Symmetry of minimizers for some nonlocal variational problems

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Abstract

We present a new approach to study the symmetry of minimizers for a large class of nonlocal variational problems. This approach which generalizes the Reflection method is based on the obtention of some integral identities. We study the identities that lead to symmetry results, the functionals that can be considered and the function spaces that can be used. Then we use our method to prove the symmetry of minimizers for a class of variational problems involving the fractional powers of Laplacian, for the generalized Choquard functional and for the standing waves of the Davey-Stewartson equation.

Keywords. Symmetry of minimizers, nonlocal functional, minimization under constraints, fractional powers of Laplacian, Choquard functional, Davey-Stewartson equation.

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1 Introduction

Many important partial differential equations arising in Physics are Euler-Lagrange equations of variational problems. Among the solutions of these equations those who correspond to a minimum of the associated functional (e.g. the "energy") subject to some constraint are of particular interest. For example in many situations the set of such solutions is orbitally stable (see [9]).

In this paper we address the general question of whether, or not, the fact that the underlying problem has some symmetries is reflected on the minimizers. Namely if a problem is invariant under the action of a group of transformations, is it true that the corresponding minimizers are also invariant under the action of this group (or, perhaps, a subgroup of it) ? As it is shown in [14], this may not be the case.

A classical approach to radial symmetry of minimizers is Schwarz symmetrization (or spherical decreasing rearrangement, see [16]). For a nonnegative function $u \in H^1(\mathbf{R}^N)$ its symmetrization u^* is a radially-decreasing function from $\mathbf{R}^N$ into $\mathbf{R}$ which has the property that $meas(\{x \in \mathbf{R}^N \mid u(x) > \lambda\} = meas(\{x \in \mathbf{R}^N \mid u^*(x) > \lambda\}$ for any $\lambda > 0$. It is well-known that u^* satisfies (among others) the following properties:

$$\int_{\mathbf{R}^N} |\nabla u^*(x)|^2\,dx \le \int_{\mathbf{R}^N} |\nabla u(x)|^2\,dx \quad \text{and} \quad \int_{\mathbf{R}^N} F(u^*(x))\,dx = \int_{\mathbf{R}^N} F(u(x))\,dx, \tag{1.1}$$

where F is, say, a smooth function from $\mathbf{R}$ into itself such that $F(u) \in L^1(\mathbf{R}^N)$ (see [16]). As a simple application of symmetrization, consider the problem of minimizing

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u(x)|^2\,dx + \int_{\mathbf{R}^N} F(u(x))\,dx$$

subject to the constraint

$$\int_{\mathbf{R}^N} G(u(x))\,dx = \lambda,$$

where F, $G \in C^1(\mathbf{R}, \mathbf{R})$ have the property that $F(u)$, $G(u) \in L^1(\mathbf{R}^N)$ whenever $u \in H^1(\mathbf{R}^N)$. If $u \in H^1(\mathbf{R}^N)$ is a nonnegative minimizer, then from (1.1) it follows that u^* also satisfies the constraint and $E(u^*) \le E(u)$; therefore, u^* is also a minimizer. To show that $u \equiv u^*$ except for translation is a more delicate question and this follows from a result in [6] and the Unique Continuation Principle.

The case of nonlocal functionals also arises in applications. For instance, the Choquard problem consists in minimizing

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u(x)|^2\,dx - \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy$$

subject to

$$\int_{\mathbf{R}^3} u^2(x)\,dx = \lambda.$$

The radial symmetry of minimizers of Choquard problem has been proved in [15] by using Riesz' inequality for rearrangements :

$$\int_{\mathbf{R}^N\times\mathbf{R}^N} f(x)g(x-y)h(y)\,dx\,dy \le \int_{\mathbf{R}^N\times\mathbf{R}^N} f^*(x)g^*(x-y)h^*(y)\,dx\,dy, \tag{1.2}$$

where f, g and h are nonnegative functions. Moreover, if g is strictly symmetric-decreasing then equality holds in (1.2) if and only if $f(x) = f^*(x-y)$ and $h(x) = h^*(x-y)$ for some $y \in \mathbf{R}^N$.

In the vector case symmetrization can also be used because of the inequality

$$\int_{\mathbf{R}^N} F(u^*(x), v^*(x))\,dx \le \int_{\mathbf{R}^N} F(u(x), v(x))\,dx, \tag{1.3}$$

which holds provided that the function F is C^2 and satisfies the cooperative condition $\frac{\partial^2 F}{\partial u \partial v}(u,v) \le 0$ for $u, v \ge 0$ (see [5]). Therefore, consider the problem of minimizing

$$E(u,v) = \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u(x)|^2 + |\nabla v(x)|^2)\,dx + \int_{\mathbf{R}^N} F(u(x), v(x))\,dx$$

subject to the constraint

$$\int_{\mathbf{R}^N} (G_1(u(x)) + G_2(v(x))\,dx = \lambda,$$

where $\frac{\partial^2 F}{\partial u \partial v}(u,v) \le 0$ for $u, v \ge 0$. If (u,v) is a nonnegative minimizer, then from (1.1) and (1.3) we see that (u^*, v^*) is also a minimizer. Notice that the function defining the constraint must have a special form because we want the value of the constraint to be preserved by symmetrization.

Another tool to prove radial symmetry of minimizers is the result by Gidas, Ni and Nirenberg [11] about the radial symmetry of positive solutions of the semilinear elliptic equation

$$-\Delta u + f(u) = 0.$$

In the case of systems, an extension of that result has been proved in [7] and [25] assuming a cooperative condition for the nonlinearity. In [11] as well as in its generalizations the nonlinearities are also allowed to depend on the space variable in a radial and monotonic way.

As we can see, in the vector case, besides the need to know in advance that the components of the minimizer are positive, both methods described above require the nonlinearity to satisfy a cooperative condition and the function defining the constraint to have a special form. To avoid these two restrictions, the Reflection method has been developed in [18] and [19]. We now briefly describe this method.

Consider the problem of minimizing

$$E(u,v) = \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u(x)|^2 + |\nabla v(x)|^2)\, dx + \int_{\mathbf{R}^N} F(u(x), v(x))\, dx$$

subject to

$$\int_{\mathbf{R}^N} G(u(x), v(x))\, dx = \lambda \neq 0.$$

To show that any minimizer (u,v) is symmetric with respect to x_1 (except possibly for a translation), we first make a translation in the x_1 variable in such way that

$$\int_{\{x_1<0\}} G(u(x), v(x))\, dx = \int_{\{x_1>0\}} G(u(x), v(x))\, dx = \frac{\lambda}{2}. \tag{1.4}$$

Next, setting $x = (x_1, x')$, where $x' \in \mathbf{R}^{N-1}$, we define the functions u_1 and u_2 by

$$u_1(x) = u_1(x_1, x') = \begin{cases} u(x_1, x') \text{ if } x_1 < 0, \\ u(-x_1, x') \text{ if } x_1 \geq 0 \end{cases} \qquad \text{and} \qquad u_2(x) = \begin{cases} u(-x_1, x') \text{ if } x_1 < 0, \\ u(x_1, x') \text{ if } x_1 \geq 0. \end{cases}$$

In a similar way we define v_1 and v_2. According to (1.4), the pairs (u_1, v_1) and (u_2, v_2) also satisfy the constraint (i.e. they are admissible). Moreover, a change of variables shows that

$$E(u_1, v_1) + E(u_2, v_2) = 2E(u,v). \tag{1.5}$$

Thus (u_1, v_1) and (u_2, v_2) are also minimizers. This shows that there exist minimizers which are symmetric with respect to x_1. In fact, by using the Euler-Lagrange equations and the Unique Continuation Principle we can show that necessarily $(u_1, v_1) = (u,v) = (u_2, v_2)$. Clearly, this implies that any minimizer (u,v) is symmetric with respect to the first variable. Replacing the x_1-direction by any other direction in $\mathbf{R}^N$ and repeating the same argument, we can show that (u,v) is radially symmetric except for translation (details will be given later). Notice that to use this argument there is no need to know the sign of components of the minimizers.

The main point of this paper is to extend the Reflection method to a class of nonlocal functionals. To be more specific, consider the problem of minimizing

$$E(u,v) = \int_{\mathbf{R}^N} (\frac{1}{2}|(-\Delta)^{\frac{s}{2}} u|^2 + \frac{1}{2}|\nabla v|^2)\, dx + \int_{\mathbf{R}^N} F(u,v)\, dx \tag{1.6}$$

subject to the constraint

$$Q(u,v) = \int_{\mathbf{R}^N} G(u,v)\,dx = \lambda \neq 0, \tag{1.7}$$

where $0 < s < 1$. Defining

$$W(u) = \frac{1}{2}\int_{\mathbf{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\,dx$$

and (u_1, u_2) and (v_1, v_2) as above, instead of (1.5) we have

$$E(u_1, v_1) + E(u_2, v_2) - 2E(u,v) = W(u_1) + W(u_2) - 2W(u).$$

Therefore, to show that the pairs (u_1, v_1) and (u_2, v_2) are also minimizers we need to know that the following inequality holds

$$W(u_1) + W(u_2) - 2W(u) \leq 0. \tag{1.8}$$

The key to the method developed here is to show that inequality (1.8) holds true (see Theorem 2.8). Moreover, we have equality in (1.8) if and only if u is symmetric with respect to x_1. As we will see, this gives the desired radial symmetry of minimizers. More general multipliers $m(\xi)$ and more regular nonlocal functionals like the one appearing in the Choquard problem above are also considered. In this article we will use this extended Reflection method to show the symmetry of all minimizers of the following problems:

- the Hamiltonian of a coupled system between a multidimensional Korteweg-de Vries equation and a Benjamin-Ono equation (this is precisely problem (1.6)-(1.7) with $s = 1/2$). Here the minimizers correspond to solitary waves;
- the generalized Choquard problem. In this case the minimizers give rise to standing waves for the generalized Hartree equation;
- the Hamiltonian of the generalized Davey-Stewartson equation. Here again, minimizers correspond to standing waves.

The existence of minimizers for these problems can be proved by using the concentration-compactness method [17] or the alternative method presented in [20] and will not be discussed here.

Notice that the symmetrization approach, in general, does not apply to the problems above. Indeed, in the first two examples, symmetrization cannot be used to prove the existence of a radially symmetric minimizer under the general assumptions on the nonlinearities made in this paper. Furthermore, with the tools available at the present time, it is not clear how to prove the radial symmetry of *all minimizers*, even in the cases where symmetrization can be used to prove the existence of a radially symmetric minimizer. Finally, in the last example, symmetrization cannot be used because one term of the Hamiltonian of the Davey-Stewartson equation is a singular integral operator whose kernel changes sign.

This paper is organized as follows: in the next section we present some integral identities for functionals of the form $W(u) = \displaystyle\int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\,d\xi$. These identities are first proved for functions $u \in C_c^\infty$ and are crucial for our approach to symmetry. It will also appear clearly what kind of symbols $m(\xi)$ we may consider. In section 3 we search for appropriate function spaces on which our method can be applied. It will be proved that we may work on $H^s(\mathbf{R}^N)$ or on $\dot{H}^s(\mathbf{R}^N)$ if $-\frac{1}{2} < s < \frac{3}{2}$. We will extend the integral identities obtained in section 2 to these function spaces. In section 4 we apply our results to the concrete problems presented above. We end this article with some open problems.

2 Some identities

In what follows, $x = (x_1, x_2, \dots, x_N) = (x_1, x')$ denotes a point of $\mathbf{R}^N$, $x' = (x_2, \dots, x_N) \in \mathbf{R}^{N-1}$, $\xi = (\xi_1, \xi_2, \dots, \xi_N) = (\xi_1, \xi') \in \mathbf{R}^N$ with $\xi' = (\xi_2, \dots, \xi_N) \in \mathbf{R}^{N-1}$. We denote the Fourier transform either by $\widehat{\ }$ or by $\mathcal{F}$.

The aim of this section is to prove an identity for some functionals of the type

$$W(u) = \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2 \, d\xi \tag{2.1}$$

which will play a very important role in proving symmetries.

Consider a function $u \in C_c^\infty(\mathbf{R}^N)$. We define the reflected functions u_1 and u_2 as follows :

$$u_1(x) = u_1(x_1, x') = \begin{cases} u(x_1, x') \text{ if } x_1 < 0, \\ u(-x_1, x') \text{ if } x_1 \geq 0 \end{cases} \quad \text{and} \quad u_2(x) = \begin{cases} u(-x_1, x') \text{ if } x_1 < 0, \\ u(x_1, x') \text{ if } x_1 \geq 0. \end{cases} \tag{2.2}$$

We also define

$$g(x) = \frac{1}{2}(u(x_1, x') + u(-x_1, x')) \quad \text{and} \quad f(x) = \frac{1}{2}(u(x_1, x') - u(-x_1, x')). \tag{2.3}$$

Clearly, $f, g \in C_c^\infty(\mathbf{R}^N)$, g is even and f is odd with respect to x_1 and $u = f + g$. Let

$$f_*(x) = \begin{cases} f(-x_1, x') = -f(x) \text{ if } x_1 < 0, \\ f(x_1, x') \text{ if } x_1 \geq 0. \end{cases} \tag{2.4}$$

Then f_* is even with respect to x_1, $u_1 = g - f_*$ and $u_2 = g + f_*$.

We want to study the quantity

$$W(u_1) + W(u_2) - 2W(u) \tag{2.5}$$

where W is given by (2.1). Later in Theorem 2.8 we specify the class of multipliers under consideration but, at this early stage, besides integrability conditions, we assume that

$$m(\xi) \text{ is real and } m(-\xi_1, \xi') = m(\xi_1, \xi'). \tag{2.6}$$

We have :

$$\begin{aligned} \widehat{g}(-\xi_1, \xi') &= \int_{\mathbf{R}^N} e^{ix_1\xi_1 - ix'.\xi'} g(x_1, x') dx = \int_{\mathbf{R}^N} e^{-iy_1\xi_1 - ix'.\xi'} g(-y_1, x') dy_1 dx' \\ &= \widehat{g}(\xi_1, \xi') \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \widehat{f}(-\xi_1, \xi') &= \int_{\mathbf{R}^N} e^{ix_1\xi_1 - ix'.\xi'} f(x_1, x') dx = \int_{\mathbf{R}^N} e^{-iy_1\xi_1 - ix'.\xi'} f(-y_1, x') dy_1 dx' \\ &= \quad -\widehat{f}(\xi_1, \xi'). \end{aligned} \tag{2.8}$$

Therefore

$$\begin{aligned} &W(u_1) + W(u_2) - 2W(u) \\ &= \int_{\mathbf{R}^N} m(\xi_1, \xi')(|\widehat{g}(\xi) - \widehat{f_*}(\xi)|^2 + |\widehat{g}(\xi) + \widehat{f_*}(\xi)|^2 - 2|\widehat{g}(\xi) + \widehat{f}(\xi)|^2) \, d\xi \\ &= \int_{\mathbf{R}^N} m(\xi_1, \xi')(2|\widehat{f_*}(\xi)|^2 - 2|\widehat{f}(\xi)|^2 - 4\mathrm{Re}(\widehat{g}(\xi)\overline{\widehat{f}(\xi)}) \, d\xi \\ &= 2\int_{\mathbf{R}^N} m(\xi_1, \xi')(|\widehat{f_*}(\xi)|^2 - |\widehat{f}(\xi)|^2) \, d\xi = 2W(f_*) - 2W(f) \end{aligned} \tag{2.9}$$

because $\int_{\mathbf{R}^N} m(\xi_1,\xi')\mathrm{Re}(\widehat{g}(\xi)\overline{\widehat{f}(\xi)}d\xi = 0$ in view of (2.6), (2.7) and (2.8).
It is easy to see that

$$\begin{aligned}
\widehat{f}(\xi_1,\xi') &= \int_{\mathbf{R}}\int_{\mathbf{R}^{N-1}} e^{-ix_1\xi_1 - ix'.\xi'} f(x_1,x')\,dx'\,dx_1 \\
&= \int_0^\infty \int_{\mathbf{R}^{N-1}} (e^{-ix_1\xi_1} - e^{ix_1\xi_1})e^{-ix'.\xi'} f(x_1,x')\,dx'\,dx_1 \\
&= -2i\int_0^\infty \int_{\mathbf{R}^{N-1}} \sin(x_1\xi_1)e^{-ix'.\xi'} f(x_1,x')\,dx'\,dx_1
\end{aligned}$$

and

$$\begin{aligned}
\widehat{f_*}(\xi_1,\xi') &= \int_{\mathbf{R}}\int_{\mathbf{R}^{N-1}} e^{-ix_1\xi_1 - ix'.\xi'} f_*(x_1,x')\,dx'\,dx_1 \\
&= \int_0^\infty \int_{\mathbf{R}^{N-1}} (e^{-ix_1\xi_1} + e^{ix_1\xi_1})e^{-ix'.\xi'} f(x_1,x')\,dx'\,dx_1 \\
&= 2\int_0^\infty \int_{\mathbf{R}^{N-1}} \cos(x_1\xi_1)e^{-ix'.\xi'} f(x_1,x')\,dx'\,dx_1.
\end{aligned}$$

We denote by $\mathcal{F}_{N-1}$ the partial Fourier transform in the last $N-1$ variables, that is

$$\mathcal{F}_{N-1}f(x_1,\xi') = \int_{\mathbf{R}^{N-1}} e^{-ix'.\xi'} f(x_1,x')\,dx'.$$

Since $f \in C_c^\infty(\mathbf{R}^N)$ we may use Fubini's theorem to get

$$\begin{aligned}
&|\widehat{f}(\xi_1,\xi')|^2 = \widehat{f}(\xi_1,\xi')\overline{\widehat{f}(\xi_1,\xi')} \\
&= 4\int_0^\infty\int_0^\infty \sin(x_1\xi_1)\sin(y_1\xi_1)(\mathcal{F}_{N-1}f)(x_1,\xi')\overline{(\mathcal{F}_{N-1}f)(y_1,\xi')}\,dx_1\,dy_1
\end{aligned}$$

and similarly

$$\begin{aligned}
&|\widehat{f_*}(\xi_1,\xi')|^2 = \widehat{f_*}(\xi_1,\xi')\overline{\widehat{f_*}(\xi_1,\xi')} \\
&= 4\int_0^\infty\int_0^\infty \cos(x_1\xi_1)\cos(y_1\xi_1)(\mathcal{F}_{N-1}f)(x_1,\xi')\overline{(\mathcal{F}_{N-1}f)(y_1,\xi')}\,dx_1\,dy_1.
\end{aligned}$$

Consequently,
(2.10)

$$\begin{aligned}
&W(f_*) - W(f) = \int_{\mathbf{R}^N} m(\xi_1,\xi')(|\widehat{f_*}(\xi_1,\xi')|^2 - |\widehat{f}(\xi_1,\xi')|^2)\,d\xi \\
&= 4\int_{\mathbf{R}^N} m(\xi_1,\xi')\int_0^\infty\int_0^\infty [\cos(x_1\xi_1)\cos(y_1\xi_1) - \sin(x_1\xi_1)\sin(y_1\xi_1)] \\
&\qquad\qquad (\mathcal{F}_{N-1}f)(x_1,\xi')\overline{(\mathcal{F}_{N-1}f)(y_1,\xi')}\,dx_1\,dy_1\,d\xi \\
&= 4\int_{\mathbf{R}^N} m(\xi_1,\xi')\int_0^\infty\int_0^\infty \cos((x_1+y_1)\xi_1)(\mathcal{F}_{N-1}f)(x_1,\xi')\overline{(\mathcal{F}_{N-1}f)(y_1,\xi')}\,dx_1\,dy_1\,d\xi.
\end{aligned}$$

For an arbitrary (but fixed) $\xi' \in \mathbf{R}^{N-1}$, we define $\varphi_{\xi'}(t) = (\mathcal{F}_{N-1}f)(t,\xi')$. Since $f \in C_c^\infty(\mathbf{R}^N)$, it is clear that $\varphi_{\xi'} \in C_c^\infty(\mathbf{R})$. If $\mathrm{supp}(f) \subset B_{\mathbf{R}^N}(0,R)$, then $\mathrm{supp}(\varphi_{\xi'}) \subset [-R,R]$. For $z \in \mathbf{C}$, we define

(2.11) $$h_{\xi'}(z) = \int_0^\infty\int_0^\infty e^{i(x_1+y_1)z}\varphi_{\xi'}(x_1)\overline{\varphi_{\xi'}(y_1)}\,dx_1\,dy_1.$$

Since $\varphi_{\xi'}$ is bounded and has compact support, $h_{\xi'}$ is well-defined and is an holomorphic function on $\mathbf{C}$. For any $z \in \mathbf{R}$ we have

$$\overline{h_{\xi'}(z)} = \int_0^\infty \int_0^\infty e^{-i(x_1+y_1)z}\overline{\varphi_{\xi'}(x_1)}\varphi_{\xi'}(y_1)\, dx_1\, dy_1 = h_{\xi'}(-z)$$

and

$$\mathrm{Re}(h_{\xi'}(z)) = \frac{1}{2}(h_{\xi'}(z) + \overline{h_{\xi'}(z)}) = \int_0^\infty \int_0^\infty \cos((x_1+y_1)z)\varphi_{\xi'}(x_1)\overline{\varphi_{\xi'}(y_1)}\, dx_1\, dy_1.$$

From (2.6), (2.9) and (2.10) we get

$$W(u_1) + W(u_2) - 2W(u) = 2W(f_*) - 2W(f) = 8\int_{\mathbf{R}^{N-1}} \int_{-\infty}^{\infty} m(\xi_1, \xi')h_{\xi'}(\xi_1)\, d\xi_1\, d\xi'. \tag{2.12}$$

Some properties of the function $h_{\xi'}$ are given in the next lemma. To simplify the notation, we shall write h instead of $h_{\xi'}$.

Lemma 2.1 *For any fixed ξ', the function $h = h_{\xi'}$ given by (2.11) has the following properties:*

i) h is bounded in the upper half-plane $\{z \in \mathbf{C} \mid Im(z) \geq 0\}$.

ii) There exists a constant $C > 0$ (depending on f and ξ') such that for any $z \neq 0$ with $\mathrm{Im}(z) \geq 0$ we have:

$$|h(z)| \leq \frac{C}{|z|^4} \qquad \textit{and} \tag{2.13}$$

$$|h'(z)| \leq \frac{C}{|z|^5}. \tag{2.14}$$

Proof. *i)* If $b \geq 0$ and $x \geq 0$ then $|e^{iax-bx}| \leq 1$ and we have

$$|h(a+ib)| = \left| \int_0^\infty \int_0^\infty e^{i(x_1+y_1)a-(x_1+y_1)b}\varphi_{\xi'}(x_1)\overline{\varphi_{\xi'}(y_1)}\, dx_1\, dy_1 \right|$$
$$\leq \left(\int_0^\infty |e^{iat-bt}| \cdot |\varphi_{\xi'}(t)|\, dt \right)^2 \leq \left(\int_0^\infty |\varphi_{\xi'}(t)|\, dt \right)^2.$$

ii) It is clear that

$$h(z) = \int_0^\infty e^{ix_1z}\varphi_{\xi'}(x_1)\, dx_1 \cdot \int_0^\infty e^{iy_1z}\overline{\varphi_{\xi'}(y_1)}\, dy_1 = \Psi_1(z)\Psi_2(z), \tag{2.15}$$

where $\Psi_1(z)$ and $\Psi_2(z)$ are defined in an obvious way. Notice that $\varphi_{\xi'}(0) = (\mathcal{F}_{N-1}f)(0, \xi') = 0$ because $f(0, x') = 0$ (recall that f is odd with respect to x_1). Moreover, for any $k \in \mathbf{N}$,

$$\frac{d^k}{dt^k}\varphi_{\xi'}(t) = \frac{d^k}{dt^k}\int_{\mathbf{R}^{N-1}} e^{-ix'.\xi'}f(t, x')\, dx'$$
$$= \int_{\mathbf{R}^{N-1}} e^{-ix'\xi'}\frac{\partial^k f}{\partial x_1^k}(t, x')\, dx' = (\mathcal{F}_{N-1}\frac{\partial^k f}{\partial x_1^k})(t, \xi')$$

is a C_c^∞ function of t, uniformly bounded for $(t, \xi') \in \mathbf{R} \times \mathbf{R}^{N-1}$. Integrating by parts, we get:

$$\begin{aligned}\Psi_1(z) &= \int_0^\infty e^{itz}\varphi_{\xi'}(t)dt = \frac{1}{iz}e^{itz}\varphi_{\xi'}(t)\Big|_{t=0}^{\infty} - \frac{1}{iz}\int_0^\infty e^{itz}\varphi'_{\xi'}(t)\, dt \\ &= -\frac{e^{itz}}{(iz)^2}\varphi'_{\xi'}(t)\Big|_{t=0}^{\infty} + \frac{1}{(iz)^2}\int_0^\infty e^{itz}\varphi''_{\xi'}(t)\, dt \\ &= -\frac{1}{z^2}\left[\varphi'_{\xi'}(0) + \int_0^\infty e^{itz}\varphi''_{\xi'}(t)\, dt\right].\end{aligned}$$

It is clear that a similar estimate is true for $\Psi_2(z)$; hence (2.13) holds.

In a similar way we have

$$\Psi_1'(z) = \int_0^\infty ite^{itz}\varphi_{\xi'}(t)\,dt = \frac{1}{z}e^{itz}t\varphi_{\xi'}(t)\Big|_{t=0}^{\infty} - \frac{1}{z}\int_0^\infty e^{itz}\frac{d}{dt}(t\varphi_{\xi'}(t))\,dt$$

$$= -\frac{1}{iz^2}e^{itz}\frac{d}{dt}(t\varphi_{\xi'}(t))\Big|_{t=0}^{\infty} + \frac{1}{iz^2}\int_0^\infty e^{itz}\frac{d^2}{dt^2}(t\varphi_{\xi'}'(t))\,dt$$

$$= -\frac{1}{z^3}e^{itz}\frac{d^2}{dt^2}(t\varphi_{\xi'}(t))\Big|_{t=0}^{\infty} + \frac{1}{z^3}\int_0^\infty e^{itz}\frac{d^3}{dt^3}(t\varphi_{\xi'}'(t))\,dt$$

$$= \frac{1}{z^3}\left[2\varphi_{\xi'}'(0) + \int_0^\infty e^{itz}\frac{d^3}{dt^3}(t\varphi_{\xi'}'(t))\,dt\right].$$

Since an analogous estimate is valid for $\Psi_2'(z)$ and $h'(z) = \Psi_1'(z)\Psi_2(z) + \Psi_1(z)\Psi_2'(z)$, inequality (2.14) holds. $\square$

Remark 2.2 In general, $\frac{\partial f}{\partial x_1}(0, x')$ does not vanish identically; hence $\mathcal{F}_{N-1}f(0,\xi') \neq 0$ for some ξ', i.e. there exists ξ' such that $\varphi_{\xi'}'(0) \neq 0$. For such ξ', the functions Ψ_1 and Ψ_2 do not decay faster than $\frac{1}{|z|^2}$ and then the estimate (2.13) is optimal.

Remark 2.3 Note that for any $t \in \mathbf{R}$ we have

$$h(it) = \left|\int_0^\infty e^{-x_1 t}\varphi_{\xi'}(x_1)\,dx_1\right|^2 \in [0,\infty).$$

Suppose that for any fixed $\xi' \in \mathbf{R}^{N-1}$, $m(\xi_1, \xi')$ admits an holomorphic extension $z \longmapsto m(z,\xi')$ to the upper half-plane $\{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$, with possibly some singularities on the imaginary axis $\{it \mid t \in [0,\infty)\}$. If $|m(z,\xi')|$ increases more slowly than $|z|^3$ as $|z| \longrightarrow \infty$, then $\int_{-\infty}^{\infty} m(\xi_1,\xi')h(\xi_1)\,d\xi_1$ should depend only on the values of h on the singular set of $m(\cdot,\xi')$. This simple idea will enable us to prove the identities that will be crucial in symmetry problems.

In order to clarify what kind of symbols may be considered, we start with some auxiliary technical results about holomorphic functions in a half-plane and their boundary values.

Given a function $\alpha \in L^p(\mathbf{R})$, $1 \le p < \infty$, we recall that its Hilbert transform is defined by

$$(H\alpha)(x) = \lim_{\varepsilon \to 0}\frac{1}{\pi}\int_{\{|y|>\varepsilon\}}\frac{\alpha(x-y)}{y}\,dy \qquad \text{or equivalently} \qquad \widehat{H\alpha}(\xi) = -i\,\mathrm{sgn}(\xi)\,\widehat{\alpha}(\xi).$$

It is well-known that H is a bounded linear mapping from $L^p(\mathbf{R})$ into $L^p(\mathbf{R})$ (see, e.g., Chapter II in [23], or inequality (2.11) p. 188 in [24]).

In the next two lemmas we collect some classical facts that will be very useful in the sequel.

Lemma 2.4 *Consider $\alpha \in L^p(\mathbf{R})$, $1 < p < \infty$, and let $\beta = H\alpha$. For $x > 0$ and $y \in \mathbf{R}$ define*

$$a(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+(y-t)^2}\alpha(t)\,dt = \int_{-\infty}^{\infty}P(y-t,x)\alpha(t)\,dt \quad \textit{and}$$

$$b(x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y-t}{x^2+(y-t)^2}\alpha(t)\,dt = -\int_{-\infty}^{\infty}Q(y-t,x)\alpha(t)\,dt,$$

where $P(s,k) = \dfrac{1}{\pi}\dfrac{k}{s^2+k^2}$ *and* $Q(s,k) = \dfrac{1}{\pi}\dfrac{s}{s^2+k^2}$ *are the Poisson kernel, respectively the conjugate Poisson kernel.*

Then we have:

i) $b(x,y) = -\displaystyle\int_{-\infty}^{\infty} P(y-t,x)\beta(t)\,dt$ *for any* $x>0$ *and* $t \in \mathbf{R}$.

ii) $||a(x,\cdot)||_{L^p(\mathbf{R})} \le ||\alpha||_{L^p(\mathbf{R})}$, $||b(x,\cdot)||_{L^p(\mathbf{R})} \le ||\beta||_{L^p(\mathbf{R})}$ *and* $||a(x,\cdot)-\alpha||_{L^p(\mathbf{R})} \longrightarrow 0$, $||b(x,\cdot)+\beta||_{L^p(\mathbf{R})} \longrightarrow 0$ *as* $x \longrightarrow 0$. *Moreover,* $a(x,y) \longrightarrow \alpha(y)$ *for any* y *in the Lebesgue set of* α *(hence almost everywhere) and* $b(x,y) \longrightarrow -\beta(y)$ *for any* y *in the Lebesgue set of* β.

iii) The functions a *and* b *are harmonic in* $\{(x,y) \in \mathbf{R}^2 \mid x>0\}$ *and* $r(z) = r(x+iy) := a(x,y)+ib(x,y)$ *is holomorphic in* $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.

iv) For any $\delta > 0$ *we have*

$$\lim_{|(x,y)|\to\infty,\, x\ge\delta} a(x,y) = 0 \qquad \text{and} \qquad \lim_{|(x,y)|\to\infty,\, x\ge\delta} b(x,y) = 0.$$

v) Suppose in addition that α *is even and there exists* $\varepsilon > 0$ *such that* $\alpha \equiv 0$ *on* $[-\varepsilon,\varepsilon]$. *Then* a *and* b *are well-defined, bounded and harmonic in the strip* $\{(x,y) \in \mathbf{R}^2 \mid -\frac{\varepsilon}{2} < y < \frac{\varepsilon}{2}\}$, r *is well-defined and holomorphic in this strip and* $r(0)=0$.

