Stationary solutions to a nonlinear Schrödinger equation with potential in one dimension

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Abstract

We study the 1-dimensional Gross-Pitaevskii-Schrödinger equation with a potential U moving at velocity v. For a fixed v less than the sound velocity, it is proved that there exist two time-independent solutions if the potential is not too big.

1 Introduction

We consider the 1-dimensional nonlinear Schrdinger (NLS) equation with an external repulsive potential U moving at velocity $v > 0$:

$$iA_t + A_{xx} + A - |A|^2 A - U(x - vt)A = 0, \quad x \in \mathbf{R}, t \in \mathbf{R}. \tag{1.1}$$

This equation arises in many physical contexts. For example, it describes the motion of an impurity (modelized by U) at constant velocity v in a NLS fluid at rest at $+\infty$. The behaviour of equation (1.1) in one dimension is similar to that in higher dimensions, vortices being replaced by propagating localized density depressions which are called gray solitons (see [4]). Equation (1.1) can be put into a hydrodynamical form using Madelung's transformation $A(x,t) = \sqrt{\rho(x,t)}e^{i\phi(x,t)}$, see [6] or [7]. This change of variables leads to the system

$$\rho_t + 2(\rho\phi_x)_x = 0, \tag{1.2}$$

$$\phi_t + |\phi_x|^2 - \frac{\rho_{xx}}{2\rho} + \frac{|\rho_x|^2}{4\rho^2} - 1 + \rho + U(x - vt) = 0. \tag{1.3}$$

Note that the Madelung transformation is singular when $A = 0$. Equation (1.2) and the derivative with respect to x of (1.3) are the equation of conservation of mass, respectively Euler's equation for a compressible inviscid fluid of density ρ and velocity $2\phi_x$. We require that the fluid be at rest at infinity with density 1.

This gives the "boundary condition" $A(x) \longrightarrow 1$ at $+\infty$. Taking the derivative with respect to t of (1.3) and substituting ρ_t from (1.2) we get

$$\phi_{tt} - 2\rho\phi_{xx} - 2\rho_x\phi_x + \frac{\partial}{\partial t}\Big(|\phi_x|^2 - \frac{\rho_{xx}}{2\rho} + \frac{|\rho_x|^2}{4\rho^2} + U(x - vt)\Big) = 0. \tag{1.4}$$

For a small oscillatory motion (i.e. a sound wave), all the nonlinear terms appearing in (1.4) except $2\rho\phi_{xx}$ may be neglected and the velocity potential ϕ essentially obeys to the wave equation $\phi_{tt} - 2\rho\phi_{xx} - vU'(x - vt) = 0$. We see that sound waves propagate with velocity $\sqrt{2\rho}$ and therefore the sound velocity at infinity is $\sqrt{2}$.

Equation (1.1) can be written in the frame of the moving impurity as

$$iA_t - ivA_x + A_{xx} + A - |A|^2A - U(x)A = 0. \tag{1.5}$$

In this context, it describes the flow of a NLS fluid past a fixed obstacle when a flow of constant density is injected at velocity v at infinity. The obstacle is modelized by the localized potential U. This problem was considered by V. Hakim in [4]. In the case of a Dirac potential, he proved the existence of a critical velocity v_c such that for $v < v_c$ there exist two stationary solutions of (1.5) (i.e. solutions which do not depend on t), one of them being stable and the other unstable. Using formal asymptotic expansions and numerical experiments, he showed that a similar phenomenon takes place for small potentials and for slowly varying potentials (i.e. potentials of the form $U(\varepsilon x)$, ε small). In all these cases, the two solutions become identical at critical velocity and no stationary solution exists for $v > v_c$. The critical velocity depends on the obstacle and is less than the sound velocity. Above the critical velocity the characteristics of the time-dependent flow were studied numerically. It was found that the obstacle emitted repeatedly gray solitons propagating downstream and sound propagating upstream.

The aim of this paper is to prove rigorously that, for a general potential U, equation (1.5) admits two stationary solutions if the velocity v is reasonably small.

Since one expects, from physical considerations, that the solutions are slowly varying and have a modulus tending to 1 at $\pm\infty$, we seek for solutions of the form $A(x) = (1 + r(x))e^{i\theta(x)}$ with $r(x) \longrightarrow 0$ and $\theta'(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Substituting this expression in (1.5) one finds that the real functions r and θ must satisfy

$$-vr_x + 2r_x\theta_x + (1 + r)\theta_{xx} = 0, \tag{1.6}$$

$$v(1 + r)\theta_x + r_{xx} - (1 + r)\theta_x^2 + (1 + r) - (1 + r)^3 - U(x)(1 + r) = 0. \tag{1.7}$$

Multiplying equation (1.6) by $1 + r$ and integrating we find

$$\theta_x = \frac{v}{2}\Big(1 - \frac{1}{(1 + r)^2}\Big). \tag{1.8}$$

This determines θ_x (half of the fluid velocity) as a function of $(1 + r)^2$ (the local fluid density). Introducing (1.8) in (1.7) we find that r satisfies the equation (also derived by V. Hakim):

$$-r_{xx} - (1 + r) + (1 + r)^3 - \frac{v^2}{4}\Big(1 + r - \frac{1}{(1 + r)^3}\Big) + (1 + r)U(x) = 0. \tag{1.9}$$

From now on, we will focus our attention on finding solutions of (1.9). Once this task accomplished, it is easy to determine the corresponding phase θ from (1.8). Then $A(x) = (1 + r(x))e^{i\theta(x)}$ will be a solution of (1.5).

