive answer. As an example, we easily get the following corollary (compare with Example 6.3).

Corollary 1.3. *A smooth plane curve is Fano visitor.*

2. Preliminaries

2.1. Notation

We work over an algebraically closed field of characteristic zero k. A vector space will be denoted by a capital letter W; the dual vector space is denoted by $W^\vee$. Suppose

$\dim(W) = N$, then the projective space of W is denoted by $\mathbb{P}W$ or simply by $\mathbb{P}^{N-1}$. We follow Grothendieck's convention, so that $\mathbb{P}W$ is the set of hyperplanes through the origin of W. The dual projective space is denoted by $\mathbb{P}W^\vee$ or by $(\mathbb{P}^{N-1})^\vee$.

We assume the reader to be familiar with the theory of semiorthogonal decompositions and exceptional objects (see [11,19,30]). Recall that the summands of a semiorthogonal decomposition of a triangulated category $\mathbf{T}$, by definition are full triangulated subcategories of $\mathbf{T}$ which are admissible, *i.e.* such that the inclusion admits a left and right adjoint.

2.2. Categorical resolutions of singularities

By a *noncommutative scheme* we mean (following Kuznetsov [26, §2.1]) a scheme X together with a coherent $\mathscr{O}_X$-algebra $\mathscr{A}$. Morphisms are defined accordingly. By definition, a noncommutative scheme $(X, \mathscr{A})$ has $\mathsf{Coh}(X, \mathscr{A})$, the category of coherent $\mathscr{A}$-modules, as category of coherent sheaves and $\mathrm{D^b}(X, \mathscr{A})$ as bounded derived category.

Following Bondal–Orlov [10, §5], a *categorical* (or *noncommutative*) *resolution of singularities* $(X, \mathscr{A})$ of a possibly singular proper scheme X is a torsion free $\mathscr{O}_X$-algebra $\mathscr{A}$ of finite rank such that $\mathsf{Coh}(X, \mathscr{A})$ has finite homological dimension (*i.e.*, is smooth in the noncommutative sense).

Definition 2.1. Let X be a scheme. An object T of $\mathrm{D^b}(X)$ is called a *compact generator* if T is perfect and, for any object S of $\mathrm{D^b}(X)$, we have that the fact that $\mathrm{Hom}_{\mathrm{D^b}(X)}(S, T[i]) = 0$ for all integers i is equivalent to $S = 0$. Notice that, if X is smooth and proper, the natural inclusion $\mathrm{Perf}(X) \subset \mathrm{D^b}(X)$ of perfect complexes into $\mathrm{D^b}(X)$ is an equivalence. Hence any object in $\mathrm{D^b}(X)$ is perfect.

In the case where X admits a full exceptional collection, there is an explicit compact generator T.

Proposition 2.2. *(See [9].) Suppose that X is smooth and proper, and that $\mathrm{D^b}(X)$ is generated by a full exceptional sequence $\mathrm{D^b}(X) = \langle E_1, \ldots, E_s \rangle$. Then $E = \oplus_{i=1}^s E_i$ is a compact generator. In particular, consider the dg-k-algebra $\mathrm{End}(E)$. Then there is an equivalence of triangulated categories $\mathrm{D^b}(X) \simeq \mathrm{D^b}(\mathrm{End}(E))$.*

2.3. Homological projective duality

Homological Projective Duality (HPD) was introduced by Kuznetsov [24] in order to study derived categories of hyperplane sections (see also [23]).

Let us first recall the basic notion of HPD from [24]. Let X be a projective scheme together with a base-point-free line bundle $\mathscr{O}_X(H)$.

Definition 2.3. A *Lefschetz decomposition* of $\mathrm{D^b}(X)$ with respect to $\mathscr{O}_X(H)$ is a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathbf{A}_0, \mathbf{A}_1(H), \ldots, \mathbf{A}_{i-1}((i-1)H) \rangle, \tag{2.1}$$

with

$$0 \subset \mathbf{A}_{i-1} \subset \ldots \subset \mathbf{A}_0.$$

Such a decomposition is said to be *rectangular* if $\mathbf{A}_0 = \ldots = \mathbf{A}_{i-1}$.

Let $W := H^0(X, \mathscr{O}_X(H))$, and $f : X \to \mathbb{P}W$ the map given by the linear system associated with $\mathscr{O}_X(H)$, so that $f^*\mathscr{O}_{\mathbb{P}W}(1) \cong \mathscr{O}_X(H)$. We denote by $\mathcal{X} \subset X \times \mathbb{P}W^\vee$ the universal hyperplane section of X

$$\mathcal{X} := \{(x, H) \in X \times \mathbb{P}W^\vee | x \in H\}.$$

Definition 2.4. Let $f : X \to \mathbb{P}W$ be a smooth projective scheme with a base-point-free line bundle $\mathscr{O}_X(H)$ and a Lefschetz decomposition as above. A scheme Y with a map $g : Y \to \mathbb{P}W^\vee$ is called *homologically projectively dual* (or the *HP-dual*) to $f : X \to \mathbb{P}W$ with respect to the Lefschetz decomposition (2.1), if there exists a fully faithful functor $\Phi : \mathrm{D^b}(Y) \to \mathrm{D^b}(\mathcal{X})$ giving the semiorthogonal decomposition

$$\mathrm{D^b}(\mathcal{X}) = \langle \Phi(\mathrm{D^b}(Y), \mathbf{A}_1(1) \boxtimes \mathrm{D^b}(\mathbb{P}W^\vee), \ldots, \mathbf{A}_{i-1}(i-1) \boxtimes \mathrm{D^b}(\mathbb{P}W^\vee)\rangle.$$

Let $N = \dim(W)$ and let $c \leq N$ be an integer. Given a c-codimensional linear subspace $L \subset W$, we define the linear subspace $\mathbb{P}_L \subset \mathbb{P}W$ of codimension c as $\mathbb{P}(W/L)$. Dually, we have a linear subspace $\mathbb{P}^L = \mathbb{P}L^\perp$ of dimension $c-1$ in $\mathbb{P}W^\vee$, whose defining equations are the elements of $L^\perp \subset W^\vee$. We define the varieties:

$$X_L = X \times_{\mathbb{P}W} \mathbb{P}_L, \qquad Y_L = Y \times_{\mathbb{P}W^\vee} \mathbb{P}^L.$$

Theorem 2.5. *(See [24, Theorem 1.1].) Let X be a smooth projective variety with a map $f : X \to \mathbb{P}W$, and a Lefschetz decomposition with respect to $\mathscr{O}_X(H)$. If Y is HP-dual to X, then:*

(i) *Y is smooth projective and admits a* dual *Lefschetz decomposition*

$$\mathrm{D^b}(Y) = \langle \mathbf{B}_{j-1}(1-j), \ldots, \mathbf{B}_1(-1), \mathbf{B}_0\rangle, \qquad \mathbf{B}_{j-1} \subset \ldots \subset \mathbf{B}_1 \subset \mathbf{B}_0$$

with respect to the line bundle $\mathscr{O}_Y(H) = g^*\mathscr{O}_{\mathbb{P}W^\vee}(1)$.

(ii) *If L is admissible, i.e. if*

$$\dim X_L = \dim X - c, \quad \textit{and} \quad \dim Y_L = \dim Y + c - N,$$

then there exist a triangulated category $\mathbf{C}_L$ and semiorthogonal decompositions:

$$\mathrm{D}^{\mathrm{b}}(X_L) = \langle \mathbf{C}_L, \mathbf{A}_c(1), \dots, \mathbf{A}_{i-1}(i-c)\rangle,$$
$$\mathrm{D}^{\mathrm{b}}(Y_L) = \langle \mathbf{B}_{j-1}(N-c-j), \dots, \mathbf{B}_{N-c}(-1), \mathbf{C}_L\rangle.$$

Remark 2.6. In general, HPD involves non-smooth varieties. Indeed, as shown by Kuznetsov [24, Theorem 7.9] the critical locus of the map $g : Y \to \mathbb{P}W^\vee$ is the classical projective dual $X^\vee$ of X, which is rarely smooth even if X is smooth. If X (resp. Y) is singular, then we have to replace $\mathrm{D}^{\mathrm{b}}(X)$ (resp. $\mathrm{D}^{\mathrm{b}}(Y)$) by a categorical resolution of singularities $\mathrm{D}^{\mathrm{b}}(X, \mathscr{A})$ (resp. $\mathrm{D}^{\mathrm{b}}(Y, \mathscr{B})$) in all the statements and definitions of this section. Theorem 2.5 holds in this more general framework, where we have to consider $\mathrm{D}^{\mathrm{b}}(X_L, \mathscr{A}_L)$ (resp. $\mathrm{D}^{\mathrm{b}}(Y_L, \mathscr{B}_L)$) for $\mathscr{A}_L$ (resp. $\mathscr{B}_L$) the restriction of $\mathscr{A}$ to X_L (resp. of $\mathscr{B}$ to Y_L) in item (ii).

2.4. Categorical representability and Fano visitors

First, let us recall the definition of categorical representability for a variety.

Definition 2.7. (See [5].) A triangulated category $\mathbf{T}$ is *representable in dimension* j if it admits a semiorthogonal decomposition

$$\mathbf{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_l\rangle,$$

and for all $i = 1, \dots, l$ there exists a smooth projective connected variety Y_i with $\dim Y_i \leq j$, such that $\mathbf{A}_i$ is equivalent to an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(Y_i)$.