Proof. *i)* is exactly Lemma 1.5 p. 219 in [24] and *ii)* follows from Theorem 2.1 p. 47 in [24]. Since the Poisson kernel is a harmonic function, it is straightforward that a and b are harmonic. It is easy to check that the Cauchy-Riemann conditions $\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$ are satisfied; then r is holomorphic in $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$ and *iii)* holds.

iv) Using Lemma 2.6 p. 51 in [24] we infer that there exists a constant $A > 0$ such that

$$|a(x,y)| \le \frac{A||\alpha||_{L^p}}{x^{\frac{1}{p}}} \qquad \text{and} \qquad |b(x,y)| \le \frac{A||\alpha||_{L^p}}{x^{\frac{1}{p}}} \tag{2.16}$$

for any $x>0$ and $y \in \mathbf{R}$.

We fix $\varepsilon > 0$. It follows from (2.16) that there exists $M>0$ such that $|a(x,y)| < \varepsilon$ and $|b(x,y)| < \varepsilon$ for any (x,y) with $x \ge M$. Let $q \in (1,\infty)$ be the conjugate exponent of p, i.e. $\frac{1}{p}+\frac{1}{q} = 1$. It is easy to see that $||P(\cdot,x)||_{L^1(\mathbf{R})} = 1$ and $||P(\cdot,x)||_{L^\infty(\mathbf{R})} = \frac{1}{\pi x}$; consequently, $||P(\cdot,x)||_{L^q(\mathbf{R})} \le ||P(\cdot,x)||_{L^1(\mathbf{R})}^{\frac{1}{q}} ||P(\cdot,x)||_{L^\infty(\mathbf{R})}^{\frac{1}{p}} = \pi^{-\frac{1}{p}} x^{-\frac{1}{p}}$. Also, for any $B>0$ we have $||P(\cdot,x)||_{L^1([B,\infty))} = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan\frac{B}{x}\right)$ and $||P(\cdot,x)||_{L^\infty([B,\infty))} = \frac{1}{\pi}\frac{x}{x^2+B^2}$, hence

$$||P(\cdot,x)||_{L^q([B,\infty))} \le \left(\frac{1}{\pi}\frac{x}{x^2+B^2}\right)^{\frac{1}{p}} \left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{B}{x}\right)^{\frac{1}{q}}. \tag{2.17}$$

A similar estimate holds on $(-\infty,-B]$. For any $x \in [\delta, M]$ and any $y \ge 2B$ we have $||P(\cdot,x)||_{L^q((y-B,y+B))} \le ||P(\cdot,x)||_{L^q([B,\infty))}$ and

$$\begin{aligned}
|a(x,y)| &\le \left| \int_{-B}^{B} P(y-t,x)\alpha(t)\,dt \right| + \left| \int_{\{|t|\ge B\}} P(y-t,x)\alpha(t)\,dt \right| \\
&\le \left| \int_{y-B}^{y+B} P(s,x)\alpha(y-s)\,ds \right| + ||P(\cdot,x)||_{L^q(\mathbf{R})} \cdot ||\alpha||_{L^p((-\infty,B]\cup[B,\infty))} \\
&\le ||P(\cdot,x)||_{L^q([y-B,y+B])} \cdot ||\alpha||_{L^p(\mathbf{R})} + ||P(\cdot,x)||_{L^q(\mathbf{R})} \cdot ||\alpha||_{L^p((-\infty,B]\cup[B,\infty))} \qquad (2.18) \\
&\le ||\alpha||_{L^p(\mathbf{R})} \left(\frac{1}{\pi}\frac{x}{x^2+B^2}\right)^{\frac{1}{p}} \left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{B}{x}\right)^{\frac{1}{q}} + ||\alpha||_{L^p((-\infty,B]\cup[B,\infty))} \pi^{-\frac{1}{p}} x^{-\frac{1}{p}} \\
&\le ||\alpha||_{L^p(\mathbf{R})} \left(\frac{M}{\pi(\delta^2+B^2)}\right)^{\frac{1}{p}} \left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{B}{M}\right)^{\frac{1}{q}} + ||\alpha||_{L^p((-\infty,B]\cup[B,\infty))} \pi^{-\frac{1}{p}} \delta^{-\frac{1}{p}}.
\end{aligned}$$

We may choose $B = B(\varepsilon)$ sufficiently large so that the right-hand side term in (2.18) is less than ε. Then for any $x \in [\delta, M]$ and $y \geq 2B(\varepsilon)$ we have $|a(x,y)| < \varepsilon$. Clearly the same inequality is true if $y \leq -2B$. Therefore $|a(x,y)| < \varepsilon$ if $x \geq M$ or if $|y| \geq 2B$ and $x \in [\delta, M]$. Since ε was arbitrary, we infer that $|a(x,y)| \longrightarrow 0$ as $|(x,y)| \longrightarrow \infty$ and $x \geq \delta$. A similar proof is valid for the function b and *iv)* is proved.

v) For any $y \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and $t \in \mathrm{supp}(a)$ we have $|t-y| \geq \frac{\varepsilon}{2}$; hence $x^2 + (y-t)^2 \geq \frac{\varepsilon^2}{4}$ and $|P(y-t,x)| = \frac{1}{\pi}|\frac{x}{x^2+(y-t)^2}| \leq \frac{1}{\pi}\frac{4}{\varepsilon^2}|x|$, therefore $|P(y-t,x)| \leq \frac{1}{\pi}\min\left(\frac{4}{\varepsilon^2}|x|, \frac{1}{2|y-t|}\right)$. Similarly $|Q(y-t,x)| = \frac{1}{\pi}|\frac{y-t}{x^2+(y-t)^2}| \leq \frac{1}{\pi}\min\left(\frac{4}{\varepsilon^2}|y-t|, \frac{1}{|y-t|}\right)$. Thus $P(y-\cdot,x)$ and $Q(y-\cdot,x)$ are uniformly bounded in $L^q(\mathbf{R})$ for $(x,y) \in [-1,1] \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. It follows that a and b are well-defined for any (x,y) with $|y| \leq \frac{\varepsilon}{2}$ and bounded near the origin. It is straightforward to check that a and b are twice continuously differentiable, $\Delta a = \Delta b = 0$ and $r(x+iy) = a(x,y)+ib(x,y)$ is holomorphic. Clearly, $a(0,y) = 0$ for any $y \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and $b(x,0) = \displaystyle\int_{-\infty}^{\infty} \frac{t}{t^2+x^2}\alpha(t)\,dt = 0$ for any $x \in \mathbf{R}$ because $t \longmapsto \frac{t}{t^2+x^2}$ is odd and $t \longmapsto \alpha(t)$ is even. Hence $r(0) = 0$. □

Lemma 2.5 *Let μ be a finite Borel measure on $\mathbf{R}$. For $x > 0$ and $y \in \mathbf{R}$ define*

$$a(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2}\,d\mu(t) = \int_{-\infty}^{\infty} P(y-t,x)\,d\mu(t) \quad \textit{and}$$

$$b(x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y-t}{x^2+(y-t)^2}\,d\mu(t) = -\int_{-\infty}^{\infty} Q(y-t,x)\,d\mu(t),$$

where $P(s,k)$ and $Q(s,k)$ are the Poisson kernel, respectively the conjugate Poisson kernel.
Then:

i) The functions a and b are harmonic in $\{(x,y) \in \mathbf{R}^2 \mid x > 0\}$ and $r(z) = r(x+iy) := a(x,y) + ib(x,y)$ is holomorphic in the right half-plane $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.

ii) For any $x > 0$ and any p, $1 \leq p \leq \infty$, we have

$$||a(x,\cdot)||_{L^p(\mathbf{R})} \leq \frac{1}{\pi^{\frac{1}{q}} x^{\frac{1}{q}}}||\mu||, \tag{2.19}$$

where q is the conjugate exponent of p and $||\mu||$ is the total variation of μ. Furthermore,

$$\lim_{x \to 0} \int_{\mathbf{R}} a(x,y)\phi(y)\,dy = \int_{\mathbf{R}} \phi(y)\,d\mu(y) \tag{2.20}$$

for any function ϕ which is continuous on $\mathbf{R}$ and tends to zero at $\pm\infty$.

iii) For any $x > 0$ we have $|b(x,y)| \leq \frac{1}{2\pi x}||\mu||$.

iv) For $x > 0$ we have $b(x,\cdot) = -Ha(x,\cdot)$ and for any x_1, $x_2 > 0$,

$$a(x_1+x_2,y) = \int_{-\infty}^{\infty} P(y-t,x_1)a(x_2,t)\,d\mu(t), \tag{2.21}$$

$$b(x_1+x_2,y) = \int_{-\infty}^{\infty} P(y-t,x_1)b(x_2,t)\,d\mu(t) = -\int_{-\infty}^{\infty} Q(y-t,x_1)a(x_2,t)\,d\mu(t). \tag{2.22}$$

v) For any $p \in (1,\infty)$ there exists $A_p > 0$ such that

$$||b(x,\cdot)||_{L^p(\mathbf{R})} \leq A_p x^{-\frac{p-1}{p}}||\mu||.$$

vi) For any $\delta > 0$,

$$\lim_{|(x,y)| \to \infty,\, x \geq \delta} a(x,y) = 0 \qquad \textit{and} \qquad \lim_{|(x,y)| \to \infty,\, x \geq \delta} b(x,y) = 0.$$

vii) Suppose in addition that $\mu(S) = \mu(-S)$ and $\mu(S \cap [-\varepsilon, \varepsilon]) = 0$ for any Borel measurable set S. Then a and b are well-defined, bounded and holomorphic in the strip $\{(x, y) \in \mathbf{R}^2 \mid -\frac{\varepsilon}{2} < y < \frac{\varepsilon}{2}\}$, the function $r(x+iy) = a(x, y) + ib(x, y)$ is holomorphic in that strip and $r(0) = 0$.

Proof. *i)* If $x > 0$, the functions $t \longmapsto P(y-t, x)$ and $t \longmapsto Q(y-t, x)$ are continuous on $\mathbf{R}$ and tend to zero at $\pm\infty$; hence $a(x, y)$ and $b(x, y)$ are well-defined. Using Lebesgue's Dominated Convergence Theorem it is easy to check that a and b are twice continuously differentiable and $\Delta a = \Delta b = 0$. Moreover, a and b satisfy the Cauchy-Riemann conditions $\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$, and then $r = a + ib$ is holomorphic in the right half-plane.

ii) It follows from Theorem 2.3 p. 49 in [24] that $||a(x, \cdot)||_{L^1(\mathbf{R})} \le ||\mu||$ and that (2.20) holds. It is obvious that $||P(y - \cdot, x)||_{L^\infty(\mathbf{R})} \le \frac{1}{\pi x}$; hence $|a(x, y)| \le ||P(y - \cdot, x)||_{L^\infty(\mathbf{R})}||\mu|| = \frac{1}{\pi x}||\mu||$. Finally, for $1 < p < \infty$ we have $||a(x, \cdot)||_{L^p} \le ||a(x, \cdot)||_{L^\infty}^{\frac{1}{q}} \cdot ||a(x, \cdot)||_{L^1}^{\frac{1}{p}} \le \pi^{-\frac{1}{q}} x^{-\frac{1}{q}} ||\mu||$.

iii) It is obvious that $|Q(y - t, x)| \le \frac{1}{2\pi x}$ and this implies
$|b(x, y)| \le ||Q(y - \cdot, x)||_{L^\infty(\mathbf{R})}||\mu|| \le \frac{1}{2\pi x}||\mu||$.

iv) We have just proved that a and b are harmonic in the right half-plane and bounded in each proper sub-half-plane $\{(x, y) \in \mathbf{R}^2 \mid x > \delta\}$, where $\delta > 0$. Then (2.21) and the first equality in (2.22) follow directly from Lemma 2.7 p. 51 in [24]. Fix $x_2 > 0$. We introduce the function

$$r_1(z) = r_1(x+iy) = \int_{-\infty}^{\infty} P(y-t, x) a(x_2, t)\, dt - i \int_{-\infty}^{\infty} Q(y-t, x) a(x_2, t)\, dt.$$

It is not hard to see that $a(x_2, \cdot) \in L^p(\mathbf{R})$ for any $p \in [1, \infty]$, $a(x_2, \cdot)$ is C^∞ and $Ha(x_2, \cdot)$ is continuous. It is clear that r_1 is bounded and by Lemma 2.4 *ii)* and *iii)* we infer that r_1 is holomorphic in the right half-plane, $\lim\limits_{x \to 0} \mathrm{Re}(r_1(x, y)) = a(x_2, y)$ and $\lim\limits_{x \to 0} \mathrm{Im}(r_1(x, y)) = -(Ha(x_2, \cdot))(y)$ for any $y \in \mathbf{R}$. Let $r_2(z) = r(x_2 + z) - r_1(z)$. It is easy to see that r_2 is well-defined, bounded and holomorphic in the right half-plane and $\lim\limits_{x \to 0} \mathrm{Re}(r_2(x, y)) = 0$. Using Schwarz' reflection principle (see, e.g., [8] p. 75), we may extend r_2 to a holomorphic function $\tilde{r}_2$ defined in the whole complex plane so that we have $\tilde{r}_2(z) = -\overline{r_2(-\overline{z})}$ for any z with $\mathrm{Re}(z) < 0$. Since $\tilde{r}_2$ is also bounded, from Liouville's theorem it follows that $\tilde{r}_2$ is constant. From *ii)* and *iii)* we infer that $\lim\limits_{x \to \infty} r(x) = 0$ and from Lemma 2.4, part *iv)*, we get $\lim\limits_{x \to \infty} r_1(x) = 0$; hence $\lim\limits_{x \to \infty} r_2(x) = 0$. Consequently $\tilde{r}_2$ is identically zero on $\mathbf{C}$, that is $r_1(z) = r(x_2 + z)$. This proves the second equality in (2.22). Moreover, we have $\mathrm{Im}(r(x_2 + iy)) = b(x_2, y)$ and $\lim\limits_{x \to 0} \mathrm{Im}(r_1(x + iy)) = -H(a(x_2, \cdot))(y)$; we conclude that $b(x_2, \cdot) = -H(a(x_2, \cdot))$.

v) We know that there exists $C_p > 0$ such that $||H\phi||_{L^p} \le C_p||\phi||_{L^p}$ for any $\phi \in L^p(\mathbf{R})$. Using *ii)* and *iv)* we get

$$||b(x, \cdot)||_{L^p} = ||Ha(x, \cdot)||_{L^p} \le C_p||a(x, \cdot)||_{L^p} \le C_p \pi^{-\frac{1}{q}} x^{-\frac{1}{q}} ||\mu||$$

for any $x > 0$, where $\frac{1}{p} + \frac{1}{q} = 1$.

vi) is a direct consequence of (2.21), (2.22) and Lemma 2.4, part *iv)*. The proof of *vii)* is very similar to the proof of part *v)* of Lemma 2.4 and we omit it. □

Remark 2.6 Under the assumptions *v)* of Lemma 2.4 (respectively *vii)* of Lemma 2.5) an easy computation gives

$$\frac{\partial a}{\partial x}(0, 0) = \frac{\partial b}{\partial y}(0, 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(t)}{t^2}\, dt, \quad \text{respectively} \quad \frac{\partial a}{\partial x}(0, 0) = \frac{\partial b}{\partial y}(0, 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2}\, d\mu(t).$$

If α is nonnegative and $\alpha \not\equiv 0$ (respectively if μ is a positive measure) we have $\frac{\partial r}{\partial z}(0) = \frac{\partial a}{\partial x}(0,0) > 0$; hence $z = 0$ is a simple zero of r.

After this preparation, we come back to the study of the integral $\int_{\mathbf{R}} m(\xi_1, \xi') h_{\xi'}(\xi_1)\, d\xi_1$ which appears in the right hand side of (2.12).

Lemma 2.7 *Suppose that for a given $\xi' \in \mathbf{R}^{N-1}$ the symbol $m(\xi_1, \xi')$ can be written as*

$$
\begin{aligned}
m(\xi_1, \xi') &= A_0(\xi') + A_1(\xi')|\xi_1| + A_2(\xi')\xi_1^2 \\
&+ \frac{1}{\pi}\left[\int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2}\, d\mu_{\xi',0}(t) + \xi_1^2 \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2}\, d\mu_{\xi',1}(t) + \xi_1^4 \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2}\, d\mu_{\xi',2}(t)\right] \qquad (2.23) \\
&+ \frac{1}{\pi} \sum_{k=0}^{4} |\xi_1|^k \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2} \alpha_{\xi',k}(t)\, dt,
\end{aligned}
$$

where :

a) $A_0(\xi'),\ \ A_1(\xi'),\ \ A_2(\xi') \in \mathbf{R}$,

b) $\mu_{\xi',i}$ *are finite Borel measures on* $\mathbf{R}$ *such that* $\mu_{\xi',i}(S) = \mu_{\xi',i}(-S)$ *for any Borel measurable set* $S \subset \mathbf{R}$, $i = 0,\ 1,\ 2.$

c) $\alpha_{\xi',k} \in L^{p_k}(\mathbf{R})$ *for some* $p_k \in (1, \infty)$ *and* $\alpha_{\xi',k}$ *are even functions,* $k = 0,\ 1,\ 2,\ 3,\ 4.$

d) There exists $\eta > 0$ *such that* $\alpha_{\xi',0} \equiv 0$ *on* $[-\eta, \eta]$ *and* $\mu_{\xi',0}(S) = 0$ *for any Borel measurable set* $S \subset [-\eta, \eta]$.

Let $\beta_{\xi',1} = H\alpha_{\xi',1}$ *and* $\beta_{\xi',3} = H\alpha_{\xi',3}$, *where* H *is the Hilbert transform. If* $h = h_{\xi'}$ *is given by (2.11) then we have the identity:*

$$
\begin{aligned}
\frac{1}{2}\int_{-\infty}^{\infty} m(\xi_1, \xi') h(\xi_1)\, d\xi_1 &= -A_1(\xi') \int_0^{\infty} t\, h(it)\, dt \\
&+ \int_0^{\infty} \frac{h(it)}{t}\, d\mu_{\xi',0}(t) - \int_0^{\infty} t\, h(it)\, d\mu_{\xi',1}(t) + \int_0^{\infty} t^3 h(it)\, d\mu_{\xi',2}(t) \qquad (2.24) \\
&+ \int_0^{\infty} \left(\frac{\alpha_{\xi',0}(t)}{t} + \beta_{\xi',1}(t) - t\alpha_{\xi',2}(t) - t^2\beta_{\xi',3}(t) + t^3\alpha_{\xi',4}(t)\right) h(it)\, dt.
\end{aligned}
$$

Proof. For $i = 0,\ 1,\ 2$ and $z = x + iy \in \mathbf{C}$ with $\mathrm{Re}(z) > 0$ we define

$$
p_i(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{x}{x^2 + (y-t)^2}\, d\mu_{\xi',i}(t) - \frac{i}{\pi}\int_{\mathbf{R}} \frac{y-t}{x^2 + (y-t)^2}\, d\mu_{\xi',i}(t).
$$

In view of Lemma 2.5, p_i are well-defined and holomorphic in the right half-plane $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$. Moreover, by assumption *d*) and Lemma 2.5, part *vii*), p_0 admits an holomorphic extension to the domain $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0 \text{ or } |\mathrm{Im}(z)| < \frac{\eta}{2}\}$, and $p_0(0) = 0$. Consequently, $\frac{p_0(z)}{z}$ is holomorphic in this domain and is bounded in a neighbourhood of zero.

For $k = 0,\ 1,\ 2,\ 3,\ 4$ we define

$$
r_k(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{x}{x^2 + (y-t)^2} \alpha_{\xi',k}(t)\, dt - \frac{i}{\pi}\int_{\mathbf{R}} \frac{y-t}{x^2 + (y-t)^2} \alpha_{\xi',k}(t)\, dt.
$$

It follows from Lemma 2.4 that r_k are well-defined and holomorphic in the right half-plane. Furthermore, r_0 admits an holomorphic extenion to $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0 \text{ or } |\mathrm{Im}(z)| < \frac{\eta}{2}\}$ and $r_0(0) = 0$; therefore, $\dfrac{r_0(z)}{z}$ is holomorphic in this domain and bounded near zero.

Finally, we define

$$m_{\xi'}(z) = A_0(\xi') + A_1(\xi')z + A_2(\xi')z^2 + \frac{p_0(z)}{z} + zp_1(z) + z^3 p_2(z) + \sum_{k=0}^{4} z^{k-1} r_k(z). \tag{2.25}$$

It is obvious that $m_{\xi'}$ is well-defined and holomorphic in the right half-plane. Since $\alpha_{\xi',k}$ and $\mu_{\xi',i}$ are "even" and $t \longmapsto \frac{t}{\xi_1^2+t^2}$ is odd, for any $\xi_1 > 0$ we have $\mathrm{Im}(m_{\xi'}(\xi_1)) = 0$ and

$$m_{\xi'}(\xi_1) = \mathrm{Re}(m_{\xi'}(\xi_1)) = m(\xi_1, \xi').$$

For ε, $R > 0$, consider the closed continuous path $\gamma_{\varepsilon,R}$ composed by the following pieces :

$$\begin{array}{ll} \gamma_{1,\varepsilon,R}(t) = t, & t \in [\varepsilon, \varepsilon + R] \\ \gamma_{2,\varepsilon,R}(\theta) = \varepsilon + Re^{i\theta}, & \theta \in [0, \frac{\pi}{2}] \\ \gamma_{3,\varepsilon,R}(t) = \varepsilon + i(R - t), & t \in [0, R]. \end{array}$$

The function $z \longmapsto m_{\xi'}(z)h(z)$ being holomorphic in the right half-plane we have $\displaystyle\int_{\gamma_{\varepsilon,R}} m_{\xi'}(z)h(z)\, dz = 0$, that is

$$\int_{\varepsilon}^{R} m(\xi_1, \xi')h(\xi_1)\, d\xi_1 + \int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z)h(z)\, dz + \int_{\gamma_{3,\varepsilon,R}} m_{\xi'}(z)h(z)\, dz = 0. \tag{2.26}$$

It follows from (2.25), Lemma 2.4 part *iv*) and Lemma 2.5 part *vi*) that $\displaystyle\lim_{|z|\to\infty,\, Re(z)\geq\varepsilon} \frac{m_{\xi'}(z)}{z^3} = 0$; hence, $\displaystyle\lim_{R\to\infty} \frac{m_{\xi'}(\varepsilon + Re^{i\theta})}{(\varepsilon + Re^{i\theta})^3} = 0$ uniformly with respect to $\theta \in [0, \frac{\pi}{2}]$. On the other hand, from Lemma 2.1 part *ii*), we have $|h(\varepsilon + Re^{i\theta})| \leq \frac{C}{|\varepsilon+Re^{i\theta}|^4}$ and then $|(\varepsilon + Re^{i\theta})^3 h(\varepsilon + Re^{i\theta}) \cdot iRe^{i\theta}| \leq \frac{CR}{|\varepsilon+Re^{i\theta}|} \leq \frac{CR}{R-\varepsilon} \leq 2C$ for any $R \geq 2\varepsilon$. We infer that $\displaystyle\lim_{R\to\infty} \int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z)h(z)\, dz = 0$.

From (2.16) and (2.19) it follows that $|m(\xi_1, \xi')| \leq C|\xi_1|^{-1+\delta_1}$ for $0 < \xi_1 < 1$ and $|m(\xi_1, \xi')| \leq C|\xi_1|^{3-\delta_2}$ for large ξ_1 and some C, δ_1, $\delta_2 > 0$. Since h is continuous and $|h(\xi_1)| \leq \frac{C}{|\xi_1|^4}$ (see(2.13)), the integral $\displaystyle\int_0^\infty m(\xi_1, \xi')h(\xi_1)\, d\xi_1$ converges absolutely.

Clearly we have

$$\int_{\gamma_{3,\varepsilon,R}} m_{\xi'}(z)h(z)\, dz = -i \int_0^R m_{\xi'}(\varepsilon + iy)h(\varepsilon + iy)\, dy.$$

Passing to the limit as $R \longrightarrow \infty$ in (2.26) we infer that $\displaystyle\int_0^\infty m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy)\, dy$ converges and

$$\int_{\varepsilon}^{\infty} m(\xi_1, \xi')h(\xi_1)\, d\xi_1 = i \int_0^\infty m_{\xi'}(\varepsilon + iy)h(\varepsilon + iy)\, dy. \tag{2.27}$$

Since $m(\xi_1, \xi')$ is real and symmetric with respect to ξ_1 we have

$$\int_{-\infty}^{-\varepsilon} m(\xi_1, \xi')h(\xi_1)\, d\xi_1 = \int_{\varepsilon}^{\infty} m(-\xi_1, \xi')h(-\xi_1)\, d\xi_1 = \int_{\varepsilon}^{\infty} m(\xi_1, \xi')\overline{h(\xi_1)}\, d\xi_1,$$

and then, taking (2.27) into account, we get

$$\int_{-\infty}^{-\varepsilon} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 + \int_{\varepsilon}^{\infty} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 = -2\int_0^{\infty} \mathrm{Im}(m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy))\,dy; \tag{2.28}$$

hence

$$\int_{-\infty}^{\infty} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 = -2\lim_{\varepsilon\to 0}\int_0^{\infty} \mathrm{Im}(m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy))\,dy. \tag{2.29}$$

Since $h(iy)\in\mathbf{R}$ for $y\in[0,\infty)$, using Lemma 2.1 and the Dominated Convergence Theorem we find

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{\infty} \mathrm{Im}\left[(A_0(\xi')+A_1(\xi')(\varepsilon+iy)+A_2(\xi')(\varepsilon+iy)^2)h(\varepsilon+iy)\right]\,dy\\ &= A_1(\xi')\int_0^{\infty} y\,h(iy)\,dy.\end{aligned} \tag{2.30}$$

Let $\chi\in C_c^{\infty}(\mathbf{R},\mathbf{R}_+)$ be such that $\mathrm{supp}(\chi)\subset[-\frac{\eta}{4},\frac{\eta}{4}]$ and $\chi\equiv 1$ on $[-\frac{\eta}{8},\frac{\eta}{8}]$. Since the function $z\longmapsto\frac{p_0(z)}{z}h(z)$ is uniformly continuous on $[-1,1]\times[-\frac{\eta}{4},\frac{\eta}{4}]$ we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{\infty}\mathrm{Im}\left[\frac{p_0(\varepsilon+iy)}{\varepsilon+iy}h(\varepsilon+iy)\chi(y)\right]\,dy = \int_0^{\infty}\mathrm{Im}\left(\frac{p_0(iy)}{iy}h(iy)\chi(y)\right)\,dy\\ &= -\int_0^{\infty}\frac{\mathrm{Re}(p_0(iy))}{y}h(iy)\chi(y)\,dy = 0.\end{aligned} \tag{2.31}$$

By Lemma 2.1 we infer that there exists $C_1>0$ such that $|h(\varepsilon+iy)-h(iy)|\le\varepsilon C_1\min(1,\frac{1}{|y|^5})$ for any $y\in(0,\infty)$ and $\varepsilon\in[0,1]$. It is easy to see that $|\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy}-\frac{h(iy)}{iy}\right)(1-\chi(y))|\le C_2\varepsilon\min(\frac{1}{y^6},1)$ for any $y\in(0,\infty)$ and some $C_2>0$. Consequently there exists $C_3>0$ such that

$$\left|\left|\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy}-\frac{h(iy)}{iy}\right)(1-\chi(y))\right|\right|_{L^p(0,\infty)}\le C_3\varepsilon \qquad \text{for any } p\in[1,\infty]. \tag{2.32}$$

Using the Cauchy-Schwarz inequality, Lemma 2.5 parts *ii*) and *v*) and (2.32), we get

$$\begin{aligned}&\left|\int_0^{\infty}p_0(\varepsilon+iy)\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy}-\frac{h(iy)}{iy}\right)(1-\chi(y))\,dy\right|\\ &\le\Big(||\mathrm{Re}(p_0(\varepsilon+i\cdot))||_{L^2(\mathbf{R})}+||\mathrm{Im}(p_0(\varepsilon+i\cdot))||_{L^2(\mathbf{R})}\Big)\\ &\qquad\qquad\left|\left|\left(\tfrac{h(\varepsilon+iy)}{\varepsilon+iy}-\tfrac{h(iy)}{iy}\right)(1-\chi(y))\right|\right|_{L^2(0,\infty)}\\ &\le C_4\varepsilon^{\frac{1}{2}}\longrightarrow 0 \text{ as } \varepsilon\longrightarrow 0.\end{aligned} \tag{2.33}$$

We also have by (2.20) and assumption *d*),

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{\infty}\mathrm{Im}\left[p_0(\varepsilon+iy)\frac{h(iy)}{iy}(1-\chi(y))\right]\,dy\\ &= -\lim_{\varepsilon\to 0}\int_0^{\infty}\mathrm{Re}(p_0(\varepsilon+iy))\frac{h(iy)}{y}(1-\chi(y))\,dy\\ &= -\int_0^{\infty}\frac{h(iy)}{y}(1-\chi(y))\,d\mu_{\xi',0}(y) = -\int_0^{\infty}\frac{h(iy)}{y}d\mu_{\xi',0}(y).\end{aligned} \tag{2.34}$$

From (2.31), (2.33) and (2.34) we get

$$\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\left[\frac{p_0(\varepsilon+iy)}{\varepsilon+iy}h(\varepsilon+iy)\right]\,dy = -\int_0^\infty \frac{h(iy)}{y}d\mu_{\xi',0}(y). \tag{2.35}$$

This proof can be slightly modified to show that

$$\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\left[\frac{r_0(\varepsilon+iy)}{\varepsilon+iy}h(\varepsilon+iy)\right]\,dy = -\int_0^\infty \frac{h(iy)}{y}\alpha_{\xi',0}(y)\,dy. \tag{2.36}$$

(All we have to do is to use Hölder's inequality to obtain an analogous of (2.33) and to use Lemma 2.4 part *ii)* instead of (2.20) to get an analogous of (2.34)). Moreover, it is easy to see that $|(\varepsilon+iy)^\ell h(\varepsilon+iy)-(iy)^\ell h(iy)|\le C_5\varepsilon\min(1,\frac{1}{y^2})$ for $y\in(0,\infty)$, $\ell\in\{0,1,2,3\}$ and $\varepsilon\in[0,1]$. Therefore, there exists $C_6>0$ such that

$$||(\varepsilon+iy)^{k-1}h(\varepsilon+iy)-(iy)^{k-1}h(iy)||_{L^p(0,\infty)}\le C_6\varepsilon \tag{2.37}$$

for any $\varepsilon\in[0,1]$, $k\in\{1,\,2,\,3,\,4\}$ and $p\in[1,\infty]$. This implies that

$$\begin{aligned}
&\left|\int_0^\infty \mathrm{Im}\Big((\varepsilon+iy)^{k-1}h(\varepsilon+iy)r_k(\varepsilon+iy)\Big)\,dy-\int_0^\infty \mathrm{Im}\Big((iy)^{k-1}h(iy)r_k(\varepsilon+iy)\Big)\,dy\right|\\
&\le \left(||\mathrm{Re}(r_k(\varepsilon+i\cdot))||_{L^{p_k}}+||\mathrm{Im}(r_k(\varepsilon+i\cdot))||_{L^{p_k}}\right)||(\varepsilon+iy)^{k-1}h(\varepsilon+iy)-(iy)^{k-1}h(iy)||_{L^{q_k}(0,\infty)}\\
&\le \left(||\alpha_{\xi',k}||_{L^{p_k}}+||H\alpha_{\xi',k}||_{L^{p_k}}\right)C_6\varepsilon\longrightarrow 0 \qquad \text{as } \varepsilon\longrightarrow 0.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\Big((\varepsilon+iy)^{k-1}r_k(\varepsilon+iy)h(\varepsilon+iy)\Big)\,dy\\
&=\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\Big((iy)^{k-1}r_k(\varepsilon+iy)h(iy)\Big)\,dy,
\end{aligned} \tag{2.38}$$

where the latter limit exists by Lemma 2.4 *ii)* and (2.13). Using (2.38) and Lemma 2.4 *ii)* we obtain :

$$\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\left(r_1(\varepsilon+iy)h(\varepsilon+iy)\right)\,dy = -\int_0^\infty (H\alpha_{\xi',1})(y)h(iy)\,dy, \tag{2.39}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\left((\varepsilon+iy)r_2(\varepsilon+iy)h(\varepsilon+iy)\right)\,dy = \int_0^\infty \alpha_{\xi',2}(y)\cdot yh(iy)\,dy, \tag{2.40}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\Big((\varepsilon+iy)^2r_3(\varepsilon+iy)h(\varepsilon+iy)\Big)\,dy = \int_0^\infty (H\alpha_{\xi',3})(y)\cdot y^2h(iy)\,dy, \tag{2.41}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\Big((\varepsilon+iy)^3r_4(\varepsilon+iy)h(\varepsilon+iy)\Big)\,dy = -\int_0^\infty \alpha_{\xi',4}(y)\cdot y^3h(iy)\,dy. \tag{2.42}$$

Similarly we find

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\left((\varepsilon+iy)p_1(\varepsilon+iy)h(\varepsilon+iy)\right)\,dy = \lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Im}\left(p_1(\varepsilon+iy)(iy)h(iy)\right)\,dy\\
&=\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{Re}\left(p_1(\varepsilon+iy)yh(iy)\right)\,dy = \int_0^\infty yh(iy)\,d\mu_{\xi',1}(y)
\end{aligned} \tag{2.43}$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\Big((\varepsilon+iy)^3 p_2(\varepsilon+iy)h(\varepsilon+iy)\Big)\,dy\\
&=\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\Big(p_2(\varepsilon+iy)(iy)^3 h(iy)\Big)\,dy \qquad (2.44)\\
&=-\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Re}\Big(p_2(\varepsilon+iy)y^3 h(iy)\Big)\,dy=-\int_0^\infty y^3 h(iy)\,d\mu_{\xi',2}(y).
\end{aligned}
$$

Since $m_{\xi'}(z)$ is given by (2.25), replacing (2.30), (2.35), (2.36) and (2.39)-(2.44) into (2.29) we obtain the conclusion of the lemma. □

Now we are ready to state and prove the main result of this section.

