

Cycles, derived categories, and rationality

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ABSTRACT. Our main goal is to give a sense of recent developments in the (stable) rationality problem from the point of view of unramified cohomology and 0-cycles as well as derived categories and semiorthogonal decompositions, and how these perspectives intertwine and reflect each other. In particular, in the case of algebraic surfaces, we explain the relationship between Bloch's conjecture, Chow-theoretic decompositions of the diagonal, categorical representability, and the existence of phantom subcategories of the derived category.

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In this text, we explore two potential measures of rationality. The first is the universal triviality of the Chow group of 0-cycles, which is related to the Chow-theoretic decomposition of the diagonal of Bloch and Srinivas. A powerful degeneration method for obstructing the universal triviality of the Chow group of 0-cycles, initiated by Voisin, and developed by Colliot-Thélène and Pirutka, combines techniques from singularity theory and unramified cohomology and has led to a recent breakthrough in the stable rationality problem.

The second is categorical representability, which is defined by the existence of semiorthogonal decompositions of the derived category into components whose dimensions can be bounded. We will give a precise definition of this notion and

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present many examples, as well as motivate why one should expect categorical representability in codimension 2 for rational varieties.

Furthermore, we would like to explore how the Chow-theoretic and derived categorical measures of rationality can contrast and reflect each other. One of the motivating topics in this circle of ideas is the relationship, for complex surfaces, between Bloch's conjecture, the universal triviality of the Chow group of 0-cycles, and the existence of phantoms in the derived categories. Another motivating topic is the rationality problem for cubic fourfolds and its connections between derived categories, Hodge theory, as well as Voisin's recent results on the universal triviality of the Chow group of 0-cycles for certain loci of special cubic fourfolds.

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1. Preliminaries on Chow groups

Let k be an arbitrary field. By scheme, we will mean a separated k -scheme of finite type. In this section, we give a quick introduction to the Chow group of algebraic cycles on a scheme up to rational equivalence. See Fulton's book [84] for more details.

Denote by $Z_i(X)$ the free \mathbb{Z} -module generated by all i -dimensional closed integral subschemes of X . The elements of Z_i are called *algebraic i -cycles*. We will also employ the codimension notation $Z^i(X) = Z_{n-i}(X)$ when X is smooth of pure dimension n . The *support* of an i -cycle $\sum_n a_n[V_n]$ is the union of the closed subschemes V_i in X ; it is *effective* if $a_n > 0$ for all n .

Given an $(i + 1)$ -dimensional closed integral subscheme W of X , and a closed integral subscheme $V \subset W$ of codimension 1, we denote by $\mathcal{O}_{W,V}$ the local ring of W at the generic point of V ; it is a local domain of dimension 1 whose field of fractions is the function field $k(W)$. For a nonzero function $f \in \mathcal{O}_{W,V}$, we define the *order* of vanishing $\text{ord}_V(f)$ of f along V to be the length of the $\mathcal{O}_{W,V}$ -module $\mathcal{O}_{W,V}/(f)$. The order extends uniquely to a homomorphism $\text{ord}_V : k(W)^\times \rightarrow \mathbb{Z}$. If W is normal, when ord_V coincides with the usual discrete valuation on $k(W)$ associated to V . We also define the *divisor* of a rational function $f \in k(W)^\times$ as an i -cycle on X given by

$$[\text{div}(f)] = \sum_{V \subset W} \text{ord}_V(f)[V],$$

where the sum is taken over all closed integral subschemes $V \subset W$ of codimension 1. An i -cycle z on X is *rationally equivalent* to 0 if there exists a finite number of closed integral $(i + 1)$ -dimensional subschemes $W_j \subset X$ and rational functions

$f_j \in K(W_j)^\times$ such that $z = \sum_j [\text{div}(f_j)]$ in $Z_i(X)$. Note that the set of i -cycles rationally equivalent to 0 forms a subgroup of $Z_i(X)$ since $[\text{div}(f^{-1})] = -[\text{div}(f)]$ for any rational function $f \in K(W)^\times$. Denote the associated equivalence relation on $Z_i(X)$ by \sim_{rat} . Define the *Chow group* of i -cycles on X to be the quotient group $\text{CH}_i(X) = Z_i(X)/\sim_{\text{rat}}$ of algebraic i -cycles modulo rational equivalence.

1.1. Morphisms. Let $f : X \rightarrow Y$ be a proper morphism of schemes. Define a *push-forward* map on cycles $f_* : Z_i(X) \rightarrow Z_i(Y)$ additively as follows. For a closed integral subscheme $V \subset X$ define

$$f_*([V]) = \begin{cases} 0 & \text{if } \dim(f(V)) < \dim(V) \\ \deg(V/f(V)) [f(V)] & \text{if } \dim(f(V)) = \dim(V) \end{cases}$$

where $\deg(V/f(V))$ denotes the degree of the finite extension of function fields $k(V)/k(f(V))$ determined by f . This map respects rational equivalence and hence induces a push-forward map on Chow groups $f_* : \text{CH}_i(X) \rightarrow \text{CH}_i(Y)$.

Let $f : X \rightarrow Y$ be a flat morphism of relative dimension r . Define a *pull-back* map $f^* : Z_i(Y) \rightarrow Z_{i+r}(X)$ additively as follows. For a closed integral subscheme $V \subset Y$ define

$$f^*([V]) = [f^{-1}(V)].$$

This map respects rational equivalence and hence induces a pull-back map on Chow groups $f^* : \text{CH}_i(Y) \rightarrow \text{CH}_{i+r}(X)$.

A special case of proper push forward is given by considering a closed immersion $\iota : Z \rightarrow X$. Letting $j : U \rightarrow X$ be the open complement of Z , we note that j is flat. There is an exact excision sequence

$$\text{CH}_i(Z) \xrightarrow{\iota_*} \text{CH}_i(X) \xrightarrow{j^*} \text{CH}_i(U) \rightarrow 0$$

which comes from an analogous exact sequence on the level of cycles.

Moreover, we have the following compatibility between the proper push-forward and flat pull-back. Given a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

where g is flat of relative dimension r and f is proper, then $g^* f_* = f'_* g'^*$ as maps $\text{CH}_i(X) \rightarrow \text{CH}_{i+r}(Y')$.

A third natural map between Chow groups is the *Gysin map*. Given a regular closed embedding $\iota : X \rightarrow Y$ of codimension r , one can define a natural map $\iota^! : \text{CH}_i(Y) \rightarrow \text{CH}_{i-r}(X)$. The precise definition of the Gysin map is quite involved, see [84, §5.2; 6.2], and in particular, is not induced from a map on cycles. In particular, this map factors through $\text{CH}_i(N)$, where $s : N \rightarrow Y$ is the normal cone of X in Y , and can be described by the composition of a Gysin map $s^!$ for vector bundles and the inverse of the pull back f^* by $f : N \rightarrow X$.

The Gysin map allows one to write the *excess intersection formula* for a regular closed embedding $\iota : X \rightarrow Y$ of codimension r

$$\iota^! \iota_*(\alpha) = c_r(h^* N_{X/Y}) \cap \alpha,$$

for any cycle α in $Z_*(Y)$, where $h : N_{X/Y} \rightarrow X$ is the normal bundle of X in Y and c_r denotes the r -th Chern class. In particular, this shows that $\iota^! \iota_* = 0$ whenever $N_{X/Y}$ is trivial.

We now define the Gysin map for any local complete intersection (lci) morphism $f : X \rightarrow Y$. Consider the factorization of f as

$$X \xrightarrow{\iota} P \xrightarrow{h} Y,$$

where ι is a regular embedding of codimension r and h is smooth of relative dimension m . Then we can define the Gysin map

$$f^! = \iota^! \circ h^* : \mathrm{CH}_i(Y) \xrightarrow{h^*} \mathrm{CH}_{i-m}(P) \xrightarrow{\iota^!} \mathrm{CH}_{i-m-r}(X).$$

Such a map is independent on the chosen factorization of f and coincides with the flat pull-back f^* whenever f is flat. As a relevant example, any morphism $f : X \rightarrow Y$ between smooth k -schemes is lci (indeed, f factors $X \rightarrow X \times Y \rightarrow Y$ into the regular graph embedding followed by the smooth projection morphism), hence induces a Gysin map $f^! : \mathrm{CH}^i(Y) \rightarrow \mathrm{CH}^i(X)$. Finally, even when f is not flat, we often denote $f^* = f^!$, so that f^* is defined for any lci morphism.

1.2. Intersections. Let X be a smooth k -scheme of pure dimension n . Then the Chow group admits an *intersection product* as follows. For closed integral subschemes $V \subset X$ and $W \subset X$ of codimension i and j , respectively, define

$$[V].[W] = \Delta^! [V \times W] \in \mathrm{CH}^{i+j}(X)$$

where $\Delta : X \rightarrow X \times X$ is the diagonal morphism, a regular embedding of codimension n . This induces a bilinear map $\mathrm{CH}^i(X) \times \mathrm{CH}^j(X) \rightarrow \mathrm{CH}^{i+j}(X)$, which makes $\mathrm{CH}(X) = \bigoplus_{i \geq 0} \mathrm{CH}^i(X)$ into a commutative graded ring with identity the class $[X] \in \mathrm{CH}^0(X)$. Gysin maps between smooth k -schemes are then ring homomorphisms for the intersection product.

One can understand the intersection product in terms of literal intersections of subschemes, via moving lemmas, see [84, §11.4] for a discussion of the technicalities involved. Assuming that X is smooth and quasi-projective, given closed integral subschemes $V \subset X$ and $W \subset X$ of codimension i and j , respectively, the moving lemma says that one can replace V by a rationally equivalent cycle $V' = \sum_l a_l [V_l]$ so that V' and W meet properly, i.e., $V_l \cap W$ has codimension $i+j$ for all l . Then one can define $[V].[W] = \sum_l a_l [V_l \cap W]$. A more refined moving lemma is then required to show that the rational equivalence class of this product is independent of the choice of cycle V' .

One easy moving lemma that we will need to use often is the moving lemma for 0-cycles: given a smooth quasi-projective k -scheme X , an open dense subscheme $U \subset X$, and a 0-cycle z on X , there exists a 0-cycle z' on X rationally equivalent to z such that the support of z' is contained in U . See, e.g., [85, §2.3], [62, p. 599] for a reference to the classical moving lemma, which implies this.

1.3. Correspondences. Let X and Y be smooth k -schemes of pure dimension n and m respectively. We recall some notions from [84, §16.1].

DEFINITION 1.3.1. A *correspondence* from X to Y is an element $\alpha \in \mathrm{CH}(X \times Y)$. The same element α , seen in $\mathrm{CH}(Y \times X)$ is called the transpose correspondence α' from Y to X .

Now assume that Y is proper over k and let Z be a smooth equidimensional k -scheme. If $\alpha \in \text{CH}(X \times Y)$ and $\beta \in \text{CH}(Y \times Z)$ are correspondences, we define the composed correspondence

$$\beta \circ \alpha = p_{X \times Z*}(p_{X \times Y}^*(\alpha) \cdot p_{Y \times Z}^*(\beta)) \in \text{CH}(X \times Z),$$

where p_\bullet denotes the projection from $X \times Y \times Z$ to \bullet , and where we use the intersection product on $\text{CH}(X \times Y \times Z)$. Taking $X = Y = Z$ smooth and proper, the operation of composition of correspondences makes $\text{CH}(X \times X)$ into an associative ring with unit $[\Delta_X]$.

Correspondences between X and Y naturally give rise to morphisms between their Chow groups as follows. If $\alpha \in \text{CH}^{m+i}(X \times Y)$ is a correspondence from X to Y , then we define

$$\begin{aligned} \alpha_* : \text{CH}_j(X) &\longrightarrow \text{CH}_{j-i}(Y) & \alpha^* : \text{CH}^j(Y) &\longrightarrow \text{CH}^{j+i}(X) \\ z &\longmapsto q_*(p^*(z) \cdot \alpha) & z &\longmapsto p_*(q^*(z) \cdot \alpha) \end{aligned}$$

where p and q denote the projections from $X \times Y$ to X and Y , respectively. If β is a correspondence from Y to Z , then we have $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$ and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$. An important special case are correspondences $\alpha \in \text{CH}^n(X \times X)$, which define maps $\alpha_* : \text{CH}_i(X) \rightarrow \text{CH}_i(X)$ and $\alpha^* : \text{CH}^i(X) \rightarrow \text{CH}^i(X)$. In particular, the map

$$\text{CH}^n(X \times X) \rightarrow \text{End}_{\mathbb{Z}}(\text{CH}_i(X))$$

is a ring homomorphism (see [84, Cor. 16.1.2]).

As an example, letting $f : X \rightarrow Y$ be a morphism with graph $\Gamma_f \subset X \times Y$, we can consider $\alpha = [\Gamma_f] \in \text{CH}^m(X \times Y)$ as a correspondence from X to Y , and then $\alpha_* = f_*$ and $\alpha^* = f^*$.

1.4. Specialization. Most of the previous intersection theoretic considerations carry over to a more general relative setting, replacing the base field k with a regular base scheme S .

Let X be a scheme that is separated and finite type over S . For a closed integral subscheme $V \subset X$, we define the *relative dimension*

$$\dim_S(V) = \text{tr.deg}(K(V)/K(W)) - \text{codim}_S(W),$$

where W is the closure of the image of V in S . A relative i -cycle on X/S is an integer linear combination of integral subschemes of X of relative dimension i . The notion of rational equivalence of relative i -cycles is as before and we denote by $\text{CH}_i(X/S)$ the group of relative i -cycles on X/S up to rational equivalence. As before, there are push-forwards for proper S -morphisms, pull-backs for flat S -morphisms, and Gysin maps for lci S -morphisms.

Now suppose that $\iota : \overline{S} \rightarrow S$ is a regular embedding of codimension r , and let $j : S^0 \rightarrow S$ be the complement of \overline{S} . Consider the following diagram of cartesian squares:

$$\begin{array}{ccccc} \overline{X} & \longrightarrow & X & \longleftarrow & X^0 \\ \downarrow & & \downarrow & & \downarrow \\ \overline{S} & \xrightarrow{\iota} & S & \xleftarrow{j} & S^0 \end{array}$$

Noting that $\mathrm{CH}_i(\overline{X}/\overline{S}) = \mathrm{CH}_{i-r}(\overline{X}/S)$ and $\mathrm{CH}_i(X^0/S^0) = \mathrm{CH}_i(X^0/S)$, then the Gysin map $\iota^! : \mathrm{CH}_i(X/S) \rightarrow \mathrm{CH}_{i-r}(\overline{X}/S) = \mathrm{CH}_i(\overline{X}/\overline{S})$ gives rise to a diagram

$$\begin{array}{ccccc} \mathrm{CH}_i(\overline{X}/\overline{S}) & \xrightarrow{\iota_*} & \mathrm{CH}_i(X/S) & \xrightarrow{j^*} & \mathrm{CH}_i(X^0/S^0) & \longrightarrow & 0 \\ \parallel & & \downarrow \iota^! & & \parallel & & \\ \mathrm{CH}_{i+r}(\overline{X}/\overline{S}) & & \mathrm{CH}_i(\overline{X}/\overline{S}) & \xleftarrow{\sigma} & \mathrm{CH}_i(X^0/S^0) & & \end{array}$$

where the top row is the relative short exact excision sequence. We see that the obstruction to defining a well-defined map *specialization map*

$$\sigma : \mathrm{CH}_i(X^0/S^0) \rightarrow \mathrm{CH}_i(\overline{X}/\overline{S})$$

fitting into the diagram is precisely the image of $\iota^! \iota_*$. By the excess intersection formula, if $N_{\overline{S}/S}$ is trivial, then $\iota^! \iota_* = 0$, in which case we arrive at a well-defined specialization map. When it exists, the specialization map is compatible with push-forwards and pull-backs.

An important special case is when $S = \mathrm{Spec}(R)$ for a discrete valuation ring R , so that $S^0 = \mathrm{Spec}(K)$ and $\overline{S} = \mathrm{Spec}(k)$, where k and K denote the residue and the fraction field of R , respectively. Given a separated R -scheme X of finite type, the k -scheme $\overline{X} = X_k$ is the special fiber, the K -scheme $X^0 = X_K$ is the generic fiber, and we arrive at specialization maps $\sigma : \mathrm{CH}_i(X_K) \rightarrow \mathrm{CH}_i(X_k)$. For more details, we refer to [84, §20.3].

2. Preliminaries on semiorthogonal decompositions

Let k be an arbitrary field. We present here the basic notions of semiorthogonal decompositions and exceptional objects for k -linear triangulated categories, bearing in mind our main application, the derived category of a projective k -variety. We refer to [104, Ch. 1, 2, 3] for an introduction to derived categories aimed at algebraic geometers. In particular, we will assume the reader to be familiar with the notions of triangulated and derived categories, and basic homological algebra as well as complexes of coherent sheaves on schemes.

However, a disclaimer here is necessary. The appropriate structure to consider to study derived categories of smooth projective varieties is the structure of k -linear differential graded (dg) category; that is, a category enriched over dg complexes of k -vector spaces (see [111] for definitions and main properties). In this perspective, morphisms between two objects in the triangulated structure can be seen as the zeroth cohomology of the complex of morphisms between the same objects in the dg structure. Considering the dg structure is natural under many point of views: above all, all categories we will consider can be endowed with a canonical dg structure (see [134]), in such a way that dg functors will correspond to Fourier–Mukai functors (see [165]). Moreover, the dg structure allows to define noncommutative motives, which give a motivic framework to semiorthogonal decompositions. Even if related to some of our considerations, we will not treat noncommutative motives in this report. The interested reader can consult [163].

2.1. Semiorthogonal decompositions and their mutations. Let T be a k -linear triangulated category. A full triangulated subcategory A of T is called *admissible* if the embedding functor admits a left and a right adjoint.

DEFINITION 2.1.1 ([45]). A *semiorthogonal decomposition* of \mathbb{T} is a sequence of admissible subcategories $\mathbf{A}_1, \dots, \mathbf{A}_n$ of \mathbb{T} such that

- $\mathrm{Hom}_{\mathbb{T}}(A_i, A_j) = 0$ for all $i > j$ and any A_i in \mathbf{A}_i and A_j in \mathbf{A}_j ;
- for every object T of \mathbb{T} , there is a chain of morphisms

$$0 = T_n \rightarrow T_{n-1} \rightarrow \dots \rightarrow T_1 \rightarrow T_0 = T$$

such that the cone of $T_k \rightarrow T_{k-1}$ is an object of \mathbf{A}_k for all $k = 1, \dots, n$.

Such a decomposition will be written

$$\mathbb{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle.$$

If $\mathbf{A} \subset \mathbb{T}$ is admissible, we have two semiorthogonal decompositions

$$\mathbb{T} = \langle \mathbf{A}^\perp, \mathbf{A} \rangle = \langle \mathbf{A}, {}^\perp \mathbf{A} \rangle,$$

where \mathbf{A}^\perp and ${}^\perp \mathbf{A}$ are, respectively, the left and right orthogonal of \mathbf{A} in \mathbb{T} (see, for example, [45, §3]).

Given a semiorthogonal decomposition $\mathbb{T} = \langle \mathbf{A}, \mathbf{B} \rangle$, Bondal [44, §3] defines left and right mutations $L_{\mathbf{A}}(\mathbf{B})$ and $R_{\mathbf{B}}(\mathbf{A})$ of this pair. In particular, there are equivalences $L_{\mathbf{A}}(\mathbf{B}) \simeq \mathbf{B}$ and $R_{\mathbf{B}}(\mathbf{A}) \simeq \mathbf{A}$, and semiorthogonal decompositions

$$\mathbb{T} = \langle L_{\mathbf{A}}(\mathbf{B}), \mathbf{A} \rangle, \quad \mathbb{T} = \langle \mathbf{B}, R_{\mathbf{B}}(\mathbf{A}) \rangle.$$

We refrain from giving an explicit definition for the mutation functors in general, which can be found in [44, §3]. In §2.2 we will give an explicit formula in the case where \mathbf{A} and \mathbf{B} are generated by exceptional objects.

2.2. Exceptional objects. Very special examples of admissible subcategories, semiorthogonal decompositions, and their mutations are provided by the theory of exceptional objects and collections. The theory of exceptional objects and semiorthogonal decompositions in the case where k is algebraically closed and of characteristic zero was studied in the Rudakov seminar at the end of the 80s, and developed by Rudakov, Gorodentsev, Bondal, Kapranov, Kuleshov, and Orlov among others, see [91], [44], [45], [48], and [154]. As noted in [12], most fundamental properties persist over any base field k .

Let \mathbb{T} be a k -linear triangulated category. The triangulated category $\langle \{E_i\}_{i \in I} \rangle$ *generated* by a class of objects $\{E_i\}_{i \in I}$ of \mathbb{T} is the smallest thick (that is, closed under direct summands) full triangulated subcategory of \mathbb{T} containing the class. We will write $\mathrm{Ext}_{\mathbb{T}}^r(E, F) = \mathrm{Hom}_{\mathbb{T}}(E, F[r])$.

DEFINITION 2.2.1. Let A be a division (not necessarily central) k -algebra (e.g., A could be a field extension of k). An object E of \mathbb{T} is called *A-exceptional* if

$$\mathrm{Hom}_{\mathbb{T}}(E, E) = A \quad \text{and} \quad \mathrm{Ext}_{\mathbb{T}}^r(E, E) = 0 \quad \text{for } r \neq 0.$$

An exceptional object in the classical sense [90, Def. 3.2] of the term is a k -exceptional object. By *exceptional* object, we mean *A-exceptional* for some division k -algebra A .

A totally ordered set $\{E_1, \dots, E_n\}$ of exceptional objects is called an *exceptional collection* if $\mathrm{Ext}_{\mathbb{T}}^r(E_j, E_i) = 0$ for all integers r whenever $j > i$. An exceptional collection is *full* if it generates \mathbb{T} , equivalently, if for an object W of \mathbb{T} , the vanishing $\mathrm{Ext}_{\mathbb{T}}^r(E_i, W) = 0$ for all $i = 1, \dots, n$ and all integers r implies $W = 0$. An exceptional collection is *strong* if $\mathrm{Ext}_{\mathbb{T}}^r(E_i, E_j) = 0$ whenever $r \neq 0$.

Exceptional collections provide examples of semiorthogonal decompositions if \mathbb{T} is the bounded derived category of a smooth projective scheme.

PROPOSITION 2.2.2 ([44, Thm. 3.2]). *Let $\{E_1, \dots, E_n\}$ be an exceptional collection on the bounded derived category $\mathbb{D}^b(X)$ of a smooth projective k -scheme X . Then there is a semiorthogonal decomposition*

$$\mathbb{D}^b(X) = \langle \mathbf{A}, E_1, \dots, E_n \rangle,$$

where $\mathbf{A} = \langle E_1, \dots, E_n \rangle^\perp$ is the full subcategory of those objects W such that $\mathrm{Ext}_{\mathbb{T}}^r(E_i, W) = 0$ for all $i = 1, \dots, n$ and all integers r . In particular, the sequence is full if and only if $\mathbf{A} = 0$.

Given an exceptional pair $\{E_1, E_2\}$ with E_i being A_i -exceptional, consider the admissible subcategories $\langle E_i \rangle$, forming a semiorthogonal pair. We can hence perform right and left mutations, which provide equivalent admissible subcategories.

Recall that mutations provide equivalent admissible subcategories and flip the semiorthogonality condition. It easily follows from the definition that the object $R_{E_2}(E_1)$ is A_1 -exceptional, the object $L_{E_1}(E_2)$ is A_2 -exceptional, and the pairs $\{L_{E_1}(E_2), E_1\}$ and $\{E_2, R_{E_2}(E_1)\}$ are exceptional. We call $R_{E_2}(E_1)$ the *right mutation* of E_1 through E_2 and $L_{E_1}(E_2)$ the *left mutation* of E_2 through E_1 .

In the case of k -exceptional objects, mutations can be explicitly computed.

DEFINITION 2.2.3 ([90, §3.4]). Given a k -exceptional pair $\{E_1, E_2\}$ in \mathbb{T} , the *left mutation* of E_2 with respect to E_1 is the object $L_{E_1}(E_2)$ defined by the distinguished triangle:

$$(2.1) \quad \mathrm{Hom}_{\mathbb{T}}(E_1, E_2) \otimes E_1 \xrightarrow{ev} E_2 \longrightarrow L_{E_1}(E_2),$$

where ev is the canonical evaluation morphism. The *right mutation* of E_1 with respect to E_2 is the object $R_{E_2}(E_1)$ defined by the distinguished triangle:

$$R_{E_2}(E_1) \longrightarrow E_1 \xrightarrow{coev} \mathrm{Hom}_{\mathbb{T}}(E_1, E_2) \otimes E_2,$$

where $coev$ is the canonical coevaluation morphism.

Given an exceptional collection $\{E_1, \dots, E_n\}$, one can consider any exceptional pair $\{E_i, E_{i+1}\}$ and perform either right or left mutation to get a new exceptional collection.

Exceptional collections provide an algebraic description of admissible subcategories of \mathbb{T} . Indeed, if E is an A -exceptional object in \mathbb{T} , the triangulated subcategory $\langle E \rangle \subset \mathbb{T}$ is equivalent to $\mathbb{D}^b(k, A)$. The equivalence $\mathbb{D}^b(k, A) \rightarrow \langle E \rangle$ is obtained by sending the complex A concentrated in degree 0 to E . The right adjoint functor is the morphism functor $\mathbf{R}\mathrm{Hom}(-, E)$.

We conclude this section by considering a weaker notion of exceptionality, which depends only on the numerical class and on the bilinear form χ .

DEFINITION 2.2.4. Let X be a smooth projective variety. A *numerically exceptional collection* is a collection E_1, \dots, E_n of exceptional objects in the derived category $\mathbb{D}^b(X)$ such that $\chi(E_i, E_j) = 0$ for $i > j$ and $\chi(E_i, E_i) = 1$ for all $i = 1, \dots, n$.

REMARK 2.2.5. It is clear that any exceptional collection is a numerically exceptional collection, while the converse need not to be true.

2.3. How to construct semiorthogonal decompositions? Examples and subtleties. Given a variety X is quite difficult to describe semiorthogonal decompositions of X . Moreover, the geometry of X plays a very important rôle in understanding whether the category $\mathrm{D}^b(X)$ has semiorthogonal decompositions and in describing semiorthogonal sets of admissible subcategories. In general, the most difficult task is to show that such sets form a generating system for the whole category.¹

The main motivation for the study of birational geometry via semiorthogonal decompositions is the following famous theorem by Orlov [146].

THEOREM 2.3.1 (Orlov). *Let X be a smooth projective variety, $Z \subset X$ a smooth subvariety of codimension $c \geq 2$, and $\sigma : Y \rightarrow X$ the blow-up of Z . Then the functor $L\sigma^* : \mathrm{D}^b(X) \rightarrow \mathrm{D}^b(Y)$ is fully faithful, and, for $i = 1, \dots, c-1$ there are fully faithful functors $\Phi_i : \mathrm{D}^b(Z) \rightarrow \mathrm{D}^b(Y)$, and a semiorthogonal decomposition*

$$\mathrm{D}^b(Y) = \langle L\sigma^* \mathrm{D}^b(X), \Phi_1 \mathrm{D}^b(Z), \dots, \Phi_{c-1} \mathrm{D}^b(Z) \rangle$$

Notice that Orlov's argument of the fully faithfulness of $L\sigma^*$ extends to the cases of a surjective morphism with rationally connected fibers between smooth and projective varieties, though the description of the orthogonal complement is in general unknown. The fact that $L\sigma^*$ is fully faithful in Theorem 2.3.1 can be seen as a special case of the following Lemma, since a blow up gives a surjective map with the required properties.