Of course it is interesting to find solutions of (1.9) under the more general possible assumptions on U. In what follows, we suppose that U is a positive Borel measure with bounded total variation. A few notations are in order: by $\int_{\mathbf{R}} f(x)U(x)dx$ we denote the integral of a function f with respect to the measure U and by $||U||$ the total variation of U, i.e. $||U|| = \int_{\mathbf{R}} U(x)dx$. If $f \in L^\infty(\mathbf{R})$, then fU is also a Borel measure of bounded total variation and therefore $fU \in \mathcal{D}'(\mathbf{R})$. In particular, if $r \in L^\infty(\mathbf{R})$ and $r \neq -1$ a.e., all quantities appearing in (1.9) make sense in $\mathcal{D}'(\mathbf{R})$.

We discuss now what happens if U vanishes on some interval I. It is easily seen that equation (1.9) can be integrated explicitly on this interval. This simple observation gives an obstruction to the existence of stationary solutions of (1.5) for v greater than $\sqrt{2}$ (which is the sound velocity at infinity) in the case of a potential with compact support.

Indeed, suppose that $U \equiv 0$ on an interval I. On this interval equation (1.9) becomes

$$-r_{xx} - (1+r) + (1+r)^3 - \frac{v^2}{4}\Big(1 + r - \frac{1}{(1+r)^3}\Big) = 0. \tag{1.10}$$

We remark that if $r > -1$ is a continuous solution, then r_{xx} is also continuous, therefore $r \in C^2(I)$. Multiplying (1.10) by $2r_x$ and integrating, it is easy to see that there exists a constant C such that

$$-r_x^2 + \frac{1}{2}((1+r)^2 - 1)^2 - \frac{v^2}{4}\Big(1 + r - \frac{1}{1+r}\Big)^2 + C = 0. \tag{1.11}$$

If I is of the form $(-\infty, a)$ or (b, ∞), the condition $r \longrightarrow 0$ at $\pm\infty$ implies $C = 0$, that is

$$r_x^2 = \frac{1}{2}((1+r)^2 - 1)^2 - \frac{v^2}{4}\Big(1 + r - \frac{1}{1+r}\Big)^2 = r^2(r+2)^2\Big(\frac{1}{2} - \frac{v^2}{4}\frac{1}{(1+r)^2}\Big). \tag{1.12}$$

Since $r^2(r+2)^2\Big(\frac{1}{2} - \frac{v^2}{4}\frac{1}{(1+r)^2}\Big) < 0$ for $r \in (-1, -1 + \frac{v}{\sqrt{2}}) \setminus \{0\}$ and $r_x^2 \geq 0$, we see that any solution r of (1.9) cannot take values in $(-1, -1 + \frac{v}{\sqrt{2}}) \setminus \{0\}$. If v is greater than $\sqrt{2}$, any solution of (1.9) that tends to zero at $\pm\infty$ must be identically zero on I (since otherwise, by continuity it would take values sufficiently close to 0, but different from 0, which is impossible).

If $v \leq \sqrt{2}$, any solution r of (1.9) must be less than or equal to 0 on $\mathbf{R}$ by the maximum principle. Indeed, the function $x \longmapsto \psi_v(x) = -(1+x) + (1+x)^3 - \frac{v^2}{4}\Big(1 + x - \frac{1}{(1+x)^3}\Big)$ is strictly increasing and positive on $(0, \infty)$. Suppose that r achieves a positive maximum at x_0. Then $r''(x_0) \leq 0$. On the other hand, from (1.9) we infer that $r''(x_0) \geq \psi_v(r(x_0)) > 0$, a contradiction.

Suppose that $U \equiv 0$ on an interval I of the form $(-\infty, a)$ or (b, ∞). If $v = \sqrt{2}$, we see from (1.12) that we have also $r \geq 0$ on I, and therefore $r \equiv 0$ on I. If $v < \sqrt{2}$, we must have $-1 + \frac{v}{\sqrt{2}} \leq r \leq 0$ on I.