Theorem 2.8 *Suppose that for any $\xi'\in\mathbf{R}^{N-1}$, $m(\xi_1,\xi')$ satisfies the assumptions of Lemma 2.7. For $u\in C_c^\infty(\mathbf{R}^N)$ define u_1, u_2, f and g as in (2.2)-(2.4) and for a given function $\varphi\in C_c^0(\mathbf{R}^N)$, let $W(\varphi)=\displaystyle\int_{\mathbf{R}^N} m(\xi)|\widehat{\varphi}(\xi)|^2\,d\xi$. Then we have the identity:*

$$
\begin{aligned}
&\frac{\pi^2}{16}\left(W(u_1)+W(u_2)-2W(u)\right)\\
&=-\int_{R^{N-1}} A_1(\xi')\int_0^\infty t\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'\\
&+\int_{R^{N-1}}\int_0^\infty \frac{1}{t}\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',0}(t)\,d\xi'\\
&-\int_{R^{N-1}}\int_0^\infty t\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',1}(t)\,d\xi' \qquad (2.45)\\
&+\int_{R^{N-1}}\int_0^\infty t^3\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',2}(t)\,d\xi'\\
&+\int_{R^{N-1}}\int_0^\infty \left[\frac{\alpha_{\xi',0}(t)}{t}+\beta_{\xi',1}(t)-t\alpha_{\xi',2}(t)-t^2\beta_{\xi',3}(t)+t^3\alpha_{\xi',4}(t)\right]\\
&\qquad\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'.
\end{aligned}
$$

Proof. Since $\mathcal{F}_{N-1}f\in\mathcal{S}(\mathbf{R}^N)$, the integral $\displaystyle\int_0^\infty e^{-x_1t}(\mathcal{F}_{N-1}f)(x_1,\xi')\,dx_1$ is well defined for all $t>0$ and $\xi'\in\mathbf{R}^{N-1}$. Using Plancherel's theorem we get

$$
\begin{aligned}
\int_0^\infty e^{-x_1t}(\mathcal{F}_{N-1}f)(x_1,\xi')\,dx_1 &=\langle \mathcal{F}_{N-1}f(\cdot,\xi')\,,\,e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\rangle_{L^2(\mathbf{R})} \qquad (2.46)\\
&=(2\pi)^{-1}\langle \mathcal{F}_1(\mathcal{F}_{N-1}f(\cdot,\xi')),\mathcal{F}_1\left(e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\right)\rangle_{L^2(\mathbf{R})}.
\end{aligned}
$$

Moreover, we have

$$
\mathcal{F}_1\left(e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\right)(\xi_1)=\int_0^\infty e^{-ix_1\xi_1}e^{-x_1t}dx_1=-\frac{1}{t+i\xi_1}e^{-(t+i\xi_1)x_1}\Big|_{x_1=0}^{\infty}=\frac{1}{t+i\xi_1}
$$

and then, using (2.46) and the oddness of $\widehat{f}$ with respect to ξ_1 we get :

$$h_{\xi'}(it) = \left| \int_0^\infty e^{-x_1 t} (\mathcal{F}_{N-1} f)(x_1, \xi')\, dx_1 \right|^2 = (2\pi)^{-2} \left| \int_{-\infty}^\infty \widehat{f}(\xi_1, \xi') \cdot \frac{1}{t - i\xi_1}\, d\xi_1 \right|^2$$

$$(2.47) \qquad = (2\pi)^{-2} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \left(\frac{1}{t - i\xi_1} - \frac{1}{t + i\xi_1} \right) d\xi_1 \right|^2$$

$$= \frac{1}{\pi^2} \left| \int_0^\infty \widehat{f}(\xi_1, \xi')\ \frac{\xi_1}{t^2 + \xi_1^2}\, d\xi_1 \right|^2 .$$

Identity (2.45) is a simple consequence of (2.12), (2.24) and (2.47) and Theorem 2.8 is proved. □

Remark 2.9 It is worth to note that we can prove an identity analogous to (2.45) whenever we work with a symbol $m(\xi) = m(\xi_1, \xi')$ symmetric with respect to ξ_1 and such that for any $\xi' \in \mathbf{R}^{N-1}$, $m(\cdot, \xi')$ admits an holomorphic extension $m_{\xi'}(z)$ to the domain $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$ having the following properties :

P1 : $\displaystyle\lim_{z \to \xi_1,\, Im(z) > 0} m_{\xi'}(z) = m(\xi_1, \xi')$.

P2 : For any $\varepsilon > 0$, $\displaystyle\lim_{|z| \to \infty,\, Re(z) \geq \varepsilon} \frac{m_{\xi'}(z)}{z^3} = 0$.

P3 : $\displaystyle\lim_{\varepsilon \to 0} \int_0^\infty m_{\xi'}(\varepsilon + it) h_{\xi'}(\varepsilon + it)\, dt$ exists (and depends on ξ' and the values taken by $h_{\xi'}$ on the imaginary axis).

Note that assumption **P1** implies that $m(\cdot, \xi')$ admits an holomorphic extension to the whole right half-plane. Indeed, it follows from Schwarz' reflection principle ([8], p. 75) that the function

$$\tilde{m}_{\xi'} = \begin{cases} m_{\xi'}(z) & \text{if} \quad \mathrm{Im}(z) \geq 0, \\ \\ \overline{m_{\xi'}(\overline{z})} & \text{if} \quad \mathrm{Im}(z) < 0 \end{cases}$$

is holomorphic in $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.

Assumption **P2** is needed in the proof of Lemma 2.7 to show that $\displaystyle\lim_{R \to \infty} \int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z) h_{\xi'}(z)\, dz = 0$ (where $\gamma_{2,\varepsilon,R}(\theta) = \varepsilon + Re^{i\theta}$, $\theta \in [0, \frac{\pi}{2}]$). We recall that $|h_{\xi'}(z)|$ behaves like $\frac{1}{|z|^4}$ as $|z| \longrightarrow \infty$ (see Lemma 2.1 and Remark 2.2). This assumption could be replaced by a weaker one that guarantees at least that $\displaystyle\lim_{n \to \infty} \int_{\gamma_{2,\varepsilon,R_n}} m_{\xi'}(z) h_{\xi'}(z)\, dz = 0$ for some sequence $R_n \longrightarrow \infty$.

In Theorem 2.8 assumption **P3** is satisfied because of the special form of $m(\cdot, \xi')$ given by (2.23).

In this context, the hypotheses of Theorem 2.8 are almost optimal. Indeed, suppose that a function $m(z)$ has the properties **P1, P2, P3** above. Let $\tilde{m}$ be the holomorphic extension of m to the right half-plane and define $q(z) = \frac{\tilde{m}(z)}{z^3}$. Clearly, q is an holomorphic function in the right half-plane and $\displaystyle\lim_{|z| \to \infty,\, Re(z) \geq \varepsilon} q(z) = 0$ for any $\varepsilon > 0$. Thus for any $x > \varepsilon$ we have the Poisson representation formulae

$$(2.48) \qquad \begin{aligned} q(x + iy) &= \frac{1}{\pi} \int_{-\infty}^\infty \frac{x - \varepsilon}{(x - \varepsilon)^2 + (t - y)^2} \mathrm{Re}(q(\varepsilon + it))\, dt \\ &+ \frac{i}{\pi} \int_{-\infty}^\infty \frac{t - y}{(x - \varepsilon)^2 + (t - y)^2} \mathrm{Re}(q(\varepsilon + it))\, dt \end{aligned}$$

and

$$
(2.49)\qquad \begin{aligned} q(x+iy) = & \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{t-y}{(x-\varepsilon)^2+(t-y)^2}\mathrm{Im}(q(\varepsilon+it))\,dt \\ & +\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{x-\varepsilon}{(x-\varepsilon)^2+(t-y)^2}\mathrm{Im}(q(\varepsilon+it))\,dt. \end{aligned}
$$

Multiplying (2.48) (respectively (2.49)) by $(x+iy)^3$, we find the expression of $m(x+iy)$ in terms of $\mathrm{Re}(q(\varepsilon+it))$ (respectively in terms of $\mathrm{Im}(q(\varepsilon+it))$). If $\mathrm{Re}(q(\varepsilon+it)) \longrightarrow \alpha(t)$ as $\varepsilon \longrightarrow 0$ and if it is possible to pass to the limit as $\varepsilon \longrightarrow 0$ in (2.48) then we obtain, at least formally,

$$
m_{\xi'}(\xi_1) = \xi_1^3 q(\xi_1) = \frac{\xi_1^4}{\pi}\int_{-\infty}^{\infty} \frac{\alpha(t)}{\xi_1^2+t^2}\,dt.
$$

However, as it will be seen later in applications, the function q may be singular at the origin. In this case it is not possible to pass to the limit as $\varepsilon \longrightarrow 0$ in (2.48) or in (2.49) in order to express the function q (hence the function m) in terms of its "boundary values" on the imaginary axis. This is the reason why we have introduced "lower order terms" in the expression of $m_{\xi'}(z)$ in (2.23).

We give now some examples illustrating several situations that may be encountered in applications. Throughout $u \in C_c^\infty(\mathbf{R}^N)$ and we keep the notation introduced in (2.2)-(2.3).

Example 2.10 If the symbol m is of the form $m(\xi_1,\xi') = A_1(\xi')|\xi_1|$, then Theorem 2.8 gives

$$
(2.50)\quad W(u_1)+W(u_2)-2W(u) = -\frac{16}{\pi^2}\int_{R^{N-1}} A_1(\xi')\int_0^\infty t\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'.
$$

This kind of symbol appears in problems involving operators of the type $H_1\frac{\partial}{\partial x_1}P(\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_N})$, where H_1 is the Hilbert transform with respect to the x_1 variable and P is a pseudo-differential operator in the last $N-1$ variables.

Example 2.11 *i*) Consider the symbol $m(\xi) = \dfrac{1}{|\xi|^2}$ appearing in Choquard's problem. It can be written as

$$
m(\xi_1,\xi') = \frac{1}{\xi_1^2+|\xi'|^2} = \frac{1}{\pi}\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',0}(t),
$$

where $\mu_{\xi',0} = \frac{\pi}{2}(\delta_{-|\xi'|}+\delta_{|\xi'|})$ and δ_a is the Dirac measure with support $\{a\}$. From Theorem 2.8 we get the identity

$$
(2.51)\qquad W(u_1)+W(u_2)-2W(u) = \frac{8}{\pi}\int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|}\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'.
$$

The same identity could be obtained by observing that the function $m_{\xi'}(z) = \dfrac{1}{z^2+|\xi'|^2}$ is meromorphic in $\mathbf{C}$ and has exactly one pole in the upper half-plane, namely $i|\xi'|$. Using Residue's Theorem it is not hard to see that

$$
\int_{-\infty}^{\infty} m_{\xi'}(z)h_{\xi'}(z)\,dz = 2\pi i\,\mathrm{Res}(m_{\xi'}h_{\xi'},\, i|\xi'|),
$$

and integrating this identity over $\mathbf{R}^{N-1}$ we get (2.51).

ii) Consider the symbol $m(\xi) = \dfrac{1}{|\xi|^2+a^2} = \dfrac{1}{\xi_1^2+|\xi'|^2+a^2}$ corresponding to the operator $(-\Delta+a^2)^{-1}$. It is obvious that

$$
m(\xi_1,\xi') = \frac{1}{\pi}\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',0}(t),
$$

where $\mu_{\xi',0} = \frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2+a^2}} + \delta_{\sqrt{|\xi'|^2+a^2}})$. From Theorem 2.8 we get the identity

$$(2.52)\ \ W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi} \int_{R^{N-1}} \frac{1}{\sqrt{|\xi'|^2 + a^2}} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{a^2 + |\xi'|^2 + \xi_1^2}\, d\xi_1 \right|^2 d\xi'.$$

The same identity could be obtained by applying Residue's Theorem to the meromorphic function $z \longmapsto \dfrac{1}{z^2 + |\xi'|^2 + a^2} h_{\xi'}(z)$.

iii) More generally, consider a symbol of the form $m(\xi_1, \xi') = \dfrac{c(\xi')}{\xi_1^2 + r^2(\xi')}$. It can be written as

$$m(\xi_1, \xi') = \frac{1}{\pi} \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2}\, d\mu_{\xi',0}(t),$$

where $\mu_{\xi',0} = \frac{\pi}{2} c(\xi')(\delta_{-r(\xi')} + \delta_{r(\xi')})$. Using Theorem 2.8 we obtain the identity

$$(2.53)\qquad W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi} \int_{R^{N-1}} \frac{c(\xi')}{r(\xi')} \cdot \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{r^2(\xi') + \xi_1^2}\, d\xi_1 \right|^2 d\xi'.$$

In particular, for the symbol $m(\xi_1, \xi') = \dfrac{\xi_j^{2k}}{\xi_1^2 + |\xi'|^2 + a^2}$, $j = 2, \ldots, N$ (corresponding to the operator $(-1)^k \dfrac{\partial^{2k}}{\partial x_j^{2k}}(-\Delta + a^2)^{-1})$), we get

$$(2.54)\ \ W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi} \int_{R^{N-1}} \frac{\xi_j^{2k}}{\sqrt{|\xi'|^2 + a^2}} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{a^2 + |\xi'|^2 + \xi_1^2}\, d\xi_1 \right|^2 d\xi'.$$

iv) The symbol $m(\xi_1, \xi') = \dfrac{\xi_1^2}{\xi_1^2 + |\xi'|^2 + a^2}$ can be expressed as

$$m(\xi_1, \xi') = \frac{\xi_1^2}{\pi} \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2}\, d\mu_{\xi',1}(t),$$

where $\mu_{\xi',1} = \frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2+a^2}} + \delta_{\sqrt{|\xi'|^2+a^2}})$. From Theorem 2.8 we find the identity

$$(2.55)\ \ W(u_1) + W(u_2) - 2W(u) = -\frac{8}{\pi} \int_{R^{N-1}} \sqrt{|\xi'|^2 + a^2} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{a^2 + |\xi'|^2 + \xi_1^2}\, d\xi_1 \right|^2 d\xi'.$$

Notice that the right-hand side in (2.55) is negative, while in (2.54) it is positive.

v) The symbol $m(\xi_1, \xi') = \dfrac{\xi_1^4}{\xi_1^2 + |\xi'|^2 + a^2}$ (corresponding to the operator $\dfrac{\partial^4}{\partial x_1^4}(-\Delta + a^2)^{-1})$ can be written as

$$m(\xi_1, \xi') = \frac{\xi_1^4}{\pi} \int_{\mathbf{R}} \frac{1}{\xi_1^2 + t^2}\, d\mu_{\xi',2}(t),$$

where $\mu_{\xi',2} = \frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2+a^2}} + \delta_{\sqrt{|\xi'|^2+a^2}})$. By Theorem 2.8 we have the identity

$$(2.56)\ \ W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi} \int_{R^{N-1}} (|\xi'|^2 + a^2)^{\frac{3}{2}} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{a^2 + |\xi'|^2 + \xi_1^2}\, d\xi_1 \right|^2 d\xi'.$$

Obviously all the identities in (2.53)-(2.56) could be obtained by using the Residue Theorem.

Example 2.12 Consider the symbol $m(\xi) = |\xi|^{2s}$, corresponding to the operator $(-\Delta)^s$.

It is well-known that the argument of a complex number, $\arg(z)$, can be defined analytically on $\mathbf{C} \setminus (-\infty, 0]$ in such a way that

$$\begin{aligned} &\forall t \in (0,\infty), \quad \arg(t) = 0, \\ &\forall t \in (-\infty, 0), \quad \lim_{\varepsilon \downarrow 0} \arg(t + i\varepsilon) = \pi \quad \text{ and } \quad \lim_{\varepsilon \uparrow 0} \arg(t+i\varepsilon) = -\pi. \end{aligned}$$

The complex logarithm $\log(z) = \ln|z| + i\arg(z)$ is well defined and holomorphic on $\mathbf{C} \setminus (-\infty, 0]$. For $z \in \Omega_{\xi'} := \mathbf{C} \setminus \{it \mid t \in (-\infty, -|\xi'|] \cup [|\xi'|, \infty)\}$, we have $z^2 + |\xi'|^2 \notin (-\infty, 0]$; hence we may define

$$m_{\xi'}(z) = e^{s\log(z^2+|\xi'|^2)} = |z^2 + |\xi'|^2|^s e^{is\arg(z^2+|\xi'|^2)}.$$

The function $m_{\xi'}$ is holomorphic in $\Omega_{\xi'}$ and $|m_{\xi'}(z)| = |z^2 + |\xi'|^2|^s$ for any $z \in \Omega_{\xi'}$.

If $s < \frac{3}{2}$ and $\xi' \neq 0$, the function $z \longmapsto \frac{m_{\xi'}(z)}{z^3}$ is holomorphic in $\Omega_{\xi'} \setminus \{0\}$, tends to zero as $|z| \longrightarrow \infty$ and has a third order pole at the origin. It is easy to see that

$$m_{\xi'}(z) = |\xi'|^{2s}\left(1 + s\frac{z^2}{|\xi'|^2} + \sum_{k=2}^{\infty} C_s^k \frac{z^{2k}}{|\xi'|^{2k}}\right), \tag{2.57}$$

where $C_s^k = \frac{s(s-1)\dots(s-k+1)}{k!}$ and the series converges in the open ball $B_{\mathbf{C}}(0, |\xi'|)$. Consider the function $r_{\xi'}(z) = \frac{1}{z^3}(m_{\xi'}(z) - |\xi'|^{2s} - s|\xi'|^{2s-2}z^2)$. According to (2.57), $r_{\xi'}$ is a holomorphic function in $\Omega_{\xi'}$. If $s < \frac{3}{2}$, we have $r_{\xi'}(z) \longrightarrow 0$ as $|z| \longrightarrow \infty$. Consequently, the Poisson representation formula (2.48) holds for $r_{\xi'}$. Since $r_{\xi'}(\overline{z}) = \overline{r_{\xi'}(z)}$, the function $t \longmapsto \mathrm{Re}(r_{\xi'}(\varepsilon + it))$ is even and we have, in particular,

$$\begin{aligned} m_{\xi'}(\xi_1) &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \xi_1^3 r_{\xi'}(\xi_1) \\ &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \frac{\xi_1^3}{\pi}\int_{-\infty}^{\infty} \frac{\xi_1 - \varepsilon}{(\xi_1-\varepsilon)^2 + (t-y)^2}\mathrm{Re}(r_{\xi'}(\varepsilon + it))\,dt. \end{aligned} \tag{2.58}$$

It is clear from the definition of $r_{\xi'}$ that for any $t \in (-|\xi'|, |\xi'|)$ we have $\lim_{\varepsilon \to 0}\mathrm{Re}(r_{\xi'}(\varepsilon + it)) = \mathrm{Re}(r_{\xi'}(it)) = 0$. For any $t > |\xi'|$ we have $\lim_{\varepsilon\downarrow 0} m_{\xi'}(\varepsilon + it) = (t^2 - |\xi'|^2)^s e^{is\pi}$ and $\lim_{\varepsilon \downarrow 0}\mathrm{Re}(r_{\xi'}(\varepsilon + it)) = -\sin(s\pi)\dfrac{(t^2 - |\xi'|^2)^s}{t^3}$.

On the other hand, it is not hard to check that for $-1 < s < \frac{3}{2}$, there exists $p_s \in (1, \infty)$ and $C_{s,\xi'} > 0$ such that

$$||r_{\xi'}(\varepsilon + i\cdot)||_{L^{p_s}(\mathbf{R})} \leq C_{s,\xi'} \qquad \text{for any } \varepsilon \in (0, \frac{|\xi'|}{2}). \tag{2.59}$$

Indeed, since $|r_{\xi'}(\varepsilon + i\cdot)|$ is even, it suffices to show that $||r_{\xi'}(\varepsilon + i\cdot)||_{L^{p_s}([0,\infty))}$ has a bound independent of ε. Since $|r_{\xi'}(\varepsilon + it)|$ is uniformly bounded for $\varepsilon \in [0, \frac{|\xi'|}{2}]$ and $t \in [0, \frac{|\xi'|}{2}]$, it suffices to show that $||r_{\xi'}(\varepsilon + i\cdot)||_{L^{p_s}([\frac{|\xi'|}{2}, \infty))} \leq C'_{s,\xi'}$.

If $s \geq 0$, we have $|m_{\xi'}(z)| = |z^2 + |\xi'|^2|^s \leq C_{1,s}(|z|^{2s} + |\xi'|^{2s})$. Thus for any $\varepsilon \in (0, \frac{|\xi'|}{2})$ and $t \geq \frac{|\xi'|}{2}$ we have

$$\begin{aligned} |r_{\xi'}(\varepsilon + it)| &\leq \tfrac{|m_{\xi'}(\varepsilon+it)|}{|\varepsilon+it|^3} + \tfrac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \tfrac{s|\xi'|^{2s-2}}{|\varepsilon+it|} \\ &\leq \tfrac{C_{1,s}}{|\varepsilon+it|^{3-2s}} + \tfrac{C_{1,s}|\xi'|^{2s}}{|\varepsilon+it|^3} + \tfrac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \tfrac{s|\xi'|^{2s-2}}{|\varepsilon+it|} \\ &\leq C_{1,s}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right)^{3-2s} + (C_{1,s}+1)|\xi'|^{2s}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right)^3 + s|\xi'|^{2s-2}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right). \end{aligned}$$

Thus it suffices to take $p_s > 1$ such that $p_s(3-2s) > 1$ to obtain the desired bound.

If $s < 0$ then for $\varepsilon \in (0, \frac{|\xi'|}{2})$ and $t \geq \frac{|\xi'|}{2}$, we have $|(\varepsilon+it)+i|\xi'||^s \leq |\varepsilon+it|^s$, and $|(\varepsilon+it)-i|\xi'||^s \leq |t-|\xi'||^s$. Since $|m_{\xi'}(\varepsilon+it)| = |(\varepsilon+it)+i|\xi'||^s\,|(\varepsilon+it)-i|\xi'||^s$, we find in this case

$$|r_{\xi'}(\varepsilon+it)| \leq \frac{|m_{\xi'}(\varepsilon+it)|}{|\varepsilon+it|^3} + \frac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \frac{s|\xi'|^{2s-2}}{|\varepsilon+it|} \leq \frac{|(\varepsilon+it)-i|\xi'||^s}{|\varepsilon+it|^{3-s}} + \frac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \frac{s|\xi'|^{2s-2}}{|\varepsilon+it|}$$

$$\leq \frac{|t-|\xi'||^s}{|\varepsilon+it|^{3-s}} + \frac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \frac{s|\xi'|^{2s-2}}{|\varepsilon+it|}$$

$$\leq |t-|\xi'||^s \min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right)^{3-s} + |\xi'|^{2s}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right)^3 + s|\xi'|^{2s-2}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right).$$

Consequently it suffices to take $p_s > 1$ such that $-sp_s < 1$ (i.e. $p_s \in (1, -\frac{1}{s})$) to obtain (2.59).

It follows from (2.59) and Theorem 2.5 p. 50 in [24] that there exists $k_{\xi'} \in L^{p_s}(\mathbf{R})$ such that $\mathrm{Re}(r_{\xi'}(x+iy)) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2} k_{\xi'}(t)\,dt$. Moreover, from Theorem 2.1 p. 47 in [24] we have $\lim\limits_{\varepsilon\downarrow 0} \mathrm{Re}(r_{\xi'}(\varepsilon+it)) = k_{\xi'}(t)$ for almost every $t \in \mathbf{R}$ and $||\mathrm{Re}(r_{\xi'}(\varepsilon+i\cdot)) - k_{\xi'}||_{L^{p_s}} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. In view of the pointwise convergence, we infer that $k_{\xi'}$ is even and

$$k_{\xi'}(t) = \begin{cases} 0 & \text{if} \quad t \in (-|\xi'|, |\xi'|) \\ -\sin(s\pi)\frac{(t^2-|\xi'|^2)^s}{|t|^3} & \text{if} \quad |t| > |\xi'| \end{cases}$$

a.e. on $\mathbf{R}$. Now it is clear that the symbol $m(\xi_1, \xi')$ can be written as

$$\begin{aligned} m(\xi_1,\xi') &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \xi_1^3 r_{\xi'}(\xi_1) \\ &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \frac{\xi_1^4}{\pi}\int_{-\infty}^{\infty} \frac{1}{\xi_1^2+t^2} k_{\xi'}(t)\,dt. \end{aligned} \tag{2.60}$$

Thus we may apply Theorem 2.8 to get, for any $u \in C_c^\infty(\mathbf{R}^N)$ and $s \in (-1, \frac{3}{2})$,

$$\begin{aligned} W(u_1)+W(u_2)-2W(u) &= \frac{16}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_0^\infty t^3 k_{\xi'}(t)\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi' \\ &= -\frac{16\sin(s\pi)}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_{|\xi'|}^\infty \left(t^2-|\xi'|^2\right)^s\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'. \end{aligned} \tag{2.61}$$

Similarly, if we consider the symbol $m(\xi) = (|\xi|^2+a^2)^s$ we get the identity

$$\begin{aligned} &W(u_1)+W(u_2)-2W(u) \\ &= -\frac{16\sin(s\pi)}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_{\sqrt{|\xi'|^2+a^2}}^\infty \left(t^2-|\xi'|^2-a^2\right)^s\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'. \end{aligned} \tag{2.62}$$

3 Symmetry and function spaces

For any $u \in C_c^\infty(\mathbf{R}^N)$ we define u_1 and u_2 as in (2.1) and we put $T_1u = u_1$, $T_2u = u_2$. Clearly, T_1 and T_2 are linear continuous mappings from $C_c^\infty(\mathbf{R}^N)$ to $C_c^0(\mathbf{R}^N)$. In this section we consider the following intimately related problems :

1°. Determine significant subspaces $\mathcal{X} \subset \mathcal{D}'(\mathbf{R}^N)$ such that T_1 and T_2 can be extended to linear continuous mappings from $\mathcal{X}$ to $\mathcal{X}$. (Or, equivalently, find the subspaces $\mathcal{X}$ such that $u \in \mathcal{X}$ implies T_1u, $T_2u \in \mathcal{X}$ and $u \longmapsto T_1u$, $u \longmapsto T_2u$ are continuous for the $\mathcal{X}$ topology).

2°. If $\mathcal{X}$ is a subspace as above, how the identities proved in the previous section can be extended to $\mathcal{X}$?