Definition 2.8. (See [5].) Let X be a projective variety. We say that X is *categorically representable* in dimension j (or equivalently in codimension $\dim(X) - j$) if there exists a categorical resolution of singularities of $\mathrm{D}^{\mathrm{b}}(X)$ representable in dimension j.

Based on Bondal's Question 1.1, we introduce the following definition.

Definition 2.9. A triangulated category $\mathbf{T}$ is *Fano-visitor* if there exists a smooth Fano variety F and a fully faithful functor $\mathbf{T} \to \mathrm{D}^{\mathrm{b}}(F)$ such that $\mathrm{D}^{\mathrm{b}}(F) = \langle \mathbf{T}, \mathbf{T}^\perp\rangle$. A smooth projective variety X is said to be a *Fano-visitor* if its derived category $\mathrm{D}^{\mathrm{b}}(X)$ is Fano-visitor.

We remark that, having a fully faithful functor $\mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(F)$ is enough to have the required semiorthogonal decomposition [8]. Relaxing slightly the hypotheses on the smoothness of the Fano variety we get the following weaker definition.

Definition 2.10. A triangulated category $\mathbf{T}$ is *weakly Fano-visitor* if there exists a (possibly singular) Fano variety F, a categorical crepant resolution of singularities $\mathbf{DF}$ of F and a fully faithful functor $\mathbf{T} \to \mathbf{DF}$ such that $\mathbf{DF} = \langle \mathbf{T}, \mathbf{T}^\perp\rangle$. Notice that this implies that the functor $\mathbf{T} \to \mathbf{DF}$ has a right and left adjoint by definition of semiorthogonal decomposition. As before, if $\mathbf{T} \cong \mathrm{D^b}(X)$ for a smooth projective variety X, then X itself is said to be *weakly Fano-visitor*.

3. Homological projective duality for determinantal varieties

We describe here homological projective duality for determinantal varieties in terms of the Springer resolution of the space of $n \times m$ matrices of rank at most r and in terms of categorical resolution of singularities.

3.1. The Springer resolution of the space of matrices of bounded rank

Let us introduce the variety $\mathscr{Z}^r_{m,n}$ of $n \times m$ matrices over our base field, having rank at most r. Let U, V be vector spaces, with $\dim U = m$, $\dim V = n$, and assume $n \geq m$. Set $W = U \otimes V$. Let r be an integer in the range $1 \leq r \leq m-1$. We define $\mathscr{Z}^r = \mathscr{Z}^r_{m,n}$ to be the variety of matrices $M : V \to U^\vee$ in $\mathbb{P}W$ cut by the minors of size $r+1$ of the matrix of indeterminates:

$$\psi = \begin{pmatrix} x_{1,1} & \dots & x_{m,1} \\ \vdots & \ddots & \vdots \\ x_{m,n} & \dots & x_{m,n} \end{pmatrix}$$

3.1.1. Springer resolution and projective bundles

Consider the Grassmann variety $\mathbb{G}(U,r)$ of r-dimensional quotient spaces of U, the tautological sub-bundle and the quotient bundle over $\mathbb{G}(U,r)$, denoted respectively by $\mathscr{U}$ and $\mathscr{Q}$, respectively of rank $m-r$ and r. We will write $\mathbb{G}$ for $\mathbb{G}(U,r)$.

The tautological (or Euler) exact sequence reads:

$$0 \to \mathscr{U} \to U \otimes \mathscr{O}_{\mathbb{G}} \to \mathscr{Q} \to 0. \tag{3.1}$$

We will use the following notation:

$$\mathscr{X}^r_{m,n} = \mathbb{P}(V \otimes \mathscr{Q}).$$

However, the dependency on m, n, r will often be omitted.

The manifold $\mathscr{X} = \mathscr{X}^r_{m,n}$ has dimension $r(n+m-r)-1$. It is the resolution of singularities of the variety of $m \times n$ matrices of rank at most r, in a sense that we will now review. Denote by p the natural projection $\mathscr{X} \to \mathbb{G}$. The space $H^0(\mathbb{G}, \mathscr{Q})$ is naturally identified with U.

Let us denote by $\mathscr{O}_{\mathscr{X}}(H_X)$ the relatively ample tautological line bundle on $\mathscr{X}$. We will often write simply H for H_X. We get natural isomorphisms:

$$H^0(\mathbb{G}, V \otimes \mathscr{Q}) \simeq H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(H)) \simeq W = U \otimes V.$$

Therefore, the map f associated with the linear system $\mathscr{O}_{\mathscr{X}}(H)$ maps $\mathscr{X}$ to $\mathbb{P}W$, and clearly $\mathscr{O}_{\mathscr{X}}(H) \simeq f^*(\mathscr{O}_{\mathbb{P}W}(1))$. This is summarized by the diagram:

$$\begin{tikzcd} \mathscr{X} \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{P}W = \mathbb{P}(U \otimes V) \\ \mathbb{G} & \end{tikzcd}$$

On the other hand, we will denote by P the pull-back to $\mathbb{P}(V \otimes \mathscr{Q})$ of the first Chern class $c_1(\mathscr{Q})$ on $\mathbb{G}$. Hence we have that $c_1(V \otimes \mathscr{Q})$ pulls back to nP and $\omega_{\mathbb{G}}$ to $-mP$. The Picard group of $\mathscr{X}$ is generated by P and H.

Notice that giving a rank-1 quotient of $W = U \otimes V$ corresponds to the choice of a linear map $M : V \to U^\vee$, so an element of $\mathbb{P}W$ can be considered as (the proportionality class of) the linear map M. On the other hand, the map f sends a rank-1 quotient of $V \otimes \mathscr{Q}$ over a point $\lambda \in \mathbb{G}$ to the quotient of W obtained by composition with the obvious quotient $U \to \mathscr{Q}_\lambda$.

Therefore, the matrix M lies in the image of f if and only if M factors through $V \to \mathscr{Q}_\lambda^\vee$, for some $\lambda \in \mathbb{G}$, *i.e.*, if and only if $\mathrm{rk}(M) \leq r$. Clearly, if M has precisely rank r then it determines λ and the associated quotient of $U \to \mathscr{Q}_\lambda$. Since this happens for a general matrix M of $\mathscr{Z}^r = \mathscr{Z}^r_{m,n}$, the map $f : \mathscr{X} \to \mathscr{Z}^r$ is birational. This map is in fact a desingularization, called the *Springer resolution*, of $\mathscr{Z}^r$. It is an isomorphism above the locus of matrices of rank exactly r.

In a more concrete way, given $\lambda \in \mathbb{G}$ we let π_λ be the linear projection from $U^\vee$ to $U^\vee/\mathscr{Q}_\lambda^\vee$. Then, the variety $\mathscr{X}$ can be thought of as:

$$\mathscr{X} = \{(\lambda, M) \in \mathbb{G} \times \mathscr{Z}^r \mid \pi_\lambda \circ M = 0\}.$$

This way, the maps p and f are just the projections from $\mathscr{X}$ onto the two factors.

Let us now look at the dual picture. We consider the projective bundle:

$$\mathscr{Y}^r_{m,n} = \mathbb{P}(V^\vee \otimes \mathscr{U}^\vee).$$

Write $\mathscr{Y} = \mathscr{Y}^r_{m,n}$ for short. Denote by q the projection $\mathscr{Y} \to \mathbb{G}$. We will denote by H_Y (or sometimes just by H) the tautological ample line bundle on $\mathscr{Y}$. This time, since $H^0(\mathbb{G}, \mathscr{U}^\vee) \simeq U^\vee$, the linear system associated with $\mathscr{O}_{\mathscr{Y}}(H)$ sends $\mathscr{Y}$ to $\mathbb{P}W^\vee \simeq \mathbb{P}(V^\vee \otimes U^\vee)$ via a map that we call g. By the same argument as above, g is a desingularization of the variety $\mathscr{W}^r$ of matrices $V^\vee \to U$ in $\mathbb{P}W^\vee$ of corank at least r. There exists an obvious isomorphism $\mathscr{Z}^{m-r} \cong \mathscr{W}^r$, which we will use without further mention.

The spaces $\mathbb{P}W$ and $\mathbb{P}W^\vee$ are equipped with tautological morphisms of sheaves, which are both identified by the matrix ψ, corresponding to the identity in $W \otimes W^\vee = U \otimes V \otimes U^\vee \otimes V^\vee$:

$$V \otimes \mathscr{O}_{\mathbb{P}W}(-1) \xrightarrow{\psi} U^\vee \otimes \mathscr{O}_{\mathbb{P}W}, \tag{3.2}$$

$$V^\vee \otimes \mathscr{O}_{\mathbb{P}W^\vee}(-1) \xrightarrow{\psi} U \otimes \mathscr{O}_{\mathbb{P}W^\vee}. \tag{3.3}$$

Definition 3.1. We will denote by $\mathscr{F}$ and $\mathscr{E}$, the cokernel of the tautological map appearing in Eq. (3.2), respectively Eq. (3.3).

Lemma 3.2. *We have isomorphisms $\mathscr{X} \simeq \mathbb{G}(\mathscr{F}, m-r)$ and $\mathscr{Y} \simeq \mathbb{G}(\mathscr{E}, r)$.*

Proof. We work out the proof for $\mathscr{Y}$, the argument for $\mathscr{X}$ being identical. Given a scheme S over our field, an S-valued point $[e]$ of $\mathbb{G}(\mathscr{E}, r)$ is given by a morphism $s : S \to \mathbb{P}W^\vee$ and the equivalence class of an epimorphism $e : s^*\mathscr{E} \to \mathscr{V}$, where $\mathscr{V}$ is locally free of rank r on S. On the other hand, an S-point $[y]$ of $\mathscr{Y}$ corresponds to a morphism $t : S \to \mathbb{G}$ together with the class of a quotient $y : V^\vee \otimes t^*\mathscr{U}^\vee \to \mathscr{L}$, with $\mathscr{L}$ invertible on S. In turn, t is given by a locally free sheaf of rank r on S and a surjection from $U \otimes \mathscr{O}_S$ onto this sheaf.