LEMMA 2.3.2. *Let X and Y be smooth and projective k -schemes and $\sigma : Y \rightarrow X$ a surjective morphism such that $\sigma_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^i \sigma_* \mathcal{O}_Y = 0$ for $i \neq 0$. Then $L\sigma^* : \mathrm{D}^b(X) \rightarrow \mathrm{D}^b(Y)$ is fully faithful.*

PROOF. For any A and B objects in $\mathrm{D}^b(X)$, we have

$$\begin{aligned} \mathrm{Hom}_Y(L\sigma^* A, L\sigma^* B) &= \mathrm{Hom}_X(A, R\sigma_* L\sigma^* B) \\ &= \mathrm{Hom}_X(A, B \otimes R\sigma_* \mathcal{O}_Y) = \mathrm{Hom}_X(A, B) \end{aligned}$$

by adjunction, projection formula and by our assumption respectively. \square

The canonical bundle and its associated invariants, like the geometric genus and the irregularity, play a central rôle in this theory. First of all it is easy to remark, using Serre duality, that if X has a trivial canonical bundle, then there is no non-trivial semiorthogonal decomposition of $\mathrm{D}^b(X)$. The results obtained by Okawa [144] and Kawatani–Okawa [110] for low dimensional varieties are also strongly related to the canonical bundle.

THEOREM 2.3.3 (Okawa). *Let C be a smooth projective k -curve of positive genus. Then $\mathrm{D}^b(C)$ has no non-trivial semiorthogonal decompositions.*

THEOREM 2.3.4 (Kawatani–Okawa). *Let k be algebraically closed and S a smooth connected projective minimal surface. Suppose that*

- either $\kappa(S) = 0$ and S is not a classical Enriques surface, or
- $\kappa(S) = 1$ and $p_g(S) > 0$, or
- $\kappa(S) = 2$, that $\dim H^1(S, \omega_S) > 1$, and for any one-dimensional connected component of the base locus of ω_S , its intersection matrix is negative definite.

¹Notice that $\mathrm{D}^b(X)$ admits a fully orthogonal decomposition if and only if X is not connected. We will only consider connected varieties.

Then there is no nontrivial semiorthogonal decomposition of $D^b(S)$.

Roughly speaking, one could say that varieties admitting semiorthogonal decompositions should have cohomological properties which are very close to the ones of a Fano (relatively over some base) variety.

If X is a Fano variety, that is if the canonical bundle ω_X is antiample, any line bundle L on X is a k -exceptional object in $D^b(X)$, and hence gives a semiorthogonal decomposition $D^b(X) = \langle A, L \rangle$, where A consists of objects right orthogonal to L . In the simpler case where X has index Picard rank 1 and index i (that is, $\omega_X = \mathcal{O}(-i)$), and $\text{char}(k) = 0$, one can use Kodaira vanishing theorems to construct a natural k -exceptional sequence, as remarked by Kuznetsov [122, Corollary 3.5].

PROPOSITION 2.3.5 (Kuznetsov). *Let X be a smooth Fano variety of Picard rank 1 with ample generator $\mathcal{O}(1)$, and index i . Then there is a semiorthogonal decomposition*

$$D^b(X) = \langle A_X, \mathcal{O}_X, \dots, \mathcal{O}_X(i-1) \rangle,$$

where $A_X = \langle \mathcal{O}_X, \dots, \mathcal{O}_X(i-1) \rangle^\perp$ is the category of objects W satisfying the condition that $\text{Ext}^r(\mathcal{O}(j), W) = 0$ for all $0 \leq j < i$ and for all integers r .

The previous result is easily generalized to the relative case of Mori fiber spaces as in [12, Proposition 2.2.2].

PROPOSITION 2.3.6. *Let $\pi : X \rightarrow Y$ be a flat surjective fibration between smooth varieties, such that $\text{Pic}(X/Y) \simeq \mathbb{Z}$ with ample generator $\mathcal{O}_{X/Y}(1)$ and such that $\omega_{X/Y} = \mathcal{O}_{X/Y}(-i)$. Set $D^b(Y)(j) := \pi^* D^b(Y) \otimes \mathcal{O}_{X/Y}(j)$. For any j , this gives a fully faithful embedding of $D^b(Y)$ into $D^b(X)$. Moreover, over a field k of characteristic 0, there is a semiorthogonal decomposition*

$$D^b(X) = \langle A_{X/Y}, D^b(Y)(0), \dots, D^b(Y)(i-1) \rangle,$$

where

$$A_{X/Y} = \langle D^b(Y)(0), \dots, D^b(Y)(i-1) \rangle^\perp$$

is the category of objects W such that $\text{Ext}^r(\pi^* A \otimes \mathcal{O}(j), W) = 0$ for all $0 \leq j < i$, for all integers r , and for all objects A in $D^b(Y)$.

PROOF. Notice that $D^b(Y)(j)$ being admissible in $D^b(X)$ is a consequence of Lemma 2.3.2. The semiorthogonality is given by a relative Kodaira vanishing. Finally, define $A_{X/Y}$ to be the complement. \square

REMARK 2.3.7. The assumption on k having characteristic zero is needed to ensure that the Kodaira vanishing theorem holds on X , but can be weakened. Indeed, Kodaira vanishing theorems hold in characteristic p for varieties that lift to a smooth variety in characteristic 0, see Deligne–Illusie [78]. For example, we could consider any complete intersection in projective space of Fano type over a field of characteristic p .

One should consider the decompositions above as the most related to the geometric structure of X , and the category $A_{X/Y}$ as the best witness of the birational behavior of X .² This idea is supported by the following results of Beilinson [27] (for the case of \mathbb{P}^n) and Orlov [146].

²Notice, that one can consider the semiorthogonal sequence $D^b(Y)(j), \dots, D^b(Y)(j+i-1)$ for any integer j . However, this would give orthogonal complements which are not only equivalent as triangulated categories, but also as dg categories.

PROPOSITION 2.3.8 (Beilinson, Orlov). *Let $\pi : X \rightarrow Y$ be a projective bundle of relative dimension r , that is, $X = \mathbb{P}_Y(E)$ for some rank $r + 1$ vector bundle E on Y . Then*

$$\mathrm{D}^b(X) = \langle \mathrm{D}^b(Y)(0), \dots, \mathrm{D}^b(Y)(r) \rangle.$$

In other words, $\mathbf{A}_{X/Y} = 0$.

The most difficult task in proving the Proposition 2.3.8, already for projective spaces, is to show that a given sequence of categories generates the whole category. This is done in [27] using a complex resolving the structure sheaf of the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. Let us list other known descriptions of $\mathbf{A}_{X/Y}$.

EXAMPLE 2.3.9. Let Y be a smooth projective k -variety and $\pi : X \rightarrow Y$ as in Proposition 2.3.6. Then $\mathbf{A}_{X/Y}$ is known in the following cases:

Projective bundles. If $\pi : X \rightarrow Y$ is a projective bundle, then $\mathbf{A}_{X/Y} = 0$, [146].

Projective fibrations. Let $\pi : X \rightarrow Y$ be a relative Brauer–Severi variety (that is, the geometric fibers of π are projective spaces, but X is not isomorphic to $\mathbb{P}(E)$ for any vector bundle E on Y), and α in $\mathrm{Br}(Y)$ the class of X and r the relative dimension. If $\omega_{X/Y}$ generates $\mathrm{Pic}(X/Y)$, then

$$\mathbf{A}_{X/Y} = \langle \mathrm{D}^b(Y, \alpha), \dots, \mathrm{D}^b(Y, \alpha^r) \rangle.$$

If $\omega_{X/Y}$ is not primitive, a similar description is possible [28].

Quadric fibrations. Let $\pi : X \rightarrow Y$ be a quadric fibration of relative dimension r and let \mathcal{C}_0 be the sheaf of even Clifford algebras associated to the quadratic form defining X . Then $\mathbf{A}_{X/Y} = \mathrm{D}^b(Y, \mathcal{C}_0)$, [116].

Fibrations in intersections of quadrics. Let $\{Q_i \rightarrow Y\}_{i=0}^s$ be quadric fibrations of relative dimension r and $\pi : X \rightarrow Y$ be their intersection (see [12] for details), and suppose that $\omega_{X/Y}$ is relatively antiample (that is, $r < 2s$). Then there is a \mathbb{P}^s -bundle $Z \rightarrow Y$ and a sheaf of Clifford algebras \mathcal{C}_0 on Z , and $\mathbf{A}_{X/Y} = \mathrm{D}^b(Z, \mathcal{C}_0)$, [12].

This list is far from being exhaustive, since many specific cases are also known (see, e.g., Table 1 for 3 and 4 dimensional cases).

In the case where k is not algebraically closed, then one can look for semiorthogonal decompositions of $\mathrm{D}^b(X_{\bar{k}})$ and understand whether they can give informations on $\mathbf{A}_{X/Y}$ or, more in general on $\mathrm{D}^b(X)$. This rather challenging problem can be tackled in the simplest case, where $\mathrm{D}^b(X_{\bar{k}})$ is generated by vector bundles, using Galois descent of such vector bundles (see [11]). With this in mind one can describe $\mathbf{A}_{X/Y}$ when X is a minimal del Pezzo, Y is a point, and k is any field [11]. Other cases of (generalized) Brauer–Severi varieties [28, 38] can be treated this way.

On the other hand, even if a geometric description of $\mathbf{A}_{X/Y}$ is not possible, one can calculate its Serre functor.

DEFINITION 2.3.10. Let \mathbf{A} be a triangulated k -linear category with finite dimensional morphism spaces. A functor $S : \mathbf{A} \rightarrow \mathbf{A}$ is a *Serre functor* if it is a k -linear equivalence inducing a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{A}}(X, Y) \simeq \mathrm{Hom}_{\mathbf{A}}(Y, S(X))^{\vee}$$

of k -vector spaces, for any object X and Y of \mathbf{A} .

A category \mathbf{A} with a Serre functor $S_{\mathbf{A}}$ is a *Calabi–Yau category* (or a *non commutative Calabi–Yau*) of dimension n if $S_{\mathbf{A}} = [n]$. It is a *fractional Calabi–Yau category* of dimension n/c if c is the smallest integer such that the iterate Serre functor is a shift functor and $S_{\mathbf{A}}^c = [n]$. Note that the fractional dimension of \mathbf{A} is not a rational number, but a pair of two integer numbers.

The Serre functor generalizes the notion of Serre duality to a more general categorical setting. Indeed, if X is a smooth projective k -variety, then we have that $S_{\mathbf{D}^b(X)}(-) = - \otimes \omega_X[\dim(X)]$ by Serre duality.

As a consequence of the work of Bondal and Kapranov [45], if X is a smooth and projective k -scheme and \mathbf{A} is an admissible subcategory of $\mathbf{D}^b(X)$, then \mathbf{A} has a Serre functor which can be explicitly calculated from the Serre functor of X using adjunctions to the embedding $\mathbf{A} \rightarrow \mathbf{D}^b(X)$. Kuznetsov performed explicitly these calculations for Fano hypersurfaces in projective spaces, see [119, Cor. 4.3].

PROPOSITION 2.3.11 (Kuznetsov). *Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree $d < n + 2$, and set c the greatest common divisor of d and $n + 2$. Then \mathbf{A}_X is a (fractional) Calabi–Yau category, that is $S_{\mathbf{A}_X}^{d/c} = [\frac{(d-2)(n+2)}{c}]$.*

REMARK 2.3.12. Notice that both d/c and $\frac{(d-2)(n+2)}{c}$ are integers. However, the fractional dimension of \mathbf{A}_X is not a simplification of the fraction $\frac{(d-2)(n+2)}{c}$ unless $c = 1$. For example, for a quartic fourfold we obtain $6/2$. However, in the case where d divides $n + 2$, \mathbf{A}_X is a Calabi–Yau category.

COROLLARY 2.3.13. *If $X \subset \mathbb{P}^5$ is a smooth cubic fourfold, then \mathbf{A}_X is a 2-Calabi–Yau category (or a noncommutative K3 surface).*

The categories $\mathbf{A}_{X/Y}$ also admit algebraic descriptions, that is, one can find an equivalence with a triangulated category which arises from purely algebraic constructions. The main examples are Orlov’s description via matrix factorizations for Fano complete intersections in projective spaces (see [148]) and a rather complicated description based on Homological Projective Duality for fibrations in complete intersections of type (d, \dots, d) (see [18]).

To tackle geometrical problems, we would like a description of $\mathbf{A}_{X/Y}$ by explicit geometric constructions. A first case, which include a lot of Fano varieties, is the case of homogeneous varieties. These are conjectured to always carry a full exceptional sequence, and one can construct a candidate sequence using vanishing theorems and representation of parabolic subgroups, see [128]. The hardest part is to prove that such a sequence is full, for which spectral sequences are needed.

The most powerful tool to construct semiorthogonal decompositions is by far Kuznetsov’s Homological Projective Duality (HPD). We refrain here to give any definition, for which we refer to the very dense Kuznetsov’s original paper [121]. In practice, HPD allows to compare semiorthogonal decompositions of dual linear sections of fixed projectively dual varieties. It is in general a hard task to show that two given varieties are HP-dual, and one of the most challenging steps is to deal with singular varieties and their noncommutative resolutions. However, HPD allows one to describe a great amount of semiorthogonal decompositions for Fano varieties or Mori fiber spaces, see [121], [123], [120], [12] just to name a few.

On the other hand, (relatively) Fano varieties are not the only class of varieties whose derived category admits a semiorthogonal decompositions. The first natural examples one should consider are surfaces with $p_g = q = 0$, in which case any

line bundle is a k -exceptional object. Hence the derived category of such surfaces always admits nontrivial semiorthogonal decompositions. On the other hand, one can argue that, if S is a such a surface, then there is no fully faithful functor $\mathbf{D}^b(C) \rightarrow \mathbf{D}^b(S)$ for C a curve of positive genus. Indeed, such a functor would give a nontrivial Albanese variety (or, equivalently, a nontrivial Pic^0), see [32]. Another way to present this argument is by noticing that $H^{p,q}(S) = 0$ if $p - q \neq 0$. It follows that the Hochschild homology $HH_i(S) = 0$ for $i \neq 0$. This last fact obstructs the existence of the functor, since $HH_{\pm 1}(C) \neq 0$ for a positive genus curve C .

It is then natural to look for semiorthogonal decompositions of the form:

$$\mathbf{D}^b(S) = \langle A_S, E_1, \dots, E_n \rangle,$$

with E_i k -exceptional objects and wonder about the maximal possible value of n and the structure of A_S . Describing A_S is a very challenging question and we will treat examples and their conjectural relation with rationality questions in §6.

Let us conclude by remarking that, studying such surfaces, Böhning, Graf von Bothmer, and Sosna have been able to show that semiorthogonal decompositions do not enjoy, in general, a Jordan–Hölder type property [41]. Notice that a further example is explained by Kuznetsov [124].

PROPOSITION 2.3.14 (Böhning–Graf von Bothmer–Sosna). *Let X be the classical Godeaux complex surface. The bounded derived category $\mathbf{D}^b(X)$ has two maximal exceptional sequences of different lengths: one of length 11 and one of length 9 which cannot be extended further.*

3. Unramified cohomology and decomposition of the diagonal

Unramified cohomology has emerged in the last four decades as a powerful tool for obstructing (stable) rationality in algebraic geometry. Much of its utility comes from the fact that the theory rests on a combination of tools from scheme theory, birational geometry, and algebraic K -theory. Used notably in the context of Noether’s problem in the work of Saltman and Bogomolov, unramified cohomology can be computed purely at the level of the function field, without reference to a specific good model.

3.1. Flavors of rationality. A variety X over a field k is *rational over k* if X is k -birational to the projective space \mathbb{P}^n , it is *unirational over k* if there is a dominant rational $\mathbb{P}^N \dashrightarrow X$ for some N , it is *retract rational over k* if there is a dominant rational $\mathbb{P}^N \dashrightarrow X$ with a rational section, it is *stably rational over k* if $X \times \mathbb{P}^N$ is rational for some N . The notion of retract rationality was introduced by Saltman in the context of Noether’s problem.

We have the following implications:

rational \Rightarrow stably rational \Rightarrow retract rational \Rightarrow unirational \Rightarrow rationally connected.

Several important motivating problems in the study of rationality in algebraic geometry can be summarized as asking whether these implications are strict.

PROBLEM 3.1.1 (Lüroth problem). *Determine whether a given unirational variety X is rational.*

PROBLEM 3.1.2 (Birational Zariski problem). *Determine whether a given stably rational variety X is rational.*

PROBLEM 3.1.3. *Does there exist a rationally connected variety X , such that $X(k) \neq \emptyset$ and that is not unirational?*

PROBLEM 3.1.4. *Does there exist a retract rational variety X that is not stably rational?*

The Lüroth question has a positive answer in dimension 1 (proved by Lüroth) over an arbitrary field and in dimension 2 over an algebraically closed field of characteristic zero (proved by Castelnuovo [55]). There exist counterexamples, i.e., unirational but nonrational surfaces, over the real numbers (as remarked by Segre [157]) and over an algebraically closed field of characteristic $p > 0$ (discovered by Zariski [181]). The first known counterexamples over \mathbb{C} were in dimension 3, discovered independently by Clemens–Griffiths [58], Iskovskih–Manin [106], and Artin–Mumford [9]. We point out that the example of Artin–Mumford also provided the first example of a unirational variety that is not stably rational over \mathbb{C} . The method of the intermediate Jacobian due to Clemens–Griffiths and the method of birational rigidity due to Iskovskih–Manin do not obstruct stable rationality. We will treat the former in more details in §4.

The first known counterexamples to the birational Zariski problem were discovered by Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer [25] in dimension 2 over non algebraically closed fields and in dimension 3 over \mathbb{C} using the method of the intermediate Jacobian, which we will recall in §4.1.

There exist retract rational tori that are not stably rational over \mathbb{Q} discovered in the context of work by Swan and Voskresenskii on Noether’s problem, see [71, §8.B, p. 223].

The last two questions are still open over an algebraically closed field!

3.2. Unramified elements. Various notions of the concept of unramified cohomology emerged in the late 1970s and 1980s [59], [60], [72], [73], [68], mostly motivated by earlier investigations of the Brauer group [17], [94] and the Gersten conjecture [36] in algebraic K -theory. The general notion of “unramified element” of a functor is developed in [61, §2]. Rost [153, Rem. 5.2] gives a different perspective in terms of cycle modules, also see Morel [139, §2]. Let k be a field and denote by Local_k the category of local k -algebras together with local k -algebra homomorphisms. Let Ab be the category of abelian groups and let $M : \text{Local}_k \rightarrow \text{Ab}$ be a functor. For any field K/k the group of *unramified elements* of M in K/k is the intersection

$$M_{\text{ur}}(K/k) = \bigcap_{k \subset \mathcal{O} \subset K} \text{im}(M(\mathcal{O}) \rightarrow M(K))$$

over all rank 1 discrete valuations rings $k \subset \mathcal{O} \subset K$ with $\text{Frac}(\mathcal{O}) = K$.

There is a natural map $M(k) \rightarrow M_{\text{ur}}(K/k)$ and we say that the group of unramified elements $M_{\text{ur}}(K/k)$ is *trivial* if this map is surjective.

For X an integral scheme of finite type over a field k , we will often write $M_{\text{ur}}(X/k)$ for $M_{\text{ur}}(k(X)/k)$. By definition, the group $M_{\text{ur}}(X/k)$ is a k -birational invariant of integral schemes of finite type over k .

We will be mostly concerned with the functor $M = H_{\text{ét}}^i(-, \mu)$ with coefficients μ either $\mu_n^{\otimes(i-1)}$ (under the assumption $\text{char}(k) \neq n$) or

$$\mathbb{Q}/\mathbb{Z}(i-1) = \varinjlim \mu_n^{\otimes(i-1)},$$

the direct limit being taken over all integers n coprime to the characteristic of k . In this case, $M_{\text{ur}}(X/k)$ is called the *unramified cohomology* group $H_{\text{ur}}^i(X, \mu)$ of X with coefficients in μ .

The reason why we only consider cohomology of degree i with coefficients that are twisted to degree $(i - 1)$ is the following well-known consequence of the norm residue isomorphism theorem proved by Voevodsky, Rost, and Weibel (previously known as the Bloch–Kato conjecture).

THEOREM 3.2.1. *Let K be a field and n a nonnegative integer prime to the characteristic. Then the natural map*

$$H^i(K, \mu_n^{\otimes(i-1)}) \rightarrow H^i(K, \mathbb{Q}/\mathbb{Z}(i-1))$$

is injective and the natural map $\varinjlim H^i(K, \mu_n^{\otimes(i-1)}) \rightarrow H^i(K, \mathbb{Q}/\mathbb{Z}(i-1))$ is an isomorphism, where the limit is taken over all n prime to the characteristic.

REMARK 3.2.2. If k is an algebraically closed field whose characteristic is invertible in μ , then $H_{\text{ur}}^i(X, \mu) = 0$ for all $i > \dim(X)$, since in this case the function field $k(X)$ has cohomological dimension $\dim(X)$.

Another important functor is the Milnor K -theory functor $M = K_i^M(-)$.

Let $\mathcal{H}_{\text{ét}}^i(\mu)$ be the Zariski sheaf on the category of k -schemes Sch_k associated to the functor $H_{\text{ét}}^i(-, \mu)$. The Gersten conjecture, proved by Bloch and Ogus [36], allows for the calculation of the cohomology groups of the sheaves $\mathcal{H}_{\text{ét}}^i(\mu)$ on a smooth proper variety X as the cohomology groups of the Gersten complex (also known as the “arithmetic resolution”) for étale cohomology:

$$0 \longrightarrow H^i(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} H^{i-1}(F(x)) \longrightarrow \bigoplus_{y \in X^{(2)}} H^{i-2}(F(y)) \longrightarrow \dots$$

where $H^i(-)$ denotes the Galois cohomology group in degree i with coefficients μ either $\mu_n^{\otimes(i-1)}$ or $\mathbb{Q}/\mathbb{Z}(i-1)$, where $X^{(i)}$ is the set of codimension i points x of X with residue field $F(x)$, and where the “residue” morphisms are Gysin boundary maps induced from the spectral sequence associated to the coniveau filtration, see Bloch–Ogus [36, Thm. 4.2, Ex. 2.1, Rem. 4.7]. In particular, we have that

$$H^0(X, \mathcal{H}_{\text{ét}}^i(\mu)) = H_{\text{ur}}^i(X, \mu).$$

This circle of ideas is generally called “Bloch–Ogus theory.”

Over \mathbb{C} , this leads to the following “geometric interpretation” of unramified cohomology, as the direct limit over all Zariski open coverings $\mathcal{U} = \{U_i\}$ of the set

$$\{\{\alpha_i\} \in H_B^i(\mathcal{U}, \mu) : \alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}\}$$

where $H_B^i(\mathcal{U}, \mu) = \prod_i H_B^i(U_i, \mu)$ is Betti cohomology, using the comparison with étale cohomology.

3.3. Purity in low degree. There is a canonical map $H_{\text{ét}}^i(X, \mu) \rightarrow H_{\text{ur}}^i(X, \mu)$. If this map is injective, surjective, or bijective we say that the *injectivity*, *weak purity*, or *purity* property hold for étale cohomology in degree i , respectively, see Colliot-Thélène [61, §2.2].

For X smooth over a field k , a general cohomological purity theorem for étale cohomology is established by Artin in [7, XVI 3.9, XIX 3.2].

THEOREM 3.3.1. *Let X be a smooth variety over a field k and $V \subset X$ a closed subvariety of pure codimension $\geq c$. Then the restriction maps*

$$H_{\text{ét}}^i(X, \mu_n^{\otimes j}) \rightarrow H_{\text{ét}}^i(X \setminus V, \mu_n^{\otimes j})$$

are injective for $i < 2c$ and are isomorphisms for $i < 2c - 1$.

An immediate consequence (taking $c = 1$) is that purity holds for étale cohomology in degree ≤ 1 , i.e.

$$H_{\text{ét}}^0(X, \mu) = H_{\text{ur}}^0(X, \mu) = \mu, \quad \text{and} \quad H_{\text{ét}}^1(X, \mu) = H_{\text{ur}}^1(X, \mu).$$

See also Colliot-Thélène–Sansuc [72, Cor. 3.2, Prop. 4.1] for an extension to any geometrically locally factorial and integral scheme.

Combining this (for $c = 2$) with a cohomological purity result for discrete valuation rings and a Mayer–Vietoris sequence, one can deduce that for X smooth over a field, weak purity holds for étale cohomology in degree 2. Moreover, there’s a canonical identification $\text{Br}(X)' = H_{\text{ur}}^2(X, \mathbb{Q}/\mathbb{Z}(1))$ by Bloch–Ogus [36] such that the canonical map $H_{\text{ét}}^2(X, \mathbb{Q}/\mathbb{Z}(1)) \rightarrow H_{\text{ur}}^2(X, \mathbb{Q}/\mathbb{Z}(1)) = \text{Br}(X)'$ arises from the Kummer exact sequence. Here, $\text{Br}(X)'$ denotes the prime-to-characteristic torsion subgroup of the (cohomological) Brauer group $\text{Br}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$. For X a smooth variety over \mathbb{C} (or in fact X any complex analytic space), there is a split exact sequence

$$0 \rightarrow (H^2(X, \mathbb{Z})/\text{im}(\text{Pic}(X) \rightarrow H^2(X, \mathbb{Z}))) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow \text{Br}(X) \rightarrow H^3(X, \mathbb{Z})_{\text{tors}} \rightarrow 0$$

arising from the exponential sequence. In particular, there is a (noncanonical) isomorphism $\text{Br}(X) \cong (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho} \oplus H^3(X, \mathbb{Z})_{\text{tors}}$, where b_2 is the second Betti number and ρ the Picard rank of X . If X satisfies $H^2(X, \mathcal{O}_X) = 0$ (e.g., X is rationally connected), then $\text{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$ is an isomorphism, and hence $\text{Br}(X) = H^3(X, \mathbb{Z})_{\text{tors}}$.

There is a beautiful interpretation of unramified cohomology in degree 3 in terms of cycles of codimension 2, going back to Barbieri-Viale [19]. Let X be a smooth projective variety over k . We say that $\text{CH}_0(X)$ is supported in dimension r if there exists a smooth projective variety Y over k of dimension r and a morphism $f : Y \rightarrow X$ such that the pushforward $f_* : \text{CH}_0(Y) \rightarrow \text{CH}_0(X)$ is surjective. For example, if $\text{CH}_0(X) = \mathbb{Z}$ (e.g., X is rationally connected) then X is supported in dimension 0.

THEOREM 3.3.2 (Colliot-Thélène–Voisin [74, Thm. 1.1]). *Let X be a smooth projective variety over \mathbb{C} . Assume that $\text{CH}_0(X)$ is supported in dimension 2. Then there is an isomorphism*

$$H_{\text{ur}}^3(X, \mathbb{Q}/\mathbb{Z}(2)) \cong \frac{H^{2,2}(X) \cap H^4(X, \mathbb{Z})}{\text{im}(\text{CH}^2(X) \rightarrow H^4(X, \mathbb{Z}))}.$$

Equivalently, the unramified cohomology of X in degree 3 is the obstruction to the validity of the integral Hodge conjecture for cycles of codimension 2.