The answer to these questions is of great importance in symmetry problems. For instance, suppose that a function space $\mathcal{X}$ has the two properties described above and that the solutions of the variational problem

$$\begin{aligned}&\text{minimize } E(u) := \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\, d\xi + \int_{\mathbf{R}^N} F(u)\, dx \\ &\text{under the constraint } \int_{\mathbf{R}^N} G(u)\, dx = \lambda\end{aligned} \tag{3.1}$$

belong to $\mathcal{X}$. As before, the symbol $m(\xi) = m(\xi_1, \xi')$ is assumed to be symmetric with respect to ξ_1. Defining $W(u) := \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\, d\xi$, we suppose also that that an identity of type (2.45) holds for $W(u)$ and it can be extended to $\mathcal{X}$ in such a way that

$$W(T_1 u) + W(T_2 u) - 2W(u) < 0 \qquad \text{whenever } T_1 u \neq u,\ T_2 u \neq u.$$

(We will see later that most of the symbols in Examples 2.10-2.12 have this property.) Then, we claim that after a translation in the x_1 direction, any solution of (3.1) is symmetric with respect to x_1. Indeed, let u be a minimizer. After a translation in the x_1 direction, we may assume that $\int_{\{x_1<0\}} G(u(x))\, dx = \int_{\{x_1>0\}} G(u(x))\, dx = \frac{\lambda}{2}$. This implies $\int_{\mathbf{R}^N} G(u_1(x))\, dx = 2\int_{\{x_1<0\}} G(u(x))\, dx = \lambda$ and $\int_{\mathbf{R}^N} G(u_2(x))\, dx = 2\int_{\{x_1>0\}} G(u(x))\, dx = \lambda$; consequently u_1 and u_2 (which belong to $\mathcal{X}$) also satisfy the constraint. It is obvious that $\int_{\mathbf{R}^N} F(u_1(x))\, dx + \int_{\mathbf{R}^N} F(u_2(x))\, dx = 2\int_{\mathbf{R}^N} F(u(x))\, dx$. Suppose by contradiction that u is not symmetric with respect to x_1. Then we get

$$E(u_1) + E(u_2) - 2E(u) = W(u_1) + W(u_2) - 2W(u) < 0,$$

and this implies that either $E(u_1) < E(u)$ or $E(u_2) < E(u)$. Therefore u cannot be a minimizer and this proves the claim.

Given the motivation above, we will study the behavior of T_1 and T_2 from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$, respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$, where

$$H^s(\mathbf{R}^N) = \{u \in \mathcal{S}'(\mathbf{R}^N) \mid \widehat{u} \in L^1_{loc}(\mathbf{R}^N) \text{ and } \int_{\mathbf{R}^N} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2\, d\xi < \infty\},$$

$$\dot{H}^s(\mathbf{R}^N) = \{u \in \mathcal{S}'(\mathbf{R}^N) \mid \widehat{u} \in L^1_{loc}(\mathbf{R}^N) \text{ and } \int_{\mathbf{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2\, d\xi < \infty\}.$$

Consider $\varphi \in C_c^\infty(\mathbf{R})$, φ odd, such that $\varphi'(0) = 1$. It is obvious that $T_1\varphi(x) = -\mathrm{sgn}(x)\varphi(x)$ and $(T_1\varphi)'(x) = \begin{cases} \varphi'(x) \text{ if } x < 0, \\ -\varphi'(x) \text{ if } x > 0 \end{cases}$ and we have (in the distributional sense) $(T_1\varphi)'' = -\mathrm{sgn}(x)\varphi''(x) - 2\delta_0$. Since $(T_1\varphi)'' \notin L^2(\mathbf{R})$, we conclude that T_1 and T_2 are not well-defined from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$ if $s \geq 2$. In fact, T_1 and T_2 are not well-defined from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $s \geq \frac{3}{2}$, as it can be seen in the following example.

Example 3.1 Define $\varphi : \mathbf{R} \longrightarrow \mathbf{R}$, $\varphi(x) = xe^{-|x|}$. An easy computation shows that $\widehat{\varphi}(\xi) = \frac{-4i\xi}{(1+\xi^2)^2}$, hence $\varphi \in H^s(\mathbf{R})$ for any $s < \frac{5}{2}$ and $\varphi \in \dot{H}^s(\mathbf{R})$ for any $s \in (-\frac{3}{2}, \frac{5}{2})$. It is clear that $(T_1\varphi)(x) = -|x|e^{-|x|}$ and $\widehat{T_1\varphi}(\xi) = \frac{2(\xi^2-1)}{(1+\xi^2)^2}$. Consequently, $T_1\varphi \in H^s(\mathbf{R})$ for $s < \frac{3}{2}$ (respectively $T_1\varphi \in \dot{H}^s(\mathbf{R})$ for $-\frac{1}{2} < s < \frac{3}{2}$), but $T_1\varphi \notin H^s(\mathbf{R})$ and $T_1\varphi \notin \dot{H}^s(\mathbf{R})$ for $s \geq \frac{3}{2}$.

In dimension $N \geq 2$ it suffices to take $\psi(x) = \varphi(x_1)\varphi_1(x_2, \dots, x_N)$, where $\varphi_1 \in C_c^\infty(\mathbf{R}^{N-1})$, to see that T_1 and T_2 are not well-defined from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $\frac{3}{2} \leq s < \frac{5}{2}$.

If $s < 0$, the elements of $H^s(\mathbf{R}^N)$ or $\dot{H}^s(\mathbf{R}^N)$ are not necessarily measurable functions. In this case we extend T_1 and T_2 to $H^s(\mathbf{R}^N)$ or $\dot{H}^s(\mathbf{R}^N)$ by duality. For u, $\varphi \in C_c^\infty(\mathbf{R}^N)$ we have

$$\langle T_1 u, \varphi\rangle_{\mathcal{S}',\mathcal{S}} = \int_{\mathbf{R}^N} (T_1 u)(x)\varphi(x)\,dx = \int_{\{x_1<0\}} u(x)\varphi(x)\,dx + \int_{\{x_1>0\}} u(-x_1, x')\varphi(x)\,dx$$

$$= \int_{\{x_1<0\}} u(x)\varphi(x)\,dx + \int_{\{x_1<0\}} u(x_1, x')\varphi(-x_1, x')\,dx = \langle u, T_1^*\varphi\rangle_{L^2,L^2},$$

where $(T_1^*\varphi)(x) = \chi_{\{x_1<0\}}(\varphi(x_1, x') + \varphi(-x_1, x'))$. Hence, for $u \in H^s(\mathbf{R}^N)$ with $s < 0$ we should define $T_1 u$ by

$$\langle T_1 u, \varphi\rangle_{H^s,H^{-s}} = \langle u, T_1^*\varphi\rangle_{H^s,H^{-s}}$$

for any test function $\varphi \in C_c^\infty(\mathbf{R}^N)$. However, the operator T_1^* does not map $H^k(\mathbf{R}^N)$ into $H^k(\mathbf{R}^N)$ if $k \geq \frac{1}{2}$ (as it can be easily seen by taking the function $\eta(x) = e^{-|x|}$ in one dimension, respectively $\eta(x_1)\eta_1(x_2, \dots, x_N)$, where $\eta_1 \in C_c^\infty(\mathbf{R}^{N-1})$ in dimension $N \geq 2$). This shows that we cannot define T_1 and T_2 on $H^s(\mathbf{R}^N)$ and on $\dot{H}^s(\mathbf{R}^N)$ if $s \leq -\frac{1}{2}$.

Example 3.2 Consider the tempered distribution u defined by $u = p.v.(\frac{1}{x})$, that is

$$\langle u, \varphi\rangle_{\mathcal{S}',\mathcal{S}} = \lim_{\varepsilon\to 0} \int_{\{|x|>\varepsilon\}} \frac{1}{x}\varphi(x)\,dx \qquad \text{for any } \varphi \in \mathcal{S}(\mathbf{R}).$$

It is well-known (and easy to check) that $\widehat{u}(\xi) = -i\pi\,\mathrm{sgn}(\xi)$; hence $u \in H^s(\mathbf{R})$ for any $s < -\frac{1}{2}$. However, $T_1 u = -\frac{1}{|x|}$ and $T_2 u = \frac{1}{|x|}$ do not define distributions on $\mathbf{R}$!

Our next goal is to prove that the operators T_1 and T_2 are well-defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $-\frac{1}{2} < s < \frac{3}{2}$. It is obvious that T_1 and T_2 are well-defined and continuous from $L^2(\mathbf{R}^N)$ to $L^2(\mathbf{R}^N)$. It is well-known that $H^1(\mathbf{R}^N) = W^{1,2}(\mathbf{R}^N) = \{\varphi \in L^2(\mathbf{R}^N) \mid \frac{\partial\varphi}{\partial x_i} \in L^2(\mathbf{R}^N),\ i = 1, \dots, N\}$ and that T_1, $T_2 : W^{1,2}(\mathbf{R}^N) \longrightarrow W^{1,2}(\mathbf{R}^N)$ are well-defined and continuous. Using interpolation theory we conclude that T_1 and T_2 are well-defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ if $0 \leq s \leq 1$. However, interpolation gives no information if either $s < 0$ or $s > 1$. Our next result deals with some values of s in this range.

Theorem 3.3 *The operators T_1 and T_2 are well-defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ and from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$ for any $s \in (-\frac{1}{2}, \frac{3}{2})$.*

Proof. We will prove that there exists $C_s > 0$ such that for any $u \in C_c^\infty(\mathbf{R}^N)$ we have

$$(3.2) \qquad ||T_i u||_{H^s} \leq C_s ||u||_{H^s}, \qquad \text{respectively} \qquad ||T_i u||_{\dot{H}^s} \leq C_s ||u||_{\dot{H}^s}, \quad s = 1, 2,$$

and then the theorem will follow by density.

Therefore, suppose $u \in C_c^\infty(\mathbf{R}^N)$. If $N \geq 2$ we have by (2.61) and (2.62)

$$(3.3) \qquad \begin{aligned} &||T_1 u||^2_{\dot{H}^s} + ||T_2 u||^2_{\dot{H}^s} - 2||u||^2_{\dot{H}^s} \\ &= -\frac{16\sin(s\pi)}{\pi^2} \int_{\mathbf{R}^{N-1}} \int_{|\xi'|}^\infty \left(t^2 - |\xi'|^2\right)^s \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2}\,d\xi_1 \right|^2 dt\,d\xi', \end{aligned}$$

respectively

$$
\begin{aligned}
&||T_1u||^2_{H^s}+||T_2u||^2_{H^s}-2||u||^2_{H^s}\\
&=-\frac{16\sin(s\pi)}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_{\sqrt{|\xi'|^2+1}}^{\infty}\left(t^2-|\xi'|^2-1\right)^s\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'.
\end{aligned}
\tag{3.4}
$$

If $N=1$ we have

$$
||T_1u||^2_{\dot{H}^s}+||T_2u||^2_{\dot{H}^s}-2||u||^2_{\dot{H}^s}=-\frac{16\sin(s\pi)}{\pi^2}\int_0^\infty t^{2s}\left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2 dt,
\tag{3.5}
$$

respectively

$$
||T_1u||^2_{H^s}+||T_2u||^2_{H^s}-2||u||^2_{H^s}=-\frac{16\sin(s\pi)}{\pi^2}\int_1^\infty \left(t^2-1\right)^s\left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2 dt.
\tag{3.6}
$$

We begin by proving that T_1 and T_2 are bounded from $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$, $-\frac{1}{2}<s<\frac{3}{2}$. The integral in the right-hand side of (3.5) can be formally written as

$$
\int_0^\infty\int_0^\infty\int_0^\infty t^{2s}\frac{\xi}{t^2+\xi^2}\cdot\frac{\eta}{t^2+\eta^2}\widehat{f}(\xi)\overline{\widehat{f}(\eta)}\,d\xi\,d\eta\,dt.
\tag{3.7}
$$

Our strategy is as follows: first we compute explicitly the integral

$$
I_s(\xi,\eta)=\int_0^\infty t^{2s}\frac{\xi}{t^2+\xi^2}\cdot\frac{\eta}{t^2+\eta^2}\,dt=\xi\eta\int_0^\infty t^{2s}\frac{1}{t^2+\xi^2}\cdot\frac{1}{t^2+\eta^2}\,dt.
\tag{3.8}
$$

Observe that $I_s(\xi,\eta)>0$ if $\xi>0$, $\eta>0$. Then we will prove that for any $s\in(-\frac{1}{2},\frac{3}{2})$ and any $\varphi,\ \psi\in L^2(0,\infty)$ we have

$$
\left|\int_0^\infty\int_0^\infty \xi^{-s}\eta^{-s}I_s(\xi,\eta)\varphi(\xi)\psi(\eta)\,d\xi\,d\eta\right|\le C(s)||\varphi||_{L^2(0,\infty)}\cdot||\psi||_{L^2(0,\infty)}.
$$

This will be done in Lemma 3.4. Thereafter it will be clear that for any $f\in\dot{H}^s(\mathbf{R})$ we have

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty I_s(\xi,\eta)|\widehat{f}(\xi)|\cdot|\overline{\widehat{f}(\eta)}|\,d\xi\,d\eta\\
&=\int_0^\infty\int_0^\infty \xi^{-s}\eta^{-s}I_s(\xi,\eta)|\xi^s\widehat{f}(\xi)|\cdot|\eta^s\overline{\widehat{f}(\eta)}|\,d\xi\,d\eta\\
&\le C(s)||\,|\cdot|^s\widehat{f}||^2_{L^2(0,\infty)}\le C(s)||f||^2_{\dot{H}^s(\mathbf{R})}.
\end{aligned}
\tag{3.9}
$$

This justifies the use of Fubini's Theorem in evaluating (3.7) and proves that the right-hand side of (3.5) is less than $C_1(s)||f||_{\dot{H}^s(\mathbf{R})}$, where $C_1(s)$ is a constant depending only on s. Thus we infer that there exists $C_s>0$ such that $||T_1u||_{\dot{H}^s(\mathbf{R})}\le C_s||u||_{\dot{H}^s(\mathbf{R})}$ and $||T_2u||_{\dot{H}^s(\mathbf{R})}\le C_s||u||_{\dot{H}^s(\mathbf{R})}$ for any $u\in C_c^\infty(\mathbf{R})$. Consequently, T_1 and T_2 can be extended as continuous linear mappings form $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$, $-\frac{1}{2}<s<\frac{3}{2}$, as claimed.

To carry out the first step of this strategy, we come back to $I_s(\xi,\eta)$ given by (3.8). The complex logarithm can be defined analytically on $\mathbf{C}\setminus\{it\mid t\in(-\infty,0]\}$. Hence, we may define the holomorphic function $z\longmapsto z^{2s}:=e^{2s\log(z)}=|z|^{2s}e^{2is\arg(z)}$ on $\mathbf{C}\setminus\{it\mid t\in(-\infty,0]\}$. With this definition the function $k(z)=\dfrac{z^{2s}}{(z^2+\xi^2)(z^2+\eta^2)}$ is meromorphic on $\mathbf{C}\setminus\{it\mid t\in(-\infty,0]\}$. If $\xi\ne\eta$, k has four simple poles, namely $\pm i\xi$ and $\pm i\eta$; if $\xi=\eta$ it has two double poles at

$\pm i\xi$. For $0<\varepsilon<\min(\xi,\eta)$, and $R>\max(\xi,\eta)$, consider the closed path $\beta_{\varepsilon,R}$ composed by the following pieces :

$$\begin{array}{ll} \beta_{1,\varepsilon,R}(t)=t, & t\in[-R,-\varepsilon] \\ \beta_{2,\varepsilon}(\theta)=\varepsilon e^{i(\pi-\theta)}, & \theta\in[0,\pi] \\ \beta_{3,\varepsilon,R}(t)=t, & t\in[\varepsilon,R] \\ \beta_{4,R}(\theta)=Re^{i\theta}, & \theta\in[0,\pi]. \end{array}$$

Using the Residue Theorem we get

$$\int_{\beta\varepsilon,R} k(z)\,dz = 2\pi i[\mathrm{Res}(k,i\xi)+\mathrm{Res}(k,i\eta)] = \pi e^{is\pi}\left[\frac{\xi^{2s}}{\xi(\eta^2-\xi^2)}+\frac{\eta^{2s}}{\eta(\xi^2-\eta^2)}\right]. \tag{3.10}$$

Since $s>-\frac{1}{2}$ we have $\lim\limits_{\varepsilon\to 0}\int_{\beta_{2,\varepsilon}} k(z)\,dz=0$. We have also $\lim\limits_{R\to\infty}\int_{\beta_{4,R}} k(z)\,dz=0$ because $s<\frac{3}{2}$. Passing to the limit as $\varepsilon\longrightarrow 0$ in (3.10) and then passing to the limit as $R\longrightarrow\infty$ in the resulting equation, we get $\int_{-\infty}^{0} k(z)\,dz+\int_{0}^{\infty} k(z)\,dz=\pi e^{is\pi}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}$, that is $(e^{2is\pi}+1)\int_0^\infty \frac{t^{2s}}{(t^2+\xi^2)(t^2+\eta^2)}\,dt=\pi e^{is\pi}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}$. For $s\neq\frac{1}{2}$ we obtain

$$\int_0^\infty \frac{t^{2s}}{(t^2+\xi^2)(t^2+\eta^2)}\,dt=\frac{\pi}{2\cos(s\pi)}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}. \tag{3.11}$$

For $s=\frac{1}{2}$ we compute directly

$$\begin{aligned} &\int_0^\infty \frac{t}{(t^2+\xi^2)(t^2+\eta^2)}\,dt=\frac{1}{\eta^2-\xi^2}\int_0^\infty \frac{t}{t^2+\xi^2}-\frac{t}{t^2+\eta^2}\,dt \\ &=\frac{1}{2}\frac{1}{\eta^2-\xi^2}\Big(\ln(t^2+\xi^2)-\ln(t^2+\eta^2)\Big)\Big|_{t=0}^{\infty}=\frac{\ln\eta-\ln\xi}{\eta^2-\xi^2}. \end{aligned} \tag{3.12}$$

Notice that $\lim\limits_{\eta\to\xi}\int_0^\infty \frac{t^{2s}}{(t^2+\xi^2)(t^2+\eta^2)}\,dt=\frac{\pi(1-2s)}{4\cos(s\pi)}\xi^{2s-3}$ if $s\neq\frac{1}{2}$ and $\lim\limits_{\eta\to\xi}\int_0^\infty \frac{t}{(t^2+\xi^2)(t^2+\eta^2)}\,dt=\frac{1}{2\xi^2}$. Hence

$$I_s(\xi,\eta)=\frac{\pi}{2\cos(s\pi)}\frac{\xi\eta(\xi^{2s-1}-\eta^{2s-1})}{\eta^2-\xi^2}\quad \text{if } s\neq\frac{1}{2},\quad \text{and}\quad I_{\frac{1}{2}}(\xi,\eta)=\frac{\xi\eta(\ln\eta-\ln\xi)}{\eta^2-\xi^2}. \tag{3.13}$$

This gives $\xi^{-s}\eta^{-s}I_s(\xi,\eta)=\frac{\pi}{2\cos(s\pi)}\frac{\xi^s\eta^{1-s}-\xi^{1-s}\eta^s}{\eta^2-\xi^2}$ if $s\neq\frac{1}{2}$ and $\xi^{-\frac{1}{2}}\eta^{-\frac{1}{2}}I_{\frac{1}{2}}(\xi,\eta)=\xi^{\frac{1}{2}}\eta^{\frac{1}{2}}\frac{\ln\eta-\ln\xi}{\eta^2-\xi^2}$. An interesting property of these functions is given by the next lemma.

Lemma 3.4 *Let* $K_s(\xi,\eta)=\dfrac{\xi^s\eta^{1-s}-\xi^{1-s}\eta^s}{\eta^2-\xi^2}$ *if* $s\neq\frac{1}{2}$, *respectively* $K_{\frac{1}{2}}(\xi,\eta)=\xi^{\frac{1}{2}}\eta^{\frac{1}{2}}\dfrac{\ln\eta-\ln\xi}{\eta^2-\xi^2}$. *For any* $s\in(-\frac{1}{2},\frac{3}{2})$ *there exists a constant* $C(s)$ *(depending only on* s*) such that for any* φ, $\psi\in L^2(0,\infty)$ *we have*

$$\left|\int_0^\infty\int_0^\infty \varphi(\xi)K_s(\xi,\eta)\psi(\eta)\,d\xi\,d\eta\right|\le C(s)||\varphi||_{L^2(0,\infty)}||\psi||_{L^2(0,\infty)}.$$

Proof. Using polar coordinates we write $\xi=r\cos(\theta)$, $\eta=r\sin(\theta)$, where $r=\sqrt{\xi^2+\eta^2}$ and $\theta=\arctan\frac{\eta}{\xi}$. It is easy to see that $K_s(\xi,\eta)=\dfrac{1}{r}L_s(\theta)$, where

$L_s(\theta) = \dfrac{(\sin\theta)^s(\cos\theta)^{1-s} - (\cos\theta)^s(\sin\theta)^{1-s}}{\cos^2\theta - \sin^2\theta}$ if $s \neq \frac{1}{2}$ and
$L_{\frac{1}{2}}(\theta) = \dfrac{-\ln\tan\theta}{(1-\tan^2\theta)\cos^2\theta}(\sin\theta)^{\frac{1}{2}}(\cos\theta)^{\frac{1}{2}}$. By a change of variables we get

$$\int_0^\infty\int_0^\infty \Big|\varphi(\xi)K_s(\xi,\eta)\psi(\eta)\Big|\,d\xi\,d\eta = \int_0^{\frac{\pi}{2}}\int_0^\infty \Big|\varphi(r\cos\theta)\psi(r\sin\theta)\Big|\,dr\,|L_s(\theta)|\,d\theta.$$

Using the Cauchy-Schwarz inequality we have

$$\int_0^\infty \Big|\varphi(r\cos\theta)\psi(r\sin\theta)\Big|\,dr \le ||\varphi(\cdot\cos\theta)||_{L^2(0,\infty)}||\psi(\cdot\sin\theta)||_{L^2(0.\infty)} = \frac{||\varphi||_{L^2(0,\infty)}||\psi||_{L^2(0,\infty)}}{\sqrt{\cos\theta\cdot\sin\theta}}.$$

Consequently,

$$\int_0^\infty\int_0^\infty \Big|\varphi(\xi)K_s(\xi,\eta)\psi(\eta)\Big|\,d\xi\,d\eta \le ||\varphi||_{L^2(0,\infty)}||\psi||_{L^2(0,\infty)}\int_0^{\frac{\pi}{2}}\frac{|L_s(\theta)|}{\sqrt{\cos\theta\cdot\sin\theta}}\,d\theta. \tag{3.14}$$

The lemma will be proved if we show that the last integral in (3.14) is finite. If $s \neq \frac{1}{2}$ we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{|L_s(\theta)|}{\sqrt{\cos\theta\cdot\sin\theta}}\,d\theta &= \int_0^{\frac{\pi}{2}}\left|\frac{(\sin\theta)^{s-\frac{1}{2}}(\cos\theta)^{\frac{1}{2}-s} - (\cos\theta)^{s-\frac{1}{2}}(\sin\theta)^{\frac{1}{2}-s}}{\cos^2\theta-\sin^2\theta}\right|d\theta\\ &= \int_0^{\frac{\pi}{2}}\left|\frac{(\tan\theta)^{s-\frac{1}{2}} - (\tan\theta)^{\frac{1}{2}-s}}{1-\tan^2\theta}\right|\cdot\frac{1}{\cos^2\theta}\,d\theta = \int_0^\infty\left|\frac{t^{s-\frac{1}{2}} - t^{\frac{1}{2}-s}}{1-t^2}\right|dt.\end{aligned} \tag{3.15}$$

Using l'Hôspital's rule it is easy to see that $\lim\limits_{t\to1}\frac{t^{s-\frac{1}{2}}-t^{\frac{1}{2}-s}}{1-t^2} = \frac{1}{2}-s$; hence the function $t \longmapsto \frac{t^{s-\frac{1}{2}}-t^{\frac{1}{2}-s}}{1-t^2}$ is bounded near 1. Since $s-\frac{1}{2}\in(-1,1)$, the last integral in (3.15) converges.

If $s = \frac{1}{2}$ we have

$$\int_0^{\frac{\pi}{2}}\frac{|L_{\frac{1}{2}}(\theta)|}{\sqrt{\cos\theta\cdot\sin\theta}}\,d\theta = \int_0^{\frac{\pi}{2}}\left|\frac{-\ln\tan\theta}{1-\tan^2\theta}\right|\cdot\frac{1}{\cos^2\theta}\,d\theta = \int_0^\infty\left|\frac{\ln y}{y^2-1}\right|dy. \tag{3.16}$$

Note that $\lim\limits_{y\to1}\frac{\ln y}{y^2-1} = \frac{1}{2}$ and this implies easily that that the last integral in (3.16) converges. This completes the proof of Lemma 3.4. $\square$

In view of (3.5), (3.7), (3.9), (3.13) and Lemma 3.4, it follows that T_1 and T_2 are well-defined and continuous from $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$, $-\frac{1}{2} < s < \frac{3}{2}$.

Next we estimate the integral in the right-hand side of (3.6). If $s\in[0,\frac{3}{2})$ we have by (3.7), (3.8) and (3.9)

$$\begin{aligned}&\int_1^\infty\left(t^2-1\right)^s\left|\int_0^\infty\widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2dt \le \int_0^\infty t^{2s}\left|\int_0^\infty\widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2dt\\ &\le C(s)||f||^2_{\dot{H}^s} \le C(s)||f||^2_{H^s}.\end{aligned} \tag{3.17}$$

If $s\in(-\frac{1}{2},0)$, using the change of variable $\tau = \sqrt{t^2-1}$ and (3.11) we get

$$\begin{aligned}&\int_1^\infty\frac{(t^2-1)^s}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \int_0^\infty\frac{\tau^{2s}}{(\tau^2+1+\xi^2)(t^2+1+\eta^2)}\cdot\frac{\tau}{\sqrt{\tau^2+1}}\,d\tau\\ &\le \int_0^\infty\frac{\tau^{2s}}{(\tau^2+1+\xi^2)(t^2+1+\eta^2)}\,d\tau = \frac{\pi}{2\cos(s\pi)}\cdot\frac{(1+\xi^2)^{\frac{2s-1}{2}}-(1+\eta^2)^{\frac{2s-1}{2}}}{\eta^2-\xi^2}.\end{aligned} \tag{3.18}$$

Consequently,

$$\int_1^\infty \left(t^2-1\right)^s \left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2 dt$$

$$\leq \int_0^\infty\int_0^\infty |\widehat{f}(\xi)|\cdot|\overline{\widehat{f}(\eta)}|\int_1^\infty (t^2-1)^s\frac{\xi\eta}{(t^2+\xi^2)(t^2+\eta^2)}\,dt\,d\xi\,d\eta$$

$$\leq \frac{\pi}{2\cos(s\pi)}\int_0^\infty\int_0^\infty |\widehat{f}(\xi)|\cdot|\overline{\widehat{f}(\eta)}|\cdot\xi\eta\frac{(1+\xi^2)^{\frac{2s-1}{2}}-(1+\eta^2)^{\frac{2s-1}{2}}}{\eta^2-\xi^2}\,d\xi\,d\eta \tag{3.19}$$

$$= \frac{\pi}{2\cos(s\pi)}\int_0^\infty\int_0^\infty (1+\xi^2)^{\frac{s}{2}}|\widehat{f}(\xi)|\cdot(1+\eta^2)^{\frac{s}{2}}|\overline{\widehat{f}(\eta)}|$$
$$\cdot\frac{\xi\eta}{\eta^2-\xi^2}\cdot\frac{(1+\xi^2)^{\frac{2s-1}{2}}-(1+\eta^2)^{\frac{2s-1}{2}}}{(1+\xi^2)^{\frac{s}{2}}(1+\eta^2)^{\frac{s}{2}}}\,d\xi\,d\eta.$$

We claim that for any ξ, $\eta>0$, $\xi\neq\eta$ we have

$$\frac{\xi\eta}{\eta^2-\xi^2}\cdot\frac{(1+\xi^2)^{\frac{2s-1}{2}}-(1+\eta^2)^{\frac{2s-1}{2}}}{(1+\xi^2)^{\frac{s}{2}}(1+\eta^2)^{\frac{s}{2}}}\leq\frac{\xi^s\eta^{1-s}-\xi^{1-s}\eta^s}{\eta^2-\xi^2}=K_s(\xi,\eta). \tag{3.20}$$

We may suppose without loss of generality that $\eta>\xi$. Then (3.20) is equivalent to

$$(1+\xi^2)^{\frac{s}{2}-\frac{1}{2}}(1+\eta^2)^{-\frac{s}{2}}-(1+\eta^2)^{\frac{s}{2}-\frac{1}{2}}(1+\xi^2)^{-\frac{s}{2}}\leq\xi^{s-1}\eta^{-s}-\eta^{s-1}\xi^{-s}. \tag{3.21}$$

Let $\alpha=\frac{\eta}{\xi}>1$, $\eta_1=\sqrt{1+\eta^2}$, $\xi_1=\sqrt{1+\xi^2}$, $\alpha_1=\frac{\eta_1}{\xi_1}>1$. It is clear that $\alpha>\alpha_1$ (because $\alpha^2-1=\frac{\eta^2-\xi^2}{\xi^2}>\frac{\eta^2-\xi^2}{\xi^2+1}=\alpha_1^2-1$). Inequality (3.21) can be written as

$$\xi_1^{s-1}\eta_1^{-s}-\eta_1^{s-1}\xi_1^{-s}\leq\xi^{s-1}\eta^{-s}-\eta^{s-1}\xi^{-s},$$

or equivalently

$$\frac{1}{\eta_1}(\alpha_1^{1-s}-\alpha_1^s)\leq\frac{1}{\eta}(\alpha^{1-s}-\alpha^s). \tag{3.22}$$

Since $s<0$, the function $x\longmapsto x^{1-s}-x^s$ is increasing on $(0,\infty)$ and then $\alpha^{1-s}-\alpha^s>\alpha_1^{1-s}-\alpha_1^s>0$. It is obvious that $\frac{1}{\eta}>\frac{1}{\eta_1}>0$ and this implies (3.22). This proves our claim.

Coming back to (3.19) and using Lemma 3.4 we obtain

$$\int_1^\infty\left(t^2-1\right)^s\left|\int_0^\infty\widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2dt\leq\frac{\pi C(s)}{2\cos(s\pi)}||(1+|\cdot|^2)^{\frac{s}{2}}\widehat{f}||^2_{L^2(0,\infty)}\leq C'(s)||f||^2_{H^s}. \tag{3.23}$$

From (3.6) and (3.17) if $s\in[0,\frac{3}{2})$, respectively from (3.6) and (3.23) if $s\in(-\frac{1}{2},0)$, we infer that T_1 and T_2 can be extended as linear continuous operators from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$.

Now we prove Theorem 3.3 in the case $N\geq 2$.