Given the point $[e]$, we compose e with the surjection $U \otimes \mathscr{O}_S \to s^*\mathscr{E}$ and denote by t_e the resulting map $U \otimes \mathscr{O}_S \to \mathscr{V}$. This way, t_e provides the required morphism $t : S \to \mathbb{G}$, and clearly $t^*\mathscr{Q} \simeq \mathscr{V}$, so the kernel of $U \otimes \mathscr{O}_S \to \mathscr{V}$ is just $t^*\mathscr{U}$. Clearly, we have $t_e \circ s^*\psi = 0$ so that $s^*\psi$ factors through a map $V^\vee \otimes \mathscr{O}_S(-1) \to t^*\mathscr{U}$. Giving this last map is equivalent to the choice of a map $V^\vee \otimes t^*\mathscr{U}^\vee \to \mathscr{O}_S(1)$, which we define to be the point $[y]$ associated with $[e]$.

Conversely, let t be represented by a locally free sheaf $\mathscr{V} = t^*\mathscr{Q}$ of rank r on S and by a quotient $U \otimes \mathscr{O}_S \to \mathscr{V}$, whose kernel is $t^*\mathscr{U}$. Then, given point $[y]$ and the quotient y, we consider the composition of y and $U^\vee \otimes \mathscr{O}_S \to \mathscr{U}^\vee$ to obtain a quotient $s_y : V^\vee \otimes U^\vee \to \mathscr{L}$. This gives the desired morphism $s : S \to \mathbb{P}W^\vee$. Moreover, the map $V^\vee \otimes \mathscr{O}_S \to t^*\mathscr{U} \otimes \mathscr{L}$ associated with y can be composed with the injection $t^*\mathscr{U} \otimes \mathscr{L} \to U \otimes \mathscr{L}$ to get a map $V^\vee \otimes \mathscr{O}_S \to U \otimes \mathscr{L}$, or equivalently $V^\vee \otimes \mathscr{L}^\vee \to U \otimes \mathscr{O}_S$, and this map is nothing but $s^*\psi$. Of course, composing this map with the projection $U \otimes \mathscr{O}_S \to t^*\mathscr{Q} = \mathscr{V}$ we get zero, so there is an induced surjective map $s^*\mathscr{E} \to \mathscr{V}$. We define the class of this map to be the point $[e]$ associated with $[y]$.

We have defined two maps from the sets of S-valued points of our two schemes, which are inverse to each other by construction. The lemma is thus proved. □

3.1.2. Linear sections and projectivized sheaves

Let now c be an integer in the range $1 \le c \le mn$, and suppose we have a c-dimensional vector subspace L of W:

$$L \subset U \otimes V = W.$$

We have thus the linear subspace $\mathbb{P}_L \subset \mathbb{P}W$ of codimension c, defined by $\mathbb{P}_L = \mathbb{P}(W/L)$. Dually, we have a linear subspace $\mathbb{P}^L = \mathbb{P}L^\perp$ of dimension $c-1$ in $\mathbb{P}W^\vee$, whose defining equations are the elements of $L^\perp \subset W^\vee$. We define the varieties:

$$X_L^r = \mathscr{X}_{m,n}^r \times_{\mathbb{P}W} \mathbb{P}_L, \qquad Y_L^r = \mathscr{Y}_{m,n}^r \times_{\mathbb{P}W^\vee} \mathbb{P}^L.$$

We also write:

$$Z_L^r = \mathscr{Z}_{m,n}^r \cap \mathbb{P}_L, \qquad Z_r^L = \mathscr{Z}_{m,n}^{m-r} \cap \mathbb{P}^L.$$

We will drop r, n and/or m from the notation when no confusion is possible. We will always assume that $L \subset W$ is an *admissible subspace* in the sense of [24], which amounts to ask that X_L and Y_L have expected dimension. This means that we have:

$$\dim Z_L = \dim X_L = \dim \mathscr{X}^r_{m,n} - c = r(n+m-r) - c - 1$$
$$\dim Z^L = \dim Y_L = \dim \mathscr{Y}^r_{m,n} - (mn-c) = r(m-n-r) + c - 1.$$

Let us now give another interpretation of the choice of our linear subspace $L \subset W$. To this purpose we consider the Grassmann variety $\mathbb{G}(V,r)$ with the its tautological rank-r quotient bundle which we denote by $\mathscr{T}$. Dually, we consider $\mathbb{G}(V^\vee, m-r)$ and denote by $\mathscr{S}^\vee$ the tautological quotient bundle of rank $m-r$. Observe that there are natural isomorphisms:

$$\begin{aligned} L^\vee \otimes W = L^\vee \otimes U \otimes V &\simeq \mathrm{Hom}(L \otimes \mathscr{O}_{\mathbb{G}}, V \otimes \mathscr{Q}) \simeq \\ &\simeq L^\vee \otimes H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(H)) \simeq \\ &\simeq \mathrm{Hom}(L \otimes \mathscr{O}_{\mathbb{G}(V,r)}, U \otimes \mathscr{T}). \end{aligned}$$

There are similar isomorphisms for $\mathbb{G}(V^\vee, m-r)$. We denote by s_L the global section of $L^\vee \otimes H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(H))$ corresponding to $L \subset W$ via these isomorphisms. The subspace L corresponds also to morphisms of bundles on the Grassmann varieties, which we write as:

$$M_L : L \otimes \mathscr{O}_{\mathbb{G}} \to V \otimes \mathscr{Q}, \qquad N_L : L \otimes \mathscr{O}_{\mathbb{G}(V,r)} \to U \otimes \mathscr{T}$$

We also write:

$$M^L : L^\perp \otimes \mathscr{O}_{\mathbb{G}} \to V^\vee \otimes \mathscr{U}^\vee, \qquad N^L : L^\perp \otimes \mathscr{O}_{\mathbb{G}(V^\vee, m-r)} \to U^\vee \otimes \mathscr{S}^\vee$$

for the morphisms corresponding to $L^\perp \subset U^\vee \otimes V^\vee$.

Proposition 3.3. *We have the following equivalent descriptions of X_L:*

(i) *the vanishing locus $\mathbb{V}(s_L)$ of the section $s_L \in L^\vee \otimes H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(H))$;*
(ii) *the projectivization of* $\mathrm{coker}(M_L)$*;*
(iii) *the projectivization of* $\mathrm{coker}(N_L)$*;*
(iv) *the Grassmann bundle $\mathbb{G}(\mathscr{F}|_{\mathbb{P}_L}, m-r)$.*

Dually, the variety Y_L is:

(i) *the vanishing locus of the section $s^L \in (W/L) \otimes H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(H))$;*
(ii) *the projectivization of* $\mathrm{coker}(M^L)$*;*
(iii) *the projectivization of* $\mathrm{coker}(N^L)$*;*
(iv) *the Grassmann bundle $\mathbb{G}(\mathscr{E}|_{\mathbb{P}^L}, r)$.*

Proof. We work out the proof for X_L, the dual case Y_L being analogous. First recall that the map $\mathscr{X} \to \mathbb{P}W$ is defined by the linear system $\mathscr{O}_{\mathscr{X}}(H)$, while the inclusion $\mathbb{P}_L \subset \mathbb{P}W$ corresponds to the projection $W \to W/L$. Hence the fibre product defining X_L is given by the vanishing of the global sections in $H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(H))$ which actually lie in L, *i.e.* by the vanishing of s_L, so (i) is clear.

For (ii) we use essentially the same proof of Lemma 3.2. Indeed, given a scheme S over our field, an S-valued point of $\mathbb{P}(\mathrm{coker}(M_L))$ is defined by a morphism $t : S \to \mathbb{G}$ together with the isomorphism class of a quotient $y : t^*(\mathrm{coker}(M_L)) \to \mathscr{L}$, with $\mathscr{L}$ invertible on S. On the other hand, an S-valued point of X_L is given by a morphism $s : S \to X_L$. Once given s, composing with $X_L \to \mathscr{X} \to \mathbb{G}$ we obtain the morphism t. By the definition of $\mathscr{X}$ as projective bundle, together with t we get a map $V \otimes t^*\mathscr{Q} \to \mathscr{L}$ with $\mathscr{L}$ invertible on S. This map composes to zero with $t^*(M_L) : L \otimes \mathscr{O}_S \to V \otimes t^*\mathscr{Q}$ since the image of s is contained in X_L, hence in the vanishing locus of the linear section s_L. Therefore this map factors through $t^*(\mathrm{coker}(M_L))$ and provides the quotient y. It is not hard to check that this procedure can be reversed, which finally proves (ii).

The statement (iii) is proved in a similar fashion, while (iv) is just Lemma 3.2, restricted to $\mathbb{P}_L$. □

3.2. The noncommutative desingularization

In [15,16], noncommutative resolutions of singularities for the affine cone over $\mathscr{Z}^r = \mathscr{Z}^r_{m,n}$ are constructed. This is done by considering the vector bundles $V \otimes \mathscr{Q}$ instead of their projectivization, and Kapranov's strong exceptional collection on the Grassmannian [21] (for the details see [16]). Here we carry on this construction to the projectivized determinantal varieties.