Suppose that $v \geq \sqrt{2}$. In the particular case $U = g\delta$ (where δ is the Dirac measure and $g \geq 0$), one has $r \equiv 0$ on $(-\infty, 0) \cup (0, \infty)$; consequently, if $g > 0$, (1.9) does not admit solutions and if $g = 0$, it admits only the trivial solution. If U has a compact support with $supp(U) \subset (a, b)$ it follows that any solution r of (1.9) that tends to zero at $\pm\infty$ must vanish on $\mathbf{R} \setminus (a, b)$. But this gives too many constraints (r and its derivatives should vanish at a and b) and so we expect that (1.9) does not possess solutions satisfying the "boundary condition" $r \longrightarrow 0$ at $\pm\infty$ if $v \geq \sqrt{2}$ and $U \neq 0$.

From now on, we will suppose throughout that $0 < v < \sqrt{2}$.

This paper is organized as follows. In the next section we give a variational formulation of equation (1.9) and we introduce our main tools. It will be seen that the solutions of (1.9) are the critical points of a functional E defined on the space $H^1(\mathbf{R})$. Section 3 is devoted to a detailed study of the particular case $U = g\delta$, where the solutions are known explicitly. It is proved that there exists a positive function $\varphi(v)$ such that if $0 < g < \varphi(v)$, there are exactly two solutions of (1.9). One of them minimizes E on an open set of $H^1(\mathbf{R})$ and the other is a critical point of E of mountain-pass type. The two solutions are the same when $g = \varphi(v)$ and no solution exists when $g > \varphi(v)$. In the general case, we show that an analogous phenomenon takes place. Our main result is:

Theorem 1.1 *a) There exists a function $\varphi_1(v) > 0$ such that if $||U|| < \varphi_1(v)$, then E admits a minimizer on an open set (which will be described later) of $H^1(\mathbf{R})$.*

b) There exists a function $\varphi_2(v) > 0$ such that if $||U|| < \varphi_2(v)$ and U has compact support, E admits a second critical point (of "mountain-pass" type).

We have $\varphi(v) > \varphi_1(v) > \varphi_2(v)$ for any $v \in [0, \sqrt{2})$. The graphs of these functions are given in Fig. 1 below. It is quite clear that the existence of nontrivial solutions for (1.9) should depend also on the shape of U, not only on its total variation. Therefore for a given potential U, we expect to have a nontrivial solution of (1.9) for values of v slightly larger than $\varphi_1^{-1}(||U||)$ and two distinct solutions for v slightly larger than $\varphi_2^{-1}(||U||)$.

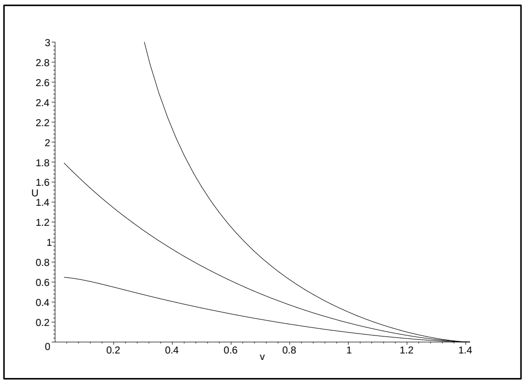

Fig. 1. The graphs of functions φ, φ_1 and φ_2.

The proof of part a) in Theorem 1.1 is rather classical and is given in Section 4. We prove part b) in Section 5. The main difficulty is that the Palais-Smale sequences of E do not converge. We use a theorem of Ghoussoub and Preiss [3] to obtain Palais-Smale sequences with a supplementary property which enables us to deduce their convergence to a solution of (1.9). We have also to impose further restriction on the total mass of U in order to be sure that this second solution is different from that one obtained in Section 4.

2 Variational formulation

Consider the set $V = \{u \in H^1(\mathbf{R}) | \inf_{s\in\mathbf{R}} u(s) > -1\}$. Clearly V is a not-empty open subset of $H^1(\mathbf{R})$ (recall that $H^1(\mathbf{R})$ is continuously embeded in $C_b^0(\mathbf{R})$). We introduce the following functionals:

$$G : V \longrightarrow \mathbf{R}, \quad G(u) = \int_{\mathbf{R}} |u'(x)|^2 + \frac{1}{4}u^2(x)(u(x)+2)^2\Big(2 - \frac{v^2}{(u(x)+1)^2}\Big)dx,$$

$$H : H^1(\mathbf{R}) \longrightarrow \mathbf{R}, \quad H(u) = \int_{\mathbf{R}} u(x)(u(x)+2)U(x)dx,$$

$$E : V \longrightarrow \mathbf{R}, \qquad E(u) = G(u) + H(u).$$

It is easy to check that the functionals G and H are well defined and of class C^1 on V, respectively on $H^1(\mathbf{R})$. A function $r \in V$ satisfies (1.9) (in the distributional sense) if and only if r is a critical point of E.