If $s\in[0,\frac{3}{2})$, arguing as in (3.7)-(3.9) and using Lemma 3.4 we have

$$\int_{|\xi'|}^\infty\left(t^2-|\xi'|^2\right)^s\left|\int_0^\infty\widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2dt\leq\int_0^\infty t^{2s}\left|\int_0^\infty\widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2dt$$

$$\leq\int_0^\infty\int_0^\infty|\widehat{f}(\xi_1,\xi')|\xi_1^s\cdot|\overline{\widehat{f}(\eta_1,\xi')}|\eta_1^s\cdot\left(\xi_1^{-s}\eta_1^{-s}I_s(\xi_1,\eta_1)\right)\,d\xi_1\,d\eta_1 \tag{3.24}$$

$$\leq C(s)||\,|\cdot|^s\widehat{f}(\cdot,\xi')||^2_{L^2(0,\infty)}\leq C(s)\int_{-\infty}^\infty\left(\xi_1^2+|\xi'|^2\right)^s|\widehat{f}(\xi_1,\xi')|^2\,d\xi_1.$$

If $s \in (-\frac{1}{2}, 0)$, using the change of variable $\tau = \sqrt{t^2 - |\xi'|^2}$, arguing as in the proof of (3.18), then taking (3.11) into account we obtain

$$\int_{|\xi'|}^{\infty} \frac{(t^2 - |\xi'|^2)^s}{(t^2 + \xi^2)(t^2 + \eta^2)} \, dt = \int_0^{\infty} \frac{\tau^{2s}}{(\tau^2 + |\xi'|^2 + \xi_1^2)(\tau^2 + |\xi'|^2 + \eta_1^2)} \cdot \frac{\tau}{\sqrt{\tau^2 + |\xi'|^2}} \, d\tau$$

$$\leq \int_0^{\infty} \frac{\tau^{2s}}{(\tau^2 + |\xi'|^2 + \xi_1^2)(\tau^2 + |\xi'|^2 + \eta_1^2)} \, d\tau = \frac{\pi}{2\cos(s\pi)} \cdot \frac{(|\xi'|^2 + \xi_1^2)^{\frac{2s-1}{2}} - (|\xi'|^2 + \eta_1^2)^{\frac{2s-1}{2}}}{\eta_1^2 - \xi_1^2}.$$

We also have

$$\frac{\xi_1 \eta_1}{\eta_1^2 - \xi_1^2} \cdot \frac{(\xi_1^2 + |\xi'|^2)^{\frac{2s-1}{2}} - (\eta_1^2 + |\xi'|^2)^{\frac{2s-1}{2}}}{(\xi_1^2 + |\xi'|^2)^{\frac{s}{2}} (\eta_1^2 + |\xi'|^2)^{\frac{s}{2}}} \leq K_s(\xi_1, \eta_1)$$

(the proof being the same as the proof of (3.20)). Arguing as in (3.19), using the two previous inequalities and Lemma 3.4 we get

$$\begin{aligned} &\int_{|\xi'|}^{\infty} \left(t^2 - |\xi'|^2\right)^s \left| \int_0^{\infty} \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 \right|^2 dt \\ &\leq \frac{\pi C(s)}{2\cos(s\pi)} ||(|\xi'|^2 + |\cdot|^2)^{\frac{s}{2}} \widehat{f}(\cdot, \xi')||^2_{L^2(0,\infty)} \leq C'(s) \int_{-\infty}^{\infty} \left(\xi_1^2 + |\xi'|^2\right)^s |\widehat{f}(\xi_1, \xi')|^2 \, d\xi_1. \end{aligned} \tag{3.25}$$

Integrating (3.24), respectively (3.25), over $\mathbf{R}^{N-1}$ we infer that the integral in the right-hand side of (3.3) is less than $C''(s)||f||^2_{\dot{H}^s}$. This proves that T_1 and T_2 can be extended by continuity from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$ for $s \in (-\frac{1}{2}, \frac{3}{2})$.

In a similar way we show that T_1 and T_2 can be extended by continuity from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ for $s \in (-\frac{1}{2}, \frac{3}{2})$. Theorem 3.3 is now proved. □

For a measurable function u defined on $\mathbf{R}^N$, we define its antisymmetric part in the x_1 direction by $Au(x_1, x') = \frac{1}{2}(u(x_1, x') - u(-x_1, x'))$. If u is a tempered distribution, we define Au by $\langle Au, \phi \rangle_{\mathcal{S}', \mathcal{S}} = \langle u, A\phi \rangle_{\mathcal{S}', \mathcal{S}}$ for any $\phi \in \mathcal{S}$. Obviously, Au is odd with respect to x_1 (for distributions, this means that $\langle Au, \phi(-x_1, x') \rangle_{\mathcal{S}', \mathcal{S}} = -\langle Au, \phi \rangle_{\mathcal{S}', \mathcal{S}}$). It is clear from the definition that A defines a linear continuous map from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) for any s. Moreover, for any tempered distribution u, the distribution $\mathcal{F}(Au)$ is odd with respect to x_1.

It follows from the proof of Theorem 3.3 that for any $s \in (-\frac{1}{2}, \frac{3}{2})$, the following complex bilinear forms are continuous :

$$B_{1,s} : \dot{H}^s(\mathbf{R}) \times \dot{H}^s(\mathbf{R}) \longrightarrow \mathbf{C},$$
$$B_{1,s}(u, v) = \int_0^{\infty} t^{2s} \int_0^{\infty} \widehat{Au}(\xi) \frac{\xi}{t^2 + \xi^2} \, d\xi \cdot \int_0^{\infty} \overline{\widehat{Av}(\eta)} \frac{\eta}{t^2 + \eta^2} \, d\eta \, dt,$$

$$\tilde{B}_{1,s} : H^s(\mathbf{R}) \times H^s(\mathbf{R}) \longrightarrow \mathbf{C},$$
$$\tilde{B}_{1,s}(u, v) = \int_1^{\infty} (t^2 - 1)^s \int_0^{\infty} \widehat{Au}(\xi) \frac{\xi}{t^2 + \xi^2} \, d\xi \cdot \int_0^{\infty} \overline{\widehat{Av}(\eta)} \frac{\eta}{t^2 + \eta^2} \, d\eta \, dt,$$

$$B_{N,s} : \dot{H}^s(\mathbf{R}^N) \times \dot{H}^s(\mathbf{R}^N) \longrightarrow \mathbf{C},$$
$$B_{N,s}(u, v) = \int_{\mathbf{R}^{N-1}} \int_{|\xi'|}^{\infty} \left(t^2 - |\xi'|^2\right)^s \int_0^{\infty} \widehat{Au}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 \int_0^{\infty} \overline{\widehat{Av}(\eta_1, \xi')} \frac{\eta_1}{t^2 + \eta_1^2} \, d\eta_1 \, dt \, d\xi',$$

$$\tilde{B}_{N,s} : H^s(\mathbf{R}^N) \times H^s(\mathbf{R}^N) \longrightarrow \mathbf{C},$$
$$\tilde{B}_{N,s}(u, v)$$
$$= \int_{\mathbf{R}^{N-1}} \int_{\sqrt{|\xi'|^2+1}}^{\infty} \left(t^2 - |\xi'|^2 - 1\right)^s \int_0^{\infty} \widehat{Au}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 \int_0^{\infty} \overline{\widehat{Av}(\eta_1, \xi')} \frac{\eta_1}{t^2 + \eta_1^2} \, d\eta_1 \, dt \, d\xi'.$$

Moreover, from (3.3) - (3.6) we have the identities

$$(3.26) \qquad ||T_1u||^2_{\dot{H}^s(\mathbf{R}^N)} + ||T_1u||^2_{\dot{H}^s(\mathbf{R}^N)} - 2||u||^2_{\dot{H}^s(\mathbf{R}^N)} = -\frac{16\sin(s\pi)}{\pi^2}B_{N,s}(Au, Au),$$

$$(3.27) \qquad ||T_1u||^2_{H^s(\mathbf{R}^N)} + ||T_1u||^2_{H^s(\mathbf{R}^N)} - 2||u||^2_{H^s(\mathbf{R}^N)} = -\frac{16\sin(s\pi)}{\pi^2}\tilde{B}_{N,s}(Au, Au)$$

for any $u \in C_c^\infty(\mathbf{R}^N)$. From Theorem 3.3, the continuity of $B_{N,s}$ and of $\tilde{B}_{N,s}$ and the density of $C_c^\infty(\mathbf{R}^N)$ in $\dot{H}^s(\mathbf{R}^N)$ and in $H^s(\mathbf{R}^N)$ we infer that we have the following :

Corollary 3.5 *Let $s \in (-\frac{1}{2}, \frac{3}{2})$. The identity (3.26) holds for any $u \in \dot{H}^s(\mathbf{R}^N)$ and (3.27) holds for any $u \in H^s(\mathbf{R}^N)$.*

Our next aim is to show that the quadratic forms $B_{N,s}$ and $\tilde{B}_{N,s}$ define norms in some spaces of odd functions. We start with the following proposition :

Lemma 3.6 *Assume that $g : \mathbf{R} \longrightarrow \mathbf{R}$ is measurable, odd and*
- *either $g \in L^p(\mathbf{R})$ for some $p \in (1, \infty)$,*
- *or $(\alpha^2 + \xi^2)^{\frac{s}{2}} g(\xi) \in L^2(\mathbf{R})$ (respectively $|\xi|^s g(\xi) \in L^2(\mathbf{R})$) for some $s \in (-\frac{1}{2}, \frac{3}{2})$.*

Suppose that the set $A = \{x > 0 \mid \int_0^\infty \frac{\xi}{x^2 + \xi^2} g(\xi)\, d\xi = 0\}$ has a limit point $x_0 > 0$.

Then $g = 0$ almost everywhere on $\mathbf{R}$.

In particular, if $\int_0^\infty \frac{\xi}{x^2 + \xi^2} g(\xi)\, d\xi = 0$ for almost every x in some open interval, then $g \equiv 0$.

Proof. We may suppose without loss of generality that g is real (otherwise we carry out the proof for its real and imaginary parts).

First we deal with the much simpler case $g \in L^p(\mathbf{R})$ for some p, $1 < p < \infty$. We define the Poisson integrals for g,

$$(3.28) \qquad a(x, y) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{x}{x^2 + (y-t)^2} g(t)\, dt \quad \text{and} \quad b(x, y) = -\frac{1}{\pi}\int_{-\infty}^\infty \frac{y-t}{x^2 + (y-t)^2} g(t)\, dt.$$

It follows from Lemma 2.4 iii) that the functions a and b are well-defined and harmonic in the right half-plane and $r(x + iy) := a(x, y) + ib(x, y)$ is holomorphic in $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$. Since g is odd, we have $a(x, 0) = 0$ for any $x > 0$. If $x \in A$, we have also $b(x, 0) = 0$. Consequently, $r(x) = 0$ for any $x \in A$. But r is holomorphic and A has a limit point $x_0 > 0$, thus necessarily $r \equiv 0$. By Lemma 2.4 ii) we know that $a(x, y) \longrightarrow g(y)$ as $x \longrightarrow 0$ for almost every y, hence $g = 0$ a.e. on $\mathbf{R}$.

Suppose that $(\alpha^2 + |\cdot|^2)^{\frac{s}{2}} g \in L^2(\mathbf{R})$ for some $s \in (-\frac{1}{2}, \frac{3}{2})$. We may assume that $\alpha = 1$. If $s \in [0, \frac{3}{2})$, then obviously $g \in L^2(\mathbf{R})$ and the conclusion of the lemma follows from the above considerations. If $s \in (-\frac{1}{2}, 0)$, then for any $x > 0$ and $y \in \mathbf{R}$ the functions $\varphi_{x,y}(t) = (1 + t^2)^{-\frac{s}{2}} \frac{x}{x^2+(y-t)^2}$ and $\psi_{x,y}(t) = (1 + t^2)^{-\frac{s}{2}} \frac{y-t}{x^2+(y-t)^2}$ belong to $L^2(\mathbf{R})$. We may write $\int_{-\infty}^\infty \frac{x}{x^2 + (y-t)^2} g(t)\; dt = \int_{-\infty}^\infty \varphi_{x,y}(t)(1 + t^2)^{\frac{s}{2}} g(t)\, dt$ and $\int_{-\infty}^\infty \frac{y-t}{x^2 + (y-t)^2} g(t)\; dt = \int_{-\infty}^\infty \psi_{x,y}(t)(1 + t^2)^{\frac{s}{2}} g(t)\, dt$. Using the Cauchy-Schwarz inequality, we see that the functions a and b in (3.28) are well-defined in the right half-plane (in particular, $\int_0^\infty \frac{\xi}{x^2 + \xi^2} g(\xi)\, d\xi$ exists for any $x > 0$). Clearly the function $r(x + iy) := a(x, y) + ib(x, y)$ is holomorphic and, as above we have $r(x) = 0$ for $x \in A$. Since A has a limit point $x_0 > 0$, we infer that $r \equiv 0$. Next, we claim that $\lim_{x\downarrow 0} a(x,y) = g(y)$ whenever y is a Lebesgue point of g. This obviously implies $g = 0$ a.e., as desired. Let y be a Lebesgue point of g and fix $\varepsilon > 0$. Then there exists $\delta = \delta(\varepsilon) > 0$ such that $\frac{1}{r}\int_{-r}^{r} |g(y-\tau) - g(y)|\,d\tau < \varepsilon$ for any $r \in (0,\delta]$. We have :

$$
\begin{aligned}
|a(x,y) - g(y)| &= \frac{1}{\pi}\left|\int_{-\infty}^{\infty} \frac{x}{x^2+t^2}(g(y-t) - g(y))\,dt\right| \\
&\le \frac{1}{\pi}\int_{-\delta}^{\delta} \frac{x}{x^2+t^2}|g(y-t)-g(y)|\,dt + \frac{1}{\pi}\int_{|t|>\delta} \frac{x}{x^2+t^2}|g(y-t)|\,dt \qquad (3.29)\\
&\quad + \frac{1}{\pi}\int_{|t|>\delta} \frac{x}{x^2+t^2}|g(y)|\,dt = I_1 + I_2 + I_3.
\end{aligned}
$$

Let $G(r) = \int_{-r}^{r} |g(y-\tau) - g(y)|\,d\tau$. It is obvious that G is nondecreasing on $[0,\infty)$ and we have $G'(r) = |g(y-r)-g(y)| + |g(y+r)-g(y)|$ almost everywhere. Using integration by parts and the fact that $0 \le G(t) \le \varepsilon t$ for any $t \in [0,\delta]$, we get :

$$
\begin{aligned}
I_1 &= \frac{1}{\pi}\int_0^{\delta} \frac{x}{x^2+t^2}|g(y-r)-g(y)| + |g(y+r)-g(y)|\,dt = \frac{1}{\pi}\int_0^{\delta}\frac{x}{x^2+t^2}G'(t)\,dt \\
&= \frac{1}{\pi}\frac{x}{x^2+\delta^2}G(\delta) + \frac{2x}{\pi}\int_0^{\delta}\frac{t}{(x^2+t^2)^2}G(t)\,dt \le \frac{G(\delta)}{2\pi\delta} + \frac{2x}{\pi}\int_0^{\delta}\frac{\varepsilon t^2}{(x^2+t^2)^2}\,dt \qquad (3.30)\\
&\le \frac{\varepsilon}{2\pi} + \frac{2x\varepsilon}{\pi}\int_0^{\delta}\frac{1}{x^2+t^2}\,dt \le \frac{\varepsilon}{2\pi} + \frac{2\varepsilon}{\pi}\arctan\frac{\delta}{x} \le \frac{\varepsilon}{2\pi} + \varepsilon.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality we have :

$$
\begin{aligned}
I_2 &= \frac{1}{\pi}\int_{|t|>\delta}\frac{x}{x^2+t^2}(1+|y-t|^2)^{-\frac{s}{2}}(1+|y-t|^2)^{\frac{s}{2}}|g(y-t)|\,dt \qquad (3.31)\\
&\le \frac{x}{\pi}||(1+|\cdot|^2)^{\frac{s}{2}}g||_{L^2(\mathbf{R})}\left(\int_{|t|>\delta}\frac{(1+|y-t|^2)^{-s}}{t^4}\,dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $s > -\frac{1}{2}$, the last integral in (3.31) converges. Let $K(y,\delta)$ be its value. We have proved that

$$
I_2 \le \frac{x}{\pi}K(y,\delta)||(1+|\cdot|^2)^{\frac{s}{2}}g||_{L^2(\mathbf{R})} \qquad \text{for any } x > 0. \qquad (3.32)
$$

Finally, the integral I_3 is easy to compute :

$$
I_3 = \frac{|g(y)|}{\pi}(\pi - 2\arctan\frac{\delta}{x}). \qquad (3.33)
$$

For x sufficiently small, the right-hand side terms in (3.32) and (3.33) are less than ε. From (3.29), (3.30), (3.32) and (3.33) we infer that $|a(x,y) - g(y)| \le 4\varepsilon$ if x is sufficiently small. Consequently $a(x,y) \longrightarrow g(y)$ as $y \longrightarrow 0$ and the claim is proved.

In the case $|\cdot|^s g \in L^2(\mathbf{R})$ and $s \in (-\frac{1}{2},\frac{1}{2})$, we may repeat almost word by word the proof above (we have only to replace the functions $\varphi_{x,y}$ and $\psi_{x,y}$ by $t \longmapsto t^{-s}\frac{x}{x^2+(y-t)^2}$, respectively by $t \longmapsto t^{-s}\frac{y-t}{x^2+(y-t)^2}$).

If $|\cdot|^s g \in L^2(\mathbf{R})$ and $s \in [\frac{1}{2}, \frac{3}{2})$, the integrals defining a and b in (3.28) do not necessarily converge. In this case we define

$$
\begin{aligned}
&a_1(x,y) = \frac{1}{\pi}\int_0^\infty \frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]} g(t)\,dt \quad \text{and} \\
&b_1(x,y) = \frac{1}{\pi}\int_0^\infty \frac{2t(t^2+x^2-y^2)}{[x^2+(y-t)^2][x^2+(y+t)^2]} g(t)\,dt.
\end{aligned} \tag{3.34}
$$

Notice that if $g \in L^1_{loc}(\mathbf{R})$ is odd and $\frac{g(t)}{t} \in L^1([1,\infty))$, then $a = a_1$ and $b = b_1$. It is obvious that for fixed $x > 0$, $y \in \mathbf{R}$ and $s \in (-\frac{1}{2}, \frac{3}{2})$, the functions $\varphi_1(t) = t^{-s}\frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]}$ and $\psi_1(t) = t^{-s}\frac{2t(t^2+x^2-y^2)}{[x^2+(y-t)^2][x^2+(y+t)^2]}$ belong to $L^2((0,\infty))$ and this implies that a_1 and b_1 are well-defined. It is straightforward that $r_1(x+iy) := a_1(x,y) + b_1(x,y)$ is holomorphic in the right half-plane. Obviously $a_1(x,0) = 0$ for any $x > 0$ and $b_1(x,0) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{t}{x^2+t^2} g(t)\,dt = 0$ for $x \in A$. Consequently $r = 0$ on A. Since A has a limit point $x_0 > 0$, we infer that $r \equiv 0$ in the right half-plane. The lemma will be proved if we show that $a_1(x,y) \longrightarrow g(y)$ as $x \longrightarrow 0$ for almost every y.

Let $y > 0$ be a Lebesgue point of g. Note that $\displaystyle\int_0^\infty \frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]}\,dt = 2\arctan\frac{y}{x}$. Proceeding as in (3.29)-(3.33), we may show that $|a_1(x,y) - \frac{2}{\pi}(\arctan\frac{y}{x})g(y)| \longrightarrow 0$ as $x \longrightarrow 0$, hence $\lim\limits_{x\downarrow 0} a_1(x,y) = g(y)$ and the lemma is proved. $\square$

We set

$$
\begin{aligned}
H^s_{1,odd}(\mathbf{R}^N) &= \{f \in H^s(\mathbf{R}^N) \mid f \text{ is odd with respect to } x_1\} = \{f \in H^s(\mathbf{R}^N) \mid f = Af\},\\
\dot{H}^s_{1,odd}(\mathbf{R}^N) &= \{f \in \dot{H}^s(\mathbf{R}^N) \mid f \text{ is odd with respect to } x_1\} = \{f \in \dot{H}^s(\mathbf{R}^N) \mid f = Af\},
\end{aligned}
$$

where, as before, Af is the antisymmetric part of f in the x_1 direction. For $f \in \dot{H}^s_{1,odd}(\mathbf{R}^N)$ we define $N_s(f) = (B_{N,s}(f,f))^{\frac{1}{2}}$ and for $f \in H^s_{1,odd}(\mathbf{R}^N)$ we define $\tilde{N}_s(f) = (\tilde{B}_{N,s}(f,f))^{\frac{1}{2}}$.

Theorem 3.7 *$\tilde{N}_s$ is a norm on $H^s_{1,odd}(\mathbf{R}^N)$, continuous with respect to the usual H^s norm, and N_s is a norm on $\dot{H}^s_{1,odd}(\mathbf{R}^N)$, continuous with respect to the $\dot{H}^s$ norm.*

Endowed with these norms, $H^s_{1,odd}(\mathbf{R}^N)$ and $\dot{H}^s_{1,odd}(\mathbf{R}^N)$ are pre-Hilbert spaces.

Proof. It is clear that $\tilde{B}_{N,s}$ and $B_{N,s}$ are complex-symmetric bilinear forms on $H^s(\mathbf{R}^N)$ (respectively on $\dot{H}^s(\mathbf{R}^N)$) and that $\tilde{B}_{N,s}(f,f) \geq 0$ and $B_{N,s}(f,f) \geq 0$ for any f (thus, in particular, $\tilde{N}_s$ and N_s are well-defined). Suppose, for instance, that $f \in H^s_{1,odd}(\mathbf{R}^N)$ and $\tilde{B}_{N,s}(f,f) = 0$. This implies that for almost every $\xi' \in \mathbf{R}^{N-1}$ we have : $\widehat{f}(\cdot,\xi')$ is odd, $(|\cdot|^2 + |\xi'|^2)^{\frac{s}{2}}\widehat{f}(\cdot,\xi') \in L^2(\mathbf{R})$ and $\displaystyle\int_{\sqrt{|\xi'|^2+1}}^\infty \left(t^2 - |\xi'|^2 - 1\right)^s \left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+|\xi'|^2}\,d\xi_1\right| dt = 0$. For such ξ' we must have $\displaystyle\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+|\xi'|^2}\,d\xi_1 = 0$ for almost every $t \in (\sqrt{|\xi'|^2+1},\infty)$ and using Lemma 3.6 we infer that $\widehat{f}(\cdot,\xi') = 0$ a.e. on $\mathbf{R}$, hence $\displaystyle\int_{\mathbf{R}} \left(\xi_1^2 + |\xi'|^2\right)^s |\widehat{f}(\xi_1,\xi')|^2\,d\xi_1 = 0$. Consequently $||f||^2_{H^s} = \displaystyle\int_{\mathbf{R}^{N-1}}\int_{\mathbf{R}} \left(\xi_1^2 + |\xi'|^2\right)^s |\widehat{f}(\xi_1,\xi')|^2\,d\xi_1\,d\xi' = 0$, i.e. $f = 0$ a.e. The proof is the same for $f \in \dot{H}^s(\mathbf{R}^N)$. Finally, the continuity of $\tilde{N}_s$ and N_s with respect to the usual norms follows from Theorem 3.3 and Corollary 3.5. $\square$

4 Applications

In this section we illustrate how the results in Sections 2 and 3 can be used to prove the symmetry of minimizers in some concrete examples.

4.1 We start with two scalar variational problems.

Theorem 4.1 *Let $s \in (0,1)$ and assume that $F, G : \mathbf{R} \to \mathbf{R}$ are such that $u \to F(u)$ and $u \to G(u)$ map $\dot{H}^s(\mathbf{R}^N)$ (or $H^s(\mathbf{R}^N)$) into $L^1(\mathbf{R}^N)$. Suppose that either*

Case A. $u \in \dot{H}^s(\mathbf{R}^N)$ and u is a solution of the minimization problem

$$\begin{aligned} &\textit{minimize} \quad E(u) := \int_{\mathbf{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbf{R}^N} F(u(x))\, dx \\ &\textit{under the constraint } I(u) = \int_{\mathbf{R}^N} G(u(x))\, dx = \lambda, \qquad \textit{or} \end{aligned}$$

Case B. $u \in H^s(\mathbf{R}^N)$ and u is a solution of the minimization problem

$$\begin{aligned} &\textit{minimize} \quad E(u) := \int_{\mathbf{R}^N} \left(1 + |\xi|^2\right)^s |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbf{R}^N} F(u(x))\, dx \\ &\textit{under the constraint } I(u) = \int_{\mathbf{R}^N} G(u(x))\, dx = \lambda. \end{aligned}$$

Then, after a translation in $\mathbf{R}^N$, u is radially symmetric.

Proof. Let us prove first that u is symmetric with respect to x_1. Making a translation in the x_1 direction if necessary, we may assume that $\int_{\{x_1<0\}} G(u(x))\, dx = \int_{\{x_1>0\}} G(u(x))\, dx = \frac{\lambda}{2}$. Let $u_1 = T_1 u$ and $u_2 = T_2 u$. It follows from Theorem 3.3 that $u_1,\, u_2 \in \dot{H}^s(\mathbf{R}^N)$ in case A, respectively $u_1,\, u_2 \in H^s(\mathbf{R}^N)$ in case B. It is obvious that we have $\int_{\mathbf{R}^N} G(u_1(x))\, dx = 2\int_{\{x_1<0\}} G(u(x))\, dx = \lambda$ and $\int_{\mathbf{R}^N} G(u_2(x))\, dx = 2\int_{\{x_1>0\}} G(u(x))\, dx = \lambda$; hence u_1 and u_2 also satisfy the constraint. From (3.26) and (3.27) we have

$$E(u_1) + E(u_2) - 2E(u) = -\tfrac{16\sin(s\pi)}{\pi^2} N_s^2(Au) \qquad \text{in case A, respectively}$$

$$E(u_1) + E(u_2) - 2E(u) = -\tfrac{16\sin(s\pi)}{\pi^2} \tilde{N}_s^2(Au) \qquad \text{in case B,}$$

where, as before, $Au(x_1, x') = \frac{1}{2}(u(x_1,x') - u(-x_1,x'))$ is the antisymmetric part of u in the x_1 direction. If $Au \not\equiv 0$, then Theorem 3.7 implies $N_s^2(Au) > 0$ (respectively $\tilde{N}_s^2(Au) > 0$) and we infer that $E(u_1) + E(u_2) - 2E(u) < 0$, contradicting the fact that u is a minimizer. Thus necessarily $Au \equiv 0$ and this means that u is symmetric with respect to x_1.

Arguing similarly with the remaining variables $x_2, \dots, x_N$, we find a new origin O' such that u is symmetric with respect to any of the variables $x_1, \dots, x_N$; in particular, $u(-x) = u(x)$ a.e. on $\mathbf{R}^N$. Now let Π be any hyperplane containing the new origin O' and let Π_+ and Π_- be the halfspaces determined by Π. Since the transformation $x \longmapsto -x$ maps Π_- into Π_+, we see that $\int_{\Pi_-} G(u(x))\, dx = \int_{\Pi_+} G(u(x))\, dx = \frac{\lambda}{2}$. Arguing as above we conclude that u must be symmetric with respect to Π. This implies that u is radially symmetric with respect to the new origin O'. □

An application of Theorem 4.1 concerns the solitary waves to the generalized Benjamin-Ono equation

$$A_t + \alpha A A_x - \beta(-\Delta)^{\frac{1}{2}} A_x = 0, \qquad (x,y) \in \mathbf{R}^2,\ t \in \mathbf{R},$$

where α, $\beta > 0$. Solitary waves are solutions of the form $A(t,x,y) = u(x-ct,y)$. After a scale change, a solitary wave $u(x,y)$ satisfies the equation

$$u + (-\Delta)^{\frac{1}{2}} u = u^2 \qquad \text{in } \mathbf{R}^2.$$

The existence of solitary waves was proved in [21] by minimizing the functional

$$V(u) = \frac{1}{2}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u|^2\,dx + \int_{\mathbf{R}^2} u^2\,dx = \frac{1}{2(2\pi)^2}\int_{\mathbf{R}^2} |\xi||\widehat{u}(\xi)|^2\,d\xi + \int_{\mathbf{R}^2} u^2\,dx$$

under the constraint $I(u) = \dfrac{1}{3}\displaystyle\int_{\mathbf{R}^2} u^3\,dx = constant$. It has been shown in [21] that any solution u_* of the above problem also minimizes

$$E(v) := \frac{1}{2}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2\,dx - \frac{1}{3}\int_{\mathbf{R}^2} v^3\,dx$$

under the constraint $Q(v) = Q(u_*)$, where $Q(v) = \dfrac{1}{2}\displaystyle\int_{\mathbf{R}^2} |u|^2\,dx$.

It follows directly from Theorem 4.1 that, except for translation, any minimizer of these problems is radially symmetric.

4.2 Next we apply our method to a variational problem involving two unknown functions (the vector case). Consider the functionals

$$E(u,v) = \frac{1}{2}\int_{\mathbf{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |\nabla v|^2)\,dx + \int_{\mathbf{R}^N} F(u,v)\,dx$$

where $0 < s < 1$, and

$$Q(u,v) = \int_{\mathbf{R}^N} G(u,v)\,dx.$$

We make the following assumptions:

A1 : F, $G : \mathbf{R}^2 \longrightarrow \mathbf{R}$ are C^2 functions satisfying $F(0,0) = \partial_1 F(0,0) = \partial_2 F(0,0) = 0$, $G(0,0) = \partial_1 G(0,0) = \partial_2 G(0,0) = 0$ and the growth conditions

$$|\partial_i F(u,v)| \le C(|u|^{p-1} + |v|^{q-1}) \quad \text{and} \quad |\partial_i G(u,v)| \le C(|u|^{p-1} + |v|^{q-1}) \quad \text{if } |(u,v)| \ge 1,$$

where $i \in \{1, 2\}$, C is a positive constant, $2 < p < \frac{2N}{N-2s}$ and $2 < q < \frac{2N}{N-2}$.

A2 : If $(u,v) \in H^s(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ and $(u,v) \not\equiv (0,0)$, then either $\partial_1 G(u,v) \not\equiv 0$ or $\partial_2 G(u,v) \not\equiv 0$ (a manifold condition).