Consider $\mathscr{X} = \mathscr{X}^r_{m,n}$ as rank-$(rn-1)$ projective bundle $p : \mathscr{X} \to \mathbb{G}$. Orlov [34] gives a semiorthogonal decomposition

$$\mathrm{D^b}(\mathscr{X}) = \langle p^*\mathrm{D^b}(\mathbb{G}), \ldots, p^*\mathrm{D^b}(\mathbb{G})((rn-1)H)\rangle. \tag{3.4}$$

On the other hand, Kapranov shows that $\mathbb{G}$ has a full strong exceptional collection [21] consisting of vector bundles. We obtain then an exceptional collection on $\mathscr{X}$ consisting of vector bundles, and hence a tilting bundle E as the direct sum of the bundles from the exceptional collection. Let us consider $M := Rf_*E$, and let $\mathscr{R} := \mathscr{E}nd(E)$ and $\mathscr{R}' := \mathscr{E}nd(M)$ (where $\mathscr{E}nd$ denotes the sheaf of endomorphisms).

Proposition 3.4. *The endomorphism algebra $\mathscr{E}nd(M)$ is a coherent $\mathscr{O}_{\mathscr{Z}^r}$-algebra Morita-equivalent to $\mathscr{R}$. In particular, $\mathrm{D^b}(\mathscr{Z}^r, \mathscr{R}) \simeq \mathrm{D^b}(\mathscr{X})$ is a categorical resolution of singularities, which is crepant if $m = n$.*

Proof. First of all, since $\mathbb{G}$ has a strong full exceptional collection, we have a tilting bundle G over it. A cohomological calculation, together with the semiorthogonal decomposition (3.4) provides a tilting bundle $E = \bigoplus_{i=0}^{nr-1} p^*G \otimes \mathscr{O}_{\mathscr{X}}(iH)$ over $\mathscr{X}$. We have thus:

$$\mathrm{D}^{\mathrm{b}}(\mathscr{X}) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{End}(E)).$$

Since the exceptional locus of f has codimension greater than one, [37, Lemma 4.2.1] implies that $f_*\mathscr{R}$ is reflexive. There is a natural map $f_*\mathscr{R} \to \mathscr{R}'$ of reflexive sheaves which explicitly reads:

$$f_*\mathscr{R} = \bigoplus_{i,j=0}^{nr-1} f_*\mathscr{E}nd(p^*G)(i-j) \to \bigoplus_{i,j=0}^{nr-1} \mathscr{E}nd(f_*p^*G)(i-j) = \mathscr{R}'.$$

Again, since the exceptional locus of f has codimension greater than one the locus where $f_*\mathscr{R}$ and $\mathscr{R}'$ may be non-isomorphic has codimension at least 2. Since both sheaves are reflexive, we obtain $f_*\mathscr{R} \cong \mathscr{R}'$ (compare with [15, Proposition 6.5]). Moreover we know from [16, Proposition 3.4] that $R^k f_*\mathscr{R} = 0$ for $k > 0$ so we actually have:

$$Rf_*\mathscr{R} \cong \mathscr{R}'.$$

Therefore:

$$\mathrm{End}(E) \simeq H^\bullet(\mathscr{R}) \simeq H^\bullet(Rf_*\mathscr{R}) \simeq H^\bullet(\mathscr{R}').$$

We have now proved:

$$\mathrm{D}^{\mathrm{b}}(\mathscr{X}) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{End}(E)) \simeq \mathrm{D}^{\mathrm{b}}(H^\bullet(\mathscr{R}')) \simeq \mathrm{D}^{\mathrm{b}}(\mathscr{Z}^r, \mathscr{R}').$$

Finally, $\mathscr{R}'$ is maximally Cohen–Macaulay by [16, Proposition 3.4] (as this property is local) and has finite global dimension since it is Morita-equivalent to the endomorphism algebra $\mathscr{R}$, which is defined over a smooth variety. If $m = n$, the variety $\mathscr{Z}^r$ has Gorenstein singularities and f is a crepant resolution, so that the noncommutative resolution is also crepant (compare with [15]). □

3.3. Homological projective duality for matrices of bounded rank

With this in mind, we can prove our main result directly from Kuznetsov's HPD for the projective bundles $\mathscr{X}^r_{m,n} = \mathscr{X}$ and $\mathscr{Y}^r_{m,n} = \mathscr{Y}$. We consider the rectangular Lefschetz decomposition (3.4) for $\mathscr{X}$ with respect to $\mathscr{O}_{\mathscr{X}}(H)$.

Theorem 3.5. *The morphism $g : \mathscr{Y} \to \mathbb{P}W^\vee$ is the homological projective dual of $f : \mathscr{X} \to \mathbb{P}W$, relatively over $\mathbb{G}$, with respect to the rectangular Lefschetz decomposition (3.4) induced by $\mathscr{O}_{\mathscr{X}}(H)$, generated by $nr\binom{m}{r}$ exceptional bundles.*

Proof. Given the setup of §3.1, we consider the vector bundles $V \otimes \mathscr{Q}$ and $V^\vee \otimes \mathscr{U}^\vee$ over $\mathbb{G}$ and recall that $\mathscr{X} = \mathbb{P}(V \otimes \mathscr{Q})$ and $\mathscr{Y} = \mathbb{P}(V^\vee \otimes \mathscr{U}^\vee)$.

Set $\mathbf{A} = p^*(\mathrm{D^b}(\mathbb{G}))$. The decomposition (3.4) of the projective bundle $\mathscr{X} \to \mathbb{G}$ then reads:

$$\mathrm{D^b}(\mathscr{X}) = \langle \mathbf{A}, \mathbf{A}(H), \dots, \mathbf{A}((rn-1)H) \rangle.$$

This is a rectangular Lefschetz decomposition with respect to $\mathscr{O}_{\mathscr{X}}(H)$, generated by nr copies of Kapranov's exceptional collection on $\mathbb{G}$, hence by $nr\binom{m}{r}$ exceptional bundles.

Clearly the vector bundles $V \otimes \mathscr{Q}$ and $V^\vee \otimes \mathscr{U}^\vee$ are generated by their global sections, so we may apply [24, Corollary 8.3] to their projectivization (actually we use the Grothendieck's notation for projectivized bundles rather than the usual notation as in [24], but this does affect the result). The evaluation map of global sections of $V \otimes \mathscr{Q}$ gives (3.1) tensored with the identity over V *i.e.*:

$$0 \to V \otimes \mathscr{U} \to W \to V \otimes \mathscr{Q} \to 0.$$

This says that $V^\vee \otimes \mathscr{U}^\vee$ is the orthogonal in Kuznetsov's sense of $V \otimes \mathscr{Q}$. Also, the morphism associated with the tautological line bundle H_X over $\mathscr{X}$ is f, while g is associated with H_Y over $\mathscr{Y}$. Therefore [24, Corollary 8.3] applies and gives the result.

Note that $\mathrm{D^b}(\mathscr{Y})$ is generated by $n(m-r)\binom{m}{r}$ exceptional vector bundles. □

We can rephrase this in terms of categorical resolutions, as a consequence of Proposition 3.4. In this way, one can state HPD as a duality between categorical resolutions of determinantal varieties given by matrices of fixed rank and corank. This leads us to prove our second main Theorem.

Theorem 3.6. *There is a $\mathscr{O}_{\mathscr{Z}^r}$-algebra $\mathscr{R}'$ such that $(\mathscr{Z}^r, \mathscr{R}')$ is a categorical resolution of singularities of $\mathscr{Z}^r$. Moreover, $\mathrm{D^b}(\mathscr{Z}^r, \mathscr{R}') \simeq \mathrm{D^b}(\mathscr{X})$ so that $(\mathscr{Z}^r, \mathscr{R}')$ is HP-dual to $(\mathscr{Z}^{m-r}, \mathscr{R}')$.*

Proof. Recall that $\mathscr{X}$ is a projective bundle over a Grassmann variety, and hence has a full exceptional sequence. By applying Proposition 3.4 to the full exceptional sequence on $\mathscr{X}$, we get the first statement. The second statement is now straightforward from Theorem 3.5, together with the isomorphism $\mathscr{X}^{m-r} \simeq \mathscr{Y}^r$. □

3.4. Semiorthogonal decompositions for linear sections

Let L be a dimension c subspace of $U \otimes V = W$, given by the choice of an element $t \in L^\vee \otimes W$. Recall that we assume that the subspace $L \subset W$ is *admissible* in the sense of [24]. This happens if L is general enough in W.

Moreover, again if L is general enough, the singularities of $Z_L = Z_L^r$ appear precisely along Z_L^{r-1}. Also, the map f, for the rank r locus, is an isomorphism when restricted to

$Z_L \setminus Z_L^{r-1}$. Furthermore, we recall from the preceding section that Z_L is a determinantal variety inside $\mathbb{P}^{mn-c-1}$ given by a $m \times n$ matrix of linear forms and $\mathrm{D^b}(Z_L, \mathscr{R}'_{\mathbb{P}_L})$ is a categorical resolution of singularities of Z_L, where $\mathscr{R}'_{\mathbb{P}_L}$ is the pull-back of $\mathscr{R}'$ from $\mathscr{Z}^r_{m,n}$ to Z_L under the natural restriction map.