More generally, without the assumption that $\text{CH}_0(X)$ is supported in dimension 2, the torsion subgroup of $(H^{2,2}(X) \cap H^4(X, \mathbb{Z}))/\text{im}(\text{CH}^2(X) \rightarrow H^4(X, \mathbb{Z}))$ is a quotient of $H_{\text{ur}}^3(X, \mathbb{Q}/\mathbb{Z}(2))$ by a divisible subgroup.

There is also a version of this result valid over more general fields, in particular over finite fields, due to Colliot-Thélène and Kahn [67] and extended to higher codimension cycles by Pirutka [151]. Finally, there is a description of $H_{\text{ur}}^4(X, \mathbb{Q}/\mathbb{Z}(3))$ in terms of torsion in $\text{CH}^3(X)$, due to Voisin [173].

3.4. Triviality. If F/k is a field extension, we write $X_F = X \times_k F$. If X is geometrically integral over k , we say that $M_{\text{ur}}(X/k)$ is *universally trivial* if $M_{\text{ur}}(X_F/F)$ is trivial for every field extension F/k . Let N be a positive integer. We say that $M_{\text{ur}}(X/k)$ is *universally N -torsion* if the cokernel of the natural map $M(F) \rightarrow M_{\text{ur}}(X_F/F)$ is killed by N for every field extension F/k .

PROPOSITION 3.4.1 ([61, §2 and Thm. 4.1.5]). *Let $M : \text{Local}_k \rightarrow \text{Ab}$ be a functor satisfying the following conditions:*

- *If \mathcal{O} is a discrete valuation ring containing k , with fraction field K and residue field κ , then $\ker(M(\mathcal{O}) \rightarrow M(K)) \subset \ker(M(\mathcal{O}) \rightarrow M(\kappa))$.*
- *If A is a regular local ring of dimension 2 containing k , with fraction field K , then $\text{im}(M(A) \rightarrow M(K)) = \bigcap_{\text{ht}(\mathfrak{p})=1} \text{im}(M(A_{\mathfrak{p}}) \rightarrow M(K))$.*
- *The group $M_{\text{ur}}(\mathbb{A}_k^1/k)$ is universally trivial.*

Then $M_{\text{ur}}(\mathbb{P}_k^n/k)$ is universally trivial. In particular, if X is a rational variety over k , then $M_{\text{ur}}(X/k)$ is universally trivial.

The functor $H_{\text{ét}}^i(-, \mu)$ satisfies the conditions of Proposition 3.4.1 (cf. [61, Thm. 4.1.5]), hence if X is a k -rational variety, then $H_{\text{ur}}^i(X, \mu)$ is universally trivial. More generally, $H_{\text{ur}}^i(X, \mu)$ is universally trivial if X is stably rational, see [68, Prop. 1.2], or even retract k -rational, which can be proved using the general machinery in [109, Cor. RC.12–13], see [138, Prop. 2.15].

3.5. Applications: Noether’s problem and Artin–Mumford. Here we describe two important examples where unramified cohomology has been used in the rationality problem.

EXAMPLE 3.5.1. Let G be a finite group, V a finite dimensional linear representation over k , and $k(V)$ the field of rational functions on the affine space associated to V . Then Noether’s question asks if the field of invariants $k(V)^G$ is purely transcendental over k , equivalently, if the variety V/G is rational. This question was posed by Emmy Noether in 1913, and has endured as one of the most challenging rationality problems in algebraic geometry.

Over the rational numbers, the problem takes on a very arithmetic flavor. Indeed, Noether’s original motivation was the inverse Galois problem, see [162] for a survey in this direction. So we will focus on the case when k is algebraically closed of characteristic zero.

In this case, the question has a positive answer when G is any abelian group but is still open for $G = A_n$ for $n \geq 6$. Saltman [155] gave the first examples of p -groups having a negative answer to Noether’s question when k is algebraically closed. While V/G often has terrible singularities and its smooth projective models are not easy to compute, nor feasible to work with, the insight of Saltman was that one could still compute unramified cohomology, in particular, $H_{\text{ur}}^2(k(V)^G/k, \mathbb{Q}/\mathbb{Z}(1))$. By the above purity results, if X were a smooth proper model of $k(V)^G$, then $H_{\text{ur}}^2(k(V)^G/k, \mathbb{Q}/\mathbb{Z}(1)) = \text{Br}(X)$. Bogomolov [39] gave a simple group theoretic

formula to compute $H_{\text{ur}}^2(k(V)^G/k, \mathbb{Q}/\mathbb{Z}(1))$ purely in terms of G , when k is algebraically closed of characteristic not dividing the order of G . Because of this, this group is often called the ‘‘Bogomolov multiplier’’ in the literature.

We point out that examples of groups G where $H_{\text{ur}}^i(k(V)^G/k, \mathbb{Q}/\mathbb{Z}(1))$ is trivial for $i = 2$ yet nontrivial for $i \geq 3$ were first constructed by Peyre [150].

EXAMPLE 3.5.2. Artin and Mumford [9] constructed a unirational threefold X over an algebraically closed field of characteristic $\neq 2$ having nontrivial 2-torsion in $\text{Br}(X) = H^3(X, \mathbb{Z})_{\text{tors}}$, which by purity (see §3.3) coincides with $H_{\text{ur}}^2(X, \mathbb{Q}/\mathbb{Z}(1))$, hence such X not retract rational (hence not stably rational). The ‘‘Artin–Mumford solid’’ X is constructed as the desingularization of a double cover of \mathbb{P}^3 branched over a certain quartic hypersurface with 10 nodes, and is unirational by construction. The solid X can also be presented as a conic bundle $X \rightarrow S$ over a rational surface S . The unramified cohomology perspective on the examples of Artin and Mumford was further investigated by Colliot-Thélène and Ojanguren [68].

Denote by Ab^\bullet the category of graded abelian groups. An important class of functors $M : \text{Local}_k \rightarrow \text{Ab}^\bullet$ arise from the general theory of *cycle modules* due to Rost [153, Rem. 5.2]. In particular, unramified cohomology arises from the étale cohomology cycle module, and to some extent, the theory of cycle modules is a generalization of the theory of unramified cohomology. Rost’s key observation is that classical Chow groups appear as the unramified elements of the Milnor K -theory cycle module. The definition of cohomology groups arising from cycle modules is very parallel to the definition of homology of a CW complex from the singular chain complex. A cycle module M comes equipped with residue maps of graded degree -1

$$M^i(k(X)) \xrightarrow{\partial} \bigoplus_{x \in X^{(1)}} M^{i-1}(k(x))$$

for any integral k -variety X . If X is smooth and proper, then the group of unramified elements $M_{\text{ur}}^i(X/k)$ is defined to be the kernel.

3.6. Decomposition of the diagonal. We say that a smooth proper variety X of dimension n over a field k has an (*integral*) *decomposition of the diagonal* if we can write

$$(3.1) \quad \Delta_X = P \times X + Z$$

in $\text{CH}^n(X \times X)$, where P is a 0-cycle of degree 1 and Z is a cycle with support in $X \times V$ for some closed subvariety $V \subsetneq X$. We say that X has a *rational decomposition of the diagonal* if there exists $N \geq 1$ such that

$$(3.2) \quad N\Delta_X = P \times X + Z$$

in $\text{CH}^n(X \times X)$, where P is a 0-cycle of degree N and Z is as before. This notion was studied by Bloch and Srinivas [37], with the idea going back to Bloch’s proof of Mumford’s result on 2-forms on surfaces (see [34, Lecture 1, Appendix]).

EXAMPLE 3.6.1. The class of the diagonal $\Delta_{\mathbb{P}^n} \in \text{CH}^n(\mathbb{P}^n \times \mathbb{P}^n)$ can be expressed in terms of the pull backs $\alpha, \beta \in \text{CH}^1(\mathbb{P}^n \times \mathbb{P}^n)$ of hyperplane classes from the two projections. The Chow ring can be presented in terms of these classes as

$$\text{CH}(\mathbb{P}^n \times \mathbb{P}^n) = \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1})$$

and one can compute that

$$\Delta_{\mathbb{P}^n} = \alpha^n + \alpha^{n-1}\beta + \cdots + \alpha\beta^{n-1} + \beta^n$$

in $\text{CH}^n(\mathbb{P}^n \times \mathbb{P}^n)$, see [84, Ex. 8.4.2]. The class α^n is the same as the class $P \times \mathbb{P}^n$, for $P \in \mathbb{P}^n$ a rational point, while the classes $\alpha^i\beta^{n-i}$ for $i > 0$ all have support on $\mathbb{P}^n \times H$, where $H \subset \mathbb{P}^n$ is the hyperplane defining β . So any projective space has an integral decomposition of the diagonal.

EXAMPLE 3.6.2. Let $f : Y \rightarrow X$ be a proper surjective generically finite morphism of degree N between smooth quasi-projective varieties. Then we have that $(f \times f)_*\Delta_Y = N\Delta_X$. Assume that Y has a decomposition of the diagonal $\Delta_Y = P \times Y + Z$, where P is a 0-cycle of degree 1 on Y and Z is a cycle with support on $Y \times V$. Then

$$N\Delta_X = (f \times f)_*\Delta_Y = (f \times f)_*(P \times Y + Z) = f_*(P) \times X + Z'$$

where $f_*(P)$ is a 0-cycle of degree N and Z' is a cycle on X with support on $X \times f(V)$. Hence X has a rational decomposition of the diagonal.

Let $f : Y \rightarrow X$ be a surjective birational morphism between smooth quasi-projective varieties. Given a decomposition of the diagonal $\Delta_X = P \times X + Z$ on X , by the moving lemma for 0-cycles, we can move P , up to rational equivalence, outside of the image of the exceptional locus of f . Then $(f \times f)^*\Delta_X - \Delta_Y$ is a sum of cycles whose projections to Y are contained in the exceptional locus of f . But $(f \times f)^*\Delta_X = (f \times f)^*(P \times X + Z) = f^{-1}(P) \times Y + (f \times f)^*(Z)$, and $(f \times f)^*(Z)$ is a cycle with support on $Y \times f^*(V)$. In total, Y has a decomposition of the diagonal.

We can use this to show that if $\mathbb{P}^n \dashrightarrow X$ is a unirational parameterization of degree N over a field of characteristic zero, then X has a rational decomposition of the diagonal $N\Delta_X = P \times X + Z$. Indeed, by resolution of singularities, we can resolve the rational map to a proper surjective generically finite morphism $Y \rightarrow X$ of degree N , where $Y \rightarrow \mathbb{P}^n$ is a sequence of blow up maps along smooth centers. By the above considerations, the decomposition of the diagonal on \mathbb{P}^n induces one on Y , which in turn induces the desired rational decomposition of the diagonal on X .

We remark that one can argue without the use of resolution of singularities, using the machinery of [109, App. RC], but this is slightly more delicate.

3.7. Decomposition of the diagonal acting on cohomology. A rational decomposition of the diagonal puts strong restrictions on the variety X . For example, the following result is well known.

PROPOSITION 3.7.1. *Let X be a smooth proper geometrically irreducible variety over a field k of characteristic zero. If X has a rational decomposition of the diagonal then $H^0(X, \Omega_X^i) = 0$ and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Over a complex surface, this result goes back to Bloch's proof [34, App. Lec. 1] of Mumford's [141] result on 2-forms on surfaces, exploiting a decomposition of the diagonal and the action of cycles on various cohomology theories (de Rham and étale). This argument was further developed in [37]. A proof over the complex numbers can be found in [175, Cor. 10.18, §10.2.2]. A variant of the argument for rigid cohomology in characteristic p is developed by Esnault [82, p. 187], in her proof that rationally connected varieties over a finite field have a rational point. A

variant of the argument using logarithmic de Rham cohomology over any field is developed by Totaro [166, Lem. 2.2] using the cycle class map of Gros.

Let $H^i(-)$ be a cohomology theory with a cycle class map

$$\mathrm{CH}^i(X) \rightarrow H^{2i}(X)$$

and a theory of correspondences (basically a Weil cohomology theory), so that for any $\alpha \in H^{2n}(X \times X)$, where $n = \dim(X)$, there is a map

$$\alpha_* = q_*(\alpha.p^*) : H^i(X) \rightarrow H^i(X).$$

for any $i \geq 0$. Here p and q are the left and right projections $X \times X \rightarrow X$, respectively. When $\alpha = [\Delta_X] \in H^{2n}(X \times X)$, then α_* is the identity map. When $\alpha = [P \times X] \in H^{2n}(X \times X)$, then α_* factors through $H^i(P)$, i.e., there is a commutative diagram

$$\begin{array}{ccc} H^i(X) & \xrightarrow{\alpha_*} & H^i(X) \\ \downarrow & & \parallel \\ H^i(P) & \xrightarrow{\alpha_*} & H^i(X) \end{array}$$

where the left hand vertical map is the pullback by the inclusion of the zero-dimensional subscheme $P \subset X$ (we have in mind N times a point). Assuming that $H^i(P) = 0$ for $i > 0$, we get that $N[\Delta_X]_* = [Z]_*$ on $H^i(X)$, assuming a rational decomposition of the diagonal as in (3.2).

On the other hand, since Z is a cycle supported on $X \times V$ for $V \subset X$ a proper closed subvariety, the restriction of $[Z]$ to $H^{2n}(X \times X \setminus V)$ is zero. Consider $\alpha = [Z] \in H^{2n}(X \times X)$ and the commutative diagram

$$\begin{array}{ccc} H^i(X) & \xrightarrow{\alpha_*} & H^i(X) \\ \parallel & & \downarrow \\ H^i(X) & \xrightarrow{0_*} & H^i(X \setminus V) \end{array}$$

where the right hand vertical arrow is the pullback by the inclusion $X \setminus V \subset X$ and the bottom horizontal arrow is the pushforward associated to the restriction of $[Z]$ to $X \times (X \setminus V)$, which is zero. Hence we have that $\alpha_* H^i(X)$ is contained in the kernel of the restriction map $H^i(X) \rightarrow H^i(X \setminus V)$. If we additionally assume that the cohomology theory has a localization sequence

$$\cdots \rightarrow H_V^i(X) \rightarrow H^i(X) \rightarrow H^i(X \setminus V) \rightarrow \cdots$$

involving cohomology with supports, then we can also conclude that $\alpha_* H^i(X)$ is contained in the image of the map $H_V^i(X) \rightarrow H^i(X)$.

Now, applying this to algebraic de Rham cohomology $H_{\mathrm{dR}}^i(-)$, we have that (by the degeneration of the Hodge-to-de Rham spectral sequence) any element $\alpha \in H_{\mathrm{dR}}^{2n}(X \times X)$ in the image of the cycle map lands in

$$H^n(X \times X, \Omega_{X \times X}^n) = H^n(X \times X, \bigoplus_j \Omega_X^j \boxtimes \Omega_X^{n-j}) = \bigoplus_{i,j} H^i(X, \Omega_X^j) \otimes H^{n-i}(X, \Omega_X^{n-j})$$

so α has a component in $H^0(X, \Omega_X^i) \otimes H^n(X, \Omega_X^{n-i})$, which is isomorphic (by Serre duality) to $\mathrm{End}(H^0(X, \Omega_X^i))$. Thus this component of the pushforward α_* defines a map $H^0(X, \Omega_X^i) \rightarrow H^0(X, \Omega_X^i)$, whose image lands in the kernel of the restriction map $H^0(X, \Omega_X^i) \rightarrow H^0(X \setminus V, \Omega_{X \setminus V}^i)$. But this kernel is trivial, since restriction

to a Zariski open set is injective on global differential forms. We thus conclude that (N times) the identity on $H^0(X, \Omega_X^i)$ coincides with the zero map, hence $H^0(X, \Omega_X^i) = 0$ for all $i > 0$.

Similarly, applying this to the cycle class map of Gros in logarithmic de Rham cohomology, Totaro shows that even in characteristic p , if X has an (integral) decomposition of the diagonal, then $H^0(X, \Omega_X^i) = 0$ for all $i > 0$.

Applying this to the transcendental part of the cohomology

$$H^2(X, \mathbb{Q}_\ell) / \text{im}(\text{NS}(X) \otimes \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^2(X, \mathbb{Q}_\ell)$$

of a surface X , Bloch shows that a rational decomposition of the diagonal implies the vanishing of the transcendental part of the cohomology. Via the Hodge-theoretic fact that $p_g(X) = 0$ is equivalent to $b_2(X) - \rho(X) = 0$, this is Bloch's new proof of Mumford's theorem stating that if $p_g(X) > 0$ then the kernel of the degree map $\text{deg} : \text{CH}_0(X) \rightarrow \mathbb{Z}$ is not representable.

Applying this to Berthelot's theory of rigid cohomology, Esnault shows that if X is defined over a field of characteristic p and has a rational decomposition of the diagonal, then the Frobenius slope $[0, 1)$ part of the rigid cohomology of $H^i(X)$ is trivial for all $i > 0$. If X is defined over a finite field \mathbb{F}_q , the Lefschetz trace formula implies that $\#X(\mathbb{F}_q) \equiv 1 \pmod{q}$, in particular, $X(\mathbb{F}_q) \neq \emptyset$.

We remark that over \mathbb{C} , an integral decomposition of the diagonal does not imply $H^0(X, \omega_X^{\otimes n}) = 0$ for all $n > 1$. Otherwise, a smooth projective surface X over \mathbb{C} with integral decomposition of the diagonal would, aside from satisfying $p_g(X) = h^0(X, \omega_X) = 0$ and $q = h^1(X, \mathcal{O}_X) = h^0(X, \Omega_X^1) = 0$, would additionally satisfy $P_2(X) = h^0(X, \omega_X^{\otimes 2}) = 0$, hence would be rational by Castelnuovo's criterion. However, there do exist (nonrational) complex surfaces X of general type (e.g., Barlow surfaces) admitting an integral decomposition of the diagonal that emerge in the context of Bloch's conjecture on 0-cycles on surfaces, see §5.3 for a more detailed discussion.

4. Cubic threefolds and special cubic fourfolds

Cubic hypersurfaces of dimension 3 and 4 are some of the most important motivating objects in birational geometry since the last half of the 20th century. An irreducible cubic hypersurface is rational as soon as it has a rational singular point, unless possibly when it is a cone over a cubic hypersurface of lower dimension, see [75, Chapter 1, Section 5, Example 1.28]. Working over an algebraically closed field of characteristic not 3, we recall that in dimension 1, a cubic hypersurface is not rational if and only if it is smooth, in which case it is a curve of genus 1. In dimension 2, smooth cubic hypersurfaces are rational, and they are realized geometrically as the blow-up of six points in general position on \mathbb{P}^2 . In dimension 3, the fact that every smooth cubic hypersurface over \mathbb{C} is not rational is a celebrated theorem of Clemens and Griffith [58]. In dimension 4, some families of smooth cubics hypersurfaces are known to be rational, while the very general one is expected to be nonrational, even though not a single one is currently provably nonrational.

Cubic hypersurfaces seem to occupy a space in the birational classification of varieties that is "very close" to rational varieties, in that their familiar cohomological and birational invariants are similar to those of projective space. Proving their nonrationality seems to require the development of much finer techniques. The nonrationality of the cubic threefold was indeed one of the first counterexamples

to the Lüroth problem (see Problem 3.1.1) in characteristic zero, and the proof of its nonrationality required a deep study of algebraic cycles and the intermediate Jacobian. The study of the (non)rationality of cubic fourfolds has already attracted Hodge and moduli-theoretic techniques, and is undoubtedly one of the most famous open question in algebraic geometry.

The aim of this section is to introduce “classical” constructions arising in the study of cubic hypersurfaces. Intermediate Jacobians will be presented in the first part. In the second part, we recall the Hodge theoretic approach to moduli spaces of cubic fourfolds, and the known examples. We work here exclusively over \mathbb{C} .

4.1. Intermediate Jacobians and cubic threefolds. Recall the definition of the intermediate Jacobians of a smooth complex variety X of dimension n (see [175, Ch. 12]). Consider the Betti cohomology group $H^i(X, \mathbb{C})$ together with the Hodge filtration $F^p H^i(X, \mathbb{C})$. If $i = 2j - 1$ is odd, the j -th filtered module yields:

$$F^j H^{2j-1}(X, \mathbb{C}) = \bigoplus_{p+q=2j-1, p \geq j} H^{p,q}(X).$$

The Hodge structure on Betti cohomology then gives that $H^{2j-1}(X, \mathbb{C})$ is the sum of $F^j H^{2j-1}(X, \mathbb{C})$ and its conjugate, so that

$$H^{2j-1}(X, \mathbb{Z})/\text{Tors} \longrightarrow H^{2j-1}(X, \mathbb{C})/F^j H^{2j-1}(X, \mathbb{C}) = \overline{F^j H^{2j-1}(X, \mathbb{C})}$$

is an injective map (via de Rham cohomology). We define the $(2j - 1)$ -st intermediate Jacobian $J^{2j-1}(X)$ as the quotient of the \mathbb{C} -vector space $\overline{F^j H^{2j-1}(X, \mathbb{C})}$ by this lattice. The Jacobian is in general a complex torus, and not an Abelian variety.

If X is a threefold with $H^1(X, \mathbb{C}) = 0$, then the only nontrivial Jacobian is $J^3(X)$. Indeed, by Poincaré duality $H^1(X, \mathbb{C}) = H^5(X, \mathbb{C}) = 0$, so that $J^1(X) = 0$ and $J^5(X) = 0$. In this case, we denote $J(X) := J^3(X)$. Moreover, assume X is a Fano, or in general, a threefold with $H^1(X, \mathbb{C}) = H^{3,0}(X) = 0$, then the only nontrivial intermediate Jacobian is

$$J(X) = J^3(X) = H^{1,2}(X)/\text{im}(H^3(X, \mathbb{Z})/\text{Tors} \rightarrow H^{1,2}(X)).$$

The key idea of Clemens and Griffiths [58] is to show that in this case the complex torus $J(X)$ is an abelian variety endowed with a canonical principal polarization. Let us briefly sketch a proof of that fact, loosely following the presentation in [180, 3.1]. The cup product gives a unimodular intersection pairing $\langle -, - \rangle$ on $H^3(X, \mathbb{Z})/\text{Tors}$. Moreover, consider any nontrivial $(2, 1)$ -cohomology classes $\alpha, \beta \in H^{2,1}(X)$. Recall we assume that $H^{1,0}(X) = 0$, so that

$$(4.1) \quad \langle \alpha, \beta \rangle = 0, \quad -\sqrt{-1}\langle \alpha, \bar{\alpha} \rangle > 0,$$

since the cup product is Hermitian and skew symmetric and respects the Hodge decomposition (see [175, 7.2.1] for more details). It follows that $\langle -, - \rangle$ can be identified with the first Chern class of an ample line bundle L on $J(X)$, via the identification $\bigwedge^2 H^1(J(X), \mathbb{Z}) \simeq H^2(J(X), \mathbb{Z})$ (see [140, Ch. I, 3] for more details), and the line bundle L is well-defined up to translation. In particular $J(X)$ is an Abelian variety. Moreover, since the cup product is unimodular, $H^0(J(X), L)$ is one dimensional. Hence L is a Theta divisor for $J(X)$, which is then principally polarized. A famous result proved by Clemens and Griffiths [58] shows that one can extract a birational invariant from this Abelian variety.

THEOREM 4.1.1 (Clemens–Griffiths [58]). *If a complex threefold X is rational, then there exist smooth projective curves $\{C_i\}_{i=1}^r$ and an isomorphism of principally polarized Abelian varieties*

$$J(X) = \bigoplus_{i=1}^r J(C_i).$$

Moreover, if X is a complex threefold with $H^{3,0}(X) = H^{1,0}(X) = 0$, there is a well-defined principally polarized Abelian subvariety $A_X \subset J(X)$ which is a birational invariant: if $X' \dashrightarrow X$ is a birational map, then $A_{X'} \simeq A_X$ as principally polarized Abelian varieties.

SKETCH OF PROOF. It is enough to define A_X and prove the second statement, which is stronger. Indeed, it is easy to see that $J(\mathbb{P}^3) = 0$, so that $A_{\mathbb{P}^3} = 0$. The splitting of the intermediate Jacobian of a rational threefold will then be evident by the definition of A_X .

Clemens and Griffiths show that the category of principally polarized Abelian varieties is semisimple, that is any injective morphism is split (see [58, §3]). We work exclusively in this category. Hence we define A_X as follows: any injective map $J(C) \rightarrow J(X)$ for C a smooth curve gives a splitting $J(X) = A \oplus J(C)$. There is hence a finite number of curves $\{C_i\}_{i=1}^r$ with $J(C_i) \neq 0$ (i.e., $g(C_i) > 0$) and a splitting $J(X) = A_X \oplus \bigoplus_{i=1}^r J(C_i)$, such that there is no nontrivial morphism $J(C) \rightarrow A_X$ for any smooth projective curve C . By semisimplicity of the category of principally polarized abelian varieties, we get that A_X is well defined.

If we consider a birational morphism $\rho : Y \rightarrow X$, we can then show that $\rho^* : J(X) \rightarrow J(Y)$ is an injective map. Then $J(Y) = J(X) \oplus A$ for some Abelian variety A . If moreover ρ is the blow-up along of a point, then $J(Y) \simeq J(X)$. If ρ is the blow-up along a smooth curve C , then $J(Y) = J(X) \oplus J(C)$.

Consider the birational map $X' \dashrightarrow X$. By Hironaka's resolution of singularities, there is a smooth projective X_1 with birational morphisms $\rho_1 : X_1 \rightarrow X'$ and $\pi_1 : X_1 \rightarrow X$, such that π_1 is a composition of a finite number of smooth blow-ups. We denote $\{C_i\}_{i=1}^s$ the curves blown-up by π_1 . Similarly, there are $\rho_2 : X_2 \rightarrow X'$ and $\pi_2 : X_2 \rightarrow X'$ birational maps with π_2 a composition of a finite number of blow-ups. We denote $\{D_i\}_{i=1}^t$ the curves blown-up by π_2 . It follows that looking at the decompositions of $J(X_1)$ and $J(X_2)$ respectively, we have:

$$\begin{aligned} J(X) &\subset J(X') \oplus J(D_1) \oplus \dots \oplus J(D_t) \\ J(X') &\subset J(X) \oplus J(C_1) \oplus \dots \oplus J(C_s) \end{aligned}$$

and we conclude, by semisimplicity of the category of principally polarized abelian varieties, that we must have $A_X = A_{X'}$. Indeed the first equation gives $A_X \subset A_{X'}$, and the second one gives $A_{X'} \subset A_X$. \square

The first statement of Theorem 4.1.1 provides the *Clemens–Griffiths nonrationality criterion*, namely that if the intermediate Jacobian of a smooth projective threefold X with $h^1 = h^{3,0} = 0$ does not factor (in the category of principally polarized abelian varieties) into a product of Jacobians of curves, then X is not rational. The first application of this criterion is the proof of the nonrationality of a smooth cubic threefold [58].

THEOREM 4.1.2 (Clemens–Griffiths). *Let X be a smooth cubic threefold. The principally polarized abelian variety $J(X)$ is not split by Jacobians of curves. In particular, X is not rational.*

We will not give here a proof of Theorem 4.1.2, but just mention that it relies on the careful study of singularities the Theta-divisor of $J(X)$, which is a five-dimensional Abelian variety. Just to mention the huge amount of interesting mathematics appearing in this context, we notice that this question is also related to the *Schottky problem*, that is the study of the moduli of Jacobians inside the moduli space of principally polarized Abelian varieties.