We want to study the behaviour of $G(u)$ in terms of the variations of the function u. For this purpose, we use the following simple observation:

Remark 2.1 Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function such that $f(0) = 0$. Put $F(x) = \displaystyle\int_0^x f(s)ds$. Then for any $u \in H^1(\mathbf{R})$ and any $a, b \in \mathbf{R},\ a < b$ we have

$$(2.1) \quad |F(u(b)) - F(u(a))| = |\int_a^b f(u(s))u'(s)ds| \le \frac{1}{2}\int_a^b |f(u(s))|^2 + |u'(s)|^2 ds.$$

If $F(u(b)) \ge F(u(a))$, we have equality in (2.1) if and only if $u'(s) = f(u(s))$ a.e. on $[a, b]$. If $F(u(b)) < F(u(a))$, equality holds if and only if $u'(s) = -f(u(s))$ a.e. on $[a, b]$. In particular, for any $a \in \mathbf{R}$ one has $|F(u(a))| \le \frac{1}{2}\displaystyle\int_{-\infty}^a |f(u(s))|^2 + |u'(s)|^2 ds$ and $|F(u(a))| \le \frac{1}{2}\displaystyle\int_a^\infty |f(u(s))|^2 + |u'(s)|^2 ds$. Hence

$$(2.2) \qquad 4|F(u(a))| \le \int_{-\infty}^{\infty} |f(u(s))|^2 + |u'(s)|^2 ds, \quad \forall a \in \mathbf{R}.$$

Moreover, equality holds in (2.2) if and only if $u' = \sigma f(u)$ a.e. on $(-\infty, a)$ and $u' = -\sigma f(u)$ a.e. on (a, ∞), where $\sigma = sgn(F(u(a)))$.

Now take $f : [-1 + \frac{v}{\sqrt{2}}, \infty) \longrightarrow \mathbf{R}$,

$$(2.3) \qquad f(x) = \frac{1}{2}x(x+2)\sqrt{2 - \frac{v^2}{(1+x)^2}}$$

and let $F(x) = \displaystyle\int_0^x f(s)ds$. Observe that f is negative on $(-1+\frac{v}{\sqrt{2}}, 0)$ and positive on $(0,\infty)$, hence F is decreasing on $[-1+\frac{v}{\sqrt{2}}, 0]$ and increasing on $[0,\infty)$, so that F is positive on $[-1+\frac{v}{\sqrt{2}}, \infty) \setminus \{0\}$.

Let $r \in H^1(\mathbf{R})$ be so that $\inf\limits_{x\in\mathbf{R}} r(x) = r(x_0) = a \in [-1+\frac{v}{\sqrt{2}}, 0]$. Applying the previous remark we obtain that

$$0 \le 4F(a) = 4F(r(x_0)) \le G(r)$$

and equality holds if and only if $r'(x) = f(r(x))$ a.e. on $(-\infty, x_0)$ and $r'(x) = -f(r(x))$ a.e. on (x_0,∞). Solving the Cauchy problem

$$\left\{ \begin{array}{l} r'(x) = f(r(x)) \text{ on } (-\infty, 0] \\ r(0) = a \end{array} \right. \tag{2.4}$$

we find the solution

(2.5)

$$r_{1,a}(x) = -1 + \sqrt{\frac{v^2}{2} + (1-\frac{v^2}{2})\tanh^2(\frac{1}{2}\sqrt{2-v^2}(x+c(a)))}, \quad a \in [-1+\frac{v}{\sqrt{2}}, 0)$$

where $c(a) = \frac{1}{\sqrt{2-v^2}} \ln \frac{\sqrt{2-v^2}-\sqrt{2(a+1)^2-v^2}}{\sqrt{2-v^2}+\sqrt{2(a+1)^2-v^2}}$, respectively $r_{1,0} \equiv 0$ if $a = 0$. It is obvious that the Cauchy problem $\left\{ \begin{array}{l} r'(x) = -f(r(x)) \text{ on } [0,\infty) \\ r(0) = a \end{array} \right.$ has the solution $r_{2,a}(x) = r_{1,a}(-x)$. We put

$$r_a(x) = \left\{ \begin{array}{ll} r_{1,a}(x) & \text{if } \ x \le 0 \\ r_{2,a}(x) & \text{if } \ x > 0. \end{array} \right.$$

The functions $(r_a)_{a\in[-1+\frac{v}{\sqrt{2}},0]}$ will be very useful in what follows. We list below some of their basic properties.

Lemma 2.2 *The following assertions hold:*

i) $r_a \in H^1(\mathbf{R})$ *and the mapping* $a \longmapsto r_a$ *is continuous from* $[-1+\frac{v}{\sqrt{2}}, 0]$ *to* $H^1(\mathbf{R})$.

ii) r_a *is symmetric about 0, decreasing on* $(-\infty, 0]$ *and increasing on* $[0,\infty)$ *and tends exponentially to zero at* $\pm\infty$.

iii) r_a *is* C^∞ *on* $\mathbf{R}\setminus\{0\}$.

iv) $c(-1+\frac{v}{\sqrt{2}}) = 0$ *and* c *is strictly decreasing on* $[-1+\frac{v}{\sqrt{2}}, 0)$ *with* $\lim\limits_{a\uparrow 0} c(a) = -\infty$.