Notice that if Z_L is smooth, then $\mathrm{D^b}(Z_L) \simeq \mathrm{D^b}(X_L)$, in fact, $Z_L \simeq X_L$ in this case. Similarly, if $Z^L = Z^L_r$ is smooth, then $\mathrm{D^b}(Z^L) \simeq \mathrm{D^b}(Y_L)$ as again $Z^L \simeq Y_L$ in this case. In particular, in the smooth case, the sheaves of algebras $\mathscr{R}'_{\mathbb{P}_L}$ are Morita-equivalent to the structure sheaf.

Our goal now is to draw consequences from the homological projective duality that we have displayed. Notably we will give in several examples a positive answer to the questions asked in the introduction, *i.e.* Bondal's Question 1.1 and Question 1.2 concerning rationality and categorical representability. Remember that $\mathscr{X}$ (respectively $\mathscr{Y}$) is the projectivization of a vector bundle of rank nr (resp. $n(m-r)$) over $\mathbb{G}$. Hence, by Orlov's result [34] on the semiorthogonal decompositions for projective bundles we have:

$$\mathrm{D^b}(\mathscr{X}) = \langle \mathbf{A}, \mathbf{A}(H), \ldots, \mathbf{A}((nr-1)(H) \rangle;$$
$$\mathrm{D^b}(\mathscr{Y}) = \langle \mathbf{B}((1-nm+nr)H), \ldots, \mathbf{B}(-H), \mathbf{B} \rangle,$$

where $\mathbf{A}$ and $\mathbf{B}$ are the respective pull-backs of $\mathrm{D^b}(\mathbb{G})$ to the projective bundles. This in turn implies that, via HPD, when we intersect $\mathscr{X}$ with $\mathbb{P}_L$ and $\mathscr{Y}$ with $\mathbb{P}^L$, we have the following

$$\mathrm{D^b}(X_L) = \langle \mathbf{C}_L, \mathbf{A}(H), \ldots, \mathbf{A}(nr-c)(H) \rangle;$$
$$\mathrm{D^b}(Y_L) = \langle \mathbf{B}((-c+nr)H), \ldots, \mathbf{B}(-H), \mathbf{C}_L \rangle.$$

Recalling that $\mathrm{D^b}(X_L) \simeq \mathrm{D^b}(Z^r_L, \mathscr{R}'_{\mathbb{P}_L})$ and $\mathrm{D^b}(Y_L) = \mathrm{D^b}(Z^L_r, \mathscr{R}'_{\mathbb{P}^L})$ are categorical resolutions of singularities of dual determinantal varieties, we get:

$$\mathrm{D^b}(Z^r_L, \mathscr{R}'_{\mathbb{P}_L}) = \langle \mathbf{C}_L, \mathbf{A}(H), \ldots, \mathbf{A}(nr-c)(H) \rangle;$$
$$\mathrm{D^b}(Z^L_r, \mathscr{R}'_{\mathbb{P}^L}) = \langle \mathbf{B}((-c+nr)H), \ldots, \mathbf{B}(-H), \mathbf{C}_L \rangle.$$

Finally, the categories $\mathbf{A}$ and $\mathbf{B}$ are both generated by $\binom{m}{r}$ exceptional objects.

Corollary 3.7. *Suppose that $L \subset W$ is admissible of dimension c.*

(i) *If $c > nr$, there is a fully faithful functor*

$$\mathrm{D^b}(Z_L, \mathscr{R}'_{\mathbb{P}_L}) \simeq \mathrm{D^b}(X_L) \longrightarrow \mathrm{D^b}(Y_L) \simeq \mathrm{D^b}(Z^L_r, \mathscr{R}'_{\mathbb{P}^L})$$

whose orthogonal complement is given by $c - nr$ copies of $\mathrm{D^b}(\mathbb{G})$, and is then generated by $(c-nr)\binom{m}{r}$ exceptional objects.

(ii) *If $nr = c$, there is an equivalence*

$$\mathrm{D^b}(Z_L, \mathscr{R}'_{\mathbb{P}_L}) \simeq \mathrm{D^b}(X_L) \simeq \mathrm{D^b}(Y_L) \simeq \mathrm{D^b}(Z_r^L, \mathscr{R}'_{\mathbb{P}^L})$$

(iii) *If $c < nr$, there is a fully faithful functor*

$$\mathrm{D^b}(Z_r^L, \mathscr{R}'_{\mathbb{P}^L}) \simeq \mathrm{D^b}(Y_L) \longrightarrow \mathrm{D^b}(X_L) \simeq \mathrm{D^b}(Z_L, \mathscr{R}'_{\mathbb{P}_L})$$

whose orthogonal complement is given by $nr - c$ copies of $\mathrm{D^b}(\mathbb{G})$, and is then generated by $\binom{m}{r}(nr - c)$ exceptional objects.

Proof. The statement is obtained applying Kuznetsov's Theorem 2.5 to the pair of HPDual varieties from Theorem 3.5, and using the resolutions of singularities described in Theorem 3.6. The functors involved can be explicitly described as Fourier–Mukai with kernels in $\mathrm{D^b}(X_L \times Y_L)$ (see the detailed description in the original Kuznetsov's paper [24, §5]). □

Using the notation introduced in Section 3.1 for the generators of the Picard group, we have the following formula for the canonical bundle of $\mathscr{X}$:

$$\omega_{\mathscr{X}} \simeq \mathscr{O}_{\mathscr{X}}(-nrH + (n - m)P).$$

A consequence of this formula is the following lemma. Call ϕ_K the canonical map of X_L, *i.e.* the rational map associated with the linear system $|\omega_{X_L}|$. Write also ϕ_{-K} for the map associated with $|\omega^\vee_{X_L}|$ (*i.e.* the anticanonical map).

Lemma 3.8. *The canonical bundle of the linear section X_L is:*

$$\omega_{X_L} \simeq \mathscr{O}_{X_L}((c - nr)H + (n - m)P).$$

i) *The variety X_L is Calabi–Yau if and only if $m = n$ and $c = nr$.*
ii) *If $c > nr$, or if $c = nr$ and $n > m$, ϕ_K is a birational morphism onto its image.*
iii) *If $c < nr$ and $m = n$, ϕ_{-K} is a birational morphism onto its image. If moreover $X_L^{r-1} = \emptyset$, ϕ_{-K} is an embedding and X_L is Fano.*

Proof. The formula for ω_{X_L} is obvious by adjunction. By this formula, $\omega_{X_L} \simeq \mathscr{O}_{X_L}$ whenever $m = n$ and $c = nr$. Conversely, remark that X_L is connected, so if X_L is CY, then there is no nontrivial semiorthogonal decomposition of $\mathrm{D^b}(X_L)$. Corollary 3.7 forces then $c \geq nr$.

Suppose $c > nr$, or $c = nr$ and $n > m$. Notice first that both P and H are nef. Then canonical divisor is a linear combination of nef divisors with positive coefficients, which is in turn nef. On the other hand, we have that ω_{X_L} is $\mathscr{O}_{X_L}$ if $c = nr$ and $m = n$, so using $c \geq nr$ we conclude the proof of (i).

Table 1
Behaviour of HPD functors according on c and nr.

HPD Functor	$c < nr$ $\mathrm{D^b}(Y_L) \to \mathrm{D^b}(X_L)$	$c = nr$ equivalence	$c > nr$ $\mathrm{D^b}(X_L) \to \mathrm{D^b}(Y_L)$
$Y_L \to Z^L$	nef canonical Fano visitor if $n = m$	nef canonical if $n \neq m$ CY if $n = m$	Fano if $n = m$
$X_L \to Z_L$	Fano if $n = m$	nef canonical if $n \neq m$ CY if $n = m$	nef canonical Fano visitor if $n = m$

For (ii), by definition H and P are base-point-free and H is very ample away from the exceptional locus of f, so the statement follows directly from the formula for ω_{X_L}. A similar argument proves (iii). $\square$

Corollary 3.9. *We have the following formulas for the canonical bundles*

$$\omega_{\mathscr{Y}} \simeq \mathscr{O}_{\mathscr{Y}}(-n(m-r)H + (n-m)Q),$$

$$\omega_{Y_L} \simeq \mathscr{O}_{Y_L}((nr-c)H + (n-m)Q).$$

In particular, Y_L is Calabi–Yau if and only if $m = n$ and $c = nr$. If $c < nr$, or if $c = nr$ and $n > m$, then the canonical map of Y_L is a birational morphism onto its image. If $c > nr$, $m = n$ and $Y_L^{r-1} = \emptyset$ then Y_L is Fano.

Proof. Everything follows from the isomorphism $\mathscr{Y}^r \simeq \mathscr{X}^{m-r}$. Indeed, we find $Y_L^r \simeq X_{L^\perp}^{m-r}$: and, since $\dim L^\perp + \dim L = \dim(W) = nm$, we get the formula for ω_{Y_L} from Lemma 3.8, recalling that the relative hyperplane section is identified with Q in this case. The other statements follow as in Lemma 3.8. $\square$

We resume in Table 1 the results of this section. The functor mentioned there is the HPD functor.

4. Birational and equivalent linear sections

As explained in Corollary 3.7 and then displayed in Table 1, the condition $c = nr$ guarantees that HPD gives an equivalence of categories. Hence our construction gives examples of derived equivalences of Calabi–Yau manifolds for any $n = m$. One first example was produced in [20]. In fact the authors of [20] take $n = m = 4$, $r = 2$, the self dual orbit of rank 2, 4×4 matrices and consider the codimension eight threefolds obtained by taking orthogonal linear sections in $\mathbb{P}^{15}$. In fact, our construction shows that these two Calabi–Yau are derived equivalent. On the other hand it is very likely that they are one the flop of the other. We can show indeed that X_L and Y_L are birational whenever $c = nr$.