Clemens–Griffiths nonrationality criterion applies to any threefold with trivial $H^{1,0}(X)$ and $H^{3,0}(X)$, and has allowed Beauville [23], and Shokurov [160] to completely classify rational conic bundles over minimal surfaces. We recall that a conic bundle is *standard* if the fiber over any irreducible curve is an irreducible surface (this is equivalent to relative minimality).

THEOREM 4.1.3 (Beauville, Shokurov). *Let $X \rightarrow S$ be a relatively minimal conic bundle, with X smooth, over a smooth minimal rational surface S with discriminant divisor $C \subset S$ having at most isolated nodal singularities. Then X is rational if and only if $J(X)$ is split by Jacobians of curves, and this happens only in five cases (besides projective bundles):*

- S is a plane, and C is a cubic, or a quartic, or a quintic and the discriminant double cover $\tilde{C} \rightarrow C$ is given by an even theta-characteristic in the latter case.
- S is a Hirzebruch surface and the fibration $S \rightarrow \mathbb{P}^1$ induces either a hyperelliptic or a trigonal structure $C \rightarrow \mathbb{P}^1$ on the discriminant divisor.

The proof of Theorem 4.1.3 relies on the isomorphism $J(X) \simeq \text{Prym}(\tilde{C}/C)$ as principally polarized Abelian varieties [23] and on the study of Prym varieties. Notice that Theorem 4.1.3 recovers Theorem 4.1.2 since the blow-up of a smooth cubic threefold X along any line $l \subset X$ gives a relatively minimal conic bundle $\tilde{X} \rightarrow \mathbb{P}^2$ whose discriminant divisor C is a smooth quintic and $\tilde{C} \rightarrow C$ is given by an odd theta-characteristic. As recalled $J(X) \simeq \text{Prym}(\tilde{C}/C)$, so one can fairly say then that cubic threefolds are (birationally) the non-rational conic bundles with the smallest intermediate Jacobian.

4.2. Intermediate Jacobians and the Zariski problem. Another important problem where the method of the intermediate Jacobian has been successful is in constructing the first counterexamples to Problem 3.1.2, posed by Zariski in 1949, see [156]. Indeed, using Prym variety considerations, Beauville, Colliot-Thélène, Sansuc, and Swinnerton-Dyer [25] used the intermediate Jacobian to construct the first example of a nonrational but stably rational variety, a fibration in Châtelet surfaces $V \rightarrow \mathbb{P}^1$ with affine model

$$y^2 - \delta(t)z^2 = P(x, t)$$

where $P(x, t) = x^3 + p(t)x + q(t)$ is an irreducible polynomial in $\mathbb{C}[x, t]$ such that its discriminant $\delta(t) = 4p(t)^3 + 27q(t)^2$ has degree ≥ 5 . They proved, using the intermediate Jacobian, that V is not rational, yet that $V \times \mathbb{P}^3$ is rational. Shepherd-Barron [159] used a slightly different construction to prove that $V \times \mathbb{P}^2$ is rational. It is unknown whether $V \times \mathbb{P}^1$ is rational.

The key point is that the Clemens–Griffiths criterion for irrationality of a threefold using the intermediate Jacobian is not a stable birational invariant. Indeed, it strictly applies to threefolds.

4.3. (Special) cubic fourfolds, Hodge theory and Fano schemes of lines. We turn our attention to smooth cubic fourfolds and the Hodge structure on their middle cohomology. The ideas we present in this section go back to Beauville–Donagi [26] and Hassett [95, 96]. Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. We denote by $h \in H^2(X, \mathbb{Z})$ the Betti cycle class of a hyperplane section of X . In particular, $h^4 = 3$ and

$$H^2(X, \mathbb{Z}) = \mathbb{Z}[h], \quad H^6(X, \mathbb{Z}) = \mathbb{Z}[h^3/3],$$

by the Lefschetz hyperplane theorem and Poincaré duality. One can, moreover, calculate the Hodge numbers: the (upper half) Hodge diamond of X has the following shape:

$$\begin{array}{cccccc} H^8(X, \mathbb{C}) & & & & & 1 \\ H^7(X, \mathbb{C}) & & & & 0 & 0 \\ H^6(X, \mathbb{C}) & & & 0 & 1 & 0 \\ H^5(X, \mathbb{C}) & & 0 & 0 & 0 & 0 \\ H^4(X, \mathbb{C}) & 0 & 1 & 21 & 1 & 0. \end{array}$$

We then focus on the cohomology lattice $H^4(X, \mathbb{Z})$, endowed with the intersection pairing $\langle -, - \rangle$, and we denote by $H_0^4(X, \mathbb{Z})$ the primitive cohomology sublattice. In particular, we have that $h^2 \in H^4(X, \mathbb{Z})$ and that $H_0^4(X, \mathbb{Z}) = \langle h^2 \rangle^\perp$.

Rational, algebraic, and homological equivalence all coincide for cycles of codimension 2 on any smooth projective rationally connected variety X over \mathbb{C} satisfying $H^3(X, \mathbb{Z}/l\mathbb{Z}) = 0$ for some prime l , cf. [63, Prop. 5.1]. Hence for a smooth cubic fourfold X , the Betti cycle class map $\text{CH}^2(X) \rightarrow H^4(X, \mathbb{Z})$ is injective. The image of the cycle class map is contained in the subgroup of Hodge classes $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$. In particular, $\text{CH}^2(X)$, with its intersection product, is a sublattice of $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$, which is positive definite by the Riemann bilinear relations. Furthermore, we have that the cycle class map induces an isomorphism $\text{CH}^2(X) = H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ by the integral Hodge conjecture for cycles of codimension 2 on smooth cubic fourfolds proved by Voisin [176, Thm. 18], building on [142] and [182].

To study the cohomology lattice, we consider the Fano variety of lines $F(X)$, defined to be the subvariety $F(X) \subset \text{Gr}(2, 6)$ parameterizing the lines contained in X . Then $F(X)$ is a smooth fourfold. Despite its name, $F(X)$ is an irreducible holomorphic symplectic (IHS) variety, as shown by Beauville and Donagi [26, Prop. 2].

The cohomology of $F(X)$ and X are related by an Abel–Jacobi map, as follows. Denote by $Z \subset X \times F(X)$ the universal line over X , and consider the diagram:

$$\begin{array}{ccc} & Z & \\ p \swarrow & & \searrow q \\ X & & F(X), \end{array}$$

where p and q denote the restrictions to Z of the natural projections from $X \times F(X)$ to X and $F(X)$ respectively. The Abel–Jacobi map $\alpha : H^4(X, \mathbb{Z}) \rightarrow H^2(F(X), \mathbb{Z})$ is defined as $\alpha = q_* p^*$. Since $F(X)$ is an IHS variety, $H^2(F(X), \mathbb{Z})$ is endowed with a bilinear form, which we will denote by $\langle -, - \rangle_{BB}$, the Beauville–Bogomolov form. Moreover, since $F(X) \subset \text{Gr}(2, 6)$, we can restrict the class on $\text{Gr}(2, 6)$ defining the Plücker embedding to a class $g \in H^2(F(X), \mathbb{Z})$. Now we define the primitive

cohomology $H_0^2(F(X), \mathbb{Z}) = \langle g \rangle^\perp \subset H^2(F(X), \mathbb{Z})$ to be the orthogonal complement of g with respect to the Beauville–Bogomolov form. One checks that $\alpha(h^2) = g$. Moreover, Beauville and Donagi establish an isomorphism of Hodge structures [26].

THEOREM 4.3.1 (Beauville–Donagi [26]). *Let X be a smooth cubic fourfold. The Abel–Jacobi map $\alpha : H_0^4(X, \mathbb{Z}) \rightarrow H_0^2(F(X), \mathbb{Z})$ satisfies:*

$$\langle \alpha(x), \alpha(y) \rangle_{BB} = -\langle x, y \rangle.$$

In other words, α induces an isomorphism of Hodge structures:

$$H_0^4(X, \mathbb{C}) \simeq H_0^2(F(X), \mathbb{C})(-1).$$

For a smooth projective surface S and a positive integer n , we write $S^{[n]}$ for the Hilbert scheme of length n subscheme on S , which is a smooth projective variety. Beauville and Donagi describe the deformation class of $F(X)$.

THEOREM 4.3.2 (Beauville–Donagi). *The Fano variety of lines $F(X)$ is an irreducible holomorphic symplectic variety deformation equivalent to $S^{[2]}$, where S is a degree 14 K3 surface.*

One possible interpretation of the results of Beauville and Donagi is that the variety $F(X)$ acts as a Hodge-theoretic analogue for the intermediate Jacobian of a cubic threefold.

The proof of Theorem 4.3.2 proceeds via a deformation argument to the case where X is a Pfaffian cubic fourfold, as follows. Let V be a 6-dimensional complex vector space and consider $\text{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$ via the Plücker embedding. The variety $\text{Pf}(4, \wedge^2 V^*) \subset \mathbb{P}(\wedge^2 V^*)$ is defined as (the projectivization of) the set of degenerate skew-symmetric forms on V , which is isomorphic to the set of skew symmetric 6×6 matrices with rank bounded above by 4. It is a (nonsmooth) cubic hypersurface of $\mathbb{P}(\wedge^2 V^*)$ defined by the vanishing of the Pfaffian. Now let $L \subset \mathbb{P}(\wedge^2 V)$ be a linear subspace of dimension 8, and denote by $L^* \subset \mathbb{P}(\wedge^2 V)$ its orthogonal subspace, which has dimension 5. Then taking L general enough, we have that $X = L^* \cap \text{Pf}(4, \wedge^2 V^*)$ is a smooth cubic fourfold in $L^* = \mathbb{P}^5$ and $S = L \cap \text{Gr}(2, V)$ is a smooth K3 surface in $L = \mathbb{P}^8$ with a degree 14 polarization l . Cubic fourfolds arising from this construction are called Pfaffian with associated K3 surface S . Then Beauville and Donagi prove Theorem 4.3.2 directly for pfaffian cubic fourfolds.³

THEOREM 4.3.3 (Beauville–Donagi). *Let X be a Pfaffian cubic fourfold with associated K3 surface S , not containing a plane. Then X is rational and $F(X)$ is isomorphic to $S^{[2]}$.*

Theorem 4.3.2 is then obtained by a deformation argument from Theorem 4.3.3. The proofs of the two facts stated in Theorem 4.3.3 both rely on the explicit geometric construction of X and S , and do not, on the face of it, seem to be related. However, this result hints at a deep relationship between the Fano variety of lines, K3 surfaces, and the birational geometry of cubic fourfolds.

Hassett’s work [95] is based on the study of the Hodge structure and the integral cohomology lattice of a smooth cubic fourfold X . A key observation of Beauville and Donagi is that being Pfaffian implies the existence of a rational normal quartic

³The fact that any Pfaffian cubic not containing a plane has the properties required by Beauville and Donagi’s proof was proved recently by Bolognesi and Russo [43].

scroll inside X , in fact a two dimensional family of such scrolls parameterized by S . In fact, cubic fourfolds containing rational normal quartic scrolls, and their rationality, were already considered by Fano [83]. Hassett's key idea is to consider the class of such a ruled surface in $H^4(X, \mathbb{Z})$ and the lattice-theoretic properties that one can deduce from its existence.

Consider the integral cohomology lattice $H^4(X, \mathbb{Z})$ and its sublattice $H_0^4(X, \mathbb{Z})$. Recall that $F(X)$ is a deformation of $S^{[2]}$ and that $H^2(S^{[2]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}[\delta]$, with $\langle \delta, \delta \rangle_{BB} = -2$, is an orthogonal decomposition. In particular,

$$H^2(F(X), \mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus (-2),$$

where U is the hyperbolic lattice, E_8 is the lattice associated to the Dynkin diagram of type E_8 , and (-2) is the rank one primitive sublattice generated by δ . This allows one to calculate the lattice $H_0^4(X, \mathbb{Z})$ via the Abel–Jacobi map.

PROPOSITION 4.3.4 (Hassett [95]). *The integral primitive cohomology lattice of a cubic fourfold is*

$$H_0^4(X, \mathbb{Z}) \simeq B \oplus U^{\oplus 2} \oplus E_8^{\oplus 2},$$

where B is a rank 2 lattice with intersection matrix:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

In particular, $H_0^4(X, \mathbb{Z})$ has signature $(20, 2)$.

It follows from Proposition 4.3.4 that, though $H_0^4(X, \mathbb{Z})$ has the same rank as a (Tate twist of a) K3 lattice, their signatures differ, since the latter has signature $(19, 3)$. However, one should be tempted to wonder whether, or under which conditions, it is possible to find a K3 surface S and isomorphic sublattices of signature $(19, 2)$ of $H_0^4(X, \mathbb{Z})$ and of $H^2(S, \mathbb{Z})$. On the surface side, there is a very natural (and geometrically relevant) candidate: if l is a polarization on S , then the primitive cohomology $H_0^2(S, \mathbb{Z}) = \langle l \rangle^\perp$ could be a candidate to consider.

For example, let X be a Pfaffian cubic fourfold and S an associated K3 surface with its polarization l of degree 14. As recalled, X contains a homology class of rational normal quartic scrolls parameterized by S . Let $T \in H^4(X, \mathbb{Z})$ be the cohomology class of this 2-cycle. In particular, T is not homologous to h^2 , hence we have a rank 2 primitive sublattice K , generated by T and h^2 , of $H^4(X, \mathbb{Z})$. As $T.T = c_2(N_{T/X}) = 10$, we have that the intersection matrix

$$\begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix}$$

of K has determinant 14, equal to the degree of the polarized K3 surface S associated to the Pfaffian construction of X . The key remark of Hassett is that $K^\perp \subset H^4(X, \mathbb{Z})$ and $l^\perp = H_0^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$ are isomorphic lattices (up to a Tate twist) in this case. This motivates the following definition.

DEFINITION 4.3.5 (Hassett). A cubic fourfold X is *special* if it contains an algebraic 2-cycle T , not homologous to h^2 , i.e., if the rank of $\text{CH}^2(X)$ is at least 2.

Given an abstract rank 2 positive definite lattice K with a distinguished element h^2 of self-intersection 3, a *labeling* of a special cubic fourfold is the choice of a primitive embedding $K \hookrightarrow \text{CH}^2(X)$ identifying the distinguished element with the double hyperplane section h^2 . The *discriminant* of a labeled special cubic fourfold (X, K) is defined to be the determinant of the intersection matrix of K ; it

is a positive integer. Note that a cubic fourfold could have labelings of different discriminants.

Let (X, K) be a labeled special cubic fourfold. A polarized K3 surface (S, l) is *associated* to (X, K) if there is an isomorphism of lattices $K^\perp \simeq H_0^2(S, \mathbb{Z})(-1)$.

EXAMPLE 4.3.6 (Hassett). If X is a Pfaffian cubic fourfold with associated polarized K3 surface (S, l) of degree 14, then X is special, has a labeling K of discriminant 14 defined by the class of the rational normal quartic scrolls parameterized by S , and (S, l) is associated to (X, K) .

On the other hand, Bolognesi and Russo [43, Thm. 0.2] have shown that any special cubic fourfold of discriminant 14 not containing a plane is Pfaffian. Even more has been proved in [10] by completely different techniques: any special cubic fourfold of discriminant 14 that is not Pfaffian must contain two disjoint planes.

EXAMPLE 4.3.7 (Hassett [95], [96]). Let X be a smooth cubic fourfold containing a plane $P \subset \mathbb{P}^5$. Such X is special, as P is not homologous to h^2 . Since $P \cdot P = c_2(N_{P/X}) = 3$, we have that the sublattice K generated by h^2 and P has intersection matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

so defines a labeling of discriminant 8. In general, there is no K3 surface associated to this labeled cubic fourfold (X, K) (see Theorem 4.3.8).

Consider the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ from the plane P . Restricting this projection to X gives rise to a rational map $X \dashrightarrow \mathbb{P}^2$ which can be resolved, by blowing up P , into a quadric surface bundle $\pi : \tilde{X} \rightarrow \mathbb{P}^2$, degenerating along a (generically smooth) sextic curve $C \subset \mathbb{P}^2$. The double cover $S \rightarrow \mathbb{P}^2$ branched along C is a K3 surface with a polarization of degree 2, which plays a rôle in the Hodge theory of X , but is not associated to (X, K) .

Using the period map and the Torelli Theorem (see [174]) for cubic fourfolds, one can construct a 20-dimensional (coarse) algebraic moduli space \mathcal{C} of smooth cubic fourfolds, as explained in [95, 2.2]. Using this algebraic structure, Hassett shows that the very general cubic fourfold is not special, and that the locus of special cubic fourfolds of fixed discriminant is a divisor of \mathcal{C} , which might be empty depending on the value of the discriminant. Hassett also finds further restrictions on the discriminant for special cubic fourfolds having associated K3 surfaces S and for which $F(X)$ is isomorphic to $S^{[2]}$.

THEOREM 4.3.8 (Hassett [95]). *Special cubic fourfolds of discriminant d form a nonempty irreducible divisor $\mathcal{C}_d \subset \mathcal{C}$ if and only if $d > 6$ and $d \equiv 0, 2 \pmod{6}$.*

Special cubic fourfolds of discriminant $d > 6$ have associated K3 surfaces if and only if d is not divisible by 4, 9, or any odd prime $p \equiv 2 \pmod{3}$.

Assume that $d = 2(n^2 + n + 1)$ where $n \geq 2$ is an integer, and let X be a generic special cubic fourfold of discriminant d , in which case X has an associated K3 surface S . Then there is an isomorphism $F(X) \simeq S^{[2]}$.

REMARK 4.3.9. The condition on $d > 6$ ensures that X is smooth. For completeness, the low discriminant cases are known: a cubic fourfold of discriminant 2 is determinantal (and hence is singular along a Veronese surface), see [95, 4.4]; a cubic fourfold of discriminant 6 has a single ordinary double point, see [95, 4.2]. The loci \mathcal{C}_6 and \mathcal{C}_2 do not lie in the moduli space \mathcal{C} , but rather in its boundary (see [131] and [132]).

The last statement in Theorem 4.3.8 can be made more precise, once one weakens it by asking that $F(X)$ is not isomorphic but just birational to $S^{[2]}$. The numerical necessary and sufficient condition for this was established by Addington [2].

THEOREM 4.3.10 (Addington). *Let X be a special cubic fourfold of discriminant d , with associated K3 surface S . Then $F(X)$ is birational to $S^{[2]}$ if and only if d is of the form:*

$$d = \frac{2n^2 + 2n + 2}{a^2},$$

for some n and a in \mathbb{Z} .

As noticed by Addington [2], having an associated K3 surface does not necessarily imply that $F(X)$ is birational to $S^{[2]}$. The numerical condition from the second statement of Theorem 4.3.8 is indeed strictly stronger than the numerical condition from Theorem 4.3.10. The smallest value of d for which a special cubic of discriminant d has an associated K3 surface S but $F(X)$ is not birational to $S^{[2]}$ is 74.

Let us recall the known examples of rational cubic fourfolds, in order to consider a Hodge-theoretic expectation about rationality.

EXAMPLE 4.3.11. Let X be a cubic fourfold. If either

- 2,6) X is singular, e.g. $X \in \mathcal{C}_6$ has a single node or $X \in \mathcal{C}_2$ is determinantal;
or
- 8) X contains a plane P , so that $X \in \mathcal{C}_8$, and the associated quadric surface fibration $\tilde{X} \rightarrow \mathbb{P}^2$ (see Example 4.3.7) admits a multisection of odd degree [96]; or
- 14) X is Pfaffian, so that $X \in \mathcal{C}_{14}$ [26];

then X is rational.⁴

In particular, all cubics in \mathcal{C}_2 or \mathcal{C}_6 , and the general cubic in \mathcal{C}_{14} are rational.⁵ The cubics in \mathcal{C}_8 satisfying condition 8) form a countable union of divisors in \mathcal{C}_8 [96].

Let X be a cubic containing a plane, and $\tilde{X} \rightarrow \mathbb{P}^2$ the associated quadric fibration. Having an odd section for \tilde{X} is a sufficient, but not necessary condition for rationality. Indeed, there exist Pfaffian cubics in \mathcal{C}_8 such that $\tilde{X} \rightarrow \mathbb{P}^2$ doesn't have any odd section. Such cubics are then rational, they lie in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{14}$ and were constructed in [13].

If one imagines that $H_0^4(X, \mathbb{Z})$, with its Hodge structure, plays the rôle that the intermediate Jacobian plays for cubic threefolds, then one would naturally expect that having no associated K3 surface should be an obstruction to rationality. For more on this perspective, see the recent survey [97, §3].⁶ On the other hand, there is no known example of a nonrational cubic fourfold, and few general families of rational ones. One should be very cautious when wondering whether having an

⁴A new class of rational cubic fourfolds X has very recently been constructed by Addington, Hassett, Tschinkel, and Várilly-Alvarado [4], these are in \mathcal{C}_{18} and are birational to a fibration $\tilde{X} \rightarrow \mathbb{P}^2$ in sextic del Pezzo surfaces admitting a multisection of degree prime to 3.

⁵In fact, every cubic fourfold in \mathcal{C}_{14} is rational, as was proved independently by [10] and [43].

⁶A sample result showing the interplay between Hodge theory and rationality is provided by Kulikov, who has shown that Hodge-indecomposability of the transcendental cohomology would be a sufficient condition of nonrationality for X , see [115]. However, such indecomposability was recently shown not to hold in [14].

associated K3 surface is a sufficient criterion of rationality. Hassett has asked about the existence of further examples of rational cubic fourfolds [97, Question 16].

On the other hand, as we will see in Section 8, Kuznetsov’s conjecture [123] is equivalent, at least for a generic cubic fourfold, to the statement that the rationality of X is equivalent to the existence of an associated K3, as shown by Addington and Thomas [3]. As we will see later, decompositions of the derived category of a cubic fourfold increase the amount of evidence motivating the expectation that having an associated K3 surface should be a necessary condition for rationality. Then one should read Kuznetsov conjecture and Hassett’s question [97, Question 16] as the two most “rational” or “nonrational” expectations for cubic fourfolds.

Let us end this section by recalling Galkin and Shinder’s construction [87], relying on motivic measures, which aims to describe a criterion of nonrationality. This construction would have given indeed a nonrationality criterion under the hypothesis that the class of the affine line \mathbb{L} in $K_0(\text{Var}(\mathbb{C}))$ is not a zero-divisor (see Chapter 8 for details on this Grothendieck group). Unfortunately, after Galkin and Shinder’s paper appeared, Borisov [50] proved that \mathbb{L} is indeed a zero-divisor. However, we recall Galkin–Shinder’s statement:

Assume that the class of the affine line \mathbb{L} is not a zero divisor in the Grothendieck ring $K_0(\text{Var}(\mathbb{C}))$. If a cubic fourfold X is rational, then $F(X)$ is birational to $S^{[2]}$, where S is a K3 surface.

Though based on a false assumption,⁷ the previous statement, together with Theorem 4.3.10, would say that having an associated K3 is not a sufficient condition to rationality, the first examples being cubic with discriminant 74 or 78 (see [2]). As a conclusion, we must admit that we are probably facing one of the most intriguing problems of birational geometry: not only proving that the general cubic is not rational, but also classifying the rational ones seems to need much more work and finer invariants.

5. Rationality and 0-cycles

One of the fundamental ingredients in the recent breakthrough in the stable rationality problem was to explicitly tie together the decomposition of the diagonal and the universal triviality of CH_0 . Such a link was certainly established in the work of Bloch and Srinivas. In this section, we want to explain this relationship and show how it is useful.

5.1. Diagonals and 0-cycles. We begin with the fact that CH_0 is a birational invariant of smooth proper irreducible varieties, proved by Colliot-Thélène and Coray [66, Prop. 6.3] using resolution of singularities and more generally by Fulton [84, Ex. 16.1.11] using the theory of correspondences.

LEMMA 5.1.1. *Let X and Y be smooth proper varieties over a field k . If X and Y are k -birationally equivalent then $\text{CH}_0(X) \cong \text{CH}_0(Y)$.*

PROOF. Let $f : Y \dashrightarrow X$ be a birational map and $\alpha \in \text{CH}^n(Y \times X)$ the closure of the graph of f , considered as a correspondence from Y to X . Let $\alpha' \in \text{CH}^n(X \times Y)$ be the transpose correspondence. To verify that α_* and α'_* define inverse bijections, we check that $\alpha' \circ \alpha$ is the sum of the identity (diagonal)

⁷In fact, less is required, only that \mathbb{L} does not annihilate any sum of varieties of dimension at most 2, a condition which has recently been shown to fail as well, see [129] and [99].

correspondence and other correspondences whose projections to Y are contained in proper subvarieties. By the moving lemma for 0-cycles, we can move any element in $\mathrm{CH}_0(Y)$, up to rational equivalence, away from any of these subvarieties, to where $(\alpha' \circ \alpha)_* = \alpha'_* \circ \alpha_*$ is the identity map. \square

If X is proper over k , then there is a well-defined degree map $\mathrm{CH}_0(X) \rightarrow \mathbb{Z}$. We say that $\mathrm{CH}_0(X)$ is *universally trivial* if $\deg : \mathrm{CH}_0(X_F) \rightarrow \mathbb{Z}$ is an isomorphism for every field extension F/k . This notion was first considered by in a paper by Merkurjev [138, Thm. 2.11]. Let N be a positive integer. We say that $\mathrm{CH}_0(X)$ is *universally N -torsion* if $\deg : \mathrm{CH}_0(X_F) \rightarrow \mathbb{Z}$ is surjective and has kernel killed by N for every field extension F/k .

Note that $\deg : \mathrm{CH}_0(\mathbb{P}_k^n) \simeq \mathbb{Z}$ over any field k , so that $\mathrm{CH}_0(\mathbb{P}^n)$ is universally trivial. By Lemma 5.1.1, if a smooth proper variety X is k -rational then $\mathrm{CH}_0(X)$ is universally trivial. In fact, the same conclusion holds if X is retract k -rational, in particular, stably k -rational, which can be proved using [109, Cor. RC.12], see also [69].

To check the triviality of $\mathrm{CH}_0(X_F)$ over every field extension F/k seems like quite a burden. However, it suffices to check it over the function field by the following theorem, proved in [15, Lemma 1.3].