v) $r_{-1+\frac{v}{\sqrt{2}}}$ *is of class* C^1 *on* $\mathbf{R}$ *with* $r'_{-1+\frac{v}{\sqrt{2}}}(0) = 0$. *Moreover, for each* a *we have* $r_a(x) = r_{-1+\frac{v}{\sqrt{2}}}(x+c(a))$ *for* $x \le 0$, *respectively* $r_a(x) = r_{-1+\frac{v}{\sqrt{2}}}(x-c(a))$ *for* $x > 0$.

vi) $G(r_a)) = 4F(a)$ *and* r_a *is the unique solution of the minimization problem: "minimize* $G(r)$ *under the constraint* $r(0) = a$."

vii) If $x < y \leq 0$ or $0 \leq x < y$, then for any function $v \in H^1_{loc}(\mathbf{R})$ verifying $v(x) = r_a(x)$, $v(y) = r_a(y)$ and $v \geq -1 + \frac{v}{\sqrt{2}}$ on (x, y) we have

$$\int_x^y |r_a'(s)|^2 + f^2(r_a(s))ds = 2|F(v(y)) - F(v(x))| \leq \int_x^y |v'(s)|^2 + f^2(v(s))ds. \tag{2.6}$$

The proof is obvious.

For each $a \in [-1 + \frac{v}{\sqrt{2}}, 0]$ define

$$h(a) = \inf\{E(u) \mid u \in H^1(\mathbf{R}), \inf_{x \in \mathbf{R}} u(x) = a\}.$$

Lemma 2.3 *The function h has the following properties:*

i) $h(a) \geq 4F(a) + a(a+2)||U||$, $\forall a \in [-1 + \frac{v}{\sqrt{2}}, 0]$.

ii) For all $k > 0$ and $a \in [-1 + \frac{v}{\sqrt{2}}, 0]$ we have

$$h(a) \leq 4F(a) + 2kf^2(a) + a(a+2)||\chi_{[-k,k]}U||.$$

iii) $h : [-1 + \frac{v}{\sqrt{2}}, 0] \longrightarrow \mathbf{R}$ is continuous, $h(0) = 0$ and

$$h(-1 + \frac{v}{\sqrt{2}}) = 4F(-1 + \frac{v}{\sqrt{2}}) + (\frac{v^2}{2} - 1)||U||. \tag{2.7}$$

Proof. $i)$ is clear because for any $u \in H^1(\mathbf{R})$ such that $\inf_{x \in \mathbf{R}} u(x) = a$, we have $G(u) \geq 4F(a)$ and $H(u) \geq a(a+2)||U||$ (note that the function $y \longmapsto y(y+2)$ is increasing on $[-1, \infty)$).

$ii)$ Define

$$u_{a,k}(x) = \begin{cases} r_a(x+k) & \text{if} \quad x < -k \\ a & \text{if} \quad -k \leq x \leq k \\ r_a(x-k) & \text{if} \quad x > k. \end{cases} \tag{2.8}$$

Obviously $u_{a,k} \in H^1(\mathbf{R})$, $\inf_{x \in \mathbf{R}} u_{a,k}(x) = a$, $G(u_{a,k}) = 4F(a) + 2kf^2(a)$ and $H(u_{a,k}) \leq \int_{-k}^k u_{a,k}(u_{a,k}+2)\chi_{[-k,k]}(x)U(x)dx = a(a+2)||\chi_{[-k,k]}U||$. Since by definition $h(a) \leq E(u_{a,k}) = G(u_{a,k}) + H(u_{a,k})$, $ii)$ follows.

$iii)$ It is clear that $h(0) = 0$. Because $f(-1 + \frac{v}{\sqrt{2}}) = 0$, $i)$ and $ii)$ give

$$\begin{aligned} &4F(-1 + \tfrac{v}{\sqrt{2}}) + (\tfrac{v^2}{2} - 1)||U|| \leq h(-1 + \tfrac{v}{\sqrt{2}}) \\ &\leq 4F(-1 + \tfrac{v}{\sqrt{2}}) + (\tfrac{v^2}{2} - 1)||\chi_{[-k,k]}U|| \end{aligned}$$

for all $k > 0$. Passing to the limit as $k \longrightarrow \infty$, we obtain (2.7).

Let $\varepsilon > 0$ be arbitrary, but fixed. Take k_ε sufficiently large so that $||\chi_{[-k_\varepsilon,k_\varepsilon]}U|| > ||U|| - \varepsilon$. Using $i)$ and $ii)$ we get

$$4F(a) + a(a+2)||U|| \leq h(a) \leq 4F(a) + 2k_\varepsilon f^2(a) + a(a+2)(||U|| - \varepsilon). \tag{2.9}$$

Letting $a \longrightarrow -1+\frac{v}{\sqrt{2}}$ (respectively $a \longrightarrow 0$) in (2.9) we obtain

$$h\Big(-1+\frac{v}{\sqrt{2}}\Big) \leq \liminf_{a\downarrow -1+\frac{v}{\sqrt{2}}} h(a) \leq \limsup_{a\downarrow -1+\frac{v}{\sqrt{2}}} h(a) \leq h\Big(-1+\frac{v}{\sqrt{2}}\Big)+\varepsilon\Big(1-\frac{v^2}{2}\Big),$$

respectively

$$0 = h(0) \leq \liminf_{a\uparrow 0} h(a) \leq \limsup_{a\uparrow 0} h(a) \leq 0.$$

Since ε was arbitrary, we infer that h is continuous at 0 and $-1+\frac{v}{\sqrt{2}}$.