Theorem 4.2 *Under assumptions* **A1** *and* **A2**, *any minimizer* $(u,v) \in H^s(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ *of* $E(u,v)$ *subject to the constraint* $Q(u,v) = \lambda$ *is radially symmetric (except for translation).*

Proof. First we prove that after a translation, (u,v) is symmetric with respect to x_1. In fact, after possibly a translation in the x_1 direction we may assume that

$$\int_{\{x_1<0\}} G(u,v)\,dx = \int_{\{x_1>0\}} G(u,v)\,dx = \frac{\lambda}{2}. \tag{4.1}$$

We put $u_1 = T_1 u$, $u_2 = T_2 u$, $v_1 = T_1 v$ and $v_2 = T_2 v$. By Theorem 3.3, the pairs (u_1, v_1) and (u_2, v_2) belong to $H^s(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ and in view of (4.1) they also satisfy the constraint

$Q(u_1, v_1) = Q(u_2, v_2) = \lambda$. Moreover, defining $W(\varphi) = \int_{\mathbf{R}^N} |\xi|^{2s} |\widehat{\varphi}(\xi)|^2 \, d\xi$ and using (3.26) we see that

$$E(u_1, v_1) + E(u_2, v_2) - 2E(u, v) = \frac{1}{2} \frac{1}{(2\pi)^N} \left(W(u_1) + W(u_2) - 2W(u) \right)$$

$$= -\frac{1}{(2\pi)^N} \frac{8 \sin(s\pi)}{\pi^2} B_{N,s}(Au, Au) \leq 0.$$

We conclude that (u_1, v_1) and (u_2, v_2) are also minimizers and we must have $B_{N,s}(Au, Au) = 0$. By Theorem 3.7 we infer that $Au = 0$, that is u is symmetric with respect to x_1, i.e. $u = u_1 = u_2$.

Since (u, v) and $(u_1, v_1) = (u, v_1)$ are minimizers, they satisfy the Euler-Lagrange equations

$$\left\{ \begin{array}{r} (-\Delta)^s u + \partial_1 F(u, v) + \alpha \partial_1 G(u, v) = 0, \\ -\Delta v + \partial_2 F(u, v) + \alpha \partial_2 G(u, v) = 0, \end{array} \right. \tag{4.2}$$

respectively

$$\left\{ \begin{array}{r} (-\Delta)^s u + \partial_1 F(u, v_1) + \beta \partial_1 G(u, v_1) = 0, \\ -\Delta v_1 + \partial_2 F(u, v_1) + \beta \partial_2 G(u, v_1) = 0. \end{array} \right. \tag{4.3}$$

From (4.2), **A1**, the elliptic regularity for the Laplacian and its fractional powers and the usual boot-strap argument we get $u \in H^{2s}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $v \in H^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Of course that the same conclusion holds for (u, v_1). Notice that the L^p elliptic regularity for fractional powers of the Laplacian and for $1 < p < \infty$ follows from the fact that the multiplier $m(\xi) = \dfrac{(1 + |\xi|^2)^s}{1 + |\xi|^{2s}}$ satisfies the estimate $|D^\alpha m(\xi)| \leq \dfrac{B(\alpha)}{|\xi|^\alpha}$ and from the theorem of Mihlin-Hörmander.

We recall the following well-known result :

Unique Continuation Principle: *Assume that $\Phi \in H^2(\mathbf{R}^N, \mathbf{R}^m)$ solves the linear system*

$$-\Delta \Phi + A(x)\Phi(x) = 0 \qquad \text{in } \mathbf{R}^N, \tag{4.4}$$

where $A(x)$ is an $m \times m$ matrix whose elements belong to $L^\infty(\mathbf{R}^N)$. If $\Phi \equiv 0$ in some open set $\omega \subset \mathbf{R}^N$, then $\Phi \equiv 0$ in $\mathbf{R}^N$.

A proof for the Unique Continuation Principle is given in [13], Chapter VIII in the scalar case and in the appendix of [18] in the vector case. Notice that the Unique Continuation Principle is essentially a local result. Although it is stated for functions $\Phi \in H^2(\mathbf{R}^N)$, it is also valid for functions $\Phi \in W^{2,p}(\mathbf{R}^N)$ with $p > 2$ because $W^{2,p}_{loc}(\mathbf{R}^N) \subset H^2_{loc}(\mathbf{R}^N)$. This observation will be useful later.

Now let us come back to the proof of Theorem 4.2.

If $(u_1, v_1) = (0, 0)$, then obviously $u = 0$ in $\mathbf{R}^N$. By assumption **A2** and the regularity of v we have $\partial_2 F(0, v) = a_1(x) v$ and $\partial_2 G(0, v) = b_1(x) v$, where a_1, $b_1 \in L^\infty(\mathbf{R}^N)$. Using the second equation (4.2), the fact that $v = v_1$ in the half-space $\{x_1 < 0\}$ and the Unique Continuation Principle, we infer that $v = 0$ in $\mathbf{R}^N$, thus (u, v) is radially symmetric in a trivial way. It is obvious that this situation cannot occur if $\lambda \neq 0$.

If $(u_1, v_1) \neq (0, 0)$, it follows from **A2** that there exists $(x_1, x') \in (-\infty, 0) \times \mathbf{R}^{N-1}$ such that $\partial_1 G(u_1, v_1)(x_1, x') \neq 0$ or $\partial_2 G(u_1, v_1)(x_1, x') \neq 0$. Since $v = v_1$ for $x_1 < 0$, we infer from (4.2) and (4.3) that $\alpha = \beta$. Moreover, using the regularity of u, v, v_1 we get $\partial_2 F(u,v) - \partial_2 F(u,v_1) = b(x)(v(x) - v_1(x))$ and $\partial_2 G(u,v) - \partial_2 G(u,v_1) = c(x)(v(x) - v_1(x))$ where b, $c \in L^\infty(\mathbf{R}^N)$. Let $w(x) = v(x) - v_1(x)$. Using the second components of (4.2) and (4.3) and the fact that $\alpha = \beta$, we see that w satisfies the linear equation $-\Delta w(x) + a(x)w(x) = 0$ in $\mathbf{R}^N$, where $a = b + \alpha c \in L^\infty(\mathbf{R}^N)$. Since w vanishes on a half-space, by the Unique Continuation Principle we conclude that w vanishes everywhere, and this implies $v = v_1$ in $\mathbf{R}^N$. Thus we have shown that (u,v) is symmetric with respect to x_1.

Repeating this argument with the variables $x_2, \ldots, x_N$, we find a new origin O' such that (u,v) is symmetric with respect to $x_1, \ldots, x_N$. Then as in the proof of Theorem 4.1 we show that (u,v) is symmetric with respect to any hyperplane Π containing O', consequently (u,v) is radially symmetric with respect to the new origin O'. □

Remark 4.3 Symmetrization inequalities for functions in the space $H^{1/2}(\mathbf{R}^N)$ have been proved in [3]. Therefore if $s = \frac{1}{2}$, the function F in Theorem 4.2 satisfies the cooperative condition $\partial^2_{1,2} F(u,v) \leq 0$ (see [5]), G has a special form and it is known in advance that the components u, v of the minimizer are nonnegative, then using symmetrization one can conclude that *there exists* a radially symmetric minimizer.

Remark 4.4 In the case $F(u,v) = u^2 + v^2$, $G(u,v) = u^2 v$, by using symmetrization and Riesz' inequality it has been proved in [3] that *there exists* a radially symmetric minimizer. The fact that F and G are homogeneous plays a crucial role in their proof.

As an example of application for Theorem 4.2, we consider the Hamiltonian system :

$$
\left\{
\begin{array}{rcl}
\dfrac{\partial u}{\partial t} & = & \dfrac{\partial}{\partial x_1}((-\Delta)^{1/2} u + \partial_1 F(u,v)) \\[2ex]
\dfrac{\partial v}{\partial t} & = & \dfrac{\partial}{\partial x_1}(-\Delta v + \partial_2 F(u,v)).
\end{array}
\right.
\tag{4.5}
$$

The generalized multidimensional Benjamin-Ono equation

$$
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}((-\Delta)^{1/2} u + g(u))
\tag{4.6}
$$

with $g(u) = u^2$ and the generalized multidimensional Korteweg-deVries equation

$$
\frac{\partial v}{\partial t} = \frac{\partial}{\partial x_1}(-\Delta v + f(v))
\tag{4.7}
$$

have been considered in [21] and in [4], respectively; in these papers, references giving the physical motivation for the above equations can also be found. System (4.5) can be considered a Hamiltonian coupling between (4.6) and (4.7).

Formally, system (4.5) has the following conserved quantities:

$$
E(u,v) = \frac{1}{2}\int_{R^N} |(-\Delta)^{1/4} u|^2 + |\nabla v|^2 \, dx + \int_{R^N} F(u,v)\, dx \quad \text{and} \quad Q(u,v) = \frac{1}{2}\int_{R^N} (u^2 + v^2)\, dx.
$$

If we minimize $E(u,v)$ subject to the constraint $Q(u,v) = \lambda$, where $\lambda > 0$, then according to [9] the set S_λ containing the elements of $H^{\frac{1}{2}}(R^N) \times H^1(R^N)$ where the minimum is achieved is invariant and orbitally stable with respect to (4.5). Since any element $(\phi, \psi) \in S_\lambda$ satisfies the Euler-Lagrange system

$$
\left\{
\begin{array}{rcl}
(-\Delta)^{1/2}\phi + \partial_1 F(\phi,\psi) + c\phi & = & 0, \\
-\Delta\psi + \partial_2 F(\phi,\psi) + c\psi & = & 0,
\end{array}
\right.
$$

we see that (ϕ, ψ) gives rise to a travelling wave solution of (4.5) of the form $(u(t,x), v(t,x)) = (\phi(x_1 - ct, x'), \psi(x_1 - ct, x'))$, $x' \in \mathbf{R}^{N-1}$. As a consequence of Theorem 4.2, the elements (ϕ, ψ) obtained in this way are radially symmetric (after a translation).

4.3 Next we consider the problem of minimizing the generalized Choquard functional

$$(4.8) \quad E(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} F(u(x))\frac{1}{|x-y|^{N-2}}F(u(y))\,dx\,dy + \int_{\mathbf{R}^N} H(u(x))\,dx$$

subject to the constraint $Q(u) = \displaystyle\int_{\mathbf{R}^N} G(u(x))\,dx = constant.$

It is worth to note that the complex version of E,

$$\tilde{E}(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \int_{R^N}\int_{R^N} F_1(|u(x)|^2)\frac{1}{|x-y|^{N-2}}F_1(|u(y)|^2)\,dx\,dy + \int_{R^N} H_1(|u(x)|^2)\,dx$$

is the Hamiltonian for the generalized Hartree equation

$$(4.9) \qquad iu_t + \Delta u + 4\left(\int_{\mathbf{R}^N}\frac{F_1(|u(y)|^2)}{|x-y|^{N-2}}\,dy\right)F_1'(|u|^2)(x)u(x) - 2H_1'(|u(x)|^2)u(x) = 0,$$

and $\tilde{Q}(u) = \displaystyle\int_{\mathbf{R}^N} |u^2(x)|\,dx$ is a conserved quantity for this evolution equation. The critical points of $\tilde{E} + \omega\tilde{Q}$ give rise to standing waves for (4.9). As far as minimization is concerned, using an argument of T. Cazenave and P.-L. Lions (see the proof of Theorem II.1 p. 555 in [9]), we can restrict ourselves to the real functionals $E(u)$ and $Q(u)$.

In the case $N = 3$, $F(u) = G(u) = u^2$ and $H(u) = 0$, the problem of minimizing $E(u)$ subject to $Q(u) = \lambda$ has been studied in [15], where the existence, the radial symmetry and the uniqueness of the minimizer have been proved. The symmetry was proved by using a sharp inequality for spherical rearrangements. This can still be used in our case if we konw that the minimizer is nonnegative and if we assume assume that F is increasing on $[0, \infty)$ (because the equality $F(u^*) = (F(u))^*$ is needed). Using the results in sections 2 and 3, we will show the radial symmetry of minimizers in dimension $N \geq 3$ under more general assumptions on F, G and H.

We begin by studying some properties of the nonlocal term appearing in (4.8) :

Lemma 4.5 *Let $N \geq 3$ and let $F : \mathbf{R} \longrightarrow \mathbf{R}$ be a function of class C^2 satisfying $F(0) = F'(0) = 0$ and*

$$|F'(x)| \leq C|x|^\sigma \qquad \textit{for } |x| \geq 1,$$

where $C > 0$ is a constant and $\sigma < \dfrac{4}{N-2}$. Then the singular integral operator

$$I(\varphi)(x) = \int_{\mathbf{R}^N} \frac{1}{|x-y|^{N-2}}\varphi(y)\,dy$$

and the functional

$$M(\varphi) = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} F(\varphi(x))\frac{1}{|x-y|^{N-2}}F(\varphi(y))\,dx\,dy$$

have the following properties :

i) I is continuous from $L^p(\mathbf{R}^N)$ to $L^q(\mathbf{R}^N)$ if $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{2}{N}$.

ii) If $1 \leq p_1 < \dfrac{N}{2} < p_2 \leq \infty$, then I is continuous from $L^{p_1}(\mathbf{R}^N) \cap L^{p_2}(\mathbf{R}^N)$ to $L^\infty(\mathbf{R}^N) \cap C^0(\mathbf{R}^N)$.

iii) If $1 \le r_1 < \dfrac{2N}{N+2} < r_2 \le 2$ *and* $\varphi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$*, then*

$$\widehat{I(\varphi)}(\xi) = \frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)} \cdot \frac{1}{|\xi|^2}\widehat{\varphi}(\xi) \qquad \text{in } \mathcal{S}'(\mathbf{R}^N).$$

iv) M *is well-defined and differentiable on* $H^1(\mathbf{R}^N)$ *and*

$$M'(u).\varphi = 2\int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} \frac{F(u(y))}{|x-y|^{N-2}}\,dy \right) F'(u(x))\varphi(x)\,dx.$$

v) For any $u \in H^1(\mathbf{R}^N)$ *we have*

$$M(u) = c_N \int_{\mathbf{R}^N} \frac{1}{|\xi|^2} |\widehat{F(u)}(\xi)|^2\,d\xi, \qquad \text{where } c_N = \frac{1}{2^{N-2}\pi^{\frac{N}{2}}\Gamma(\frac{N}{2}-1)}.$$

Proof. *i)* follows directly from Theorem 1 pp. 119-120 in [23].

ii) We write $\frac{1}{|x|^{N-2}}$ as $a_1(x)+a_2(x)$, where $a_1(x) = \frac{1}{|x|^{N-2}}\chi_{\{|x|>1\}}$ and $a_2(x) = \frac{1}{|x|^{N-2}}\chi_{\{|x|\le 1\}}$. Then we have $I(\varphi) = a_1 * \varphi + a_2 * \varphi$. It is obvious that $a_1 \in L^q(\mathbf{R}^N)$ for $q \in (\frac{N}{N-2}, \infty]$ and $a_2 \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{N}{N-2})$. Let p_1' and p_2' be the conjugate exponents of p_1 and p_2. Then $p_1' > \frac{N}{N-2}$ and $p_2' < \frac{N}{N-2}$, so that $a_1 \in L^{p_1'}(\mathbf{R}^N)$ and $a_2 \in L^{p_2'}(\mathbf{R}^N)$. We infer that $I(\varphi)$ is continuous and by Young's inequality we get

$$||I(\varphi)||_{L^\infty}|| \le ||a_1||_{L^{p_1'}} \cdot ||\varphi||_{L^{p_1}} + ||a_2||_{L^{p_2'}} \cdot ||\varphi||_{L^{p_2}}.$$

iv) First we consider the bilinear form

$$P(\varphi,\psi) = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \varphi(x)\frac{1}{|x-y|^{N-2}}\overline{\psi(y)}\,dx\,dy.$$

Notice that P is well-defined and continuous on $L^{\frac{2N}{N+2}}(\mathbf{R}^N) \times L^{\frac{2N}{N+2}}(\mathbf{R}^N)$. Indeed, it follows from *i)* that I is well-defined and continuous from $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ to $L^{\frac{2N}{N-2}}(\mathbf{R}^N)$ and we have

$$|P(\varphi,\psi)| = \left| \int_{\mathbf{R}^N} I(\varphi)(x)\overline{\psi(x)}\,dx \right| \le ||I(\varphi)||_{L^{\frac{2N}{N-2}}} \cdot ||\psi||_{L^{\frac{2N}{N+2}}} \le A_N ||\varphi||_{L^{\frac{2N}{N+2}}} ||\psi||_{L^{\frac{2N}{N+2}}}.$$

Without loss of generality we may assume that $\sigma > \frac{2}{N}$. From the assumptions on F we have $|F(u)| \le C|u|^2$ if $|u| \le 1$ and $|F(u)| \le C|u|^{1+\sigma}$ if $|u| > 1$. It is well-known that $H^1(\mathbf{R}^N)$ is continuously embedded in $L^p(\mathbf{R}^N)$ for $p \in [2, \frac{2N}{N-2}]$ and then it is standard (see, e.g. [26], Appendix A) that $u \longmapsto F(u)$ is continuously differentiable from $H^1(\mathbf{R}^N)$ to $L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{1+\sigma}), \frac{2N}{(N-2)(1+\sigma)}]$. In particular, $u \longmapsto F(u)$ is continuously differentiable from $H^1(\mathbf{R}^N)$ to $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ (because $\frac{2}{1+\sigma} < \frac{2N}{N+2} < \frac{2N}{(N-2)(1+\sigma)}$). Since $M(u) = P(F(u), F(u))$, *iv)* follows.

iii) and *v)* Let $K(x) = \frac{1}{|x|^{N-2}}$. Then $K \in \mathcal{S}'(\mathbf{R}^N)$ and it follows from Theorem 4.1 p. 160 in [24] or from Lemma 1 p. 117 in [23] that $\widehat{K}(\xi) = \frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)} \cdot \frac{1}{|\xi|^2}$. From Lemma 1 p. 117 in [23] we have

$$P(\varphi,\psi) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{I(\varphi)}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi = c_N \int_{\mathbf{R}^N} \frac{1}{|\xi|^2}\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi \tag{4.10}$$

whenever φ, $\psi \in \mathcal{S}(\mathbf{R}^N)$. We claim that (4.10) holds for any φ, $\psi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ with $1 \le r_1 < \frac{2N}{N+2} < r_2 \le 2$. This assertion implies both *iii)* and *v)*.

Now let us prove the claim. Since (4.10) holds on $\mathcal{S} \times \mathcal{S}$, the bilinear form P is continuous on $L^{\frac{2N}{N+2}}(\mathbf{R}^N) \times L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ and $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ is continuously embedded into $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$, all we have to do is to show that the quadratic form

$$P_1(\varphi, \psi) = \int_{\mathbf{R}^N} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}\, d\xi$$

is continuous on $\Big(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)\Big) \times \Big(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)\Big)$; then the claim follows by density of $\mathcal{S}$ in $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$.

Let r_1', r_2' be the conjugate exponents of r_1, r_2 and let q_1, q_2 be such that $\frac{1}{r_1'} + \frac{1}{q_1} = \frac{1}{2}$, respectively $\frac{1}{r_2'} + \frac{1}{q_2} = \frac{1}{2}$. Let $b_1(\xi) = \frac{1}{|\xi|} \chi_{\{|\xi| \le 1\}}$ and $b_2(\xi) = \frac{1}{|\xi|} \chi_{\{|\xi| > 1\}}$. We have $2 \le q_1 < N$ and $q_2 > N$, so that $b_1 \in L^{q_1}(\mathbf{R}^N)$ and $b_2 \in L^{q_2}(\mathbf{R}^N)$. Since the Fourier transform maps continuously $L^{r_1}(\mathbf{R}^N)$ into $L^{r_1'}(\mathbf{R}^N)$ and $L^{r_2}(\mathbf{R}^N)$ into $L^{r_2'}(\mathbf{R}^N)$, we have :

$$|P_1(\varphi, \psi)| \le \left| \int_{\{|\xi| \le 1\}} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}\, d\xi \right| + \left| \int_{\{|\xi| > 1\}} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}\, d\xi \right|$$

$$\le ||b_1 \widehat{\varphi}||_{L^2} \cdot ||b_1 \widehat{\psi}||_{L^2} + ||b_2 \widehat{\varphi}||_{L^2} \cdot ||b_2 \widehat{\psi}||_{L^2}$$

$$\le ||b_1||^2_{L^{q_1}} ||\widehat{\varphi}||_{L^{r_1'}} ||\widehat{\psi}||_{L^{r_1'}} + ||b_2||^2_{L^{q_2}} ||\widehat{\varphi}||_{L^{r_2'}} ||\widehat{\psi}||_{L^{r_2'}}$$

$$\le C(N, r_1, r_2) \left(||\varphi||_{L^{r_1}} ||\psi||_{L^{r_1}} + ||\varphi||_{L^{r_2}} ||\psi||_{L^{r_2}} \right).$$

This proves the continuity of P_1 and our claim. Thus the proof of Lemma 4.5 is complete. □

Theorem 4.6 *Let $N \ge 3$ and let F, G, $H : \mathbf{R} \longrightarrow \mathbf{R}$ be C^2 functions satisfying the following assumptions :*

a) $F(0) = F'(0) = 0$ and there exists $\sigma < \frac{4}{N-2}$ and $C > 0$ such that

$$|F'(u)| \le C|u|^\sigma \qquad \textit{if } |u| \ge 1.$$

b) There exists $\sigma_1 \in [1, \frac{N+2}{N-2})$ and $C_1 > 0$ such that

$$|G'(u)| \le C_1 |u|^{\sigma_1} \quad \textit{and} \quad |H'(u)| \le C_1 |u|^{\sigma_1} \quad \textit{for any } u \in \mathbf{R}.$$

Moreover, if $\sigma_1 < 2$ then we assume that $\sigma_1 \ge \max(\frac{(N-2)(1+2\sigma)-4}{N}, 1)$.

c) For any $\varepsilon > 0$, $G' \not\equiv 0$ on $(-\varepsilon, 0)$ and on $(0, \varepsilon)$.

Then any minimizer $u \in H^1(\mathbf{R}^N)$ of the functional E given by (4.8) subject to the constraint $Q(u) = \lambda$ is radially symmetric (after a translation in $\mathbf{R}^N$).

Proof. First of all, notice that the functionals E and Q are well-defined and of class C^1 on $H^1(\mathbf{R}^N)$. Let $u \in H^1(\mathbf{R}^N)$ be a minimizer. We will show that, except for translation, u is symmetric with respect to x_1. The same proof is valid for any other direction in $\mathbf{R}^N$ and the radial symmetry of u follows as in the proof of Theorem 4.1.

After a translation in the x_1 direction we may suppose that

$$\int_{\{x_1 < 0\}} G(u(x))\, dx = \int_{\{x_1 > 0\}} G(u(x))\, dx = \frac{\lambda}{2}.$$

As before, we define $u_1 = T_1 u$ and $u_2 = T_2 u$. We know that $u_1,\, u_2 \in H^1(\mathbf{R}^N)$. In view of assumption *a*), it is obvious that $F(u) \in L^1(\mathbf{R}^N)$ and we have $T_1(F(u)) = F(u_1)$, $T_2(F(u)) = F(u_2)$, $Q(u_1) = Q(u_2) = \lambda$. Defining $W(\varphi) = \displaystyle\int_{\mathbf{R}^N} \frac{1}{|\xi|^2} |\widehat{\varphi}(\xi)|^2\, d\xi$, from Lemma 4.5 *v*) we get

$$
\begin{aligned}
E(u_1) + E(u_2) - 2E(u) &= -[M(u_1) + M(u_2) - 2M(u)] \\
&= -c_N [W(T_1(F(u))) + W(T_2(F(u))) - 2W(F(u))].
\end{aligned} \tag{4.11}
$$

Recall that by (2.51) we have for any $\varphi \in C_c^\infty(\mathbf{R}^N)$,

$$
W(T_1\varphi) + W(T_2\varphi) - 2W(\varphi) = \frac{8}{\pi} \int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \left| \int_0^\infty \widehat{A\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \right|^2 d\xi'. \tag{4.12}
$$

To show that this identity also holds for $F(u)$ we need the following lemma :

Lemma 4.7 *Let $N \geq 3$ and let r_1, r_2 be such that $1 < r_1 < \frac{2N}{N+2} < r_2 < 2$. The bilinear form*

$$
R(\varphi, \psi) = \int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}}(\eta_1, \xi') \frac{\eta_1}{|\xi'|^2 + \eta_1^2}\, d\eta_1\, d\xi'
$$

is continuous on $\Big(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)\Big) \times \Big(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)\Big)$.

Proof. Consider $\varphi,\, \psi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. Then $\widehat{\varphi},\, \widehat{\psi} \in L^{r_1'}(\mathbf{R}^N) \cap L^{r_2'}(\mathbf{R}^N)$, where r_1' and r_2' are the conjugate exponents of r_1 and r_2. Using Hölder's inequality and the change of variable $\xi_1 = t|\xi'|$, we get for $\xi' \neq 0$ and $i = 1, 2$,

$$
\begin{aligned}
\left| \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \right| &\leq \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_i'}\, d\xi_1 \right)^{\frac{1}{r_i'}} \left(\int_0^\infty \frac{\xi_1^{r_i}}{(|\xi'|^2 + \xi_1^2)^{r_i}}\, d\xi_1 \right)^{\frac{1}{r_i}} \\
&= |\xi'|^{\frac{1-r_i}{r_i}} \left(\int_0^\infty \frac{t^{r_i}}{(1+t^2)^{r_i}}\, dt \right)^{\frac{1}{r_i}} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_i'}\, d\xi_1 \right)^{\frac{1}{r_i'}} \\
&= C_i |\xi'|^{\frac{1-r_i}{r_i}} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_i'}\, d\xi_1 \right)^{\frac{1}{r_i'}}.
\end{aligned} \tag{4.13}
$$

A similar estimate holds for ψ. Let q_i be the conjugate exponent of $\frac{r_i'}{2}$, i.e. $q_i = \frac{r_i}{2 - r_i}$. Using (4.13), Hölder's inequality and the estimate $||\widehat{\varphi}||_{L^{r_i'}} \leq A_i ||\varphi||_{L^{r_i}}$ we have

$$
\begin{aligned}
&\left| \int_{B_{\mathbf{R}^{N-1}}(0,1)} \frac{1}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}}(\eta_1, \xi') \frac{\eta_1}{|\xi'|^2 + \eta_1^2}\, d\eta_1\, d\xi' \right| \\
&\leq C_1^2 \int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{2-2r_1}{r_1} - 1} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_1'}\, d\xi_1 \right)^{\frac{1}{r_1'}} \left(\int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^{r_1'}\, d\eta_1 \right)^{\frac{1}{r_1'}} d\xi' \\
&\leq C_1^2 \left(\int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{q_1(2-3r_1)}{r_1}}\, d\xi' \right)^{\frac{1}{q_1}} \left(\int_{B_{\mathbf{R}^{N-1}}(0,1)} \int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_1'}\, d\xi_1\, d\xi' \right)^{\frac{1}{r_1'}} \\
&\qquad \left(\int_{B_{\mathbf{R}^{N-1}}(0,1)} \int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^{r_1'}\, d\eta_1\, d\xi' \right)^{\frac{1}{r_1'}} \\
&\leq C_1^2 A_1^2 \left(\int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{q_1(2-3r_1)}{r_1}}\, d\xi' \right)^{\frac{1}{q_1}} ||\varphi||_{L^{r_1}} ||\psi||_{L^{r_1}}
\end{aligned} \tag{4.14}
$$

and

$$\Big| \int_{\{|\xi'|>1\}} \frac{1}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}}(\eta_1, \xi') \frac{\eta_1}{|\xi'|^2 + \eta_1^2}\, d\eta_1\, d\xi' \Big|$$

$$\leq C_2^2 \int_{\{|\xi'|>1\}} |\xi'|^{\frac{2-2r_2}{r_2} - 1} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_2'}\, d\xi_1 \right)^{\frac{1}{r_2'}} \left(\int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^{r_2'}\, d\eta_1 \right)^{\frac{1}{r_2'}} d\xi'$$

$$\leq C_2^2 \left(\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_1(2-3r_2)}{r_2}}\, d\xi' \right)^{\frac{1}{q_2}} \left(\int_{\{|\xi'|>1\}} \int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_2'}\, d\xi_1\, d\xi' \right)^{\frac{1}{r_2'}} \left(\int_{\{|\xi'|>1\}} \int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^{r_2'}\, d\eta_1\, d\xi' \right)^{\frac{1}{r_2'}} \tag{4.15}$$

$$\leq C_2^2 A_2^2 \left(\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_2(2-3r_2)}{r_2}}\, d\xi' \right)^{\frac{1}{q_2}} ||\varphi||_{L^{r_2}} ||\psi||_{L^{r_2}}.$$

Since $1 < r_1 < \frac{2N}{N+2} < r_2 < 2$, a direct computation shows that $\int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{q_1(2-3r_1)}{r_1}}\, d\xi'$ and $\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_2(2-3r_2)}{r_2}}\, d\xi'$ are finite. From (4.14) and (4.15) we have

$$|R(\varphi, \psi)| \leq K \left(||\varphi||_{L^{r_1}} ||\psi||_{L^{r_1}} + ||\varphi||_{L^{r_2}} ||\psi||_{L^{r_2}} \right)$$

and Lemma 4.7 is proved. $\square$

Let r_1 and r_2 be as in Lemma 4.7. Since the maps $\varphi \longmapsto T_1\varphi$ and $\varphi \longmapsto T_2\varphi$ are obviously continuous from $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ into itself and we have shown in the proof of Lemma 4.5 that the bilinear form $P_1(\varphi, \psi) = \int_{\mathbf{R}^N} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}\, d\xi$ is continuous on this space, it follows that the left-hand side of (4.12) is continuous on $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. By Lemma 4.7, the right-hand side of (4.12) also defines a continuous functional on $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. Since (4.12) is valid for any $\varphi \in C_c^\infty(\mathbf{R}^N)$, by density we infer that (4.12) holds for any $\varphi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. Recall that $u \in H^1(\mathbf{R}^N)$ and by the Sobolev embedding and assumption *a)* we have $F(u) \in L^q(\mathbf{R}^N)$ for any $q \in [\max(1, \frac{2}{1+\sigma}), \frac{2N}{(N-2)(1+\sigma)}]$; hence (4.12) is valid for $F(u)$.