THEOREM 5.1.2. *Let X be a geometrically irreducible smooth proper variety over a field k . Then the following are equivalent:*

- (i) *The group $\mathrm{CH}_0(X)$ is universally trivial.*
- (ii) *The variety X has a 0-cycle of degree 1 and $\deg : \mathrm{CH}_0(X_{k(X)}) \rightarrow \mathbb{Z}$ is an isomorphism.*
- (iii) *The variety X has an (integral) decomposition of the diagonal.*

PROOF. If $\mathrm{CH}_0(X)$ is universally trivial then $\mathrm{CH}_0(X_{k(X)}) = \mathbb{Z}$ and X has a 0-cycle of degree 1, by definition. Let us prove that if X has a 0-cycle P of degree 1 and $\mathrm{CH}_0(X_{k(X)}) = \mathbb{Z}$ then X has a decomposition of the diagonal. Write $n = \dim(X)$. Let $\xi \in X_{k(X)}$ be the $k(X)$ -rational point which is the image of the “diagonal morphism” $\mathrm{Spec}k(X) \rightarrow X \times_k \mathrm{Spec}k(X)$. By hypothesis, we have $\xi = P_{k(X)}$ in $\mathrm{CH}_0(X_{k(X)})$. The closures of $P_{k(X)}$ and ξ in $X \times X$ are $P \times X$ and the diagonal Δ_X , respectively. By the closure in $X \times X$ of a 0-cycle on $X_{k(X)}$, we mean the sum, taken with multiplicity, of the closures of each closed point in the support of the 0-cycle on $X_{k(X)}$. Hence the class of $\Delta_X - P \times X$ is in the kernel of the map $\mathrm{CH}^n(X \times X) \rightarrow \mathrm{CH}^n(X_{k(X)})$. Since $\mathrm{CH}^n(X_{k(X)})$ is the inductive limit of $\mathrm{CH}^n(X \times_k U)$ over all nonempty open subvarieties U of X , we have that $\Delta_X - P \times X$ vanishes in some $\mathrm{CH}^n(X \times U)$. We thus have a decomposition of the diagonal

$$\Delta_X = P \times X + Z$$

in $\mathrm{CH}^n(X \times X)$, where Z is a cycle with support in $X \times V$ for some closed subvariety $X \setminus U = V \subsetneq X$.

Now we prove that if X has a decomposition of the diagonal, then $\mathrm{CH}_0(X)$ is universally trivial. This argument is similar in spirit to the proof of 3.7.1 presented in §3.7. The action of correspondences (from §1.3) on 0-cycles has the following properties: $[\Delta_X]_*$ is the identity map and $[P \times X]_*(z) = \deg(z)P$ for any 0-cycle $z \in \mathrm{CH}_0(X)$. By the easy moving lemma for 0-cycles on a smooth variety recalled at the end of §1.2, for a closed subvariety $V \subsetneq X$, every 0-cycle on X is rationally equivalent to one with support away from V . This implies that $[Z]_* = 0$ for any

n -cycle with support on $X \times V$ for a proper closed subvariety $V \subset X$. Thus a decomposition of the diagonal $\Delta_X = P \times X + Z$ as in (3.1) implies that the identity map restricted to the kernel of the degree map $\deg : \mathrm{CH}_0(X) \rightarrow \mathbb{Z}$ is zero. For any field extension F/k , we have the base-change $\Delta_{X_F} = P_F \times X_F + Z_F$ of the decomposition of the diagonal (3.1), hence the same argument as above shows that $\mathrm{CH}_0(X_F) = \mathbb{Z}$. We conclude that $\mathrm{CH}_0(X)$ is universally trivial. \square

This result is useful because often statements about CH_0 are easier to prove than statements about CH_n . There is also a version with universal N -torsion.

THEOREM 5.1.3. *Let X be a geometrically irreducible smooth proper variety over a field k . Then the following are equivalent:*

- (i) *The group $\mathrm{CH}_0(X)$ is universally N -torsion.*
- (ii) *The variety X has a 0-cycle of degree 1 and $\deg : \mathrm{CH}_0(X_{k(X)}) \rightarrow \mathbb{Z}$ has kernel killed by N .*
- (iii) *The variety X has a rational decomposition of the diagonal of the form $N\Delta_X = N(P \times X) + Z$ for a 0-cycle P of degree 1 on X .*

Now we mention a result of Merkurjev that helped to inspire the whole theory. Recall, from 3.2, the definition of the group of unramified elements $M_{\mathrm{ur}}(X)$ of a cycle module M and that we say $M_{\mathrm{ur}}(X)$ is trivial when the natural map $M(k) \rightarrow M_{\mathrm{ur}}(X)$ is an isomorphism.

THEOREM 5.1.4 (Merkurjev [138, Thm. 2.11]). *Let X be a smooth proper variety over a field k . Then the following are equivalent:*

- (i) *$\mathrm{CH}_0(X)$ is universally trivial.*
- (ii) *$M_{\mathrm{ur}}(X)$ is universally trivial for any cycle module M .*

There is also an analogous version of Merkurjev's result for universal N -torsion. The triviality of unramified elements in cycle modules is quite useful.

COROLLARY 5.1.5. *If X is a proper smooth retract rational (e.g., stably rational) variety, then $M_{\mathrm{ur}}(X)$ is universally trivial for all cycle modules M . In particular, all unramified cohomology is universally trivial, e.g., $H_{\mathrm{\acute{e}t}}^1(X, \mu)$ and $\mathrm{Br}(X)$ are universally trivial and, if $k = \mathbb{C}$, then the integral Hodge conjecture for codimension 2 cycles holds for X .*

EXAMPLE 5.1.6. In the spirit of Mumford's theorem on 2-forms on surfaces, if X is an algebraic surface with $p_g(X) > 0$ (more generally, $\rho(X) < b_2(X)$), then $\mathrm{CH}_0(X)$ is not universally trivial and X does not have a decomposition of the diagonal. Here we use the fact, from §3.3, that $\mathrm{Br}(X) \cong (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho} \oplus H$, for H a finite group.

5.2. Rationally connected varieties. A smooth projective variety X over a field k is called *rationally connected* if for every algebraically closed field extension K/k , any two K -points of X can be connected by the image of a K -morphism $\mathbb{P}_K^1 \rightarrow X_K$.

For example, smooth geometrically unirational varieties are rationally connected. It is a theorem of Campana [52] and Kollár–Miyaoka–Mori [114] that any smooth projective Fano variety over a field of characteristic zero is rationally connected.

If X is rationally connected, then $\mathrm{CH}_0(X_K) = \mathbb{Z}$ for any algebraically closed field extension K/k . While a standard argument then proves that the kernel of

the degree map $\deg : \mathrm{CH}_0(X_F) \rightarrow \mathbb{Z}$ is torsion for every field extension F/k , the following more precise result is known.

PROPOSITION 5.2.1. *Let X be a smooth proper connected variety over a field k . Assume that X is rationally connected, or more generally, that $\mathrm{CH}_0(X_K) = \mathbb{Z}$ for all algebraically closed extensions K/k .*

- (i) (Bloch–Srinivas [37, Prop. 1]) *Then X has a rational decomposition of the diagonal.*
- (ii) (Colliot-Thélène [62, Prop. 11]) *Then there exists an integer $N > 0$ such that $\mathrm{CH}_0(X)$ is universally N -torsion.*

Of course, both of these are equivalent by Theorem 5.1.3.

In fact, over \mathbb{C} , something more general can be proved.

LEMMA 5.2.2. *Let X be a smooth proper connected variety over an algebraically closed field k of infinite transcendence degree over its prime field (e.g., $k = \mathbb{C}$). If $\mathrm{CH}_0(X) = \mathbb{Z}$ then there exists an integer $N > 0$ such that $\mathrm{CH}_0(X)$ is universally N -torsion.*

PROOF. The variety X is defined over an algebraically closed subfield $L \subset k$, with L algebraic over a field finitely generated over its prime field. That is, there exists a variety X_0 over L with $X \cong X_0 \times_L k$. Let η be the generic point of X_0 . Let P be an L -point of X_0 . One may embed the function field $F = L(X_0)$ into k , by the transcendence degree hypothesis on k . Let K be the algebraic closure of F inside k . By Lemma 5.2.3 (below) and the hypothesis that $\mathrm{CH}_0(X) = \mathbb{Z}$, we have that $\mathrm{CH}_0(X_0 \times_L F) = \mathbb{Z}$. This implies that there is a finite extension E/F of fields such that $\eta_E - P_E = 0$ in $\mathrm{CH}_0(X_0 \times_L E)$. Taking the corestriction (i.e., pushforward) to F , one finds that $N(\eta_F - P_F) = 0$ in $\mathrm{CH}_0(X_0 \times_L F)$, hence in $\mathrm{CH}_0(X)$ as well. As in the proof of Theorem 5.1.2, we conclude that $\mathrm{CH}_0(X)$ is universally N -torsion. \square

LEMMA 5.2.3. *Let X be a smooth projective connected variety over k . If K/k is an extension of fields, then the kernel of the natural map $\mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(X_K)$ is torsion. If k is algebraically closed, then $\mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(X_K)$ is injective.*

PROOF. Let z be a 0-cycle on X that becomes rationally equivalent to zero on X_K . Then there exists a subextension L of K/k that is finitely generated over k , such that z becomes rationally equivalent to zero on X_L . In fact, we can find a finitely generated k -algebra A with fraction field L such that z maps to zero under $\mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(X \times_k U)$ where $U = \mathrm{Spec} A$. When k is algebraically closed, there exists a k -point of U , defining a section of $\mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(X \times_k U)$, showing that z is zero in $\mathrm{CH}_0(X)$. In general, we can find a rational point of U over a finite extension k'/k , so that $z_{k'}$ is zero in $\mathrm{CH}_0(X_{k'})$, from which we conclude that a multiple of z is zero in $\mathrm{CH}_0(X)$ by taking corestriction. \square

There exist rationally connected varieties X over an algebraically closed field of characteristic zero with $\mathrm{CH}_0(X)$ not universally trivial. Indeed, let X be a unirational threefold with $H_{\mathrm{ur}}^2(X, \mathbb{Q}/\mathbb{Z}(1)) \cong \mathrm{Br}(X) \neq 0$, see e.g., [9]. Then by Theorem 5.1.3, $\mathrm{CH}_0(X)$ is not universally trivial.

However, such examples do not disprove the natural universal generalization of the result of Campana [52] and Kollár–Miyaoaka–Mori [114], and this was posed as a question in [15, §1].

QUESTION 5.2.4. *Does there exist a smooth Fano variety X over an algebraically closed field of characteristic 0 with $\mathrm{CH}_0(X)$ not universally trivial?*

After this question was posed, Voisin [177] constructed the first examples of (smooth) Fano varieties over \mathbb{C} with $\mathrm{CH}_0(X)$ not universally trivial, see §7.3 for more details.

5.3. Surfaces. We briefly recall Bloch's conjecture for a complex surface. Let X be a smooth projective variety. The Albanese morphism $\mathrm{alb}_X : X \rightarrow \mathrm{Alb}(X)$ is universal for morphisms from X to an abelian variety. It extends to the Albanese map

$$\mathrm{alb}_X : A_0(X) \rightarrow \mathrm{Alb}(X)$$

where $A_0(X)$ denotes the kernel of the degree map $\mathrm{CH}_0(X) \rightarrow \mathbb{Z}$. The Albanese map is surjective on geometric points. In characteristic zero, the dimension of $\mathrm{Alb}(X)$ is $q(X) = h^1(X, \mathcal{O}_X)$. Recall that $p_g(X) = h^0(X, \Omega_X^n)$ where $n = \dim(X)$.

CONJECTURE 5.3.1 (Bloch's conjecture). *Let X be a smooth projective surface over \mathbb{C} . If $p_g(X) = 0$ then the Albanese map $\mathrm{alb}_X : A_0(X) \rightarrow \mathrm{Alb}(X)$ is injective. In particular, if $p_g(X) = q(X) = 0$, then $A_0(X) = 0$, i.e., $\mathrm{CH}_0(X) = \mathbb{Z}$.*

In fact, Bloch's conjecture is proved for all surfaces that are not of general type by Bloch, Kas, and Lieberman [35].

Of course, rational surfaces satisfy $p_g = q = 0$ and have $A_0(X) = 0$. There do exist nonrational surfaces with $p_g = q = 0$ and for which $A_0(X) = 0$. Enriques surfaces were the first examples, extensively studied in [80], [81, p. 294] with some examples considered earlier in [152], see also [55]. An Enriques surface has Kodaira dimension 0. We remark that for an Enriques surface X , we have that

$$H_{\mathrm{ur}}^1(X, \mathbb{Z}/2\mathbb{Z}) = H_{\mathrm{ét}}^1(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \text{ and } H_{\mathrm{ur}}^2(X, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(X) = \mathbb{Z}/2\mathbb{Z}.$$

Hence $\mathrm{CH}_0(X)$ is not universally trivial and X does not have a decomposition of the diagonal by Theorem 5.1.4.

The first surfaces of general type with $p_g = q = 0$ were constructed in [54] and [89]. Simply connected surfaces X of general type for which $p_g = 0$ were constructed by Barlow [20], who also proved that $\mathrm{CH}_0(X) = \mathbb{Z}$ for some of them. See also the recent work on Bloch's conjecture by Voisin [172].

We want to explore the universal analogue of Bloch's conjecture, i.e., to what extent does $p_g = q = 0$ imply universal triviality of $\mathrm{CH}_0(X)$.

The following result was stated without detailed proof as the last remark of [37]. The first proof appeared in [15, Prop. 1.19] using results of [70] and a different proof appear later in [178, Cor. 2.2].

PROPOSITION 5.3.2. *Let X be a smooth proper connected surface over \mathbb{C} . Suppose that all groups $H_{\mathbb{B}}^i(X, \mathbb{Z})$ are torsion free and that $\mathrm{CH}_0(X) = \mathbb{Z}$. Then $\mathrm{CH}_0(X)$ is universally trivial and admits a decomposition of the diagonal.*

PROOF. By Lemma 5.2.2, we have that $\mathrm{CH}_0(X)$ is universally N -torsion. Hence by Lemma 3.7.1, we have that $H^i(X, \mathcal{O}_X) = 0$ for all $i \geq 1$. Thus $p_g(X) = q(X) = 0$ and hence $b_3(X) = b_1(X) = 2q(X) = 0$, so that $H_{\mathbb{B}}^i(X, \mathbb{Z})$ consists purely of cycle classes of algebraic cycles.

The torsion-free hypothesis on cohomology allows one to use the work of Colliot-Thélène and Raskind [70, Thm. 3.10(d)] on the cohomology of the Milnor K -theory sheaf, to conclude that $\mathrm{CH}_0(X)$ is universally trivial.

The torsion-free hypothesis on cohomology allows Voisin [178, Cor. 2.2] to argue using the integral Künneth decomposition of the diagonal (see Remark 5.3.3), that X admits a decomposition of the diagonal. \square

We remark that if X is a smooth proper connected surface over \mathbb{C} with torsion free Néron–Severi group $\text{NS}(X)$, then all Betti cohomology groups are torsion free, hence Proposition 5.3.2 applies. Indeed, the torsion in $H_{\mathbb{B}}^1(X, \mathbb{Z})$ is clearly trivial and is dual to the torsion in $H_{\mathbb{B}}^3(X, \mathbb{Z})$, while the torsion in $\text{NS}(X)$ is isomorphic to the torsion in $H_{\mathbb{B}}^2(X, \mathbb{Z})$.

REMARK 5.3.3. If $H_{\mathbb{B}}^i(X, \mathbb{Z})$ is torsion free for all $0 \leq i \leq n$, then there is an integral Künneth decomposition

$$H_{\mathbb{B}}^n(X \times X, \mathbb{Z}) = \bigoplus_{i+j=n} H_{\mathbb{B}}^i(X, \mathbb{Z}) \otimes H_{\mathbb{B}}^j(X, \mathbb{Z}).$$

This follows from the degeneration of the Künneth spectral sequence with coefficients in \mathbb{Z} .

We point out that the simply connected surfaces X of general type satisfying $p_g = q = 0$ and $\text{CH}_0(X) = 0$ mentioned above, e.g., Barlow surfaces, satisfy $\text{Pic}(X) = \text{NS}(X)$ is torsion free, hence Proposition 5.3.2 applies. While the group $\text{CH}_0(X)$ is universally trivial, these surfaces are far from being rational, since they are of general type.

The interested reader can find how to adapt Proposition 5.3.2 over an algebraically closed field of infinite transcendence degree over its prime field.

Finally, we mention that Proposition 5.3.2 has been generalized by Kahn [107], and independently by Colliot-Thélène using [70], to a determination of the minimal N for which $\text{CH}_0(X)$ is universally N -torsion, which turns out to be the exponent of $\text{NS}(X)$. In general, the minimal $N \geq 1$ for which $\text{CH}_0(X)$ is universally N -torsion is a stable birational invariant of smooth proper varieties; its properties are explored in [56], where it is called the *torsion order* of X .

6. Categorical representability and rationality, the case of surfaces

This section consists of two main parts. In the first part, we define the notion of categorical representability and begin to classify (or at least, give criteria to discriminate) categories which are representable in low dimension. The second part is devoted to the applications in the case of surfaces.

6.1. Categorical representability. Using semiorthogonal decompositions, one can define a notion of *categorical representability* for a triangulated category. In the case of smooth projective varieties, this is inspired by the classical notions of representability of cycles, see [30].

DEFINITION 6.1.1. A k -linear triangulated category \mathbb{T} is *representable in dimension* m if it admits a semiorthogonal decomposition

$$\mathbb{T} = \langle \mathbf{A}_1, \dots, \mathbf{A}_r \rangle,$$

and for each $i = 1, \dots, r$ there exists a smooth projective connected k -variety Y_i with $\dim Y_i \leq m$, such that \mathbf{A}_i is equivalent to an admissible subcategory of $\text{D}^b(Y_i)$.

We use the following notation

$$\text{rdim } \mathbb{T} := \min\{m \in \mathbb{N} \mid \mathbb{T} \text{ is representable in dimension } m\},$$

whenever such a finite m exists.

DEFINITION 6.1.2. Let X be a smooth projective k -variety. We say that X is *categorically representable* in dimension (resp. codimension) m if $D^b(X)$ is representable in dimension m (resp. dimension $\dim(X) - m$).

We will use the following notations:

$$\mathrm{rdim}(X) := \mathrm{rdim} D^b(X) \quad \mathrm{rcodim}(X) := \dim(X) - \mathrm{rdim} D^b(X),$$

and notice that they are both integer numbers.

We notice that, by definition, if $\mathrm{rdim} T = n$, then T is representable in any dimension $m \geq n$.

LEMMA 6.1.3. *Let T be a k -linear triangulated category. If T is representable in dimension n , then T is representable in dimension m for any $m \geq n$.*

REMARK 6.1.4. **Warning!** Suppose that T is representable in dimension n via a semiorthogonal decomposition $T = \langle A_1, \dots, A_r \rangle$, and let $T = \langle B_1, \dots, B_s \rangle$ be another semiorthogonal decomposition (that is, B_i is not admissible in A_j and A_j is not admissible in B_i for any i and j). As recalled in Proposition 2.3.14 the Jordan–Hölder property for semiorthogonal decompositions does not hold in general. It follows that one does not know in general whether the B_i are also representable in dimension n , and counterexamples are known: in Bondal–Kuznetsov [124] there is a threefold X with a full exceptional sequence $\langle E_1, \dots, E_6 \rangle$, and another exceptional object F whose complement cannot be generated by exceptional objects.

Let us record a simple corollary of Theorem 2.3.1.

LEMMA 6.1.5. *Let $X \rightarrow Y$ be the blow-up of a smooth projective k -variety along a smooth center. Then $\mathrm{rcodim}(X) \geq \max\{\mathrm{rcodim}(Y), 2\}$. In particular, if $\mathrm{rcodim}(Y) \geq 2$, then $\mathrm{rcodim}(X) \geq 2$.*

PROOF. This is a consequence of Theorem 2.3.1. Denoting by $Z \subset Y$ the center of the blow-up we have $\mathrm{rdim}(X) \leq \max\{\mathrm{rdim}(Y), \mathrm{rdim}(Z)\}$ and the statement follows since $\dim(X) = \dim(Y)$ and by the fact that Z has codimension at least 2 in Y . \square

Inspired by the proof of Theorem 4.1.1, one can consider a birational map $X \dashrightarrow X'$ and its resolutions: by Hironaka’s resolution of singularities, there is a smooth projective X_1 with birational morphisms $\rho_1 : X_1 \rightarrow X'$ and $\pi_1 : X_1 \rightarrow X$, such that π_1 is a composition of a finite number of smooth blow-ups. Similarly, there are $\rho_2 : X_2 \rightarrow X$ and $\pi_2 : X_2 \rightarrow X'$ birational morphisms with π_2 a composition of a finite number of blow-ups. By Lemma 2.3.2 we have that $D^b(X')$ is admissible in $D^b(X_1)$, and $D^b(X)$ is admissible in $D^b(X_2)$. Lemma 6.1.5 gives bounds for $\mathrm{rcodim}(X_1)$ and $\mathrm{rcodim}(X_2)$ in terms of $\mathrm{rcodim}(X)$ and $\mathrm{rcodim}(X')$ respectively.

Based on these considerations, Kuznetsov [126] argues that if one could properly define an admissible subcategory GK_X of $D^b(X)$, maximal (with respect to the inclusion ordering) with respect to the property $\mathrm{rdim} \mathrm{GK}_X \geq \dim(X) - 1$, then such a category would be a birational invariant, which we would call *the Griffiths–Kuznetsov component* of X . In particular, since $\mathrm{rdim}(\mathbb{P}^n) = 0$, we would have that the Griffiths–Kuznetsov component of a rational variety is trivial.

Even if the Griffiths–Kuznetsov component is not well-defined, we have that if X is rational, then $D^b(X)$ is admissible in a category with $\mathrm{rdim} \leq \dim(X) - 2$. As we

recalled in Remark 6.1.4, there is no known reason to deduce that $\text{rcodim}(X) \geq 2$. However, in the small dimensional cases, we have a stronger understanding of these phenomena. We will come back to this question, giving more detailed arguments for threefolds and examples for fourfolds, in §8.

Representability in dimension 0.

PROPOSITION 6.1.6. *Let \mathbb{T} be a k -linear triangulated category. $\text{rdim } \mathbb{T} = 0$ if and only if there exists a semiorthogonal decomposition*

$$\mathbb{T} = \langle \mathbb{A}_1, \dots, \mathbb{A}_r \rangle,$$

such that for each i , there is a k -linear equivalence $\mathbb{A}_i \simeq \mathbb{D}^b(K_i/k)$ for an étale k -algebra K_i .

An additive category \mathbb{T} is *indecomposable* if for any product decomposition $\mathbb{T} \simeq \mathbb{T}_1 \times \mathbb{T}_2$ into additive categories, we have that $\mathbb{T} \simeq \mathbb{T}_1$ or $\mathbb{T} \simeq \mathbb{T}_2$. Equivalently, \mathbb{T} has no nontrivial completely orthogonal decomposition. We remark that for a k -scheme X the category $\mathbb{D}^b(X)$ is indecomposable if and only if X is connected (see [51, Ex. 3.2]). More is known if X is the spectrum of a field or a product of fields, see [11].

LEMMA 6.1.7. *Let K be a k -algebra.*

- (i) *If K is a field and \mathbb{A} is a nonzero admissible k -linear triangulated subcategory of $\mathbb{D}^b(k, K)$, then $\mathbb{A} = \mathbb{D}^b(k, K)$.*
- (ii) *If $K \cong K_1 \times \dots \times K_n$ is a product of field extensions of k and \mathbb{A} is a nonzero admissible indecomposable k -linear triangulated subcategory of $\mathbb{D}^b(k, K)$, then $\mathbb{A} \simeq \mathbb{D}^b(k, K_i)$ for some $i = 1, \dots, n$.*
- (iii) *If $K \cong K_1 \times \dots \times K_n$ is a product of field extensions of k and \mathbb{A} is a nonzero admissible k -linear triangulated subcategory of $\mathbb{D}^b(k, K)$, then $\mathbb{A} \simeq \prod_{j \in I} \mathbb{D}^b(k, K_j)$ for some subset $I \subset \{1, \dots, n\}$.*

PROOF OF PROPOSITION 6.1.6 (SEE ALSO [11]). The smooth k -varieties of dimension 0 are precisely the spectra of étale k -algebras. Hence the semiorthogonal decomposition condition is certainly sufficient to get $\text{rdim } \mathbb{T} = 0$. On the other hand, if $\text{rdim } \mathbb{T} = 0$, we have such a semiorthogonal decomposition with each \mathbb{A}_i an admissible subcategory of the derived category of an étale k -algebra. By Lemma 6.1.7(iii), we have that \mathbb{A}_i is thus itself such a category. \square

We have the following corollary of Proposition 6.1.6.

LEMMA 6.1.8. *Let \mathbb{T} be a k -linear triangulated category. If $\text{rdim } \mathbb{T} = 0$, then $K_0(\mathbb{T})$ is a free \mathbb{Z} -module of finite rank. In particular, if X is smooth and projective and $\text{rdim}(X) = 0$, we have that $\text{CH}^1(X)$ is torsion-free of finite rank.*

PROOF. The only non-trivial statement is the last one, which can be proved, as in [88, Lemma 2.2], using the topological filtration on $K_0(X)$. \square

REMARK 6.1.9. Lemma 6.1.8 gives useful criterion: if $K_0(\mathbb{T})$ has torsion elements or if it is not of finite rank, then $\text{rdim } \mathbb{T} > 0$.

In the cases where $\mathbb{T} = \mathbb{D}^b(X)$ for a smooth projective X , if $\text{rdim}(X) = 0$, there are much more consequences that can be obtained using non-commutative motives. For example, when $k \subset \mathbb{C}$ is algebraically closed, the even de Rham cohomology and all the Jacobians are trivial [135, 32] and the (rational) Chow motive is of Lefschetz type [135].

Representability in dimension 1.

PROPOSITION 6.1.10. *Let \mathbb{T} be a k -linear triangulated category. $\text{rdim } \mathbb{T} \leq 1$ if and only if \mathbb{T} admits a semiorthogonal decomposition whose components belong to the following list:*

- (i) *categories representable in dimension 0, or*
- (ii) *categories of the form $\text{D}^b(k, \alpha)$, for α in $\text{Br}(k)$ the Brauer class of a conic, or*
- (iii) *categories equivalent to $\text{D}^b(C)$ for some smooth k -curve C .*

The main tool in the proof of the previous statement is the indecomposability result for curves due to Okawa that we recalled in Theorem 2.3.3.

PROOF OF PROPOSITION 6.1.10. We already classified triangulated categories representable in dimension 0. We are hence looking for categories \mathbb{A} with $\text{rdim } \mathbb{A} = 1$. Of course, if $\mathbb{A} = \text{D}^b(C)$ for some curve, then $\text{rdim } \mathbb{A} \leq 1$ and we are done.

Using Theorem 2.3.3, if \mathbb{A} is a nontrivial triangulated category with a full and faithful functor $\phi : \mathbb{A} \rightarrow \text{D}^b(C)$ with right and left adjoints, then either ϕ is an equivalence, or $g(C) = 0$. In the latter case, let α be the class of C in $\text{Br}(k)$, which is trivial if and only if $C = \mathbb{P}^1$. By [28], there is a semiorthogonal decomposition $\text{D}^b(C) = \langle \text{D}^b(k), \text{D}^b(k, \alpha) \rangle$. It is not difficult to see that this is the only possible semiorthogonal decomposition up to mutations, so we get the proof. \square

Let us sketch a criterion of representability in dimension 1, based on the Grothendieck group, in the case where $k = \mathbb{C}$. Given a \mathbb{Z} -module M and an integer number $n > 0$, we will denote by $M[n] \subset M$ the kernel of the multiplication by n map $M \xrightarrow{\times n} M$. Such $M[n]$ has a natural structure of $\mathbb{Z}/n\mathbb{Z}$ -module. We notice that, if X is a smooth (connected) projective variety of dimension ≤ 1 , the modules $K_0(X)[n]$ are well known. Indeed, either X is a point, or X is \mathbb{P}^1 , or X is a curve of positive genus g . In the first two cases, $K_0(X)$ is free of finite rank, hence $K_0(X)[n] = 0$ for any n . In the latter case, $K_0(X) \simeq \mathbb{Z} \oplus \text{Pic}(X) = \mathbb{Z} \oplus \text{Pic}^0(X) \oplus \mathbb{Z}$ by the Grothendieck–Riemann–Roch Theorem, and the fact that the 1st Chern class is integral. Then $K_0(X)[n] = \text{Pic}^0(X)[n]$. Since $\text{Pic}^0(X)$ is a complex torus of dimension g , we have that $\text{Pic}^0(X)[n] = (\mathbb{Z}/n\mathbb{Z})^{2g}$ (see, e.g., [140, §I.1(3)]).