It remains to prove that h is continuous at any point $a \in (-1+\frac{v}{\sqrt{2}}, 0)$. Fix such an a and let $a_n \longrightarrow a$. All we have to do is to show that $h(a_n) \longrightarrow h(a)$.

Let $\varepsilon > 0$ be arbitrary, but fixed. Consider $u \in H^1(\mathbf{R})$ such that $\inf_{x\in\mathbf{R}} u(x) = a$ and $E(u) < h(a)+\varepsilon$. By continuity of E, $E(\frac{a_n}{a}u) \longrightarrow E(u)$ as $n \longrightarrow \infty$, so $E(\frac{a_n}{a}u) < h(a)+\varepsilon$ if n is sufficiently large. Since $\inf_{x\in\mathbf{R}} \frac{a_n}{a}u(x) = a_n$, it follows that $h(a_n) \leq E(\frac{a_n}{a}u) < h(a)+\varepsilon$ for all n sufficiently large. Thus $\limsup_{n\to\infty} h(a_n) \leq h(a)+\varepsilon$.

Now fix $\delta \in (-1+\frac{v}{\sqrt{2}}, a)$. For each n sufficiently large (so that $a_n > \delta$), choose $u_n \in H^1(\mathbf{R})$ verifying $\inf_{x\in\mathbf{R}} u_n(x) = a_n$, $a_n \leq u_n \leq 0$ and $E(u_n) < h(a_n)+\varepsilon$ (this is possible because $E(-u^-) \leq E(u)$, $\forall u \in V$, where $u^- = -\min(u,0)$). Note that f is a Lipschitz function on $[\delta, 0]$; let L_δ be its Lipschitz constant. Observe that there exists $C_\delta > 0$ such that $f^2(x) \geq C_\delta x^2$, $\forall x \in [\delta, 0]$. It follows that

$$\int_{\mathbf{R}} |u_n'|^2 dx + C_\delta \int_{\mathbf{R}} u_n^2 dx \leq G(u_n) = E(u_n) - H(u_n) < h(a_n)+\varepsilon - a_n(a_n+2)||U||.$$

It is seen from *i)* and *ii)* that h is bounded on $[-1+\frac{v}{\sqrt{2}}, 0]$, hence (u_n) is a bounded sequence in $H^1(\mathbf{R})$. Then we have

$$\int_{\mathbf{R}} \frac{a^2}{a_n^2}|u_n'|^2 dx - \int_{\mathbf{R}} |u_n'|^2 dx \longrightarrow 0 \text{ as } n \longrightarrow \infty;$$

$$\begin{aligned}
&\Big|\int_{\mathbf{R}} f^2(\tfrac{a}{a_n}u_n) - f^2(u_n)dx\Big| \\
&\leq L_\delta\Big(\int_{\mathbf{R}} |\tfrac{a}{a_n}u_n - u_n|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbf{R}} |f(\tfrac{a}{a_n}u_n)+f(u_n)|^2 dx\Big)^{\frac{1}{2}} \\
&\leq L_\delta|\tfrac{a}{a_n}-1|\Big(\int_{\mathbf{R}} u_n^2 dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbf{R}} 2f^2(\tfrac{a}{a_n}u_n)+2f^2(u_n)dx\Big)^{\frac{1}{2}} \longrightarrow 0 \text{ as } n \longrightarrow \infty;
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_{\mathbf{R}} \Big(\tfrac{a}{a_n}u_n(\tfrac{a}{a_n}u_n+2) - u_n(u_n+2)\Big)U(x)dx\Big| \\
&\leq \Big(|\tfrac{a^2}{a_n^2}-1|\delta^2 + 2|\tfrac{a}{a_n}-1|\cdot|\delta|\Big)||U|| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Consequently, $\lim_{n\to\infty}(E(\frac{a}{a_n}u_n) - E(u_n)) = 0$. But $\inf_{x\in\mathbf{R}} \frac{a}{a_n}u_n(x) = a$ and so

$$h(a) \leq E(\frac{a}{a_n}u_n) < h(a_n)+\varepsilon + \Big(E(\frac{a}{a_n}u_n) - E(u_n)\Big).$$