Since u is a minimizer, we must have $E(u_1) + E(u_2) - 2E(u) \geq 0$. From (4.11) and (4.12) we infer that necessarily

$$\int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \Big| \int_0^\infty \mathcal{F}(A(F(u)))(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \Big|^2 d\xi' = 0. \tag{4.16}$$

Contrary to our previous examples, (4.16) *does not* imply directly $AF(u) \equiv 0$. To see this, consider a function $\psi \in C_c^\infty(0, \infty)$ such that $\mathrm{supp}(\psi) \subset [1, \infty)$, $\psi \not\equiv 0$ and $\int_0^\infty \frac{t}{1+t^2} \psi(t)\, dt = 0$. (Such a function exists: for example, take two nonnegative functions ψ_0, $\psi_1 \in C_c^\infty(1, \infty)$ with disjoint supports and put $\psi_\tau = (1-\tau)\psi_0 - \tau\psi_1$. There is some $\tau \in (0,1)$ such that $\int_0^\infty \frac{t}{1+t^2} \psi_\tau(t)\, dt = 0$.) Extend ψ to an odd function defined on $\mathbf{R}$. Take $\alpha \in C_c^\infty(\mathbf{R}^{N-1})$ such that $\alpha \not\equiv 0$ and $\mathrm{supp}(\alpha) \subset \mathbf{R}^{N-1} \setminus B(0,1)$ and put $\widehat{f}(\xi_1, \xi') = \alpha(\xi')\psi(\frac{\xi_1}{|\xi'|})$. Then $\widehat{f} \in C_c^\infty(\mathbf{R}^N)$ (hence $f \in \mathcal{S}$), $f \not\equiv 0$ and f is odd with respect to the first variable. However, we have

$\int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 = 0$ for any $\xi' \neq 0$ and consequently

$$\int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 \right|^2 d\xi' = 0.$$

To show that u is symmetric with respect to x_1, we argue as follows: since u and u_1 minimize E under the constraint $Q = \lambda$, these functions satisfy the Euler-Lagrange equations $E'(u) + \alpha Q'(u) = 0$, respectively $E'(u_1) + \beta Q'(u_1) = 0$ for some constants α and β, that is

$$(4.17) \qquad -\Delta u - 2I(F(u))F'(u) + H'(u) + \alpha G'(u) = 0 \qquad \text{in } \mathbf{R}^N,$$

$$(4.18) \qquad -\Delta u_1 - 2I(F(u_1))F'(u_1) + H'(u_1) + \beta G'(u_1) = 0 \qquad \text{in } \mathbf{R}^N.$$

We will show in the next lemma that u and u_1 are smooth functions. Then we prove that $I(F(u))(x) = I(F(u_1))(x)$ in the half-space $\{x_1 < 0\}$. Together with assumption *c*), this implies that $\alpha = \beta$ in (4.17)-(4.18). Then we will be able to apply the Unique Continuation Principle to prove that $u = u_1$.

Lemma 4.8 *Let $u \in H^1(\mathbf{R}^N)$ be a solution of (4.17), where $F, G, H \in C^2(\mathbf{R})$ satisfy the assumptions a) and b) in Theorem 4.6. Then $u \in W^{3,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. In particular, $u \in C^2(\mathbf{R}^N)$ and $D^\alpha u$ are continuous and bounded on $\mathbf{R}^N$ if $\alpha \in \mathbf{N}^N$, $|\alpha| \leq 2$.*

Proof. The proof is rather classical and relies on a boot-strap argument. For the convenience of the reader, we give it here.

We show first that $u \in L^\infty(\mathbf{R}^N)$. By the Sobolev embedding we have $u \in L^q(\mathbf{R}^N)$ for $q \in [2, \frac{2N}{N-2}]$. We will improve this estimate by a bootstrap argument to get the desired conclusion.

Let us consider first the case $N = 3$. We may assume without loss of generality that $3 \leq \sigma < \frac{4}{N-2} = 4$ (if $\sigma < 3$, we replace σ by 3 and this gives no supplementary constraint on σ_1 in assumption *b*)). Suppose that $u \in L^q(\mathbf{R}^3)$ for any $q \in [2, \beta]$, where $\beta \geq 6$. Together with assumption *a*), this implies $F(u) \in L^q(\mathbf{R}^3)$ for $q \in [1, \frac{\beta}{1+\sigma}]$. We distinguish two cases :

Case A. If $\frac{\beta}{1+\sigma} > \frac{3}{2}$, then Lemma 4.5 *i*)-*ii*) implies $I(F(u)) \in L^q(\mathbf{R}^N)$ for $q \in (3, \infty]$. By assumption *a*) we have $F'(u)\chi_{\{|u| \leq 1\}} \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, hence $I(F(u))F'(u)\chi_{\{|u|\leq 1\}} \in L^q(\mathbf{R}^N)$ for $q \in (\frac{6}{5}, \infty]$ and $F'(u)\chi_{\{|u|>1\}} \in L^1(\mathbf{R}^N) \cap L^{\frac{\beta}{\sigma}}(\mathbf{R}^N)$, thus $I(F(u))F'(u)\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{\beta}{\sigma}]$. Consequently, $I(F(u))F'(u) \in L^q(\mathbf{R}^N)$ for $q \in (\frac{6}{5}, \frac{\beta}{\sigma}]$. Assumption *b*) implies that $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}]$. Note that $\frac{\beta}{\sigma_1} \geq \frac{6}{\sigma_1} > \frac{6}{5}$ and $\frac{\beta}{\sigma} \geq \frac{6}{\sigma} \geq \frac{2}{\sigma_1}$ by the second part of assumption *b*). From equation (4.17) we find $\Delta u \in L^q(\mathbf{R}^N)$ for any $q \in (\frac{6}{5}, \min(\frac{\beta}{\sigma}, \frac{\beta}{\sigma_1})]$ if $\frac{2}{\sigma_1} \leq \frac{6}{5}$, respectively for any $q \in [\frac{2}{\sigma_1}, \min(\frac{\beta}{\sigma}, \frac{\beta}{\sigma_1})]$ if $\frac{2}{\sigma_1} > \frac{6}{5}$. Let $q_* := \min(\frac{\beta}{\sigma}, \frac{\beta}{\sigma_1})$. If $q_* \geq 2$, we have $\Delta u \in L^{q_*}(\mathbf{R}^3)$ and obviously $u \in L^{q_*}(\mathbf{R}^N)$ (because $2 \leq q_* \leq \frac{\beta}{\sigma_1} \leq \beta$), hence $u \in W^{2,q_*}(\mathbf{R}^3)$ and by the Sobolev embedding we infer that $u \in L^\infty(\mathbf{R}^3)$. If $\frac{3}{2} < q_* < 2$, again by the Sobolev embedding we have $|\nabla u| \in L^{p_*}(\mathbf{R}^N)$, where $\frac{1}{p_*} = \frac{1}{q_*} - \frac{1}{3}$ (thus $p_* \in (3, 6)$), hence $u \in W^{1,p_*}(\mathbf{R}^3) \subset L^\infty(\mathbf{R}^3)$. If $q_* = \frac{3}{2}$, we obtain $\Delta u \in L^{\frac{3}{2}}(\mathbf{R}^3)$, which implies $|\nabla u| \in L^3(\mathbf{R}^3)$, hence $u \in W^{1,3}(\mathbf{R}^3)$ so that $u \in L^q(\mathbf{R}^3)$ for any $q \in [2, \infty)$. Repeating the above argument for some $\tilde{\beta} > \beta$, we get $u \in L^\infty(\mathbf{R}^3)$. It remains to study the case $q_* < \frac{3}{2}$. It is clear that in this case we have $q_* = \frac{\beta}{\sigma_1}$ (because $\frac{\beta}{\sigma} > \frac{3}{2}$). Since $\Delta u \in L^{q_*}(\mathbf{R}^3)$, by the Sobolev embedding we get $u \in L^{\beta_1}(\mathbf{R}^3)$, where $\frac{1}{\beta_1} = \frac{1}{q_*} - \frac{2}{3}$. Notice that $\frac{1}{\beta_1} - \frac{1}{\beta} = \frac{\sigma_1 - 1}{\beta} - \frac{2}{3} \leq \frac{\sigma_1 - 5}{6} < 0$, hence $\beta_1 > \beta$. We repeat the previous reasoning with β_1 instead of β. We obtain that either $u \in L^\infty(\mathbf{R}^3)$, or $u \in L^{\beta_2}(\mathbf{R}^3)$, where $\beta_2 > \beta_1$ and

$\frac{1}{\beta_2} - \frac{1}{\beta_1} \leq \frac{\sigma_1 - 5}{6} < 0$. In the latter case we continue with β_2 instead of β and we get that either $u \in L^\infty(\mathbf{R}^3)$, or $u \in L^{\beta_3}(\mathbf{R}^3)$, where $\beta_3 > \beta_2$ and $\frac{1}{\beta_3} - \frac{1}{\beta_2} \leq \frac{\sigma_1-5}{6}$, and so on. After a finite number of steps we get $u \in L^\infty(\mathbf{R}^3)$ (since otherwise we would obtain a positive increasing sequence $(\beta_n)_{n\geq 1}$ such that $\frac{1}{\beta_n} - \frac{1}{\beta} \leq \frac{n(\sigma_1-5)}{6} \longrightarrow -\infty$, which is impossible).

Case B. If $\frac{\beta}{1+\sigma} \leq \frac{3}{2}$, we may suppose that $\frac{\beta}{1+\sigma} < \frac{3}{2}$ (otherwise we take β a little bit smaller). By Lemma 4.5 *i)* we have $I(F(u)) \in L^q(\mathbf{R}^3)$ for $q \in (3, \left(\frac{1+\sigma}{\beta} - \frac{2}{3}\right)^{-1}]$. As in case A we obtain $I(F(u))F'(u)\chi_{\{|u|\leq 1\}} \in L^q(\mathbf{R}^N)$ for $q \in (\frac{6}{5}, \left(\frac{1+\sigma}{\beta} - \frac{2}{3}\right)^{-1}]$ and $I(F(u))F'(u)\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ for $q \in \left[1, \left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1}\right]$, so that $I(F(u))F'(u) \in L^q(\mathbf{R}^3)$ for $q \in \left(\frac{6}{5}, \left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1}\right]$. Notice that $\left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1} > \frac{6}{5}$ (because $\beta \geq 6$ and $\sigma < 4$) and $\left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1} \geq \frac{2}{\sigma_1}$ by assumption *b)*. Since obviously $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}]$, using equation (4.17) we infer that $\Delta u \in L^q(\mathbf{R}^3)$ for any $q \in \left[\max(\frac{6}{5}, \frac{2}{\sigma_1}), \min\left(\left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1}, \frac{\beta}{\sigma_1}\right)\right]$, $q \neq \frac{6}{5}$. Let $q_2 = \min\left(\left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1}, \frac{\beta}{\sigma_1}\right)$. If $q_2 \geq \frac{3}{2}$, arguing as in case A we get $u \in L^\infty(\mathbf{R}^3)$. If $q_2 < \frac{3}{2}$, by the Sobolev embedding we have $u \in L^{\beta_1}(\mathbf{R}^3)$, where $\frac{1}{\beta_1} = \frac{1}{q_2} - \frac{2}{3}$, hence $\frac{1}{\beta_1} - \frac{1}{\beta} \leq \max\left(\frac{\sigma-4}{3}, \frac{\sigma_1-5}{6}\right) < 0$, so that $\beta_1 > \beta$. Repeating the preceeding arguments for β_1 we obtain either $u \in L^\infty(\mathbf{R}^3)$, or $\frac{\beta_1}{1+\sigma} > \frac{3}{2}$ (so that we are in case A, consequently $u \in L^\infty(\mathbf{R}^3)$), or $u \in L^{\beta_2}(\mathbf{R}^3)$, where $\beta_2 > \beta_1$ and $\frac{1}{\beta_2} - \frac{1}{\beta_1} \leq \max\left(\frac{\sigma-4}{3}, \frac{\sigma_1-5}{6}\right)$. In the latter case we repeat the same reasoning, and so on. As in case A, after a finite number of steps we get $u \in L^\infty(\mathbf{R}^3)$.

Now we consider the case $N \geq 4$ and we assume that $u \in L^q(\mathbf{R}^N)$ for any $q \in [2, \beta]$, where $\beta \geq \frac{2N}{N-2}$. It is clear that $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in \left[\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}\right]$ and $F(u) \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{\beta}{1+\sigma}]$. Once again, we distinguish two cases :

Case A. If $\frac{\beta}{1+\sigma} > \frac{N}{2}$, then $I(F(u)) \in L^q(\mathbf{R}^N)$ for any $q \in (\frac{N}{N-2}, \infty]$. We have $F'(u)\chi_{\{|u|\leq 1\}} \in L^q(\mathbf{R}^N)$ for $q \in [2, \infty]$, hence $I(F(u))F'(u))\chi_{\{|u|\leq 1\}} \in L^q(\mathbf{R}^N)$ for $q \in (1, \infty]$ if $N = 4$, respectively for $q \in [1, \infty]$ if $N \geq 5$ and $F'(u)\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{\beta}{\sigma}]$, hence $I(F(u))F'(u))\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ if $q \in [1, \frac{\beta}{\sigma}]$. Consequently $I(F(u))F'(u)) \in L^q(\mathbf{R}^N)$ for $q \in (1, \frac{\beta}{\sigma}]$ if $N = 4$, respectively for $q \in [1, \frac{\beta}{\sigma}]$ if $N \geq 5$. Notice that $\beta \geq \frac{2N}{N-2}$ and the second part of assumption b) imply $\frac{\beta}{\sigma} \geq \frac{2}{\sigma_1}$. Using equation (4.17) we infer that $\Delta u \in L^q(\mathbf{R}^N)$ for $q \in \left[\max(1, \frac{2}{\sigma_1}), \min(\frac{\beta}{\sigma_1}, \frac{\beta}{\sigma})\right]$, $q \neq 1$ if $N = 4$. Let $q_3 = \min(\frac{\beta}{\sigma_1}, \frac{\beta}{\sigma})$. Notice that $q_3 \leq \beta$ because $\sigma_1 \geq 1$ and $\Delta u \in L^{q_3}(\mathbf{R}^N)$. If $q_3 > \frac{N}{2} \geq 2$, then $u \in L^{q_3}(\mathbf{R}^N)$, hence $u \in W^{2,q_3}(\mathbf{R}^N)$ and by the Sobolev embedding we get $u \in L^\infty(\mathbf{R}^N)$. If $q_3 = \frac{N}{2}$, then $u \in W^{2,\frac{N}{2}}(\mathbf{R}^N)$, consequently $u \in L^q(\mathbf{R}^N)$ for any $q \in [2, \infty)$ and repeating the above proof with $\tilde{\beta} > \beta$ we find $u \in L^\infty(\mathbf{R}^N)$. If $q_3 < \frac{N}{2}$, then necessarily $q_3 = \frac{\beta}{\sigma_1}$ (recall that $\frac{\beta}{\sigma} > \frac{\beta}{1+\sigma} > \frac{N}{2}$ because we are in case A). By the Sobolev embedding we get $u \in L^{\beta_1}(\mathbf{R}^N)$, where $\frac{1}{\beta_1} = \frac{1}{q_3} - \frac{2}{N} = \frac{\sigma_1}{\beta} - \frac{2}{N}$, thus $\frac{1}{\beta_1} - \frac{1}{\beta} = \frac{\sigma_1 - 1}{\beta} - \frac{2}{N} \leq \frac{(\sigma_1-1)(N-2)-4}{2N} < 0$ by b). Repeating the previous arguments with β replaced by β_1, we find that either $u \in L^\infty(\mathbf{R}^N)$ or $u \in L^{\beta_2}(\mathbf{R}^N)$, where $\beta_2 > \beta_1$ and $\frac{1}{\beta_2} - \frac{1}{\beta_1} \leq \frac{(\sigma_1-1)(N-2)-4}{2N}$, and so on. As previously, after a finite number of steps we get $u \in L^\infty(\mathbf{R}^N)$.

Case B. If $\frac{\beta}{1+\sigma} \leq \frac{N}{2}$, we may suppose that $\frac{\beta}{1+\sigma} < \frac{N}{2}$. By Lemma 4.5 *i)*, $I(F(u)) \in L^q(\mathbf{R}^N)$ for $q \in \left(\frac{N}{N-2}, \left(\frac{1+\sigma}{\beta} - \frac{2}{N}\right)^{-1}\right]$. As in case A we get $I(F(u))F'(u) \in L^q(\mathbf{R}^N)$ for

$q \in \left[1, \left(\frac{1+2\sigma}{\beta} - \frac{2}{N}\right)^{-1}\right]$, $q \neq 1$ if $N = 4$. By a), b) and the fact that $\beta \geq \frac{2N}{N-2}$ we have $\left(\frac{1+2\sigma}{\beta} - \frac{2}{N}\right)^{-1} \geq \frac{2}{\sigma_1}$. Since $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}]$, using (4.17) we get $\Delta u \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), q_4]$, $q \neq 1$ if $N = 4$, where $q_4 = \min\left(\frac{\beta}{\sigma_1}, \left(\frac{1+2\sigma}{\beta} - \frac{2}{N}\right)^{-1}\right)$. If $q_4 \geq \frac{N}{2}$ then, as above, we obtain $u \in L^\infty(\mathbf{R}^N)$. Otherwise by the Sobolev embedding we find $u \in L^{\beta_1}(\mathbf{R}^N)$, where $\frac{1}{\beta_1} = \frac{1}{q_4} - \frac{2}{N}$, thus $\frac{1}{\beta_1} - \frac{1}{\beta} \leq \max\left(\frac{(\sigma_1-1)(N-2)-4}{2N}, \frac{\sigma(N-2)-4}{N}\right) < 0$. Then we restart the process with β_1 instead of β. Continuing in this way, after a finite number of steps we obtain $u \in L^\infty(\mathbf{R}^N)$.

Up to now we have proved that $u \in L^q(\mathbf{R}^N)$ for any $q \in [2, \infty]$. Thus $F(u) \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $I(F(u)) \in L^q(\mathbf{R}^N)$ for $q \in (\frac{N}{N-2}, \infty]$, $F'(u) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, hence $I(F(u))F'(u) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Clearly $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \infty]$. Using (4.17) we have $\Delta u \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, thus $u \in W^{2,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. In particular, $u \in C^1(\mathbf{R}^N)$ and $\frac{\partial u}{\partial x_i}$ are continuous and bounded on $\mathbf{R}^N$. Differentiating (4.17) with respect to x_i we get

$$-\Delta(\frac{\partial u}{\partial x_i}) - 2I(F'(u)\frac{\partial u}{\partial x_i})F'(u) - 2I(F(u))F''(u)\frac{\partial u}{\partial x_i} + G''(u)\frac{\partial u}{\partial x_i} + \alpha H''(u)\frac{\partial u}{\partial x_i} = 0 \quad \text{in } \mathbf{R}^N.$$

It follows that $-\Delta(\frac{\partial u}{\partial x_i}) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Since obviously $\frac{\partial u}{\partial x_i} \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, we get $\frac{\partial u}{\partial x_i} \in W^{2,p}(\mathbf{R}^N)$, which implies $u \in W^{3,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. $\square$

It follows from Lemma 4.8 that $F(u) \in C^2(\mathbf{R}^N)$ and $F(u) \in W^{2,p}(\mathbf{R}^N)$ for $p \in [1, \infty]$. Using Lemma 4.5 *i)* and *ii)*, it is easy to check that $I(F(u)) \in C^2(\mathbf{R}^N)$ and $I(F(u)) \in W^{2,p}(\mathbf{R}^N)$ for $p \in (\frac{N}{N-2}, \infty]$. In particular, $I(F(u)) \in \mathcal{S}'(\mathbf{R}^N)$ and Lemma 4.5 *iii)* implies $\mathcal{F}(I(F(u)))(\xi) = d_N \frac{1}{|\xi|^2} \widehat{F(u)}(\xi)$, where $d_N = \frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)}$. Setting $U = I(F(u))$ we have $-\Delta U = d_N F(u)$.

Next we show that $\frac{\partial U}{\partial x_1}(0, x') = \frac{\partial}{\partial x_1} I(F(u))(0, x') = 0$ for any $x' \in \mathbf{R}^{N-1}$. From (4.16) we infer that $\int_0^\infty \mathcal{F}(A(F(u)))(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 = 0$ for almost every $\xi' \in \mathbf{R}^{N-1}$, that is $\int_{-\infty}^\infty \widehat{F(u)}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 = 0$ a.e. $\xi' \in R^{N-1}$, or equivalently

$$\int_{-\infty}^\infty \xi_1 \mathcal{F}(I(F(u)))(\xi_1, \xi') \, d\xi_1 = 0 \qquad \text{for almost every } \xi' \in R^{N-1}. \tag{4.19}$$

If $\frac{\partial}{\partial x_1} I(F(u))$ and $\mathcal{F}(\frac{\partial}{\partial x_1} I(F(u)))$ are in $L^1(\mathbf{R}^N)$, by the Fourier inversion theorem (4.19) is equivalent to $\frac{\partial}{\partial x_1} I(F(u))(0, x') = 0$, as desired.

Since we do not know whether $\frac{\partial}{\partial x_1} I(F(u)) \in L^1(\mathbf{R}^N)$ and $\mathcal{F}(\frac{\partial}{\partial x_1} I(F(u))) \in L^1(\mathbf{R}^N)$, we argue as follows : we take an arbitrary test function $\psi \in \mathcal{S}(\mathbf{R}^{N-1})$ and we put $\varphi_n(x_1) = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2 x_1^2}{2}}$. Clearly, $\varphi_n(x_1) = n\varphi_1(nx_1)$, $||\varphi_n||_{L^1(\mathbf{R})} = 1$ and $\widehat{\varphi}_n(\xi_1) = e^{-\frac{\xi_1^2}{2n^2}}$. On one hand we have, by using Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbf{R}^N} \varphi_n(x_1)\psi(x') \left[\frac{\partial}{\partial x_1} I(F(u))\right](x_1, x') \, dx \\ &= \lim_{n\to\infty} \int_{\mathbf{R}^N} \varphi_1(y_1)\psi(x') \left[\frac{\partial}{\partial x_1} I(F(u))\right](\frac{y_1}{n}, x') \, dy_1 \, dx' \\ &= \int_{\mathbf{R}^{N-1}} \psi(x') \frac{\partial}{\partial x_1}(I(F(u)))(0, x') \, dx'. \end{aligned} \tag{4.20}$$

On the other hand we have

$$\int_{\mathbf{R}^N} \varphi_n(x_1)\psi(x') \left[\frac{\partial}{\partial x_1} I(F(u))\right](x_1, x')\, dx = \langle \frac{\partial}{\partial x_1}(I(F(u))), \varphi_n(x_1)\psi(x')\rangle_{\mathcal{S}',\mathcal{S}}$$

$$= \langle \mathcal{F}\left(\frac{\partial}{\partial x_1} I(F(u))\right), \mathcal{F}^{-1}\left(\varphi_n(x_1)\psi(x')\right)\rangle_{\mathcal{S}',\mathcal{S}} \tag{4.21}$$

$$= \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \frac{i d_N \xi_1}{|\xi|^2} \widehat{F(u)}(\xi) e^{-\frac{\xi_1^2}{2n^2}} \widehat{\psi}(-\xi')\, d\xi_1\, d\xi'.$$

Since $F(u) \in L^2(\mathbf{R}^N)$, for almost every $\xi' \in \mathbf{R}^{N-1}$ we have $\widehat{F(u)}(\cdot, \xi') \in L^2(\mathbf{R})$. For any such ξ', arguing as in (4.13) we get

$$\int_{\mathbf{R}} \left| e^{-\frac{\xi_1^2}{2n^2}} \cdot \frac{\xi_1}{|\xi|^2} \widehat{F(u)}(\xi_1, \xi') \right| d\xi_1 \leq \int_{\mathbf{R}} \left| \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \widehat{F(u)}(\xi_1, \xi') \right| d\xi_1 \leq \frac{C}{|\xi'|^{\frac{1}{2}}} ||\widehat{F(u)}(\cdot, \xi')||_{L^2(\mathbf{R})},$$

where C does not depend on ξ'. Moreover, Cauchy-Schwarz inequality gives

$$\int_{\mathbf{R}^{N-1}} \frac{C|\widehat{\psi}(-\xi')|}{|\xi'|^{\frac{1}{2}}} ||\widehat{F(u)}(\cdot, \xi')||_{L^2(\mathbf{R})}\, d\xi' \leq C\left(\int_{\mathbf{R}^{N-1}} \frac{|\widehat{\psi}(-\xi')|^2}{|\xi'|}\, d\xi'\right)^{\frac{1}{2}} ||\widehat{F(u)}||_{L^2(\mathbf{R}^N)} < \infty.$$

By the Dominated Convergence Theorem, we have for almost any $\xi' \in \mathbf{R}^{N-1}$

$$\int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \widehat{F(u)}(\xi_1, \xi') e^{-\frac{\xi_1^2}{2n^2}}\, d\xi_1 \longrightarrow \int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \widehat{F(u)}(\xi_1, \xi')\, d\xi_1 = 0 \qquad \text{as } n \longrightarrow \infty.$$

Thus we may use Fubini's Theorem, then the Dominated Convergence Theorem on $\mathbf{R}^{N-1}$ to obtain

$$\int_{\mathbf{R}^N} \frac{\xi_1}{|\xi|^2} \widehat{F(u)}(\xi_1, \xi') e^{-\frac{\xi_1^2}{2n^2}} \psi(-\xi')\, d\xi_1\, d\xi'$$

$$= \int_{\mathbf{R}^{N-1}} \psi(-\xi') \int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \widehat{F(u)}(\xi_1, \xi') e^{-\frac{\xi_1^2}{2n^2}}\, d\xi_1\, d\xi' \tag{4.22}$$

$$\longrightarrow \int_{\mathbf{R}^{N-1}} \psi(-\xi') \cdot 0\, d\xi' = 0. \qquad \text{as } n \longrightarrow \infty.$$

From (4.20), (4.21) and (4.22) we infer that $\int_{\mathbf{R}^{N-1}} \psi(x') \frac{\partial}{\partial x_1}(I(F(u)))(0, x')\, dx' = 0$. Since $\psi \in \mathcal{S}(\mathbf{R}^{N-1})$ was arbitrary, we have $\frac{\partial}{\partial x_1}(I(F(u)))(0, \cdot) = 0$ in $\mathcal{S}'(\mathbf{R}^{N-1})$, hence $\frac{\partial}{\partial x_1}(I(F(u)))(0, x') = 0$ for any $x' \in \mathbf{R}^{N-1}$ because $\frac{\partial}{\partial x_1}(I(F(u)))$ is a continuous function.

We know that $F(u_1)$ is symmetric with respect to x_1 and a simple change of variables shows that the function $U_1 := I(F(u_1))$ is also symmetric with respect to x_1. Clearly U_1 also belongs to $C^2(\mathbf{R}^N)$ and satisfies $-\Delta U_1 = -\Delta(I(F(u_1))) = d_N F(u_1)$. By symmetry we have $\frac{\partial U_1}{\partial x_1}(0, x') = 0$ for any $x' \in \mathbf{R}^{N-1}$. Since $u_1(x_1, x') = u(x_1, x')$ if $x_1 < 0$, we have proved that the functions U and U_1 are both solutions of the problem

$$\begin{cases} -\Delta W = d_N F(u) & \text{in } \{(x_1, x') \in \mathbf{R}^N \mid x_1 < 0\} \\ W \in C^2(\mathbf{R}^N) \cap W^{2,p}(\mathbf{R}^N) & \text{for } p > \frac{N}{N-2}, \\ \frac{\partial W}{\partial x_1}(0, x') = 0 & \text{for any } x' \in \mathbf{R}^{N-1}. \end{cases} \tag{4.23}$$

It is not hard to see that the solution of (4.23) is unique. Consequently, $U(x_1, x') = U_1(x_1, x')$ if $x_1 < 0$. It is obvious that (u, U) and (u_1, U_1) solve the system

$$(4.24) \qquad \begin{cases} -\Delta u - 2UF'(u) + H'(u) + \alpha G'(u) = 0 \\ \\ -\Delta U - d_N F(u) = 0 \end{cases} \qquad \text{in } \mathbf{R}^N,$$

respectively

$$(4.25) \qquad \begin{cases} -\Delta u_1 - 2U_1 F'(u_1) + H'(u_1) + \beta G'(u_1) = 0 \\ \\ -\Delta U_1 - d_N F(u_1) = 0 \end{cases} \qquad \text{in } \mathbf{R}^N.$$

Next we show that if $u \equiv 0$ in the half-space $\{x_1 < 0\}$, then $u \equiv 0$ in $\mathbf{R}^N$. Indeed, if $u = 0$ in $\{x_1 < 0\}$, then from (4.23) it follows that $U = 0$ on that half-space. Now from (4.24) and the Unique Continuation Principle we infer that $(u, U) = (0, 0)$ on $\mathbf{R}^N$. In this case u trivially has a radial symmetry. Clearly, we cannot have $u \equiv 0$ if $\lambda \neq 0$.

If $u \not\equiv 0$ in $(-\infty, 0) \times \mathbf{R}^{N-1}$, then $u((-\infty, 0) \times \mathbf{R}^{N-1}) = u_1((-\infty, 0) \times \mathbf{R}^{N-1})$ contains an interval of the form $(-\varepsilon, 0)$ or $(0, \varepsilon)$ for some $\varepsilon > 0$. Now assumption c), (4.24), (4.25) and the fact that $(u, U) = (u_1, U_1)$ on $(-\infty, 0) \times \mathbf{R}^{N-1}$ imply that $\alpha = \beta$ in (4.24)-(4.25). As a consequence, we see that $(u - u_1, U - U_1)$ solves a linear system whose coefficients belong to $L^\infty(\mathbf{R}^N)$. Since $(u, U) = (u_1, U_1)$ for $x_1 < 0$ and (u, U), $(u_1, U_1) \in W^{2,p}(\mathbf{R}^N, \mathbf{R}^2)$ if $p \geq 2$ and $p > \frac{N}{N-2}$, by using the Unique Continuation Principle we infer that $u = u_1$ (and $U = U_1$) in $\mathbf{R}^N$, that is u is symmetric with respect to x_1.

Similarly we show that u is symmetric with respect to any other hyperplane Π which has the property that $\int_{\Pi_-} G(u(x))\, dx = \int_{\Pi_+} G(u(x))\, dx$, where Π_- and Π_+ are the two half-spaces determined by Π. As in the proof of Theorem 4.1 it follows that after a translation, u is radially symmetric. The proof of Theorem 4.6 is complete. □

4.4 Our last application concerns the Davey-Stewartson system

$$(4.26) \qquad \begin{cases} iu_t + \Delta u &= f(|u|^2)u - uv_{x_1}, \\ \\ \Delta v &= (|u|^2)_{x_1} \end{cases} \qquad \text{in } \mathbf{R}^3,$$

which can be written as

$$(4.27) \qquad iu_t = -\Delta u + f(|u|^2)u + R_1^2(|u|^2)u,$$

where R_1 is the Riesz transform defined by $\widehat{R_1 \varphi} = \dfrac{i\xi_1}{|\xi|} \widehat{\varphi}(\xi)$. Let $F_1(t) = \int_0^t f(\tau)\, d\tau$. It is easy to check that

$$\tilde{E}(u) = \frac{1}{2} \int_{\mathbf{R}^3} |\nabla u|^2\, dx + \frac{1}{2} \int_{\mathbf{R}^3} F_1(|u|^2)\, dx - \frac{1}{4} \int_{\mathbf{R}^3} |R_1(|u|^2)|^2\, dx$$

is a Hamiltonian for (4.27) and $\tilde{Q}(u) = \int_{\mathbf{R}^3} |u(x)|^2\, dx$ is a conserved quantity for the same equation. The standing waves for (4.27) are precisely the critical points of $\tilde{E} + \omega \tilde{Q}$. As in the previous example, when we minimize $\tilde{E}(u)$ subject to $\tilde{Q}(u) = \textit{constant}$, we may restrict ourselves to real functions u and to the real version of $\tilde{E}$,

$$E(u) = \frac{1}{2} \int_{\mathbf{R}^3} |\nabla u|^2\, dx + \int_{\mathbf{R}^3} F(u)\, dx - \frac{1}{4} \int_{\mathbf{R}^3} |R_1(u^2)|^2\, dx.$$

We will consider a more general functional than $\tilde{Q}$, namely $Q(u) = \int_{\mathbf{R}^3} G(u)\,dx$. If $G(u) = u^2$, in order to guarantee the boundedness from below of the functional E on the set of functions satisfying $Q(u) = \lambda$, the function $F(u)$ is required to behave as $a|u|^\gamma$ for u large, with $a > 0$ and $\gamma > 4$. In the case $F(u) = a|u|^\gamma$, the Cauchy problem for the evolution equation (4.27) has been analysed in [12]. The global existence of solutions was proved in the case $a > 0$ and $\gamma > 4$, while in the case $\gamma = 4$ the global existence was proved if a is sufficiently large.