LEMMA 6.1.11. *Suppose that \mathbb{T} is \mathbb{C} -linear and $\text{rdim } \mathbb{T} \leq 1$. Then, for any integer n , we have that $K_0(\mathbb{T})[n]$ is a free $\mathbb{Z}/n\mathbb{Z}$ -module of finite even rank.*

PROOF. Proposition 6.1.10 gives us all the possible components of a semiorthogonal decomposition of \mathbb{T} . If \mathbb{A} is one of such components, it follows that $K_0(\mathbb{A})[n]$ is either trivial or $(\mathbb{Z}/n\mathbb{Z})^{2g}$ if $\mathbb{A} = \text{D}^b(C)$ and $g = g(C)$. \square

Finally, let us just record a very simple remark, as a corollary of Theorem 2.3.3.

COROLLARY 6.1.12. *A smooth projective curve C is k -rational if and only if $\text{rdim}(C) = 0$.*

A glimpse on representability in dimension 2. It is more difficult to classify categories that are representable in dimension 2. Of course, for any surface S , $\mathbb{T} = \text{D}^b(S)$ satisfies $\text{rdim } \mathbb{T} \leq 2$, but it is a quite challenging question to understand which categories can occur as proper admissible subcategories of surfaces. Using the results of Kawatani–Okawa recalled in Theorem 2.3.4, we need essentially only consider surfaces S that are either ruled or with $p_g = q = 0$, at least in the case

where $k = \mathbb{C}$. In the ruled case, say $S \rightarrow C$, we have $\text{rdim}(S) = \text{rdim}(C) \leq 1$ with strict inequality holding only for $C = \mathbb{P}^1$. Hence we will focus on surfaces with $p_g = q = 0$.

Notice that, for a smooth projective surface S , any line bundle is k -exceptional if and only if S satisfies $p_g = q = 0$. This is a simple calculation using that line bundles are invertible and the definition of p_g and q . It is then natural to study exceptional collections on such surfaces and describe the corresponding semiorthogonal decompositions. These decompositions are conjecturally related to rationality criteria for surfaces, and we will treat them extensively in §6.2. We also remark that an exceptional collection on a variety X gives a free subgroup of finite rank of $K_0(X)$. Surfaces of general type with $p_g = q = 0$ containing torsion elements in $K_0(S)$ are known, and hence they cannot have a full exceptional sequence. This remark gives rise to the definition of phantom and quasi-phantom categories.

DEFINITION 6.1.13. Let X be a smooth projective variety over k . An admissible subcategory $\mathsf{T} \subset \mathsf{D}^b(X)$ is called a *quasi-phantom* if its Hochschild homology $HH_*(\mathsf{T}) = 0$ vanishes and $K_0(\mathsf{T})$ is a finite abelian group. A quasi-phantom T is a *phantom* if $K_0(\mathsf{T}) = 0$. A phantom is a *geometric phantom* if T_K is still a phantom for any algebraically closed field K containing k . A phantom is a *universal phantom* if, for any smooth and projective variety Y , the admissible subcategory $\mathsf{T} \boxtimes \mathsf{D}^b(Y)$ of $\mathsf{D}^b(X \times Y)$ is a phantom.

Notice that if T is a quasi-phantom (or a phantom or universal phantom), then $\text{rdim } \mathsf{T} > 1$.

EXAMPLE 6.1.14 (Categories representable in dimension 2). Here is a list of categories T such that $\text{rdim } \mathsf{T} = 2$, though T is not equivalent to $\mathsf{D}^b(S)$ for any smooth projective surface S .

Phantoms, $k = \mathbb{C}$. Let S be a determinantal Barlow surface and T the orthogonal complement to an exceptional collection of length 11 [42]. Let S be a Dolgachev surface of type $X_9(2, 3)$ and T the orthogonal complement to an exceptional collection of length 12 [57] (we refer to [79] for the notations on Dolgachev surfaces).

Quasi-phantoms, $k = \mathbb{C}$. Since the first example of the classical Godeaux surface [40], there are now many examples of quasi-phantoms as orthogonal complements of an exceptional sequence of line bundles of maximal length on surfaces of general type, see [86], [88], [112], [133], [6], [76].

Not quasi-phantoms, $k = \mathbb{C}$. Let S be an Enriques surface [105] and T the orthogonal complement to an exceptional collection of length 10. Let S be a classical Godeaux surface [41] and T the orthogonal complement to an exceptional collection of length 9. Both categories have $K_0(\mathsf{T}) \otimes \mathbb{Q} \simeq \mathbb{Q}^2$, but $K_0(\mathsf{T})$ not free, and do not admit any exceptional object [41, 168].

Not quasi-phantoms, general k . Let $\alpha \in \text{Br}(k)$ be the class of a Severi–Brauer surface or the class of a Severi–Brauer threefold with an involution surface and $\mathsf{T} = \mathsf{D}^b(k, \alpha)$. Let \mathcal{C}_0 be the Clifford algebra of an involution surface and $\mathsf{T} = \mathsf{D}^b(k, \mathcal{C}_0)$. Let Q and B be the simple algebras (quadratic over a degree 3 extension of k and cubic over a degree 2 extension of k , respectively) associated to a minimal del Pezzo surface of degree 6 and $\mathsf{T} = \mathsf{D}^b(k, Q)$ or $\mathsf{T} = \mathsf{D}^b(k, B)$. For all these examples, $K_0(\mathsf{T})$ is free of finite rank, see [11]. If S is a minimal del Pezzo surface of degree

$d < 5$, then \mathbb{T} is the semiorthogonal complement of $\langle \mathcal{O}_S \rangle$, and is not of the form $D^b(K, \alpha)$ for any K/k étale and α in $\text{Br}(K)$, see [11].

The result of Kawatani–Okawa, cf. Theorem 2.3.4, suggests that, for $k = \mathbb{C}$, categories representable in dimension exactly 2, and not equivalent to any $D^b(S)$, should occur only in the case where $p_g = q = 0$. In this case, Bloch’s conjecture (see §5.3) would imply that $K_0(S) \otimes \mathbb{Q}$ is a finite vector space, so it is natural to raise the following.

CONJECTURE 6.1.15. *Suppose that $\text{rdim } \mathbb{T} = 2$, and that \mathbb{T} is not equivalent to $D^b(S)$ for any surface S . Then $K_0(\mathbb{T})_{\mathbb{Q}}$ is a finite-dimensional vector space.*

6.2. Rationality questions for surfaces. We turn our attention to the possibility of characterizing rational surfaces via categorical representability. A folklore conjecture by D. Orlov states that a complex surface with a full exceptional collection is rational. We provide here a version for any field k in terms of categorical representability.

CONJECTURE 6.2.1 (Orlov). *A smooth projective surface S over a field k is k -rational if and only if $\text{rdim}(S) = 0$.*

If k is algebraically closed, then being representable in dimension zero is equivalent to having a full exceptional collection. Combining Theorem 2.3.1 and Proposition 2.3.8 it is easy to check that a rational surface has a full exceptional collection, since it is a blow-up along smooth points of either a projective space or a Hirzebruch surface.

If k is not algebraically closed, it is easy to construct rational surfaces without a full exceptional collection, for example a k -rational quadric surface of Picard rank 1, or by blowing up a closed point of degree > 1 on a k -rational surface. In fact, the “only if” part of Conjecture 6.2.1 remains true, thanks to the results of [11] for del Pezzo surfaces.

THEOREM 6.2.2. *Let S be any smooth k -rational surface. Then $\text{rdim}(S) = 0$.*

The converse is more difficult. Let us first recall the following result based on a base change formula by Orlov [147].

LEMMA 6.2.3 ([12], Lemma 2.9). *Let X be a smooth projective variety over k , and K a finite extension of k . Suppose that A_1, \dots, A_n are admissible subcategories of $D^b(X)$ such that $D^b(X_K) = \langle A_{1K}, \dots, A_{nK} \rangle$. Then $D^b(X) = \langle A_1, \dots, A_n \rangle$.*

Using Lemma 6.2.3 and the classification from Proposition 6.1.6, we deduce that, in the case $k \subset \mathbb{C}$, it is enough to check Conjecture 6.2.1 for geometrically rational surfaces and for complex surfaces with $p_g = q = 0$. As remarked earlier, these are the only surfaces where a line bundle is k -exceptional.

Geometrically rational surfaces. The first case is handled in [11] for del Pezzo surfaces and in [168] for geometrically rational surfaces with a numerically k -exceptional collection of maximal length.

THEOREM 6.2.4. *Let S be a geometrically rational surface. If either S*

- *is a blow-up of a del Pezzo and $\text{rdim}(S) = 0$, or*
- *has a (numerically) k -exceptional collection of maximal length,*

then S is k -rational.

The results of [11] also provide a categorical birational invariant. Recall, from Proposition 2.3.5, the definition of the category \mathbf{A}_X for a Fano variety X of Picard rank 1.

THEOREM 6.2.5 ([11]). *Let S be a minimal del Pezzo surface of degree d . If $d < 5$, then \mathbf{A}_S is a birational invariant. If $d \geq 5$, then the product of components \mathbf{T} of \mathbf{A}_S with $\text{rdim } \mathbf{T} > 0$ is a birational invariant. In particular, there is a well-defined Griffiths–Kuznetsov component.*

Let us conclude by showing that having a full k -exceptional collection is a stronger property than having a decomposition of the diagonal. The following result is a slight generalization of a result of Vial [168]. As we will see, nonrational surfaces satisfying the assumptions of Theorem 6.2.6 exist, and are known to have a decomposition of the diagonal, as recalled in Theorem 5.3.2. Here we give a direct proof, adapted from Vial, providing an explicit decomposition of the diagonal from the exceptional objects.

THEOREM 6.2.6. *Let S be a smooth projective surface with $p_g = q = 0$ and admitting a semiorthogonal decomposition*

$$(6.1) \quad \mathbf{D}^b(S) = \langle \mathbf{A}, E_1, \dots, E_r \rangle,$$

where E_i are k -exceptional and \mathbf{A} is a geometric phantom. Then S has a decomposition of the diagonal and the integral Chow motive of X is of Lefschetz type.

PROOF. If $\mathbf{A} = 0$, the second statement is a result of Vial [168, Thm. 2.7], while the first statement is shown in the course of Vial’s proof. We will detail the main steps of Vial’s argument to show that having a nontrivial phantom (which Vial does not address) does not affect the proof.

1st Step. First of all, the semiorthogonal decomposition implies that $K_0(S)$ is free of finite rank. Using the topological filtration on K_0 , one can show that the integral Chow ring $\text{CH}^*(S)$ is then also free of finite rank (cf. [168, Lemma 2.6]). Indeed, we have that $\text{CH}^0(S) = \mathbb{Z}$, $\text{CH}^1(S) = \text{Pic}(S)$ is free of finite rank by Lemma 6.1.8, and $\text{CH}^2(S)$ is free of finite rank since it coincides with the second graded piece of the topological filtration, which is a subgroup of $K_0(S)$, as S has dimension 2. In particular, the group of 0-cycles $A_0(S)$ of degree 0, is free of finite rank. Since \mathbf{A} is a geometric phantom, all the above remarks hold for $S_{\bar{k}}$, in particular, $A_0(S_{\bar{k}})$ is free of finite rank. However, it is well known that $A_0(S_{\bar{k}})$ is divisible (cf. [34, Lec. 1, Lemmas 1.3 and 1.4]), which implies that $A_0(S_{\bar{k}}) = 0$. Since the kernel of the scalar extension map $A_0(S) \rightarrow A_0(S_{\bar{k}})$ is torsion (see Lemma 5.2.3), we conclude that $A_0(S) = 0$.

Moreover, for any algebraically closed field extension K/k , the admissible subcategory \mathbf{A}_K of $\mathbf{D}^b(S_K)$ is a phantom in the base change of the decomposition (6.1). As before, this implies that all remarks above hold for S_K , in particular, $K_0(S_K)$ and $\text{CH}^*(S_K)$ are free of finite rank and $r = \text{rk}(K_0(S_K)) = \rho + 2$, where ρ is the Picard rank of S_K .

2nd Step. Perling [149] shows that any numerically k -exceptional collection of maximal length on a surface with $\chi(\mathcal{O}_S) = 1$ can be mutated into a numerically k -exceptional collection of maximal length consisting of objects of rank 1. Given such an exceptional collection, Vial [168, Prop. 2.3] provides a \mathbb{Z} -basis D_1, \dots, D_ρ of $\text{CH}^1(S_K) = \text{Pic}(S_K)$ with unimodular intersection matrix M and dual basis

$D_1^\vee, \dots, D_\rho^\vee$. This is accomplished using Chern classes and the Riemann–Roch formula to compare χ with the intersection pairing.

3rd Step. As shown by Vial [168, Cor. 2.5], a surface with $\chi(\mathcal{O}_S) = 1$ and a numerically k -exceptional collection of maximal length has a zero-cycle of degree 1. Hence S_K has always a zero-cycle a of degree 1, then we set $\pi^0 := a \times S_K$ and $\pi^4 := S_K \times a$ as idempotent correspondences in $\mathrm{CH}^2(S_K \times S_K)$. Moreover, Vial defines the correspondences $p_i := D_i \times D_i^\vee$ in $\mathrm{CH}^2(S_K \times S_K)$, which are idempotent since the intersection product is unimodular. It is not difficult to see that all the above correspondences are mutually orthogonal. Set $\Gamma_K := \Delta_{S_K} - \pi^0 - \pi^4 - \sum_{i=1}^{\rho} p_i$.

4th Step. Since π^i and p_i are mutually orthogonal idempotents, Γ_K is idempotent. Moreover, Γ_K acts trivially on $\mathrm{CH}^*(S_K)$, since we have $\mathrm{CH}^2(S_K) = \mathbb{Z}a$ and $\mathrm{CH}^1(S_K)$ generated (over \mathbb{Z}) by the D_i 's. By [158, Prop. 3.7], since K is an algebraically closed extension (universal domain) of k , it follows that Γ is nilpotent. Since Γ is also idempotent, we have $\Gamma = 0$ and the claim holds. \square

REMARK 6.2.7. In the case where $k = \mathbb{C}$, we can weaken the assumptions on \mathbf{A} by requiring it to be a phantom. Indeed, it is easy to show that such a decomposition implies that Bloch conjecture holds for S . In this case, Sosna Sosna [161, Cor. 4.8] (using results of [92]) remarked that any phantom category in $\mathrm{D}^b(S)$ is a universal phantom.

Complex surfaces with $p_g = q = 0$. In this paragraph, S has $p_g = q = 0$ and $k = \mathbb{C}$. The study of such surfaces is very rich and already very challenging in the case where $k = \mathbb{C}$. On one hand, it is easy to see that if S has a full exceptional collection, then $K_0(S)$ is a free \mathbb{Z} -module of finite rank. This automatically exclude surfaces with torsion line bundles, which would give rise to a torsion class in $K_0(S)$. However, a full understanding of the rôle of (quasi)-phantoms, categories representable in dimension 2, and Conjecture 6.2.1, requires a full understanding of any such surface, independently on the obstruction mentioned above.

First of all, Vial classifies all such S which have a numerically exceptional sequence of maximal length. This is based on the study of the Picard lattice which can be deduced by the numerically exceptional sequence via Riemann–Roch theorem (we refer to [79] for the notations on Dolgachev surfaces).

THEOREM 6.2.8 (Vial [168]). *Let S be as above. Then S has a numerically exceptional collection of maximal length if and only if it has a numerically exceptional collection of maximal length consisting of line bundles. Moreover, this is the cases if and only if either:*

- S is not minimal, or
- S is rational, or
- S is a Dolgachev surface of type $X_9(2, 3)$, $X_9(2, 4)$, $X_9(3, 3)$ or $X_9(2, 2, 2)$,
or
- $\kappa(S) = 2$.

REMARK 6.2.9. There are cases where such a numerically exceptional collection is actually an exceptional collection, namely rational surfaces [102] (in which case it is full), Dolgachev surfaces of type $X_9(2, 3)$ [57] and many examples of surfaces of general type, see [86], [42] [88], [112], [133], [6], [76].

The classification of complex surfaces of general type with these invariants is quite wild, and probably still incomplete, see [22] for a recent survey. Proceeding

example-wise, one considers many interesting cases to study, but it is not a realistic way to attack Conjecture 6.2.1. On the other hand, such surfaces often come in positive-dimensional moduli, and have ample anticanonical bundle, so that $D^b(S)$ identifies the isomorphism class of S by the reconstruction theorem of Bondal and Orlov [47]. Hence we have a positive dimensional family of equivalence classes of (dg enhanced) triangulated categories, while categories generated by exceptional collections depend on a finite number of countable parameters.

If a dg enhanced triangulated category T is not trivial, then its Hochschild cohomology $HH^*(T)$ is also nontrivial.⁸ Moreover, the second Hochschild cohomology encodes deformations of the dg enhanced category T (see [111, § 5.4] for a survey).

Suppose then that we have a family of surfaces S_t of general type depending on a continuous parameter t , and that we can produce, for any of these surfaces, an exceptional collection $\{L_1^t, \dots, L_n^t\}$ of maximal length consisting of line bundles with A_t its semiorthogonal complement. Since S_t is of general type, the equivalence class of $D^b(S_t)$ identifies the isomorphism class of S_t by [47]. On the other hand, A_t is determined by a dg-quiver with n vertices such that the arrows and relations are determined by the dimensions of the Ext-spaces between the objects L_i^t by [46]. Hence, only a countable number of such collections can be full, that is, we could have $A_t = 0$ only for a discrete set of parameters. It would be natural to expect that the exceptional collection does not vary, while information about the deformation of S_t (hence also about $D^b(S_t)$) as t varies must be parameterized by A_t . This reasoning is supported, even in cases of Kodaira dimension 1, by two examples, namely a family of Barlow surfaces [42] and a family of Dolgachev surfaces of type $X_9(2, 3)$ [57], and justifies the following conjectural question.

QUESTION 6.2.10. *Let S_t be a family of minimal non-rational smooth projective surfaces with $p_g = q = 0$ depending on a continuous parameter t . Assume that S_t admits an exceptional sequence $E_t = \{E_1^t, \dots, E_n^t\}$ of maximal length for any t .*

- 1) *Is $\langle E_t \rangle$ constant? That is, is $\langle E_t \rangle$ equivalent to $\langle E_{t'} \rangle$ for any t and t' ?*
- 2) *Is $A_t = \langle E_t \rangle^\perp$ nontrivial for all t ?*

Notice that, in the case when S_t is of general type, a positive answer to 1) in Question 6.2.10 would imply that A_t is nontrivial for all but possibly one value of t , and that A_t would parameterize the deformations of $D^b(S_t)$ which reflect the deformations of S_t ; a positive answer to 2) would imply Conjecture 6.2.1 for all surfaces of general type with positive dimensional moduli.

7. 0-cycles on cubics

In §5.3, we ended with an essentially complete classification of smooth projective complex universally CH_0 -trivial surfaces. In this section we will move from dimension 2 to higher dimension, and discuss the universal CH_0 -triviality of complex cubic threefolds and fourfolds. Throughout, we work over \mathbb{C} .

⁸Denote by \mathcal{T} a dg enhancement of T . The Hochschild cohomology is naturally interpreted as the homology of the complex $\mathcal{H}om(\mathbf{1}_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}})$ computed in the dg category $\mathcal{R}\mathcal{H}om(\mathcal{T}, \mathcal{T})$ where $\mathbf{1}_{\mathcal{T}}$ denotes the identity functor of \mathcal{T} , see [111, §5.4]. It follows that whenever \mathcal{T} is nontrivial, the class of the identity is a nontrivial element of $HH^*(\mathcal{T})$.

7.1. Cubic threefolds. We first provide some background about minimal curve classes on principally polarized abelian varieties. Let C be a smooth projective curve and $(J(C), \Theta)$ its Jacobian with the principal polarization arising from the theta divisor. Choosing a rational point on C , there is an embedding $C \hookrightarrow J(C)$ giving rise to a class $[C] \in \text{CH}_1(J(C))$. This class is related to the theta divisor by means of the *Poincaré formula*

$$[C] = \frac{\Theta^{g-1}}{(g-1)!} \in H^{2g-2}(J(C), \mathbb{Z}),$$

in particular, the class $\Theta^{g-1}/(g-1)!$ is represented by an effective algebraic class in $\text{CH}_1(J(C))$. To some extent, the validity of this formula gives a characterization of Jacobians of curves among principally polarized abelian varieties by the following result of Matsusaka.

Let (A, ϑ) be an irreducible principally polarized abelian variety of dimension g . The class $\vartheta^{g-1}/(g-1)!$ is always an integral Hodge class in $H^{2g-2}(A, \mathbb{Z})$. We will say that this class is algebraic (resp. effective) if it is homologically equivalent to an algebraic (resp. effective) class in $\text{CH}_1(A)$, i.e., is equal to the image of an algebraic (resp. effective) cycle in the image of the class map $\text{CH}_1(A) \rightarrow H^{2g-2}(A, \mathbb{Z})$.

THEOREM 7.1.1 (Matsusaka [137]). *Let (A, ϑ) be an irreducible principally polarized abelian variety of dimension g . Then there exists a smooth projective curve C such that $(A, \vartheta) \cong (J(C), \Theta)$ as principally polarized abelian varieties if and only if the class $\vartheta^{g-1}/(g-1)!$ is effective.*

If the principally polarized abelian variety (A, ϑ) is not irreducible, then the result of Matsusaka gives a characterization of when (A, ϑ) is a product of Jacobians of curves. In [58], this condition is equivalent to (A, ϑ) being of “level one.” This characterization gives a nice reformulation of the Clemens–Griffiths criterion for nonrationality (see Theorem 4.1.1) of a smooth projective threefold X satisfying $h^1 = h^{3,0} = 0$ and with intermediate Jacobian $(J(X), \Theta)$:

If $\Theta^{g-1}/(g-1)!$ is not effective, then X is not rational.

One can even interpret the proof of Clemens and Griffiths as showing that $\Theta^4/4!$ is not effective when X is a cubic threefold.

We know that the Clemens–Griffiths criterion for nonrationality can fail to detect stable rationality, in particular, can fail to ensure universal CH_0 -triviality, see §4.2. In hindsight, a natural question is whether there is a strengthening of the Clemens–Griffiths criterion for obstructing universally CH_0 -nontriviality. Voisin [178] provides such a strengthened criterion. In the case of cubic threefolds, where universal CH_0 -triviality is still an open question, her results are particularly beautiful.

THEOREM 7.1.2 (Voisin [178]). *Let X be a smooth cubic threefold with intermediate Jacobian $(J(X), \Theta)$. Then X is universally CH_0 -trivial if and only if $\Theta^4/4! \in H^8(J(X), \mathbb{Z})$ is algebraic.*

REMARK 7.1.3. The problem of whether the very general cubic threefold is not stably rational is still open. Voisin [178, Thm. 4.5] proves that cubic threefolds are universally CH_0 -trivial over a countable union of closed subvarieties of codimension at most 3 in the moduli space. Colliot-Thélène [65] has provided a different construction of loci in the moduli space parameterizing universally CH_0 -trivial cubic threefolds. Voisin also points out the striking open problem that there is not even

a single principally polarized abelian variety (A, ϑ) of dimension $g \geq 4$ known for which $\vartheta^{g-1}/(g-1)!$ is not algebraic!

To give an idea of the ingredients in the proof, we will start with some results of Voisin on the decomposition of the diagonal in various cohomology theories.

PROPOSITION 7.1.4 (Voisin [178, Prop. 2.1]). *Let X be a smooth projective variety over a field of characteristic 0. If X admits a decomposition of the diagonal modulo algebraic equivalence, then it admits a decomposition of the diagonal.*

The proof uses a special case of the nilpotence conjecture, proved independently by Voevodsky [169] and Voisin [170], stating that correspondences in $\mathrm{CH}(X \times X)$ algebraically equivalent to 0 are nilpotent for self-composition.

We point out that the general nilpotence conjecture, for cycles homologically equivalent to 0, is known to imply, in particular, Bloch's conjecture for surfaces with $p_g = 0$, see [179, Rem. 3.31].

PROPOSITION 7.1.5. *Let X be a smooth cubic hypersurface such that the group $H^{2i}(X, \mathbb{Z})/\mathrm{im}(\mathrm{CH}^i(X) \rightarrow H^{2i}(X, \mathbb{Z}))$ has no 2-torsion for all $i \geq 0$ (e.g., X has odd dimension or dimension 4, or is very general of any dimension). If X admits a decomposition of the diagonal modulo homological equivalence, then it admits a decomposition of the diagonal modulo algebraic equivalence.*

The proof uses the relationship between a decomposition of the diagonal on X and on $X \times X$ and the Hilbert scheme of length 2 subschemes $X^{[2]}$, as well as the fact that $X^{[2]}$ is birational to the total space of a projective bundle over X , cf. [87]. There is also a purely topological approach to this result due to Totaro [167].

Finally, Voisin provides a general necessary and sufficient condition for the decomposition of the diagonal modulo homological equivalence of a rationally connected threefold.

We first recall the Abel–Jacobi map for codimension 2 cycles on a smooth projective threefold X with intermediate Jacobian $J(X)$. The Griffiths Abel–Jacobi map

$$\alpha_X : \mathrm{CH}^2(X)_{\mathrm{hom}} \rightarrow J(X)(\mathbb{C})$$

is an isomorphism by the work of Bloch and Srinivas [37], since $\mathrm{CH}_0(X) = \mathbb{Z}$.

DEFINITION 7.1.6. We say that X admits a *universal codimension 2 cycle* if there exists $Z \in \mathrm{CH}^2(J(X) \times X)$ such that $Z_a = Z_{a \times X}$ is homologous to 0 for any $a \in J(X)$ and that the morphism $\Phi_Z : J(X) \rightarrow J(X)$, induced by $a \mapsto \alpha_X(Z_a)$, is the identity on $J(X)$.

The existence of a universal codimension 2 cycle is equivalent to the tautological class in $\mathrm{CH}^2(X_{\overline{F}})$ being in the image of the map $\mathrm{CH}^2(X_F) \rightarrow \mathrm{CH}^2(X_{\overline{F}})$, where $F = \mathbb{C}(J(X))$ is the function field of the intermediate Jacobian, see [63, §5.2].

THEOREM 7.1.7 (Voisin [178, Thm. 4.1]). *Let X be a rationally connected threefold and $(J(X), \Theta)$ its intermediate Jacobian of dimension g . Then X admits a decomposition of the diagonal modulo homological equivalence if and only if the following properties are satisfied:*

- (i) $H^3(X, \mathbb{Z})$ is torsion free.
- (ii) X admits a universal codimension 2 cycle.
- (iii) $\Theta^{g-1}/(g-1)!$ is algebraic.