Thus $h(a) \leq \liminf_{n\longrightarrow\infty} h(a_n)+\varepsilon$. Therefore we proved that

$$h(a)-\varepsilon \leq \liminf_{n\longrightarrow\infty} h(a_n) \leq \limsup_{n\longrightarrow\infty} h(a_n) \leq h(a)+\varepsilon.$$

Since ε was arbitrary, it follows that $\lim\limits_{n\longrightarrow\infty} h(a_n) = h(a)$. This proves the continuity of h at $a \in (-1 + \frac{v}{\sqrt{2}}, 0)$. $\Box$

Remark 2.4 It can be proved that if U has compact support, there exists $u_a \in H^1(\mathbf{R})$ such that $\inf\limits_{x\in\mathbf{R}} u_a(x) = a$ and $E(u_a) = h(a)$ (that is, there exists a "minimizer at level a"). We do not give here the proof because we do not make use of this result.

If u_a could be chosen in order to have a continuous map $a \longmapsto u_a$ from $[-1 + \frac{v}{\sqrt{2}}, 0]$ into $H^1(\mathbf{R})$, then the proofs in Section 5 can be considerably simplified and the results slightly strengthened. We were not able to prove that a continuous path of "minimizers at level a" exists for a general U.

3 The case $U = g\delta$ $(g > 0)$

The case $U = g\delta$ $(g > 0)$ is very simple and one can find explicitly the solutions of (1.9) (see [4]); however, it is quite instructive and gives a good feeling of what kind of result can be expected when U is a positive Borel measure.

Consider the functions r_a, $a \in [-1 + \frac{v}{\sqrt{2}}, 0]$ introduced in the previous section. On $(-\infty, 0)$ we have $r_a'' = (r_a')' = f(r_a)' = f'(r_a)r_a' = f(r_a)f'(r_a) = \frac{1}{2}(f^2)'(r_a)$, that is $r_a'' = -(1 + r_a) + (1 + r_a)^3 - \frac{v^2}{4}(1 + r_a - \frac{1}{(1+r_a)^3})$. Obviously the same is true on $(0, \infty)$. Moreover,

$$\lim_{x\uparrow 0} r_a'(x) = \lim_{x\uparrow 0} f(r_a(x)) = f(a),$$
$$\lim_{x\downarrow 0} r_a'(x) = \lim_{x\downarrow 0} -f(r_a(x)) = -f(a).$$

We obtain that r_a satisfies (1.9) for $U = -\frac{2f(a)}{a+1}\delta$ (note that $-\frac{2f(a)}{a+1} \geq 0$).

Conversely, let $r \in H^1(\mathbf{R})$ be a solution of (1.9) for $U = g\delta$, $g \geq 0$. From the discussion in Introduction it follows that $-1 + \frac{v}{\sqrt{2}} \leq r(x) \leq 0$, $\forall x \in \mathbf{R}$, $r \in C^2(\mathbf{R} \setminus \{0\})$ and (1.12) is true, i.e. $r_x^2 = f^2(r)$ on $(-\infty, 0) \cup (0, \infty)$.

Observe that 0 is not a solution of (1.9) if $U \neq 0$. Let I be a maximal interval such that $I \subset \mathbf{R} \setminus \{0\}$ and $r \neq 0$, $r \neq -1 + \frac{v}{\sqrt{2}}$ on I. Since r_x is continuous on I and $f(r) \neq 0$ if $r \notin \{0, -1 + \frac{v}{\sqrt{2}}\}$, we have either $r_x = f(r)$ on I or $r_x = -f(r)$ on I.

Let $a = r(0)$. If $a = 0$ or $a = -1 + \frac{v}{\sqrt{2}}$, it follows from (1.12) that $\lim\limits_{x\to 0} r'(x) = 0$, hence r_x may be extended by continuity at 0. Moreover, since $\lim\limits_{x\to 0} r''(x)$ exists, the continuous extension of r_x is differentiable at $x = 0$ and consequently r satisfies (1.9) for $U = 0$, that is we must have $g = 0$. So if $g > 0$, then necessarily $a = r(0) \in (-1 + \frac{v}{\sqrt{2}}, 0)$. Let

$$x_1 = \inf\{x < 0 \mid r \neq 0, r \neq -1 + \tfrac{v}{\sqrt{2}} \text{ on } (x, 0)\} \text{ and}$$
$$y_1 = \sup\{y > 0 \mid r \neq 0, r \neq -1 + \tfrac{v}{\sqrt{2}} \text{ on } (0, y)\}.$$

Clearly $x_1 < 0$, $y_1 > 0$ and the sign of r' does not change on $(x_1, 0)$ and on $(0, y_1)$. If $r' = f(r)$ or if $r' = -f(r)$ on $(x_1, 0) \cup (0, y_1)$, then r satisfies (1.9) with $U = 0$ on (x_1, y_1), a contradiction. If $r' = -f(r)$ on $(x_1, 0)$ and $r' = f(r)$ on $(0, y_1)$, then r satisfies (1.9) with $U = \frac{2f(a)}{a+1}\delta$ and $g = \frac{2f(a)}{a+1} < 0$, again a contradiction. It remains that $r' = f(r)$ on $(x_1, 0)$ and $r' = -f(r)$ on $(0, y_1)$. By a standard argument we infer that $x_1 = -\infty$, $y_1 = \infty$ and $r = r_a$. Thus we have proved that (1.9) has no other solutions than the functions r_a introduced in Section 2. Obviously we must have $g = -\frac{2f(a)}{a+1}$ if r_a is a solution.