Assume now that $c = nr$. Remark that the two vector bundles appearing in the map M_L of Proposition 3.3 have the same rank, namely nr. Let us denote by D_L the hypersurface in $\mathbb{G}$ defined by the vanishing of determinant of M_L:

$$M_L : L \otimes \mathscr{O}_{\mathbb{G}} \to V \otimes \mathscr{Q}.$$

The degree of D_L is n. Dually, we write D^L the hypersurface in $\mathbb{G}$ whose equation is the determinant of:

$$M^L : L^\perp \otimes \mathscr{O}_{\mathbb{G}} \to V^\vee \otimes \mathscr{U}^\vee.$$

Proposition 4.1. *If $c = nr$ then $D^L = D_L$, and X_L is birational to Y_L.*

Proof. To see this, we write the following exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & V^\vee \otimes \mathscr{Q}^\vee \arrow[r, equal] \arrow[d] & V^\vee \otimes \mathscr{Q}^\vee \arrow[d, "(M_L)^*"] & \\
0 \arrow[r] & L^\perp \otimes \mathscr{O}_{\mathbb{G}} \arrow[r] \arrow[d, equal] & U^\vee \otimes V^\vee \otimes \mathscr{O}_{\mathbb{G}} \arrow[r] \arrow[d] & L^\vee \otimes \mathscr{O}_{\mathbb{G}} \arrow[r] \arrow[d] & 0 \\
 & L^\perp \otimes \mathscr{O}_{\mathbb{G}} \arrow[r, "M^L"] & V^\vee \otimes \mathscr{U}^\vee \arrow[r] \arrow[d] & \mathscr{K} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Here, $\mathscr{K}$ is the cokernel both of M^L and of $(M_L)^*$. This says that:

$$D^L = \mathbb{V}(\det(M^L)) = \mathbb{V}(\det(M_L^\vee)) = \mathbb{V}(\det(M_L)) = D_L.$$

Now let us look at X_L and Y_L. The sheaf $\mathscr{K}$ is supported on $D = D^L$, and is actually of the form $\iota_*(\mathscr{K}_r)$, where $\mathscr{K}_r$ is a reflexive sheaf of rank 1 on D and $\iota : D \to \mathbb{G}$ is the natural embedding. The cokernel of M_L is also of the form $\iota_*(\mathscr{K}^r)$, with $\mathscr{K}^r$ reflexive of rank 1 on D. By Grothendieck duality, since D has degree n, the previous diagram says that $\mathscr{K}^r \simeq \mathscr{K}_r^\vee(n)$. On the (open and dense) locus of D where $\mathscr{K}^r$ and $\mathscr{K}_r$ are locally free, the variety D coincides with X_L and Y_L. Therefore, by Proposition 3.3, these varieties are both birational to D. □

A priori, X_L is not isomorphic to Y_L, as the projectivization of the two sheaves $\mathscr{K}_r$ and $\mathscr{K}_r^\vee$ gives in principle non-isomorphic varieties (cf. Example 4.3 below). This does not happen if $\mathscr{K}_r$ is locally free of rank 1 on D, which in turn is the case if D is smooth. Also, when the singularities of D are isolated points, then in order for $\mathbb{P}(\mathscr{K}_r)$ to be isomorphic to $\mathbb{P}(\mathscr{K}_r^\vee)$, it suffices to check that the rank of $\mathscr{K}_r^\vee$ and $\mathscr{K}_r$ is the same at those points, and this is of course true. Then we have:

Remark 4.2. Suppose that D is smooth or has isolated singularities, then X_L is isomorphic to Y_L.

If we assume that X_L is Calabi–Yau, then $m = n$ and $c = nr$ so we are in a sub-case of our description above, and birationality still holds. Thus, in dimension 3, the derived equivalences would follow also from the work of Bridgeland [14].

Example 4.3. Let us describe an example of two determinantal varieties X_L and Y_L which are derived equivalent, birational, but not isomorphic. Actually one can describe infinitely many examples this way, all of dimension at least 5. In all of them the canonical system is birational onto a hypersurface of general type in $\mathbb{G}$.

Take $(r, m, n) = (3, 5, 7)$, $c = 21$ and consider a general subspace $L \subset W$. Then X_L and Y_L are both smooth projective 5-folds. The Picard group $\mathrm{Pic}(X_L)$ is isomorphic to $\mathbb{Z}^2$, generated by (the restriction of) H_X and P, while $\mathrm{Pic}(Y_L)$ is also isomorphic to $\mathbb{Z}^2$, generated by H_Y and Q. Note that $\omega_{X_L} \simeq \mathscr{O}_{X_L}(2P)$ while $\omega_{Y_L} \simeq \mathscr{O}_{Y_L}(2Q)$.

We claim that X_L and Y_L are not isomorphic. Indeed, if there was an isomorphism $f : X_L \to Y_L$, we should have $f^*(Q) = P$ because of the expression of the canonical bundle. Since $(f^*(Q), f^*(H_{Y_L}))$ should form a $\mathbb{Z}$-basis of $\mathrm{Pic}(X_L)$, we have $f^*(H_{Y_L}) = H_{X_L} + aP$, for some $a \in \mathbb{Z}$. But a straightforward computation shows that $(H_{X_L} + aP)^5$ is never equal to $H^5_{Y_L}$, for any choice of a.

So the 5-folds X_L and Y_L are not isomorphic. They are however derived equivalent via HPD and both birational by projection to a determinantal hypersurface D of degree 7 in $\mathbb{G}(5, 3)$. The canonical bundle of this hypersurface is $\mathscr{O}_D(2)$. The determinantal model of X_L (respectively, of Y_L) is the fivefold of degree 490 (respectively, 1176) cut in $\mathbb{P}^{13}$ (respectively, in $\mathbb{P}^{20}$) by the 4×4 minors (respectively, the 3×3 minors) of a sufficiently general 5×7 matrix of linear forms.

Concerning rationality of determinantal varieties, we have the following result.

Proposition 4.4. *The variety X_L is rational if $nr > c$; Y_L is rational if $c > nr$.*

Proof. By Proposition 3.3, Y_L is the projectivization of the cokernel sheaf of the map M^L. Recall that $\dim(L^\perp) = nm - c$ and that $V^\vee \otimes \mathscr{U}^\vee$ has rank $n(m - r)$.

So if $c > nr$, *i.e.* if $n(m - r) > nm - c$, there is a Zariski dense open subset of $\mathbb{G}(U, r)$ where M^L has constant rank $mn - c$. Hence an open piece of Y_L is the projectivization of a locally free sheaf over a rational variety, so Y_L is rational.

The same argument works for X_L. $\square$

A side remark is that, using N^L instead of M^L we would rationality of Y_L if $c > m(n - m + r)$. However, one immediately proves that $m(n - m + r) \geq nr$.

5. The Segre-determinantal duality

In this section, we give a more detailed description of the case $r = 1$ (we suppress r from our notation for this section). In this case $\mathscr{X} \simeq \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is just a product of two projective spaces and X_L is a linear section of a Segre variety. On the other hand, $\mathscr{Y}$ is the Springer desingularization of the space of degenerate matrices.

For this section and the following ones, we make use of the standard notation $\mathscr{O}_{X_L}(a, b)$ for the restriction to X_L of $\mathscr{O}_{\mathbb{P}^{n-1}}(a) \boxtimes \mathscr{O}_{\mathbb{P}^{m-1}}(b)$, so that $\mathscr{O}_{X_L}(1, 1) = \mathscr{O}_{X_L}(H)$ and $\mathscr{O}_{X_L}(0, 1) = \mathscr{O}_{X_L}(P)$. Proposition 3.3 and Lemma 3.8 become:

Corollary 5.1. *The variety X_L can be described in two following ways:*

(i) *as the projectivization of the cokernel of:*

$$L \otimes \mathscr{O}_{\mathbb{P}U}(-1) \to V \otimes \mathscr{O}_{\mathbb{P}U};$$

(ii) *as the projectivization of the cokernel of:*

$$L \otimes \mathscr{O}_{\mathbb{P}V}(-1) \to U \otimes \mathscr{O}_{\mathbb{P}V}.$$

Also, we have the formulas for the canonical bundle:

$$\omega_{X_L} = \mathscr{O}_{X_L}(c - n, c - m).$$

In particular X_L is Fano if and only if $c < m$, and rational for $c < n$.

Proof. Since $X_L^0 = \emptyset$ the condition for X_L to be Fano descends directly for the formula for the canonical bundle. The statement on rationality is Proposition 4.4. □

The variety Y_L is itself a linear section of a Segre variety, by Proposition 3.3, as the following Lemma shows.

Lemma 5.2. *The variety Y_L is isomorphic to the complete intersection of n hyperplanes in $\mathbb{P}U \times \mathbb{P}^L$ determined by $L \subset W$. So the canonical bundle ω_{Y_L} equals $\mathscr{O}_{Y_L}(n - m, n - c)$. Moreover, for generic $L \subset W$, the determinantal variety Z^L is smooth if and only if $c < 2n - 2m + 5$.*

Proof. The first statement follows from the very last item of Proposition 3.3. Indeed, since $r = 1$, Y_L the projectivization of the sheaf $\mathscr{E}$, restricted to $\mathbb{P}^L$. Therefore, just as in the proof of Proposition 3.3, Y_L is the vanishing locus of the global section of $\mathscr{O}_{\mathbb{P}U \times \mathbb{P}^L}(1, 1)$ determined by the subspace $L^\perp \subset W^\vee$, *i.e.* by $L \subset W$.