REMARK 7.1.8. In fact, Theorem 7.1.7 has the following possible generalizations ([178, Thm. 4.2, Rem. 4.3]). If $N\Delta_X$ admits a decomposition modulo homological equivalence then $N^2\Theta^{g-1}/(g-1)!$ is algebraic. Furthermore, if X admits a unirational parameterization of degree N , then $N\Theta^{g-1}/(g-1)!$ is effective.

Finally, we outline the proof of Theorem 7.1.2 due to Voisin. Let X be a cubic threefold with intermediate Jacobian $(J(X), \Theta)$. Propositions 7.1.4 and 7.1.5 imply that a decomposition of the diagonal on X is equivalent to a decomposition of the diagonal modulo homological equivalence. Since $H^3(X, \mathbb{Z})$ is torsion free, by Theorem 7.1.7, it would then suffice to show that X admits a universal codimension 2 cycle. However, this is not known. Instead, Voisin uses results of Markushevich and Tikhomirov [136] on parameterizations of $J(X)$ with rationally connected fibers, which implies, using results from [171], that if $\Theta^{g-1}/(g-1)!$ is algebraic then X admits a universal codimension 2 cycle.

We point out that Hassett and Tschinkel [101] have proved that for every family of smooth Fano threefold not birational to a cubic threefold, either every element in the family is rational or the very general element is not universally CH_0 -trivial. They use the degeneration method outlined in §7.3.

7.2. Cubic fourfolds. Cubic fourfolds are rationally connected. Indeed, they are Fano hypersurfaces, hence their rational connectivity is a consequence of the powerful results of [114]. A more elementary reason is that they are unirational. This fact that was likely known to M. Noether (cf. [58, App. B]).

PROPOSITION 7.2.1. *Let $X \subset \mathbb{P}^{n+1}$ be a cubic hypersurface of dimension $n \geq 2$ containing a line. Then X admits a unirational parameterization of degree 2.*

PROOF. Blowing up a line $\ell \subset X$, we arrive at a conic bundle $\text{Bl}_\ell X \rightarrow \mathbb{P}^{n-1}$, for which the exceptional divisor E is a multisection of degree 2. Thus the fiber product $\text{Bl}_\ell X \times_{\mathbb{P}^{n-1}} E \rightarrow E$ is a conic bundle with a section, hence a rational variety since E is rational. The map $\text{Bl}_\ell X \times_{\mathbb{P}^{n-1}} E \rightarrow X$ is generically finite of degree 2. Thus X admits a unirational parameterization of degree 2. \square

Of course every smooth cubic hypersurface of dimension at least 2 over an algebraically closed field contains a line, as one can reduce, by taking hyperplane sections, to the case of a cubic surface. In particular, if X is a cubic fourfold, $\text{CH}_0(X)$ is universally 2-torsion. One way that X could be universally CH_0 -trivial is if X admitted a unirational parameterization of odd degree. Indeed, if a variety X admits unirational parameterizations of coprime degrees, then X is universally CH_0 -trivial. In fact, the following question is still open.

QUESTION 7.2.2. *Does there exist a nonrational variety with unirational parameterizations of coprime degrees?*

While the existence of cubic fourfolds with unirational parameterizations of odd degree is currently limited, a beautiful result of Voisin [178, Thm. 5.6] states that in fact many classes of special cubic fourfolds are universally CH_0 -trivial.

We recall, from §4.3, the divisors $\mathcal{C}_d \subset \mathcal{C}$ of special cubic fourfolds of discriminant d in the coarse moduli space \mathcal{C} of cubic fourfolds. These are Noether–Lefschetz type divisors, which are nonempty for $d > 6$ and $d \equiv 0, 2 \pmod{6}$, see Theorem 4.3.8. We recall that Voisin [176, Thm. 18] has shown the integral Hodge conjecture for cubic fourfolds, i.e., that the cycle class gives rise to an isomorphism

$\mathrm{CH}^2(X) = H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$, and moreover, that every class in $\mathrm{CH}^2(X)$ can be represented by a (possibly singular) rational surface. For X very general in the moduli space, $\mathrm{CH}^2(X)$ is generated by the square of the hyperplane class h^2 , and the rank of $\mathrm{CH}^2(X)$ is > 1 if and only if X lies on one of the divisors \mathcal{C}_d . For small values of d , the geometry of additional 2-cycles $T \in \mathrm{CH}^2(X)$, for general $X \in \mathcal{C}_d$, is well understood. For $d \leq 20$, this was understood classically. For $d \leq 38$, Nuer [143] provides explicit smooth models of the rational surfaces that arise. It is still an open question as to whether $\mathrm{CH}^2(X)$ is always generated by classes of smooth rational surfaces, cf. [97, Question 14]. Nuer's approach for $d \leq 38$ provides a unirational parameterization of \mathcal{C}_d , while it is known that for $d \gg 0$, the divisors \mathcal{C}_d become of general type, see [164].

We are interested in representations of cycle classes for $X \in \mathcal{C}_d$ because of the following result on the existence of unirational parameterizations of odd degree of certain special cubic fourfolds (see [97, Cor. 35]), which was initiated in [100, §7.5], with corrections by Voisin (see [97, Ex. 38]).

PROPOSITION 7.2.3. *Let $X \in \mathcal{C}_d$ be a special cubic fourfold whose additional 2-cycle $T \subset X$ is a rational surface. Assume T has isolated singularities and a smooth normalization. If d is not divisible by 4 then X admits a unirational parameterization of odd degree.*

In general, we do not know if the required rational surface $T \subset X$ can always be chosen with isolated singularities and smooth normalization. The construction of Nuer [143] provides smooth rational $T \subset X$ for $d \leq 38$.⁹

However, not assuming the existence of unirational parameterizations of co-prime degree, Voisin has the following result.

THEOREM 7.2.4 (Voisin [178, Thm. 5.6]). *If $4 \nmid d$ then any $X \in \mathcal{C}_d$ is universally CH_0 -trivial.*

PROOF. We give a sketch of the proof. The additional class $T \in \mathrm{CH}^2(X)$, such that the discriminant of the sublattice generated by h^2 and T is d , can be represented (after adding multiples of h^2) by a smooth surface (which by abuse of terminology we denote by) $T \subset X$.

First, Voisin proves that (at least under the hypothesis that X is very general in \mathcal{C}_d), if there exists any closed subvariety $Y \subsetneq X$ such that $\mathrm{CH}_0(Y) \rightarrow \mathrm{CH}_0(X)$ is universally surjective, then X is universally CH_0 -trivial.

Second, for a smooth surface $T \subset X$, consider the rational map $T \times T \dashrightarrow X$ defined by sending a pair of points (x, y) to the point residual to the line joining x and y . Voisin proves that if this map is dominant of even degree not divisible by 4, then $\mathrm{CH}_0(T) \rightarrow \mathrm{CH}_0(X)$ is universally surjective.

Finally, a calculation with Chern classes shows that if $T \subset X$ is a smooth surface in general position then the rational map $T \times T \dashrightarrow X$ is dominant of degree $\equiv d \pmod{4}$. Indeed, the degree is equal to twice the number of double points acquired by T after a generic projection from a point, so computations must be made comparing the numerology of the double point formula and the intersection product on X . \square

We remark that the universal CH_0 -triviality is still open in one of the most interesting classes of special cubic fourfolds, namely that of cubic fourfolds containing

⁹Recently, Lai [130] verified that the required rational surface $T \subset X$ is nodal for $d = 42$.

a plane, i.e., $X \in \mathcal{C}_8$. One nontrivial consequence of the universal CH_0 -triviality would be the universal triviality of the unramified cohomology in degree 3. For cubic fourfolds containing a plane, this was first proved in [15]. For arbitrary cubic fourfolds, this was then proved by Voisin [177, Ex. 3.2], with a different proof given by Colliot-Thélène [63, Thm. 5.8] (which still relies on Voisin’s proof of the integral Hodge conjecture).

7.3. The degeneration method. The degeneration method, initiated by Voisin [177, §2] and developed by Colliot-Thélène and Pirutka [69], has emerged as a powerful tool for obstructing universal CH_0 -triviality for various families of varieties. The idea is that universal CH_0 -triviality specializes well in families whose central fiber is mildly singular.

Analogous results for the specialization of rationality in families of threefolds was established by de Fernex and Fusi [77]. Already, Beauville [23, Lemma 5.6.1] had proved an analogous result for the specialization of the Clemens–Griffiths criterion for nonrationality relying on the Satake compactification of the moduli space of abelian varieties. Also, Kollár [113] used a specialization method (to characteristic p) for the existence of differential forms to prove nonrationality of hypersurfaces of large degree, a result that was generalized by Totaro [166] using the degeneration method for universally CH_0 -triviality. We will outline the degeneration method and some of its applications.

First we define a condition on the resolution of singularities of a singular variety.

DEFINITION 7.3.1. Let X_0 be a proper geometrically integral variety over a field k . We say that a proper birational morphism $f : \tilde{X}_0 \rightarrow X_0$ with \tilde{X}_0 smooth is a *universally CH_0 -trivial resolution* if $f_* : \mathrm{CH}_0(\tilde{X}_{0,F}) \rightarrow \mathrm{CH}_0(X_F)$ is an isomorphism for all field extensions F/k , and, is a *totally CH_0 -trivial resolution* if for every scheme-theoretic point x of X_0 , the fiber $(\tilde{X}_0)_x$ is a universally CH_0 -trivial variety over the residue field $k(x)$.

The notions of universally and totally CH_0 -trivial resolutions are due to Colliot-Thélène and Pirutka [69] and define a new class of singularities that should be classified in the spirit of the minimal model program. For example, in characteristic zero, one might ask whether X_0 has rational singularities if it admits a totally CH_0 -trivial resolution. It is proved (see [69, Prop. 1.8]) that every totally CH_0 -trivial resolution is universally CH_0 -trivial, but not conversely.

EXAMPLE 7.3.2. Let $X \rightarrow \mathbb{P}^2$ be a conic bundle of Artin–Mumford type. Then X has isolated ordinary double points, and the (universally CH_0 -trivial) resolution has nontrivial Brauer group.

Let X be a smooth proper geometrically integral variety over k . The *degeneration method* proceeds as follows:

- (i) Fit X into a proper flat family $\mathcal{X} \rightarrow B$ over a scheme B of finite type, and let X_0 be a possibly singular fiber. Assume for simplicity that the generic fiber is regular.
- (ii) Prove that X_0 admits a universally CH_0 -trivial resolution $f : \tilde{X}_0 \rightarrow X_0$.
- (iii) Prove that \tilde{X}_0 is not universally CH_0 -trivial.

The outcome is that the very general fiber of the family $\mathcal{X} \rightarrow B$ (though perhaps not X itself) will not be universally CH_0 -trivial.

Part (i) is, to a large extent, informed by the possibility of achieving part (iii). To this end, one is mostly concerned with finding good singular varieties X_0 whose resolutions have nontrivial unramified cohomological invariants or differential forms. Then one hopes that (ii) can be verified for these singular varieties. For example, conic bundles of Artin–Mumford type have been used quite a lot. Kollár [113] has constructed hypersurfaces in characteristic p with nontrivial global differential forms.

EXAMPLE 7.3.3.

- In [177], a quartic double solid with ≤ 7 nodes is shown to degenerate to an Artin–Mumford example.
- In [69], a quartic threefold degenerates to a singular quartic hypersurface model birational to an Artin–Mumford example.
- In [98], a conic bundle over a rational surface, whose discriminant curve degenerates to a union of curves of positive genus, is shown to degenerate to an Artin–Mumford example.
- In [101], smooth Fano threefolds in a family whose general element is nonrational and not birational to a cubic threefold, are shown to degenerate to an Artin–Mumford example.
- In [166], hypersurfaces of large degree were already shown by Kollár to degenerate to singular hypersurfaces in characteristic p with nonzero global differential forms.

We find it striking that all successful instances of the equicharacteristic degeneration method for threefolds over \mathbb{C} use singular central fibers that are birational to threefolds of Artin–Mumford type. However, over arbitrary fields, there are other methods, see [56], [64].

We shall say a few words about the proof of the degeneration method by Colliot-Thélène and Pirutka [69]. First there is a purely local statement about schemes faithfully flat and proper over a discrete valuation ring, to the extent that if the special fiber admits a universally CH_0 -trivial resolution, then universal CH_0 -triviality of the generic fiber implies the universal CH_0 -triviality of the special fiber. This purely local statement uses the specialization homomorphism as developed in §1.4. To get the statement about the very general fiber of a family over a base B , there is a “standard” argument using Chow schemes, see [69, App. B].

8. Categorical representability in higher dimension

In this section, we turn to higher dimensional varieties, in particular to varieties of dimension 3 and 4. Even though the constructions work over any field, and considerations related to weak factorization hold over any field of characteristic zero (see [1]), we consider here only the case where $k = \mathbb{C}$ (or k algebraically closed of characteristic zero), where the depth of the categorical questions we consider are already quite rich.

The aim of this section is to motivate, by examples and motivic arguments, the following question:

QUESTION 8.0.4. *Is categorical representability in codimension 2 a necessary condition for rationality? That is, if X is rational, do we have $\mathrm{rcodim}(X) \geq 2$?*

Let us first notice that, as soon as we consider varieties of dimension at least 3, we easily find examples of non-rational varieties X with $\mathrm{rcodim}(X) \geq 2$, the easiest

example being a projective bundle $X \rightarrow C$ of relative dimension at least 2 over a curve C with $g(C) > 0$. We thus restrict our attention to Mori fiber spaces $X \rightarrow Y$ over varieties of negative Kodaira dimension. In this case Proposition 2.3.6 gives a natural subcategory $\mathbf{A}_{X/Y}$ as a complement of a finite number of copies of $\mathbf{D}^b(Y)$. We now present some evidence and motivic considerations in order to argue that $\mathrm{rdim} \mathbf{A}_{X/Y}$ should witness obstruction to rationality.

Let $\pi : X \rightarrow Y$ be a Mori fiber space of relative dimension m , and let n be the dimension of X . First of all, we remark that if $\mathrm{rdim} \mathbf{A}_{X/Y} \leq d$, then $\mathrm{rdim}(X) \leq \max\{\dim(Y), d\}$. It then follows that to study $\mathrm{rcodim}(X)$ we should focus on $\mathbf{A}_{X/Y}$ and its representability.¹⁰

8.1. Motivic measures and a rational defect. Let us quickly recall some general motivation for Question 8.0.4. Bondal, Larsen, and Lunts [49] defined the Grothendieck ring $PT(k)$ of (dg enhanced) triangulated k -linear categories, by considering the free \mathbb{Z} -module generated by equivalence classes of such categories, denoted by $I(-)$, and introducing a scissor-type relation: $I(\mathbf{T}) = I(\mathbf{A}) + I(\mathbf{B})$ if there is a semiorthogonal decomposition $\mathbf{T} = \langle \mathbf{A}, \mathbf{B} \rangle$. The product of this ring is a convolution product, in such a way that the product of $I(\mathbf{D}^b(X))$ and $I(\mathbf{D}^b(Y))$ coincides with $I(\mathbf{D}^b(X \times Y))$. See [49] for more details. We notice that the unit \mathbf{e} of $PT(k)$ is the class of $\mathbf{D}^b(k)$ and that if \mathbf{T} is generated by r k -exceptional objects, then $I(\mathbf{T}) = r\mathbf{e}$.

One can consider the following subsets of $PT(k)$:

$$PT_d(k) := \langle I(\mathbf{T}) \in PT(k) \mid \mathrm{rdim} \mathbf{T} \leq d \rangle^+,$$

where $\langle - \rangle^+$ is the smallest subset closed under summands. One can show that these subsets are indeed subgroups providing a ring filtration of $PT(k)$.

Notice that, by definition, if $\mathrm{rdim} \mathbf{T} \leq d$, then $I(\mathbf{T})$ is in $PT_d(k)$, but the converse is not true in general, even for $d = 0$, as the following example shows.

EXAMPLE 8.1.1. Recall from Remark 6.1.4 that Kuznetsov has constructed a complex threefold X generated by exceptional objects not satisfying the Jordan–Hölder property [124]. In particular, this is based on the description, originally due to Bondal, of a quiver Q with three vertexes and relations, so that there are exceptional objects E_1, E_2, E_3, F in $\mathbf{D}^b(Q)$ and semiorthogonal decompositions:

$$\mathbf{D}^b(Q) = \langle E_1, E_2, E_3 \rangle, \quad \mathbf{D}^b(Q) = \langle \mathbf{T}, F \rangle,$$

such that $\mathbf{T} = F^\perp$ has no exceptional object. It follows that $I(\mathbf{T})$ lies in $PT_0(k)$, since $I(\mathbf{D}^b(Q))$ does, but $\mathrm{rdim} \mathbf{T} > 0$.

It would thus be very interesting to give conditions under which a category \mathbf{T} , admissible in some category generated by k -exceptional objects (and hence such that $I(\mathbf{T}) \in PT_0(k)$), admits a full k -exceptional collection.

One can consider the Grothendieck ring $K_0(\mathrm{Var}(k))$ of k -varieties whose unit $1 = [\mathrm{Spec}(k)]$ is the class of the point. If weak factorization holds, then this

¹⁰If $m > 1$, this is obvious. If $m = 1$, then the rationality of X implies the rationality of $Y \times \mathbb{P}^1$, and hence the stable rationality of Y . For surfaces over \mathbb{C} , stable rationality implies rationality, so if X is a 3-dimensional conic bundle over a rational surface S , then the obstruction is exactly contained in $\mathbf{A}_{X/Y}$, see [29] or Theorem 8.2.2. For threefolds, note that if Y is nonrational generic Fano threefold, but not a cubic threefold, then Y is not stably rational, see [101]. It follows that if $X \rightarrow Y$ is a conic bundle over a stably rational threefold, then the obstruction is again contained in $\mathbf{A}_{X/Y}$, unless perhaps if Y is a cubic threefold.

can be seen as the \mathbb{Z} -module generated by isomorphism classes of smooth proper varieties with the relation $[X] - [Z] = [Y] - [E]$ whenever $Y \rightarrow X$ is the blow-up along the smooth center Z with exceptional divisor E , see [33]. Larsen and Lunts have then shown that there is a surjective ring morphism (a *motivic measure*) $\mu : K_0(\text{Var}(k)) \rightarrow \mathbb{Z}[SB]$ to the ring generated by stable birational equivalence classes. That is, $\mathbb{Z}[SB]$ is the quotient of the Grothendieck ring $K_0(\text{Var}(k))$ by the stable birational equivalence relation.

Moreover, $\ker \mu = \langle \mathbb{L} \rangle$, the ideal generated by the class \mathbb{L} of the affine line. It follows that (as remarked in [87]), if X is rational of dimension n , then:

$$(8.1) \quad [X] = [\mathbb{P}^n] + \mathbb{L}M_X$$

in $K_0(\text{Var}(k))$, where M_X is a \mathbb{Z} -linear combination of classes of varieties of dimension bounded above by $n - 2$. Galkin and Shinder [87] define then

$$([X] - [\mathbb{P}^n])/\mathbb{L} \in K_0(\text{Var}(k))[\mathbb{L}^{-1}]$$

as the *rational defect* of X .

On the other hand, Bondal, Larsen, and Lunts [49] show that the assignment

$$\nu : K_0(\text{Var}(k)) \rightarrow PT(k), \quad [X] \mapsto I(\mathbb{D}^b(X))$$

also defines a motivic measure. Moreover, since $I(\mathbb{D}^b(\mathbb{P}^1)) = 2\mathbf{e}$ and $[\mathbb{P}^1] = 1 + \mathbb{L}$ in $K_0(\text{Var}(k))$, we have that $\nu(\mathbb{L}) = \mathbf{e}$. It follows from (8.1) that if X is rational of dimension n , then $I(\mathbb{D}^b(X))$ is in $PT_{n-2}(k)$. We can state the following result, motivating Question 8.0.4.

PROPOSITION 8.1.2. *If X is a smooth projective variety of dimension n such that $I(\mathbb{D}^b(X))$ is not in $PT_{n-2}(k)$, then X is not rational.*

DEFINITION 8.1.3. If X is a smooth and projective variety of dimension n , the class of the element $I(\mathbb{D}^b(X))$ in the group $PT(k)/PT_{n-2}(k)$ is called the *noncommutative motivic rational defect* of X .

We end by remarking that Proposition 8.1.2 is a rather weak result. Indeed, as mentioned above, we have an implication $\text{rdim}(X) \leq i \implies I(\mathbb{D}^b(X)) \in PT_i(k)$, but the converse implication is in general not known, even for $i = 0$.

However, Proposition 8.1.2 indicates that in the case of Mori fiber spaces, the category $\mathbf{A}_{X/Y}$ should be the object to consider.

COROLLARY 8.1.4. *Let $X \rightarrow Y$ be a Mori fiber space of relative dimension m , and let $n = \dim(X)$. Assume that either Y is rational or $m > 1$. Then $I(\mathbb{D}^b(X))$ is in $PT_{n-2}(k)$ if and only if $I(\mathbf{A}_{X/Y})$ is in $PT_{n-2}(k)$.*

PROOF. The assumptions on Y imply that $I(\mathbb{D}^b(Y))$ is in $PT_{n-2}(k)$. Indeed, first of all note that $\dim(Y) \leq n - 1$. Second, if Y is rational, we have $I(\mathbb{D}^b(Y))$ in $PT_{n-3}(k)$. On the other hand, if $\dim(Y) \leq n - 2$, we have $I(\mathbb{D}^b(Y))$ in $PT_{n-2}(k)$. We then conclude using the relation $I(\mathbb{D}^b(X)) = mI(\mathbb{D}^b(Y)) + I(\mathbf{A}_{X/Y})$ and the definition of $PT_i(k)$. \square

QUESTION 8.1.5. *Let X be a smooth projective variety and $X \rightarrow Y$ and $X \rightarrow Z$ two Mori fiber space structures. Is $\text{rdim}(\mathbf{A}_{X/Y}) = \text{rdim}(\mathbf{A}_{X/Z})$?*

We can extend our analysis to $X \dashrightarrow Y$, a rational map whose resolution $\tilde{X} \rightarrow Y$ is a Mori fiber space. By abuse of notation, we denote $\mathbf{A}_{X/Y,\rho} := \mathbf{A}_{\tilde{X}/Y}$ (even though $\mathbf{A}_{X/Y,\rho}$ is not necessarily a subcategory of $\mathbb{D}^b(X)$). For example, X is

a cubic threefold and $X \dashrightarrow \mathbb{P}^2$ the projection along any line in X , which is resolved into $\tilde{X} \rightarrow \mathbb{P}^2$, a conic bundle.

COROLLARY 8.1.6. *Suppose that there is a rational map $\rho : X \dashrightarrow Y$ and a commutative diagram:*

$$\begin{array}{ccc} \tilde{X} & & \\ \epsilon \downarrow & \searrow \pi & \\ X & \xrightarrow{\rho} & Y, \end{array}$$

where $\pi : \tilde{X} \rightarrow Y$ is a Mori fiber space of relative dimension m and $\epsilon : \tilde{X} \rightarrow X$ is a blow up along a smooth center. Assume that either Y is rational or $m > 1$. Then $I(\mathrm{D}^b(X))$ is in $PT_{n-2}(k)$ if and only if $I(\mathbf{A}_{X/Y,\rho})$ is in $PT_{n-2}(k)$.

PROOF. We notice that $\mathrm{D}^b(\tilde{X})$ has two decompositions, one given by the Mori fiber space map $\tilde{X} \rightarrow Y$ and the other given by the blow-up of X hence containing a copy of X and a finite number of copies of the blown-up loci. Corollary 8.1.4 applies then again, once we write the two decompositions of $I(\mathrm{D}^b(\tilde{X}))$ \square

QUESTION 8.1.7. *Let X be a smooth projective Fano variety of dimension n , and $\rho : X \dashrightarrow Y$ and $\sigma : X \dashrightarrow Z$ be rational Mori fiber spaces as above. Is $\mathrm{rdim} \mathbf{A}_{X/Y,\rho} > n - 2$ if and only if $\mathrm{rdim} \mathbf{A}_{X/Z,\sigma} > n - 2$?*

8.2. Threefolds. In this section, we consider Questions 8.0.4, 8.1.5 and 8.1.7 for threefolds. Let us first notice that we only consider Mori fiber spaces $X \rightarrow Y$ with Y of negative Kodaira dimension. Moreover, being interested in rationality, we can exclude the cases where Y is a ruled surface over a curve of positive genus. It follows that we only consider Fano threefolds of Picard rank one, relatively minimal del Pezzo fibrations over \mathbb{P}^1 and relatively minimal (or *standard*) conic bundles over rational surfaces.

On one hand, there are now many examples of known semiorthogonal decompositions describing $\mathbf{A}_{X/Y}$ for such varieties, especially for Fano and conic bundles. We recall the most of them in Table 1.

On the other hand, recall from §4.1, that such a threefold X has a unique principally polarized intermediate Jacobian $J(X)$, and that one can define the Griffiths component $A_X \subset J(X)$ to be the maximal component not split by Jacobians of curves. We consider a stronger assumption on $J(X)$, namely that it carries an *incidence polarization* (see [23, Déf. 3.2.3]), defined as follows. For any algebraic variety T , and z a cycle in $\mathrm{CH}_{\mathbb{Q}}^2(T \times X)$, the *incidence correspondence* $I(z)$ associated to z is the equivalence class of the cycle $r_*(p^*(z) \cdot q^*(z)) \in \mathrm{CH}_{\mathbb{Q}}^1(T \times T)$, where p, q and r are the projections from $T \times T \times X$ away from the first, the second and the third factor respectively. Recall that $J(X)$ carries a principal polarization, which can be seen as an element θ in $\mathrm{CH}^1(J(X) \times J(X))$. Moreover, we assume that $J(X)$ represents the group $A^2(X) \subset \mathrm{CH}^2(X)$ of (integral) algebraically trivial codimension 2 cycles on X , that is there is a universal regular map $G : A^2(X) \rightarrow J(X)$, such that for every regular map $g : A^2(X) \rightarrow B$ to an abelian variety B , there is a unique morphism of abelian varieties $u : J(X) \rightarrow B$ such that $u \circ G = g$. Finally, if all these properties are satisfied, we say that the principal polarization θ of $J(X)$ is an *incidence polarization* if for any algebraic map $f : T \rightarrow A^2(X)$ defined by a cycle $z \in \mathrm{CH}^2(T \times X)$ the equality $(G \circ f)^*(\theta) = I(z)$ holds.

Having a principally polarized intermediate Jacobian carrying an incidence polarization may seem a rather restrictive assumption, but this is actually satisfied by most of the (general) Mori fiber spaces over rational bases we are considering in this paragraph. Indeed, in these cases $J(X)$ is known to carry such a polarization, unless X is a Fano of index 2 and degree 1 or a del Pezzo fibration of degree 1 (see [30, Rmk. 3.8] for more details).

THEOREM 8.2.1 ([30]). *Suppose that X is a threefold with principally polarized intermediate Jacobian $J(X)$ carrying an incidence polarization. Then, assuming $\mathrm{rcodim}(X) \geq 2$, the Griffiths component $A_X \subset J(X)$ is trivial.*

SKETCH OF PROOF. The first step in the proof is the classification of categories representable in dimension at most 1, see Proposition 6.1.10. In the complex case, this means that there exist smooth and projective curves $\{C_i\}_{i=1}^s$ and a semiorthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathrm{D}^b(C_1), \dots, \mathrm{D}^b(C_s), E_1, \dots, E_r \rangle,$$

where E_i are exceptional. This implies, via Grothendieck–Riemann–Roch, that

$$\mathrm{CH}^*(X)_{\mathbb{Q}} \simeq \mathbb{Q}^{\oplus r} \oplus \bigoplus_{i=1}^s \mathrm{CH}^*(C_i)_{\mathbb{Q}},$$

and that the correspondences giving the maps $\phi_i : \mathrm{CH}^i(C_i)_{\mathbb{Q}} \rightarrow \mathrm{CH}^*(X)_{\mathbb{Q}}$ are obtained via the kernel of the Fourier–Mukai functors $\Phi_i : \mathrm{D}^b(C_i) \rightarrow \mathrm{D}^b(X)$, which are full and faithful.