Still in the case of pure power $F(u) = a|u|^\gamma$, with $a > 0$ and $\gamma > 4$, the existence of minimizers of E subject to the constraint $Q(u) = \int_{\mathbf{R}^3} |u|^2\,dx = \lambda$ can be proved by using the Concentration-Compactness Principle (see [17]) if λ is large enough (this assumption is needed to prevent vanishing).

In [10] the existence of ground states related to the problem (4.26) has been studied. However, our method cannot be used to prove the symmetry of these ground states because the nonlocal term appears in the constraint.

It is well-known that R_1 is a linear continuous map from $L^p(\mathbf{R}^3)$ to $L^p(\mathbf{R}^3)$ for $1 < p < \infty$ (see [23]). If $u^2 \in L^2(\mathbf{R}^3)$, then $R_1(u^2) \in L^2(\mathbf{R}^3)$ and by Plancherel's theorem we get

$$\int_{\mathbf{R}^3} |R_1(u^2)|^2\,dx = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} |\widehat{R_1(u^2)}(\xi)|^2\,d\xi = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}|\widehat{u^2}(\xi)|^2\,d\xi. \tag{4.28}$$

We have the following symmetry result :

Theorem 4.9 *Let $u \in H^1(\mathbf{R}^3)$ be a solution of the minimization problem*

$$\textit{minimize } E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \int_{\mathbf{R}^3} F(u)\,dx - \frac{1}{4}\int_{\mathbf{R}^3} |R_1(u^2)|^2\,dx$$

$$\textit{subject to } Q(u) = \int_{\mathbf{R}^3} G(u(x))\,dx = \lambda$$

under the following assumptions :

a) F, $G : \mathbf{R} \longrightarrow \mathbf{R}$ are C^2 functions, $F(0) = F'(0) = 0$, $G(0) = G'(0) = 0$ and there exist $C > 0$, $\sigma < 5$ such that

$$|F'(u)| \le C|u|^\sigma \qquad \textit{and} \qquad |G'(u)| \le C|u|^\sigma \qquad \textit{for } |u| \ge 1.$$

b) For any $\varepsilon > 0$, $G' \neq 0$ on $(-\varepsilon, 0)$ and on $(0, \varepsilon)$.

Then, after a translation, u is radially symmetric in the variables (x_2, x_3) (i.e. u is axially symmetric).

Proof. Making a translation in the x_2 direction if necessary, we may assume that $\int_{\{x_2<0\}} G(u(x))\,dx = \int_{\{x_2>0\}} G(u(x))\,dx = \frac{\lambda}{2}$. As before, we define u_1 and u_2 by

$$u_1(x_1,x_2,x_3) = \begin{cases} u(x_1,x_2,x_3) & \text{if } x_2 < 0, \\ u(x_1,-x_2,x_3) & \text{if } x_2 \ge 0 \end{cases} \qquad u_2(x_1,x_2,x_3) = \begin{cases} u(x_1,-x_2,x_3) & \text{if } x_2 < 0, \\ u(x_1,x_2,x_3) & \text{if } x_2 \ge 0. \end{cases}$$

It is obvious that $Q(u_1) = Q(u_2) = \lambda$. Moreover, using (4.28) we get

$$\begin{aligned} &E(u_1) + E(u_2) - 2E(u) \\ &= -\frac{1}{4}\frac{1}{(2\pi)^3}\left[\int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}|\widehat{u_1^2}(\xi)|^2\,d\xi + \int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}|\widehat{u_2^2}(\xi)|^2\,d\xi - 2\int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}|\widehat{u^2}(\xi)|^2\,d\xi\right]. \end{aligned} \tag{4.29}$$

Recall that by (2.53) and (2.54) we have the equality

$$
\begin{aligned}
&\int_{\mathbf{R}^N} \frac{\xi_j^2}{|\xi|^2} |\widehat{T_1\varphi}(\xi)|^2 \, d\xi + \int_{\mathbf{R}^N} \frac{\xi_j^2}{|\xi|^2} |\widehat{T_2\varphi}(\xi)|^2 \, d\xi - 2\int_{\mathbf{R}^N} \frac{\xi_j^2}{|\xi|^2} |\widehat{\varphi}(\xi)|^2 \, d\xi \\
&= \frac{8}{\pi} \int_{\mathbf{R}^{N-1}} \frac{\xi_j^2}{|\xi'|} \left| \int_0^\infty \widehat{A\varphi}(\xi_1, \xi') \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \, d\xi_1 \right|^2 d\xi'
\end{aligned} \tag{4.30}
$$

for any $\varphi \in C_c^\infty(\mathbf{R}^N)$, where $j \in \{2, \ldots, , N\}$. It is obvious that the left-hand side of (4.30) defines a continuous functional on $L^2(\mathbf{R}^N)$. By the next lemma, it follows that the right-hand side of (4.30) also defines a continuous functional on $L^2(\mathbf{R}^N)$. Then the density of $C_c^\infty(\mathbf{R}^N)$ in $L^2(\mathbf{R}^N)$ implies that (4.30) holds for any $\varphi \in L^2(\mathbf{R}^N)$.

Lemma 4.10 *Let $j \in \{2, \ldots, N\}$. The bilinear form*

$$
S_1(\varphi, \psi) = \int_{\mathbf{R}^{N-1}} \frac{\xi_j^2}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}(\eta_1, \xi')} \frac{\eta_1}{\eta_1^2 + |\xi'|^2} \, d\eta_1 \, d\xi'
$$

is continuous on $L^2(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$.

Proof. As in (4.13) we have

$$
\left| \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \, d\xi_1 \right| \leq K \frac{1}{|\xi'|^{\frac{1}{2}}} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^2 \, d\xi_1 \right)^{\frac{1}{2}},
$$

where $K = \left(\int_0^\infty \frac{t^2}{(1+t^2)^2} \, dt \right)^{\frac{1}{2}}$. Consequently

$$
\begin{aligned}
|S_1(\varphi, \psi)| &\leq K^2 \int_{\mathbf{R}^{N-1}} \frac{\xi_j^2}{|\xi'|^2} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^2 \, d\xi_1 \right)^{\frac{1}{2}} \left(\int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^2 \, d\eta_1 \right)^{\frac{1}{2}} d\xi' \\
&\leq K^2 \int_{\mathbf{R}^{N-1}} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^2 \, d\xi_1 \right)^{\frac{1}{2}} \left(\int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^2 \, d\eta_1 \right)^{\frac{1}{2}} d\xi' \\
&\leq K^2 \left(\int_{\mathbf{R}^{N-1}} \int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^2 \, d\xi_1 \, d\xi' \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbf{R}^{N-1}} \int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^2 \, d\eta_1 \, d\xi' \right)^{\frac{1}{2}} \\
&\leq K_1 ||\varphi||_{L^2(\mathbf{R}^N)} ||\psi||_{L^2(\mathbf{R}^N)}.
\end{aligned}
$$

□

Since u^2, u_1^2, $u_2^2 \in L^2(\mathbf{R}^3)$ (recall that $H^1(\mathbf{R}^3) \subset L^2(\mathbf{R}^3) \cap L^6(\mathbf{R}^3)$), by exchanging the roles of x_1 and x_2 and using (4.29) and (4.30) we find

$$
\begin{aligned}
&E(u_1) + E(u_2) - 2E(u) \\
&= -\frac{1}{4} \frac{1}{(2\pi)^3} \frac{8}{\pi} \int_{\mathbf{R}^2} \frac{\xi_1^2}{\sqrt{\xi_1^2 + \xi_3^2}} \left| \int_0^\infty \widehat{A_2(u^2)}(\xi_1, \xi_2, \xi_3) \frac{\xi_2}{\xi_1^2 + \xi_2^2 + \xi_3^2} \, d\xi_2 \right|^2 d\xi_1 \, d\xi_3,
\end{aligned} \tag{4.31}
$$

where $A_2\varphi = \frac{1}{2}(\varphi(x_1, x_2, x_3) - \varphi(x_1, -x_2, x_3))$.

Since u is a minimizer, we must have $E(u_1) + E(u_2) - 2E(u) \geq 0$, consequently the integral in the right-hand side of (4.31) must be zero, which is equivalent to

$$
\int_0^\infty \widehat{A_2(u^2)}(\xi_1, \xi_2, \xi_3) \frac{\xi_2}{\xi_1^2 + \xi_2^2 + \xi_3^2} \, d\xi_2 = 0 \qquad \text{a.e. } (\xi_1, \xi_3) \in \mathbf{R}^2. \tag{4.32}
$$

In particular, u_1 and u_2 are also minimizers. However, as in the previous example, (4.32) is not sufficient to prove that $A_2(u^2) = 0$. In order to accomplish this task, we will use the Euler-Lagrange equation of u : since u minimizes E under the constraint $Q(u) = \lambda$, there exists a constant α such that $E'(u) + \alpha Q'(u) = 0$, that is

$$-\Delta u + F'(u) + R_1^2(u^2)u + \alpha G'(u) = 0. \tag{4.33}$$

Lemma 4.11 *If F and G satisfy assumption a) in Theorem 4.9 and $u \in H^1(\mathbf{R}^3)$ is a solution of (4.33), then $u \in W^{3,p}(\mathbf{R}^3)$ for any $p \in [2, \infty)$. In particular, $u \in C^2(\mathbf{R}^3)$.*

Since R_1 and R_1^2 are linear continuous mappings from $L^p(\mathbf{R}^3)$ to $L^p(\mathbf{R}^3)$ for $1 < p < \infty$, the proof of Lemma 4.11 is standard, so we omit it.

Let $I(\varphi)(x) = \displaystyle\int_{\mathbf{R}^3} \frac{\varphi(y)}{|x-y|}\, dy$. Using Lemma 4.5 it is easy to see that $I(u^2) \in W^{2,p}(\mathbf{R}^3)$ for any $p \in (3, \infty]$ and $I(u^2)$ is a C^2 function. Moreover, we have

$$\mathcal{F}(R_1^2(u^2))(\xi) = -\frac{\xi_1^2}{|\xi|^2}\widehat{u^2}(\xi) = -\frac{1}{d_3}\xi_1^2\widehat{I(u^2)}(\xi),$$

where $d_3 = \frac{4\pi^{\frac{3}{2}}}{\Gamma(\frac{1}{2})}$, thus $R_1^2(u^2) = \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2} I(u^2)$. Equation (4.33) can be written as

$$-\Delta u + F'(u) + \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\left(I(u^2)\right)u + \alpha G'(u) = 0. \tag{4.34}$$

Arguing exactly as in the proof of Theorem 4.6, (4.32) implies that $\frac{\partial}{\partial x_2}\left(I(u^2)\right)(x_1, 0, x_3) = 0$ for any $(x_1, x_3) \in \mathbf{R}^2$.

Since u_1 is also a minimizer, it satisfies the Euler-Lagrange equation

$$-\Delta u_1 + F'(u_1) + \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\left(I(u_1^2)\right)u_1 + \beta G'(u_1) = 0. \tag{4.35}$$

The conclusion of Lemma 4.11 is obviously valid for u_1. Since u_1 is symmetric with respect to x_2, $I(u_1^2)$ is also symmetric with respect to x_2 and, consequently, $\frac{\partial}{\partial x_2}\left(I(u_1^2)\right)(x_1, 0, x_3) = 0$ for any $(x_1, x_3) \in \mathbf{R}^2$. We set $U = I(u^2)$ and $U_1 = I(u_1^2)$. Recall that $u(x_1, x_2, x_3) = u_1(x_1, x_2, x_3)$ if $x_2 < 0$; thus U and U_1 are both solutions of

$$\begin{cases} -\Delta W = u^2 & \text{in } \mathbf{R} \times (-\infty, 0) \times \mathbf{R}, \\ W \in C^2(\mathbf{R}^3) \cap W^{2,p}(\mathbf{R}^3) & \text{for } 3 < p \le \infty \\ \frac{\partial W}{\partial x_2}(x_1, 0, x_3) = 0 & \text{for any } (x_1, x_3) \in \mathbf{R}^2. \end{cases} \tag{4.36}$$

It is not hard to see that the solution of (4.36) is unique. Hence we must have $I(u^2) = I(u_1^2)$ in $\mathbf{R} \times (-\infty, 0] \times \mathbf{R}$. In the same way we obtain $I(u^2) = I(u_2^2)$ in $\mathbf{R} \times [0, \infty) \times \mathbf{R}$.

Now we focus our attention on u_1. Making a translation in the x_3 direction if necessary, we may assume that $\displaystyle\int_{\{x_3<0\}} G(u_1(x))\, dx = \int_{\{x_3>0\}} G(u_1(x))\, dx = \frac{\lambda}{2}$. We define

$$w_1(x_1, x_2, x_3) = \begin{cases} u_1(x_1, x_2, x_3) & \text{if } x_3 < 0, \\ u_1(x_1, x_2, -x_3) & \text{if } x_3 \ge 0, \end{cases}$$

$$w_2(x_1,x_2,x_3) = \begin{cases} u_1(x_1,x_2,-x_3) & \text{if } x_3 < 0, \\ u_1(x_1,x_2,x_3) & \text{if } x_3 \geq 0. \end{cases}$$

It is obvious that $Q(w_1) = Q(w_2) = \lambda$. Proceeding as above, we find the identity

$$(4.37)\qquad \begin{aligned} &E(w_1) + E(w_2) - 2E(u_1) \\ &= -\frac{1}{4}\frac{1}{(2\pi)^3}\frac{8}{\pi}\int_{\mathbf{R}^2} \frac{\xi_1^2}{\sqrt{\xi_1^2+\xi_2^2}} \left| \int_0^\infty \widehat{A_3(u_1^2)}(\xi_1,\xi_2,\xi_3)\frac{\xi_3}{\xi_1^2+\xi_2^2+\xi_3^2}\,d\xi_3 \right|^2 d\xi_1\,d\xi_2, \end{aligned}$$

where $A_3\varphi = \frac{1}{2}(\varphi(x_1,x_2,x_3) - \varphi(x_1,x_2,-x_3))$. Since u_1 is a minimizer, it follows from (4.37) that w_1 and w_2 are also minimizers of E under the constraint $Q = \lambda$; hence w_1 and w_2 satisfy the conclusion of Lemma 4.11 and $I(w_1)$, $I(w_2) \in C^2(\mathbf{R}^3) \cap W^{2,p}(\mathbf{R}^3)$ for $p \in (3,\infty]$. Moreover, the integral in the right-hand side of (4.37) must be zero. As previously, this gives $\frac{\partial}{\partial x_3} I(u_1^2)(x_1,x_2,0) = 0$ for any $(x_1,x_2) \in \mathbf{R}^2$. Proceeding as above, we find $I(u_1^2) = I(w_1^2)$ in $\mathbf{R}^2 \times (-\infty,0]$ and $I(u_1^2) = I(w_2^2)$ in $\mathbf{R}^2 \times [0,\infty)$.

Now let us consider the function w_1. It is clear that $w_1(x_1,-x_2,-x_3) = w_1(x_1,-x_2,x_3) = w_1(x_1,x_2,x_3)$, i.e. w_1 is symmetric with respect to x_2 and with respect to x_3. Consider a plane Π in $\mathbf{R}^3$ containing the line $\{(x_1,0,0) \mid x_1 \in \mathbf{R}\}$ and let Π_+ and Π_- be the two half-spaces determined by Π. Since $(x_1,x_2,x_3) \longmapsto (x_1,-x_2,-x_3)$ maps Π_+ onto Π_-, using the symmetry of w_1 we get $\displaystyle\int_{\Pi_+} G(w_1(x))\,dx = \int_{\Pi_-} G(w_1(x))\,dx = \frac{\lambda}{2}$. Let s_Π denote the symmetry in $\mathbf{R}^3$ with respect to Π. We define

$$r_1(x) = \begin{cases} w_1(x) & \text{if } x \in \Pi_-, \\ w_1(s_\Pi(x)) & \text{if } x \in \Pi_+ \end{cases} \qquad \text{and} \qquad r_2(x) = \begin{cases} w_1(s_\Pi(x)) & \text{if } x \in \Pi_-, \\ w_1(x) & \text{if } x \in \Pi_+. \end{cases}$$

Repeating the above arguments we obtain an integral identity analogous to (4.31) and (4.37) which implies that r_1 and r_2 also minimize E subject to the constraint $Q = \lambda$. Furthermore, using the fact that the integral in the right-hand side of this identity must vanish we find

$$(4.38)\qquad \frac{\partial}{\partial n} I(w_1^2)(x_1,x_2,x_3) = 0 \qquad \text{whenever } (x_1,x_2,x_3) \in \Pi,$$

where n is the unit normal to Π. Passing to cylindrical coordinates we write $I(w_1^2)(x_1,x_2,x_3) = I(w_1^2)(x_1, r\cos\theta, r\sin\theta) = \Phi(x_1,r,\theta)$, where $r = \sqrt{x_2^2+x_3^2}$. Since $I(w_1^2)$ is a C^2 function and (4.38) is valid for any plane Π containing $\{(x_1,0,0) \mid x_1 \in \mathbf{R}\}$, (4.38) is equivalent to $\dfrac{\partial\Phi}{\partial\theta} = 0$, that is Φ does not depend on θ, i.e. $I(w_1^2)(x_1,x_2,x_3) = \Phi_1(x_1, \sqrt{x_2^2+x_3^2})$ for some function Φ_1. In other words, we have proved that $I(w_1^2)$ is radially symmetric in the variables (x_2,x_3). In the same way we show that $I(w_2^2)(x_1,x_2,x_3) = \Phi_2(x_1, \sqrt{x_2^2+x_3^2})$ for some function Φ_2. Since $I(u_1^2)$ is continuous on $\mathbf{R}^3$, $I(u_1^2) = I(w_1^2)$ in the half-space $\{x_3 < 0\}$ and $I(u_1^2) = I(w_2^2)$ in the half-space $\{x_3 > 0\}$, we have necessarily $\Phi_1 = \Phi_2$, and then $I(u_1^2)$ is radially symmetric in the variables (x_2,x_3). Similarly, $I(u_2^2)$ is radially symmetric in (x_2,x_3). Recall that $I(u^2) = I(u_1^2)$ in the half-space $\{x_2 < 0\}$ and $I(u^2) = I(u_2^2)$ in the half-space $\{x_2 > 0\}$. But $I(u^2)$ is a continuous function on $\mathbf{R}^3$, thus we must have $I(u^2) = I(u_1^2) = I(u_2^2)$ on $\mathbf{R}^3$, consequently $I(u^2)$ is radially symmetric with respect to (x_2,x_3).

If $u \equiv 0$ in the half-space $\{x_2 < 0\}$, it follows that $u_1 \equiv 0$ in $\mathbf{R}^3$ and then $I(u_1^2) \equiv 0$ which implies $I(u^2) = 0$ in $\mathbf{R}^3$. In this case (4.34) becomes $-\Delta u + F'(u) + \alpha G'(u) = 0$ and from the Unique Continuation Principle we infer that $u \equiv 0$ in $\mathbf{R}^3$, thus u is radially symmetric in a trivial way. Obviously, the case $u \equiv 0$ is excluded if $\lambda \neq 0$.

If $u \not\equiv 0$ in the half-space $\{x_2 < 0\}$, by assumption *b*) there exists $(x_1, x_2, x_3) \in \mathbf{R}^3$, $x_2 < 0$ such that $G'(u(x_1, x_2, x_3)) \neq 0$. Since $u = u_1$ on $\{x_2 < 0\}$ and $I(u^2) = I(u_1^2)$ on $\mathbf{R}^3$, from (4.34) and (4.35) we infer that $\alpha = \beta$. Let $a(x) = \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\left(I(u^2)\right)(x) = \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\left(I(u_1^2)\right)(x)$. We know that a is a continuous and bounded function on $\mathbf{R}^3$. The functions u and u_1 both satisfy the equation $-\Delta w + F'(w) + a(x)w + \alpha G'(w) = 0$ in $\mathbf{R}^3$ and using the Unique Continuation Principle again we conclude that $u \equiv u_1$ in $\mathbf{R}^3$, i.e. u is symmetric with respect to x_2.

In the same way we prove that u is symmetric with respect to x_3 (after possibly a translation). Proceeding as in the proof of Theorem 4.1 we can show that u is symmetric with respect to any plane containing the line $\{(x_1, 0, 0) \mid x_1 \in \mathbf{R}\}$, consequently u is radially symmetric with respect to (x_2, x_3) variables. $\square$

Remark 4.12 *i*) We have stated and proved Theorem 4.9 in dimension $N = 3$ only for simplicity. Replacing the term $\int_{\mathbf{R}^3} |R_1(u^2)|^2(x)\,dx$ in $E(u)$ by $\int_{\mathbf{R}^N} |R_1(H(u))|^2(x)\,dx$ and making suitable assumptions on the function H, this result admits a straightforward generalization to $\mathbf{R}^N$, $N \geq 3$.

ii) We do not know whether the minimizers in Theorem 4.9 are symmetric or not with respect to x_1. Recall that by (2.55) we have

$$\begin{aligned}(4.39)\qquad &\int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2}|\widehat{T_1\varphi}(\xi)|^2\,d\xi + \int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2}|\widehat{T_2\varphi}(\xi)|^2\,d\xi - 2\int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2}|\widehat{\varphi}(\xi)|^2\,d\xi \\ &= -\frac{8}{\pi}\int_{\mathbf{R}^{N-1}} |\xi'|\left|\int_0^\infty \widehat{A\varphi}(\xi)\frac{\xi_1}{\xi_1^2 + |\xi'|^2}\,d\xi_1\right|^2 d\xi'\end{aligned}$$

for any $\varphi \in C_c^\infty(\mathbf{R}^N)$. Clearly, the left-hand side of (4.39) is continuous on $L^2(\mathbf{R}^N)$. Proceeding as in Lemma 4.10, it is easy to see that the right-hand side of (4.39) also defines a continuous functional on $L^2(\mathbf{R}^N)$. Consequently, (4.39) holds for any $\varphi \in L^2(\mathbf{R}^N)$. Using (4.28) and (4.39) we have

$$(4.40)\qquad E(T_1u) + E(T_2u) - 2E(u) = \frac{2}{\pi}\frac{1}{(2\pi)^N}\int_{\mathbf{R}^{N-1}} |\xi'|\left|\int_0^\infty \mathcal{F}(A(H(u)))(\xi)\frac{\xi_1}{|\xi|^2}\,d\xi_1\right|^2 d\xi'.$$

The right-hand side in this integral identity is always nonnegative and (4.40) does not imply the symmetry of minimizers with respect to x_1.

iii) Let us change the sign of the nonlocal term appearing in Theorem 4.9, i.e. let us consider the minimization problem

$$(4.41)\qquad \begin{gathered}\text{minimize } E_*(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \int_{\mathbf{R}^3} F(u)\,dx + \frac{1}{4}\int_{\mathbf{R}^3} |R_1(u^2)|^2\,dx \\ \text{under the constraint } Q(u) := \int_{\mathbf{R}^3} G(u(x))\,dx = \lambda.\end{gathered}$$

The minimizers of this problem (when they exist) give rise to standing waves for equation (4.27) where the sign of the nonlocal term $R_1^2(|u|^2)u$ has been reversed. Clearly, the integral identities that we have do not imply the symmetry of solutions of (4.41) with respect to x_2 and x_3.

The symmetry of minimizers of (4.41) with respect to x_1 is also an open problem. As above, in this case we have the identity

$$(4.42)\quad E_*(T_1u) + E_*(T_2u) - 2E_*(u) = -\frac{2}{\pi}\frac{1}{(2\pi)^3}\int_{\mathbf{R}^2} |\xi'|\left|\int_0^\infty \mathcal{F}(A(u^2))(\xi)\frac{\xi_1}{|\xi|^2}\,d\xi_1\right|^2 d\xi_2\,d\xi_3.$$

If u is a minimizer, the right-hand side of (4.42) must vanish. As in the proof of Theorem 4.9, this implies $\frac{\partial}{\partial x_1} I(u^2)(0, x_2, x_3) = 0$ for any $(x_2, x_3) \in \mathbf{R}^2$. Repeating the argument already used in Theorem 4.9 we get $I(u^2) = I((T_1 u)^2)$ on $\{x_1 \le 0\}$ and $I(u^2) = I((T_2 u)^2)$ on $\{x_1 \ge 0\}$. Moreover, if $\lambda \neq 0$ then u and $u_1 := T_1 u$ satisfy the same Euler-Lagrange equation, namely

$$-\Delta w + F'(w) - \frac{1}{d_3} \frac{\partial^2}{\partial x_1^2} \left(I(w^2) \right) w + \alpha G'(w) = 0. \tag{4.43}$$

Equivalently, defining $U = I(u^2)$ and $U_1 = I(u_1^2)$, we see that (u, U) and (u_1, U_1) are both solutions to the system

$$\begin{cases} -\Delta w + F'(w) - \frac{1}{d_3} \frac{\partial^2 W}{\partial x_1^2} w + \alpha G'(w) = 0, \\ -\Delta W = w^2. \end{cases} \tag{4.44}$$

Moreover, $(u, U) = (u_1, U_1)$ on $\{x_1 \le 0\}$ and u, u_1 satisfy the conclusion of Lemma 4.11. We do not know whether this information together with the boundary condition $\frac{\partial U}{\partial x_1}(0, x_2, x_3) = \frac{\partial U_1}{\partial x_1}(0, x_2, x_3) = 0$ imply that $u \equiv u_1$.

Remark 4.13 If $N = 3$, the nonlocal term in Theorem 4.9 can be written as

$$\int_{\mathbf{R}^3} |R_1(u^2)|^2 \, dx = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2} |\widehat{u^2}(\xi)|^2 \, d\xi = -\frac{1}{d_3 (2\pi)^3} \int_{\mathbf{R}^3} \mathcal{F} \left(\frac{\partial^2}{\partial x_1^2} I(u^2) \right) (\xi) \overline{\widehat{u^2}(\xi)} \, d\xi$$

$$= -\frac{1}{d_3} \int_{\mathbf{R}^3} \frac{\partial^2}{\partial x_1^2} I(u^2)(x) \overline{u^2(x)} \, dx = -\frac{1}{d_3} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} u^2(x) K(x-y) u^2(y) \, dx \, dy,$$

where $K(x) = \frac{\partial^2}{\partial x_1^2} \left(\frac{1}{|x|} \right) = \frac{2x_1^2 - x_2^2 - x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{\frac{5}{2}}}$. Since this kernel changes sign, spherical rearrangements in the variables (x_2, x_3) combined with Riesz' inequality cannot be used to prove the symmetry of minimizers.

Remark 4.14 It is worth to note the following simple idea : let u_* be a minimizer for a variational problem like those studied in this paper. Suppose that one can prove that u_* is a C^1 function and that $\frac{\partial u_*}{\partial n} = 0$ whenever $x \in \Pi$, where Π is any hyperplane in $\mathbf{R}^N$ having the property $\int_{\Pi_-} G(u_*(x)) \, dx = \int_{\Pi_+} G(u_*(x)) \, dx$ (here Π_- and Π_+ are the two half-spaces determined by Π, n is the unit normal to Π and G is the function appearing in the constraint). Proceeding as we did for $I(u^2)$ in in the proof of Theorem 4.9, one can prove that after a translation, u_* is radially symmetric. This method should be useful in problems where the integral identities that one can obtain are not sufficient to deduce the symmetry of minimizers and an unique continuation theorem is unavailable. Unfortunately it cannot give symmetry with respect to only one direction.

5 Some open problems

We close this paper speaking about several problems for which the methods described above (including ours) seem to fail.

First, let us come back to the two minimization problems considered in Theorem 4.1. As before, if u is a minimizer of any of these problems, we may assume that $\int_{\{x_1 < 0\}} G(u) \, dx =$

$\int_{\{x_1>0\}} G(u)\,dx$ and we set $u_1 = T_1 u$ and $u_2 = T_2 u$. Assume that $s \in (1, \frac{3}{2})$. Then the identities (3.26) and (3.27) are still valid (see Corollary 3.5) and we get

$$E(u_1) + E(u_2) - 2E(u) = -\frac{16\sin(s\pi)}{\pi^2} N_s^2(Au) \geq 0 \quad \text{in case A, respectively}$$

$$E(u_1) + E(u_2) - 2E(u) = -\frac{16\sin(s\pi)}{\pi^2} \tilde{N}_s^2(Au) \geq 0 \quad \text{in case B.}$$

It is easy to see that these integral identities work in the wrong direction. Are the minimizers still radially symmetric for $s \in (1, \frac{3}{2})$?

Another problem is to study the symmetry of minimizers of

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{1}{|x-y|} u(x)^2 u(y)^2\,dx\,dy + \int_{\mathbf{R}^3} F(u(x))\,dx$$

subject to the constraint

$$\int_{\mathbf{R}^3} u^2(x)\,dx = \lambda > 0.$$

In the particular case $F(u) = -C|u|^{8/3}$, this problem arises in connection with the Schrödinger-Poisson-Slater system ([22]). Due to the repulsive effect of the nonlocal term, Riesz' inequality as well as the Reflection method work in the wrong direction.

A last problem concerns the symmetry of minimizers of

$$E(u) = \int_{-\infty}^{+\infty} (u_x^2(x) + u^3(x))\,dx - \gamma \int_{-\infty}^{+\infty} |\xi||\hat{u}(\xi)|^2\,d\xi,$$

where $\gamma > 0$, subject to the constraint $\int_{-\infty}^{+\infty} u^2(x)\,dx = \lambda > 0$. These two functionals are conserved quantities for the Benjamin equation (see [1]). Symmetrization and reflection cannot be used due to the sign of the nonlocal term. Oscillating travelling waves for this equation have been found numerically; perhaps this is an indication that the minimizers of the problem above may change sign.

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