Note that $\mathscr{O}_{Y_L}(0, 1) \simeq \mathscr{O}_{Y_L}(H)$ and $\mathscr{O}_{Y_L}(1, 0) = \mathscr{O}_{Y_L}(Q)$. The canonical bundle formula follows by adjunction and agrees with Corollary 3.9.

Table 2
The Segre-determinantal duality.

	$c < m$	$m \leq c < n$	$c = n$	$n < c$
HPD Functor	$\mathrm{D^b}(Y_L) \to \mathrm{D^b}(X_L)$		equivalence	$\mathrm{D^b}(X_L) \to \mathrm{D^b}(Y_L)$
Y_L	Fano visitor		CY if $n = m$	Rational Fano if $n = m$
X_L	Rational Fano	Rational	CY if $n = m$	Fano visitor if $n = m$

The codimension in $\mathbb{P}^L$ of the singular locus of Z^L is $2n - 2m + 4$ for a general choice of $L \subset W$. So Z^L is smooth if and only if $c < 2n - 2m + 5$, which gives the last statement. □

Remark 5.3. Here, since $\mathscr{U}^\vee = T_{\mathbb{P}U}(-1)$. By Proposition 3.3, the variety Y_L can also be described as the projectivization of the cokernel sheaf of

$$L^\perp \otimes \mathscr{O}_{\mathbb{P}U} \to V^\vee \otimes T_{\mathbb{P}U}(-1). \tag{5.1}$$

The map appearing in (5.1) in the remark above, corresponds once again to the choice of $L \subset W$. Dually, for Y_L, Proposition 4.4 gives:

Lemma 5.4. *The variety Y_L is rational if $c > n$.*

Proof. This is just Proposition 4.4. □

Thanks to the constructions of section 4, we obtain the following corollary.

Corollary 5.5. *If $c = n$, then X_L and Y_L are birational $(m-2)$-folds. If $m = n$ they are Calabi–Yau and have nef canonical divisor otherwise.*

We resume the results of this section in Table 2.

6. Fano and rational varieties

6.1. Representability into Fano varieties

In this section, we consider Question 1.1. We start by stating a straightforward consequence of Corollary 3.7 and Lemma 3.8 (see also Table 1), which provides a large class of examples of weakly Fano-visitor (see Definition 2.10) varieties, up to categorical resolutions of singularities.

Proposition 6.1. *Suppose that $n = m$. If $c < rn$, then Y_L^r and $(Z_r^L, \mathscr{R}'_{\mathbb{P}^L})$ are weakly Fano visitors. If $c > nr$, then X_L^r and $(Z_L^r, \mathscr{R}'_{\mathbb{P}_L})$ are weakly Fano visitors.*

If $r = 1$ we have an interpretation of Proposition 6.1 for determinantal varieties.

Corollary 6.2. *Let $Z \subset \mathbb{P}^k$ be a determinantal variety associated with a generic $m \times n$ matrix. If $k < m-1$ then the categorical resolution of singularities of Z is Fano visitor.*

Proof. The determinantal variety Z is $Z^L_{m-1} = \mathscr{Z}^{m-1}_{m,n} \cap \mathbb{P}^L$ for a subspace $L \subset U \otimes V$ of codimension $k+1$. Then we use results from Table 2 and conclude. $\square$

Corollary 6.2 gives a positive answer to Question 1.1 for almost every curve.

Example 6.3 *(Plane curves).* Let $C \subset \mathbb{P}^2$ be a plane curve of degree $d \geq 4$. Then, it is well known (see [4, §3]) that C can be written as the determinant of a $d \times d$ matrix of linear forms. In other words, we put $m = n = d$, $k = 2$ and the inequality of Corollary 6.2 is respected. Hence any plane curve of degree at least four is a Fano-visitor, up to resolution of singularities.

On the other hand, one can check that the blow-up of $\mathbb{P}^3$ along a plane cubic is Fano (see, *e.g.*, [7, Proposition 3.1, (i)]). Hence any plane curve of positive genus is a Fano-visitor.

Example 6.4 *(More curves of general type).* Determinantal varieties with $n \neq m$ provide a wealth of examples of (even non-plane) curves of general type that are Fano-visitors.

Let us make the case where $\dim(Y^1_L) = \dim(Z^L) = 1$ explicit. We have $c = n-m+3$. From Table 2 it is straightforward to see that Y^1_L is an elliptic curve (the Calabi–Yau case) if $m = n = c = 3$; this yields indeed a plane cubic. On the other hand, we see that if $m = 2$ then the curve is rational for any value of n since $c = n+1$, and if $m > 3$ it is forced to be a curve of general type in $\mathbb{P}^{c-1}$, which is Fano visitor if $c < m$.

The dual X_L is a smooth variety of dimension $2m-5$. If $m = 3$, we have that Z_L is an elliptic curve. If $m > 3$, we have $\dim Z_L \geq 3$. This gives quite a lot of examples of space curves of general type that are Fano visitors. Take for example $c = 4$, $n = 6$ and $m = 5$. This gives a curve of genus 4 in $\mathbb{P}^3$, complete intersection of two degree 5 determinantal hypersurfaces, whose derived category is fully faithfully embedded in the derived category of a rational Fano 5-fold in $\mathbb{P}^{25}$.

6.2. Rationality and categorical representability

In this subsection, we consider Question 1.2. The second consequence of Corollary 3.7 is a large class of examples of rational varieties which are categorically representable in codimension at least 2. For simplicity, let us assume that $r = 1$, so that we already discussed in section 5 the rationality of the sections. We state the following Proposition in terms of Segre and determinantal varieties.

Corollary 6.5. *The categorical resolution of a rational determinantal variety is categorically representable in codimension at least 2. A rational linear section of the Segre variety $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \subset \mathbb{P}^{nm-1}$ is categorically representable in codimension at least 2.*

Proof. First we observe that the Segre linear section X_L is rational for $c < n$ and the determinantal linear section Y_L for $c > n$ by Table 2. Then we recall from Corollary 3.7 that, assuming $r = 1$, it is exactly in these ranges that we have the required functors and semiorthogonal decompositions. A computation of the dimensions of the linear sections, following the formulas in section 3.1.2, proves the claim. □

6.3. Categorical resolution of the residual category of a determinantal Fano hypersurface

The Segre-determinantal HPD involves categorical resolutions for determinantal varieties, which is crepant if $n = m$. In this subsection we consider the cases where such resolution gives a crepant categorical resolution for nontrivial components of a semiorthogonal decomposition. For simplicity, we will consider only determinantal *hypersurfaces*, hence we need to assume $r = 1$ and $m = n$. We will drop all the useless indexes.

Let F be a smooth Fano variety such that $\mathrm{Pic}(F) = \mathbb{Z}[\mathscr{O}_F(1)]$. The index of F is the integer i such that $\omega_F = \mathscr{O}_F(-i)$. Kuznetsov observed that this kind of varieties has a Lefschetz-type semiorthogonal decomposition.

Lemma 6.6. *(See [27, Lemma 3.4].) Let F be a smooth Fano variety of index i, then the collection $\mathscr{O}_F(-i+1), \dots, \mathscr{O}_F$ in $\mathrm{D^b}(F)$ is exceptional.*

Corollary 6.7. *(See [27, Corollary 3.5].) For any smooth Fano variety F of Picard rank 1 and index i we have the following semiorthogonal decomposition*

$$\mathrm{D^b}(F) = \langle \mathscr{O}_F(-i+1), \dots, \mathscr{O}_F, \mathbf{T}_F \rangle, \tag{6.1}$$

where $\mathbf{T}_F = \{E \in \mathrm{D^b}(V) | H^\bullet(V, E(-k)) = 0$ for all $0 \le k \le i-1\}$.

The main technical tools used in the proof of Lemma 6.6 are Kodaira vanishing Theorem and Serre duality. Before we proceed, we first need to broaden slightly the class of varieties for which the semiorthogonal decomposition (6.1) holds. In fact, we recall that Kodaira vanishing holds also for varieties with rational singularities (for example, see [32, I, Example 4.3.13]), and the well-known fact that the canonical divisor of a Gorenstein variety is Cartier.

Proposition 6.8. *Let F be a projective Gorenstein variety with rational singularities. Suppose that $\mathrm{Pic}(F) = \mathbb{Z}$, $\mathscr{O}_F(1)$ is its (ample) generator and $K_F = \mathscr{O}_F(-i)$, with $i > 0$. Then there is a semiorthogonal decomposition*

$$\mathrm{D^b}(F) = \langle \mathscr{O}_F(-i+1), \dots, \mathscr{O}_F, \mathbf{T}_F \rangle.$$

This holds in particular if $F \subset \mathbb{P}^k$ is an hypersurface of degree $d < k$ with rational singularities (in which case, $i = k - d$).