One can show that ϕ_i induces an isogeny $J(C_i) \rightarrow J(X)$ onto an Abelian subvariety (i.e., its kernel is finite), since it has to send algebraically trivial cycles to algebraically trivial cycles, and since the adjoint to the Fourier–Mukai functor provides a retraction ψ_i of ϕ_i up to torsion. Finally, one can check, using the explicit description of the kernel of the adjoint and the incidence property of the polarization of $J(X)$, that ϕ_i actually preserves the principal polarization. \square

Notice that Theorem 8.2.1 has a much stronger generalization relying on the theory of noncommutative motives [32] which allows one to define the Jacobian of any admissible subcategory of the derived category $\mathrm{D}^b(X)$ of a smooth projective variety. As recalled, Theorem 8.2.1 applies to almost all general threefolds X under examination, except Fano of index two and degree one or del Pezzo fibrations of degree 1 over \mathbb{P}^1 , whose polarization of the intermediate Jacobian is not known to be incidence.

Secondly, by a classification argument, and thanks to the work of many authors (see Table 1), we can state a converse statement for Theorem 8.2.1.

THEOREM 8.2.2. *Let X be a complex threefold. Assume that X is either:*

- *a Fano threefold of Picard rank one, very general in its moduli space, not of index 2 and degree 1; or*
- *a del Pezzo fibration $X \rightarrow Y = \mathbb{P}^1$ of degree 4; or*
- *a standard conic bundle $X \rightarrow Y$ over a minimal rational surface.*

Then X is rational if and only if $\mathrm{rcodim}(X) \geq 2$. In particular, this is the case if and only if $\mathrm{rdim} A_{X/Y} \leq 1$.

PROOF. Suppose that such X is not rational. Then it is known—see, for example, [24, Table 1] for the Fano cases (very general quartics and sextic double solids can be treated by the same degeneration arguments as in [23, §5]), [5] for the del Pezzo fibrations, and [23, 160] for the conic bundles—that the rationality defect of X is detected by a nontrivial Griffiths invariant $A_X \subset J(X)$. These Jacobians carry an incidence polarization, so that Theorem 8.2.1 implies that $\text{rcodim}(X) \leq 1$ and also that $\text{rdim } A_X \geq 2$.

Conversely, we know a list of the rational general such varieties: \mathbb{P}^3 ; quadric hypersurfaces; Fano varieties of index 2 and degree 5 or 4 (the latter are intersections of two quadrics); Fano varieties of index 1 and degree 22, 18, 16 or 12; conic bundles over \mathbb{P}^2 with discriminant divisor of degree ≤ 4 or of degree 5 and even theta-characteristic; conic bundles over Hirzebruch surfaces with trigonal or hyperelliptic discriminant divisor; del Pezzo fibrations of degree 4 over \mathbb{P}^1 that are birational to a conic bundle over a Hirzebruch surface. For those varieties, we recall all the known semiorthogonal decompositions and descriptions of $A_{X/Y}$, with the corresponding references, in Table 1.

Notice that \mathbb{P}^1 -bundles over rational surfaces, \mathbb{P}^2 -bundles and quadric fibrations over \mathbb{P}^1 also (trivially) fit the statement. \square

Recall that there exist nonrational threefolds with trivial Griffiths invariant, and even with trivial intermediate Jacobian. For example, if X is the Artin–Mumford double solid [9] recalled in Example 3.5.2, the obstruction to rationality is not given by a nontrivial Griffiths component, but rather by a nontrivial unramified cohomology class. In this case, X is singular but can be resolved by blowing-up its ten double points $\tilde{X} \rightarrow X$.

PROPOSITION 8.2.3. *Let X be the Artin–Mumford quartic double solid and $\tilde{X} \rightarrow X$ be the blow-up of its ten double points. Then $J(\tilde{X}) = 0$ and \tilde{X} is not rational. Moreover, $D^b(\tilde{X})$ is a noncommutative resolution of singularities of X and $\text{rcodim}(\tilde{X}) = 1$.*

PROOF. The fact that \tilde{X} is nonrational and has trivial Jacobian goes back to the original paper of Artin and Mumford [9], where the cohomology groups of X are explicitly calculated. In particular $h^{1,2}(X) = 0$, so that it is easy to get $h^{1,2}(\tilde{X}) = 0$ which implies $J(\tilde{X}) = 0$. Finally, just recall that $D^b(\tilde{X})$ is a noncommutative resolution of $D^b(X)$ since $\tilde{X} \rightarrow X$ is a resolution of singularities.

We are going to prove that $\text{rdim}(\tilde{X}) = 2$ by using an explicit semiorthogonal decomposition: first by showing that $\text{rdim}(\tilde{X}) \leq 2$, then by showing that the inequality cannot hold strictly.

Hosono and Takagi [103] consider the Enriques surface S associated to X (the so-called Reye congruence), and show that there is a semiorthogonal decomposition

$$D^b(\tilde{X}) = \langle D^b(S), E_1, \dots, E_{12} \rangle,$$

where E_i are exceptional objects. This implies first that $\text{rdim}(\tilde{X}) \leq 2$.

We want to prove that the inequality cannot hold strictly. First of all, since $J(\tilde{X}) = 0$, we cannot have admissible subcategories of $D^b(\tilde{X})$ equivalent to $D^b(C)$ for some positive genus curve C . It follows by Proposition 6.1.10 (and $k = \mathbb{C}$) that $\text{rdim}(\tilde{X}) \leq 2$ implies either $\text{rdim}(\tilde{X}) = 2$ or $\text{rdim}(\tilde{X}) = 0$.

Let us exclude the second case. Notice that we have $K_0(\tilde{X}) = \mathbb{Z}^{12} \oplus K_0(S)$. Moreover, the 2-torsion subgroup $K_0(S)[2]$ of $K_0(S)$ is nontrivial. Indeed, we have $K_0(S)[2] = \mathbb{Z}/2\mathbb{Z}$. Indeed, if S is an Enriques surface, the Chern character is integral and gives an isomorphism between $K_0(S)$ and the singular cohomology of S (similarly, one can argue by using the Bloch conjecture, which is true for S , and the topological filtration of the Grothendieck group of S). In particular, $K_0(S) = \mathbb{Z} \oplus \text{Pic}(S) \oplus \mathbb{Z}$ and $\text{Pic}(S) \simeq \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z}$ (see, e.g., [21, VIII Prop. 15.2]). We conclude that

$$K_0(\tilde{X}) \simeq \mathbb{Z}^{\oplus 22} \oplus \mathbb{Z}/2\mathbb{Z}.$$

But if $\text{rdim}(\tilde{X}) = 0$, then $K_0(\tilde{X})$ is free of finite rank, so we conclude.¹¹ \square

Finally notice that there could be other noncommutative resolution of singularities X , such as a small resolution $X^+ \rightarrow X$; note that X^+ is a non projective Moishezon manifold. Similar arguments, based on the semiorthogonal decomposition described by Ingalls and Kuznetsov [105], show that $\text{rdim } \mathbb{D}^b(X^+) = 2$ as well. In that case, the component $\mathbb{T} \subset \mathbb{D}^b(X^+)$ with $\text{rdim } \mathbb{T} = 2$ is a crepant categorical resolution of singularities of \mathbf{A}_X . Notice also that, despite the fact that X^+ is not an algebraic variety, it is however an algebraic space, see Artin [8], so that the category $\mathbb{D}^b(X^+)$ makes sense.

8.3. Cubic Fourfolds. This section is completely devoted to complex cubic fourfolds. From now on, let X denote a smooth hypersurface of degree 3 in complex projective space \mathbb{P}^5 . As we have seen in §4.3, one of the most useful tools to investigate the geometry of X is Hodge theory. In particular, we have seen how to construct Noether–Lefschetz type divisors \mathcal{C}_d , and to relate the numerical properties of d to the existence of some K3 surface S whose geometry is intimately related to the one of X .

We present results showing how derived categories and semiorthogonal decompositions, in the spirit of Question 8.0.4 and even beyond, provide a new language which superposes, and, hopefully, extends the Hodge theoretic approach. We start with an observation by Kuznetsov that we recalled in Corollary 2.3.13:

The category \mathbf{A}_X is a noncommutative K3 surface.

Kuznetsov states the following conjecture [123], which we will try to motivate and explore in this section.

CONJECTURE 8.3.1 (Kuznetsov). *A cubic fourfold X is rational if and only if there is a K3 surface S and an equivalence $\mathbb{D}^b(S) \simeq \mathbf{A}_X$.*

We notice that Conjecture 8.3.1 is stronger than Question 8.0.4. Moreover, Addington and Thomas have shown [3] that the existence of the equivalence requested in Conjecture 8.3.1 is equivalent to asking X to have an associated K3 surface (in the sense of §4.3), at least if X is general on its Noether–Lefschetz divisor.

THEOREM 8.3.2 (Addington–Thomas [3]). *Let X be a smooth cubic fourfold. If there exists a K3 surface S and an equivalence $\mathbb{D}^b(S) \simeq \mathbf{A}_X$, then X is special and has an associated K3 surface. Conversely, if X is special and has an associated*

¹¹Notice that we can also use Proposition 6.1.10 since $K_0(X)[2]$ is a one dimensional free $\mathbb{Z}/2\mathbb{Z}$ -module.

Y	X	Explicit description of $\mathbf{A}_{X/Y}$	Ref.
Threefolds			
pt	\mathbb{P}^3	0	[27]
	quadric 3fold	$\langle S \rangle$, a spinor bundle	[108]
	index 2, degree 5	$\langle U, V \rangle$, two vector bundles	[145]
	int. of 2 quadrics	$\mathbf{D}^b(C)$ with $g(C) = 2$ and $J(X) = J(C)$	[48]
	cubic 3fold	fractional Calabi-Yau	[119]
	nodal cubic 3fold*	$\langle \mathbf{D}^b(C), L \rangle$, L a l.bd., $g(C) = 2$ and $J(X) = J(C)$	[30]
	det'l. cubic 3fold*	four exceptional objects	[31]
	quartic double solid	noncommutative Enriques surface	[127]
	Artin-Mumford*	complement of exc. collection on Enriques surface	[105]
	index 1, degree 22	$\langle U, V, W \rangle$, three vector bundles	[117]
	index 1, degree 18	$\langle \mathbf{D}^b(C), U \rangle$, U a v.b., $g(C) = 2$ and $J(C) = J(X)$	[120]
	index 1, degree 16	$\langle \mathbf{D}^b(C), U \rangle$, U a v.b., $g(C) = 3$ and $J(C) = J(X)$	[120]
	index 1, degree 14	$\langle B, U \rangle$, U a v.b., B fractional Calabi-Yau	[119]
	index 1, degree 12	$\langle \mathbf{D}^b(C), U \rangle$, U a v.b., $g(C) = 7$ and $J(C) = J(X)$	[118]
	Hyperell. Gushel–Mukai	noncommutative Enriques surface	[127]
int. of 3 quadrics	$\mathbf{D}^b(\mathbb{P}^2, \mathcal{E}_0)$, \mathcal{E}_0 a Clifford algebra	[116]	
quartic 3fold	fractional Calabi–Yau	[119]	
\mathbb{P}^1	\mathbb{P}^2 -bundle	0	[146]
	quadric bundle	$\mathbf{D}^b(C)$, C hyperelliptic and $J(X) = J(C)$	[116]
	dP4 fibration	$\mathbf{D}^b(S, \mathcal{E}_0)$, S a Hirz. surf, \mathcal{E}_0 a Cliff. algebra	[12]
	rational dP4 fibration	$\mathbf{D}^b(C)$, with $J(C) = J(X)$	[12]
rat'l. surf.	\mathbb{P}^1 -bundle	0	[146]
	conic bundle	$\mathbf{D}^b(Y, \mathcal{E}_0)$, \mathcal{E}_0 a Clifford algebra	[116]
\mathbb{P}^2	rational conic bundle with deg. curve Γ	$\langle \mathbf{D}^b(C), V \rangle$, V v.b and $J(C) = J(X)$ if $g(\Gamma) > 1$	[29]
		$\langle L_1, L_2, L_3 \rangle$, L_i line bd. if $g(\Gamma) = 1$	[29]
Hirz.	rational conic bundle with deg. curve Γ	$\langle \mathbf{D}^b(C), V_1, V_2 \rangle$, V_i v.b and $J(C) = J(X)$ if $g(\Gamma) = 3$	[29]
		$\langle \mathbf{D}^b(C_1), \mathbf{D}^b(C_2) \rangle$, with $J(X) = J(C_1) \oplus J(C_2)$ if $g(\Gamma) = 2$	[29]
Fourfolds			
any	\mathbb{P} -bundle	0	[146]
	quadric bundle	$\mathbf{D}^b(Y, \mathcal{E}_0)$, \mathcal{E}_0 Cliff. algebra	[116]
pt	quadric 4fold	$\langle S_1, S_2 \rangle$, spinor bundles	[108]
	cubic fourfold	a noncommutative K3	[123]
	Pfaffian cubic	$\mathbf{D}^b(S)$, with S a degree 14 K3	[123]
	cubic with a plane	$\mathbf{D}^b(S, \alpha)$, with S a degree 2 K3 and $\alpha \in \text{Br}(S)$ $\alpha = 0$ iff the associated quadric fibr. has a section	[123]
	nodal cubic*	$\mathbf{D}^b(S)$, with S a degree 6 K3	[123]
	determinantal cubic*	six exceptional objects	[31]
	general cubic in \mathcal{C}_d with associated K3	$\mathbf{D}^b(S)$, with S a K3 surface of degree d	[3]
deg 10 ind 2 in $G(2, 5)$	a noncommutative K3	[125]	
\mathbb{P}^1	fibration in intersections of 2 quadrics	$\mathbf{D}^b(T, \alpha)$, $T \rightarrow \mathbb{P}^1$ hyperell. fib. $\alpha \in \text{Br}(T)$, $\alpha = 0$ if S has a line/ \mathbb{P}^1	[12]

TABLE 1. Known descriptions of $\mathbf{A}_{X/Y}$, for $X \rightarrow Y$ a Mori fiber space, and $\dim(X) = 3$ or $\dim(X) = 4$ and $\kappa(Y) = -\infty$. In the nonsmooth cases, indicated by *, the description refers to a categorical resolution of $\mathbf{A}_{X/Y}$

K3 surface, and if it is general on some \mathcal{C}_d , then there exists a K3 surface S and an equivalence $\mathrm{D}^b(S) \simeq \mathbf{A}_X$.

The above result can be stated in terms of d , since, as recalled in Theorem 4.3.8, being special with an associated K3 surface is equivalent to lie on \mathcal{C}_d for $d > 6$ not divisible by 4, 9, or any odd prime p which is not 2 modulo 3.

Notice that Theorem 8.3.2 tells that, in Hodge-theoretic terms, Conjecture 8.3.1 could be phrased as “ X is rational if and only if it has an associated K3”. As we have seen at the end of §4.3, this seems to be the most “rational” expectation for such varieties. Conjecture 8.3.1 seems then, so far, out of reach. But we want to give, in the last part of this section, a quick idea of the very interesting interplay of information that one can extract this categorical point of view.

First of all, let us briefly sketch how to prove Theorem 8.3.2. The main idea is to have a categorical way to reconstruct the Hodge lattice. This is done by considering the topological K -theory $K_0(X)_{top}$, with the intersection pairing χ , given by the Euler characteristic. Using the semiorthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathbf{A}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle,$$

it is possible to split off $K_0(\mathbf{A}_X)_{top} \subset K_0(X)_{top}$ as the χ -semiorthogonal complement to the classes $[\mathcal{O}(i)]$, for $i = 0, 1, 2$. If $\mathrm{D}^b(S) \simeq \mathbf{A}_X$ is an equivalence, it is then of Fourier–Mukai type and gives an isomorphism $K_0(S)_{top} \simeq K_0(\mathbf{A}_X)_{top}$ of \mathbb{Z} -modules respecting the Euler pairing.

The first observation is obtained using the fact that the Chern character is integral for S , so that $K_0(S)_{top} \otimes \mathbb{C}$ with the pairing χ has a Hodge structure of weight 2 induced by the so-called Mukai lattice structure on cohomology. This is (up to identifying K_0 and the cohomology via the Chern character) the following Hodge structure $\tilde{H}^{p,q}(S)$:

$$\begin{aligned} \tilde{H}^{2,0}(S) &= H^{2,0}(S) \\ \tilde{H}^{1,1}(S) &= H^{2,2}(S) \oplus H^{1,1}(S) \oplus H^{0,0}(S) \\ \tilde{H}^{0,2}(S) &= H^{0,2}(S). \end{aligned}$$

For the cubic fourfold X , Addington and Thomas define the Mukai lattice as follows:

$$\begin{aligned} \tilde{H}^{2,0}(X) &= H^{3,1}(X) \\ \tilde{H}^{1,1}(X) &= H^{4,4}(X) \oplus H^{3,3}(X) \oplus H^{2,2}(X) \oplus H^{1,1}(X) \oplus H^{0,0}(X) \\ \tilde{H}^{0,2}(X) &= H^{1,3}(X), \end{aligned}$$

and obtain the corresponding weight 2 Hodge structure on $K_0(\mathbf{A}_X)_{top}$ (up to identifying K_0 and the cohomology via the Chern character). It follows then that if $\mathrm{D}^b(S) \simeq \mathbf{A}_X$, there is a Hodge isometry $K_0(S)_{top} \simeq K_0(\mathbf{A}_X)_{top}$.

For the general X , the numerical properties of $K_0(\mathbf{A}_X)_{top}$ are explored, and the lattice $(K_0(\mathbf{A}_X)_{top}, \chi)$ is related to the Hodge lattice $H^4(X, \mathbb{Z})$. Of particular interest is the result that X has an associated K3 surface if and only if the Mukai lattice on the numerical $K_0(\mathbf{A}_X)_{num}$ ¹² contains a hyperbolic plane [3, Thm. 3.1]. Moreover, one can characterize classes of skyscraper sheaves of points $[\mathcal{O}_x]$ and $[\mathcal{O}_y]$ inside $K_0(S)_{top}$, purely using their behavior under the Euler pairing: the sublattice they generate is a hyperbolic lattice. Hence, it follows that if X has no associated K3, such classes do not exist and hence there cannot be any equivalence

¹²The numerical K_0 is obtained by taking the quotient of the algebraic K_0 by the kernel of the Euler form.

$D^b(S) \simeq A_X$. A similar result for cubic fourfolds containing a plane with two dimensional group of algebraic 2-cycles was obtained by Kuznetsov [123].

On the other hand, consider a divisor \mathcal{C}_d where d is such that X has an associated K3 surface. Then consider the intersection $\mathcal{C}_d \cap \mathcal{C}_8$ with the locus of cubics containing a plane. As we will see in Example 8.3.3 (see also Table 1), in this case there is a degree 2 K3 surface S with a Brauer class α and an equivalence $A_X \simeq D^b(S, \alpha)$. Moreover, one can find an X on $\mathcal{C}_d \cap \mathcal{C}_8$ such that α vanishes. This gives a K3 surface S of degree 2 and an equivalence $A_X \simeq D^b(S)$ in this particular case. Now, though one would like to deform this K3 surface, it is not the right thing to do. Instead, one has to construct an appropriate K3 surface S' of degree d as a moduli space of vector bundles on S , so that $D^b(S') \simeq D^b(S) \simeq A_X$. Then the existence of a K3 surface S' and of the equivalence $A_X \simeq D^b(S')$ for the general X in \mathcal{C}_d is obtained by a degeneration method.

We turn now to explicit cases supporting Conjecture 8.3.1. Indeed, in all the cases where a cubic fourfold is known to be rational, there is an explicit realization of the K3 surface S and of the equivalence $D^b(S) \simeq A_X$.

EXAMPLE 8.3.3 (Kuznetsov [123]). Let X be a cubic fourfold containing a plane P . As explained in Example 4.3.7, the blow-up $\tilde{X} \rightarrow X$ of the plane P has a structure of quadric surface bundle $\tilde{X} \rightarrow \mathbb{P}^2$.

Recall, from Example 2.3.9, that there is an identification $A_{\tilde{X}/\mathbb{P}^2} \simeq D^b(\mathbb{P}^2, \mathcal{C}_0)$ (see [116]). On the other hand, there is a semiorthogonal decomposition induced by the blow-up $\tilde{X} \rightarrow X$. Comparing the two decompositions via explicit mutations, gives an equivalence $A_X \simeq D^b(\mathbb{P}^2, \mathcal{C}_0)$.

Suppose that the degeneration divisor of the quadric fibration $\tilde{X} \rightarrow \mathbb{P}^2$ is smooth, and consider the double cover $S \rightarrow \mathbb{P}^2$ ramified along it. Kuznetsov shows [116, Prop. 3.13] that \mathcal{C}_0 lifts to a sheaf of Clifford algebras with Brauer class α in $\text{Br}(S)$, see also [16, Prop. 1.11]. It follows that $A_X \simeq D^b(S, \alpha)$.

On the other hand, the classical theory of quadratic forms says that the class α is trivial if and only if $\tilde{X} \rightarrow \mathbb{P}^2$ has an odd section. As described in Example 4.3.7 this is a necessary condition for rationality.

Secondly, Kuznetsov shows that if S has Picard rank one, then α is nontrivial and $D^b(S, \alpha)$ cannot be equivalent to any $D^b(S')$, for S' a K3 surface (this can be seen as a special case of Addington–Thomas result). In particular, one should expect that cubics with a plane with such an S are not rational. On the other hand, notice that there exist rational cubics with a plane, such that α is not trivial but $D^b(S, \alpha) \simeq D^b(S')$ for some other K3 surface S' . Example of such cubics are constructed in [13].

EXAMPLE 8.3.4 (Kuznetsov [123]). Let X be a Pfaffian cubic fourfold. As explained in Example 4.3.6, there is a classical duality construction providing a degree 14 K3 surface S associated to X . In this case, the powerful theory of Homological Projective Duality allows to show that $A_X \simeq D^b(S)$.

EXAMPLE 8.3.5 (Kuznetsov [123]). Let X be a cubic fourfold with a single node x . The projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ from the point x gives a rational parametrization $X \dashrightarrow \mathbb{P}^4$. The resolution of the latter map is obtained by blowing-up x and is realized as a blow-up $\tilde{X} \rightarrow \mathbb{P}^4$ of a degree 6 K3 surface S , obtained as a complete intersection of a cubic and a quadric. Then one can show that there is a crepant resolution of singularities \tilde{A}_X and an equivalence $\tilde{A}_X \simeq D^b(S)$.

EXAMPLE 8.3.6 ([31]). Let X be a determinantal cubic fourfold. In this case, Homological Projective Duality can be used to show that there is a crepant resolution of singularities \tilde{A}_X which is generated by six exceptional objects. Roughly speaking, one should think of the latter as a crepant resolution of a degeneration of the K3 surface from the previous case.

We can then rephrase Example 4.3.11 listing all known rational cubic fourfolds in terms of Conjecture 8.3.1.

EXAMPLE 8.3.7. Let X be a cubic fourfold. If either

- 2,6) X is singular, e.g. $X \in \mathcal{C}_6$ has a single node or $X \in \mathcal{C}_2$ is determinantal;
- 8) X contains a plane P , so that $X \in \mathcal{C}_8$, and the associated quadric surface fibration $\tilde{X} \rightarrow \mathbb{P}^2$ (see Example 4.3.7) admits an odd section [96]; or
- 14) X is Pfaffian, so that $X \in \mathcal{C}_{14}$ [26];

then X is rational and (a crepant categorical resolution of) A_X is equivalent to (a crepant categorical resolution of) $D^b(S)$, for some K3 surface S .

8.4. Other fourfolds. Let us quickly conclude with another example, described in [12], of fourfolds whose rationality or nonrationality is conjecturally related to categorical representability via an explicit semiorthogonal decomposition.

Let X be a fourfold with a Mori Fiber Space structure $X \rightarrow \mathbb{P}^1$ such that the fibers are complete intersections of two quadrics. This means that there is a projective bundle $\mathbb{P}(E) \rightarrow \mathbb{P}^1$ with $\text{rk}(E) = 5$, and two line bundle valued non-degenerate quadratic forms $q_i : L_i \rightarrow S_2(E)$, such that $X \subset \mathbb{P}(E)$ is the complete intersection of the two quadric fibrations $Q_i \rightarrow \mathbb{P}^1$ given by the forms q_i . Moreover, working over \mathbb{C} , we know that $X \rightarrow \mathbb{P}^1$ has a smooth section (see, e.g. [12, Lemma 1.9.3] for a direct argument, or use [53] or [93]).

Setting $F := L_1 \oplus L_2$, one has that the linear span q of q_1 and q_2 gives a quadric fibration $Q \rightarrow \mathbb{P}(F)$ of relative dimension 4 over a Hirzebruch surface $\mathbb{P}(F)$. The smooth section of $X \rightarrow \mathbb{P}^1$ gives a smooth section of $Q \rightarrow \mathbb{P}(F)$, along which we can perform reduction by hyperbolic splitting. This means that we can split off the form q a hyperbolic lattice, whose complement gives a quadric fibration $Q' \rightarrow \mathbb{P}(F)$ of dimension two less than Q . That is, $Q' \rightarrow \mathbb{P}(F)$ is a quadric surface fibration.

Homological Projective Duality and Morita equivalence of Clifford algebras under hyperbolic splitting (see [12] for details) show that $A_{X/\mathbb{P}^1} \simeq A_{Q'/\mathbb{P}(F)}$. The latter is known to be equivalent to $D^b(\mathbb{P}(F), \mathcal{C}_0)$, where \mathcal{C}_0 is the sheaf of even Clifford algebras of the quadric surface fibration $Q' \rightarrow \mathbb{P}(F)$. Finally, assuming the degeneration divisor of $Q' \rightarrow \mathbb{P}(F)$ to be smooth, we have a smooth double cover $S \rightarrow \mathbb{P}(F)$ and a Brauer class α in $\text{Br}(S)$, such that $A_{X/\mathbb{P}^1} \simeq D^b(S, \alpha)$. Notice that the composition $S \rightarrow \mathbb{P}(F) \rightarrow \mathbb{P}^1$ endows S with a fibration into genus 2 curves, since there are 6 degenerate quadrics for each fiber of $\mathbb{P}(F) \rightarrow \mathbb{P}^1$.

The following conjecture is inspired by Question 8.0.4 and Kuznetsov's conjecture 8.3.1 for cubic fourfolds.

CONJECTURE 8.4.1 ([12], Conj. 5.1.2). *Let $X \rightarrow \mathbb{P}^1$ be a fibration in complete intersections of two four-dimensional quadrics.*

- **Weak version.** *The fourfold X is rational if and only if $\text{rcdim}(X) \geq 2$.*
- **Strong version.** *The fourfold X is rational if and only if $\text{rdim } A_{X/\mathbb{P}^1} \leq 2$.*

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