Proof. It is straightforward to check that the line bundle $\mathscr{O}_F(i)$ is exceptional for any i. To show the semiorthogonality, we use a vanishing theorem for varieties with rational singularities (see [32, I, Example 4.3.13]), which states that

$$\mathrm{Ext}^j(\mathscr{O}_F(s), \mathscr{O}_F(t)) \simeq \mathrm{Ext}^j(\mathscr{O}_F, \mathscr{O}_F(t-s)) \simeq H^j(F, \mathscr{O}_F(t-s))$$

vanishes for $j < \dim(F)$, and $s > t$. Thanks to Serre duality

$$\mathrm{Ext}^{\dim(F)}(\mathscr{O}_F(s), \mathscr{O}_F(t)) \simeq H^{\dim(F)}(F, \mathscr{O}_F(t-s)) \simeq H^0(F, \mathscr{O}_F(s+i-t))$$

and the latter group vanishes if $s + i - t < 0$. □

Homological Projective Duality allows us to describe a resolution of singularities of $\mathbf{T}_F$ in the case where F is determinantal. This means that we consider $Z^L \subset \mathbb{P}^L$ for some integers $m = n$ and for some linear subspace $L \subset U \otimes V$ of Fano type (that is, of degree $d < k+1$). The Springer resolution of Z^L is then Y_L and the dual section of the Segre variety is X_L. Let us fix L, and drop it from the notations from now on. We want to describe a categorical resolution of the category $\mathbf{T}_Z$ described in Proposition 6.8.

We constructed a crepant categorical resolution of singularities $\mathrm{D^b}(Z, \mathscr{R}')$ of Z. The category $\mathrm{D^b}(Z, \mathscr{R}')$ is equivalent to $\mathrm{D^b}(Y)$, for Y the corresponding fibre product of the linear section of the Springer resolution (see Theorem 3.6). In particular, Y is a (the fibre product over a) linear section of a projective bundle over $\mathbb{P}^{d-1}$, since $d = n = m$ is the degree of Z. Let us denote by X the dual linear section of the Segre variety (notice in fact that X is smooth). Numerical computations provide a semiorthogonal decomposition

$$\mathrm{D^b}(Z, \mathscr{R}') \simeq \mathrm{D^b}(Y) = \langle k-d+1 \text{ copies of } \mathrm{D^b}(\mathbb{P}^{d-1}), \mathrm{D^b}(X)\rangle.$$

Hence $\mathrm{D^b}(Z, \mathscr{R}')$ is generated by $d(k-d+1)$ exceptional objects and $\mathrm{D^b}(X)$.

More precisely, the j-th occurrence of $\mathrm{D^b}(\mathbb{P}^{d-1})$ can be generated by the exceptional sequence $(\mathscr{O}_Y(j,1), \ldots, \mathscr{O}_Y(j,d))$, where we use the same notation $\mathscr{O}_Y(a,b)$ as in Section 5, *i.e.* $\mathscr{O}_{Y_L}(0,1) \simeq \mathscr{O}_{Y_L}(H)$ and $\mathscr{O}_{Y_L}(1,0) = \mathscr{O}_{Y_L}(Q)$.

This allows one to calculate a categorical resolution of singularities of $\mathbf{T}_Z$ which is decomposed into $\mathrm{D^b}(X)$ and exceptional objects.

Proposition 6.9. *Let Z be a Fano determinantal hypersurface of $\mathbb{P}^k$, and X the dual section of the Segre variety. There is a strongly crepant categorical resolution $\widetilde{\mathbf{T}}_Z$ of $\mathbf{T}_Z$, admitting a semiorthogonal decomposition by $\mathrm{D^b}(X)$ and $(d-1)(k-d+1)$ exceptional objects.*

Proof. Consider the resolution $p : Y \to Z$, and denote by D its exceptional divisor. We have proved that $\mathrm{D^b}(Y) \simeq \mathrm{D^b}(Z, \mathscr{R}')$ is a categorical resolution of singularities of $\mathrm{D^b}(Z)$. In particular (see [25]), this comes equipped with a functor $p^\vee : \mathrm{Perf}(Z) \to \mathrm{D^b}(X)$ admitting a right adjoint. Indeed, according to [25], to get such a pair for a variety M

with rational singularities, one needs to consider a desingularization $q : N \to M$ with exceptional divisor E, such that $\mathrm{D}^{\mathrm{b}}(E)$ admits a Lefschetz decomposition with respect to the conormal bundle. In our case, we can just consider the Lefschetz decomposition with one component $\mathbf{B}_0 = \mathrm{D}^{\mathrm{b}}(D)$. Now we will check that all the hypotheses of [25, Theorem 1] for the existence of such a categorical resolution are satisfied by the category generated by $\mathrm{D}^{\mathrm{b}}(X)$ and the exceptional objects. So, in order to get a categorical resolution of singularities for $\mathbf{T}_Z$, let us consider the functor p^* introduced above and its action on the semiorthogonal decomposition from Proposition 6.8.

Let $\mathbb{P}^L \simeq \mathbb{P}^k$. There is a commutative diagram:

$$\begin{tikzcd} Y \arrow[d, "p"] \arrow[dr, "f"] & \\ Z \arrow[r, "g"] & \mathbb{P}^k, \end{tikzcd}$$

where the map f is given by the restriction linear system $|\mathscr{O}_Y(1,1)|$, and the map g is defined by $|\mathscr{O}_Z(1)|$. It follows that $p^*\mathscr{O}_Z(k) = \mathscr{O}_Y(k,k)$, so that the exceptional sequence $\mathscr{O}_Z(-k+d), \dots, \mathscr{O}_Z$ pulls back to the exceptional sequence $\mathscr{O}_Y(-k+d,-k+d), \dots, \mathscr{O}_Y$.

Now recall that $\mathscr{Y}^1_{d,d}$ is a projective bundle $s : \mathscr{Y}^1_{d,d} \simeq \mathbb{P}(V^\vee \otimes T_{\mathbb{P}(U)}(-1)) \to \mathbb{P}U$. The Lefschetz decomposition of $\mathrm{D}^{\mathrm{b}}(\mathscr{Y}^1_{d,d})$ giving the HP-duality of Theorem 3.5 is:

$$\mathrm{D}^{\mathrm{b}}(\mathscr{Y}^1_{d,d}) = \langle \mathbf{A}_{-j} \otimes \mathscr{O}_{\mathbb{P}(V\otimes\mathscr{Q})}(-j), \dots, \mathbf{A}_0 \rangle,$$

with $-j = 1-d^2+d$, where $\mathbf{A}_0 = \ldots = \mathbf{A}_j = s^*\mathrm{D}^{\mathrm{b}}(\mathbb{P}U)$. In particular, we can choose, for each occurrence of $s^*\mathrm{D}^{\mathrm{b}}(\mathbb{P}U)$, an appropriate exceptional collection generating $\mathrm{D}^{\mathrm{b}}(\mathbb{P}U)$ in order to get, after taking the linear sections (recall that $Y := Y^1_L$, and $X := X^1_L$):

$$\begin{aligned} \mathrm{D}^{\mathrm{b}}(Y) = \langle & \mathscr{O}_Y(-k+d,-k+d), \dots, \mathscr{O}_Y(-k+d,-k+2d-1), \\ & \mathscr{O}_Y(-k+d+1,-k+d+1), \dots, \mathscr{O}_Y(-k+d+1,-k+2d), \\ & \dots \\ & \mathscr{O}_Y(0,0), \dots, \mathscr{O}_Y(0,d-1), \mathrm{D}^{\mathrm{b}}(X) \rangle. \end{aligned}$$

Now we can mutate all the exceptional objects which are not of the form $\mathscr{O}_Y(-t,-t)$, for some t, to the right until we get

$$\begin{aligned} \mathrm{D}^{\mathrm{b}}(Y) = \langle & \mathscr{O}_Y(-k+d,-k+d), \dots, \mathscr{O}_Y(-1,-1), \mathscr{O}_Y, \\ & E_1, \dots, E_{(d-1)(k-d+1)}, \mathrm{D}^{\mathrm{b}}(X) \rangle, \end{aligned}$$

where the E_i are the exceptional objects resulting from the mutations. Hence, the first block is the pull-back from Z of the exceptional sequence $(\mathscr{O}_Z(-k+d), \dots, \mathscr{O}_Z)$, then by definition we get that the second block is the categorical resolution of singularities for $\mathbf{T}_Z$. $\square$

Remark 6.10. A particular and interesting case is given by determinantal cubics in $\mathbb{P}^4$ and $\mathbb{P}^5$. In both cases, the dual linear section X is empty. So, the numeric values give explicitly:

- If Z is a determinantal cubic threefold, then the category $\mathbf{T}_Z$ admits a crepant categorical resolution of singularities generated by 4 exceptional objects.
- If Z is a determinantal cubic fourfold, then the category $\mathbf{T}_Z$ admits a crepant categorical resolution of singularities generated by 6 exceptional objects.

In the case of cubic threefolds and fourfolds with only one node, categorical resolution of singularities of $\mathbf{T}_Z$ is described (see resp. [5] and [28]). One should expect that these geometric descriptions carry over to the more degenerate case of determinantal cubics – which are all singular. We haven't developed the (very long) calculations, but nevertheless we outline expectations about the geometrical nature of these categorical resolutions.

In the 3-dimensional case, the 4 exceptional objects should correspond to a disjoint union of two rational curves, arising as the geometrical resolution of singularities of the discriminant locus of a projection $Z \to \mathbb{P}^3$ from one of the six singular points. This discriminant locus is composed by two twisted cubics intersecting in five points, and turns out to be a degeneration of the $(3, 2)$ complete intersection curve appearing in the one-node case (see [5, Proposition 4.6]).

In the 4-dimensional case, the 6 exceptional objects should correspond to a disjoint union of two Veronese-embedded planes (isomorphically projected to $\mathbb{P}^4$), arising as the geometrical resolution of singularities of the discriminant locus of a projection $Z \to \mathbb{P}^4$ from one of the singular points. This discriminant locus is composed by two cubic scrolls intersecting along a quintic elliptic curve, and turns out to be a degeneration of the degree 6 K3 surface appearing in the one-node case (see [28, §5]).

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