Note that in the case $U \equiv 0$, the problem is translation invariant. Following the above discussion, one easily proves that the only solutions of (1.9) are 0 and $r_{-1+\frac{v}{\sqrt{2}}}(\cdot - z)$, $z \in \mathbf{R}$.

It is natural to ask then: for a given $g > 0$, how many solutions are there? The answer is: exactly as many as the roots of the equation

$$g = -\frac{2f(a)}{a+1} \tag{3.1}$$

are. Let $k_v(a) = -\frac{2f(a)}{a+1}$. Obviously k_v is differentiable on $(-1 + \frac{v}{\sqrt{2}}, 0]$ and a straightforward computation shows that $k_v'(a) > 0$ on $(-1+\frac{v}{\sqrt{2}}, a_\star(v))$ and $k_v'(a) < 0$ on $(a_\star(v), 0)$, where $a_\star(v) = -1 + \sqrt{\frac{-1+\sqrt{1+4v^2}}{2}}$. So k_v is increasing on $[-1 + \frac{v}{\sqrt{2}}, a_\star(v)]$, decreasing on $[a_\star(v), 0]$, $k_v(-1+\frac{v}{\sqrt{2}}) = k_v(0) = 0$ and k_v has a maximum at $a_\star(v)$. Let

$$\varphi(v) = k_v(a_\star(v)) = \frac{(1 + \sqrt{1+4v^2} - 2v^2)\sqrt{2 - v^2}}{2v\sqrt{1 + v^2 + \sqrt{1+4v^2}}}. \tag{3.2}$$

Thus, if $g < \varphi(v)$, equation (3.1) has exactly two roots $a_1 \in (a_\star(v), 0)$ and $a_2 \in (-1 + \frac{v}{\sqrt{2}}, a_\star(v))$. Clearly $a_1 \downarrow a_\star(v)$ and $a_2 \uparrow a_\star(v)$ as $g \uparrow \varphi(v)$. When $g = \varphi(v)$, we have the double root $a_\star(v)$. If $g > \varphi(v)$, (3.1) has no roots. Consequently, if $g < \varphi(v)$ the equation (1.9) with $U = g\delta$ has two solutions, namely r_{a_1} and r_{a_2}. These solutions are "merging" when $g = \varphi(v)$. For $g > \varphi(v)$, equation (1.9) does not admit solutions.

Note that the function φ is continuous and strictly decreasing on $(0, \sqrt{2}]$, $\lim_{v \downarrow 0} \varphi(v) = \infty$ and $\varphi(\sqrt{2}) = 0$. Therefore φ^{-1} exists, is strictly decreasing, $\varphi^{-1}(0) = \sqrt{2}$ and $\lim_{g \to \infty} \varphi^{-1}(g) = 0$. We summarize the above discussion in the following

Proposition 3.1 *Consider the equation (1.9) with the potential $U = g\delta$.*

i) For a fixed velocity $v \in (0, \sqrt{2})$, the equation has exactly two solutions if $g \in (0, \varphi(v))$, where $\varphi(v)$ is given by (3.2). If $g = \varphi(v)$, there exists only one solution. If $g > \varphi(v)$, the equation does not admit solutions.

ii) Conversely, fix $g > 0$. If $v < \varphi^{-1}(g)$, we have exactly two solutions of velocity v. There is only one solution of velocity $v = \varphi^{-1}(g)$ and there are no solutions of velocity $v > \varphi^{-1}(g)$.

Remark 3.2 It is obvious that in the case $U = g\delta$ one has

$$h(a) = E(r_a) = 4F(a) + a(a+2)g.$$

So the function h is differentiable and

$$h'(a) = 4f(a) + 2(a+1)g = 2(a+1)(g - k_v(a)).$$

If $g > \varphi(v)$, then h is strictly increasing on $[-1+\frac{v}{\sqrt{2}}, 0]$ and it does not admit critical points. If $g = \varphi(v)$, it is still strictly increasing, but it has one critical point $a_\star(v)$. Finally, if $g < \varphi(v)$, we see that the function h is increasing on $[-1+\frac{v}{\sqrt{2}}, a_2(v)]$, decreasing on $[a_2(v), a_1(v)]$ and increasing on $[a_1(v), 0]$, where $a_1(v)$ and $a_2(v)$ are the two roots of equation (3.1). We have already seen that the two solutions of (1.9) are $r_{a_1(v)}$ and $r_{a_2(v)}$. Note that $r_{a_1(v)}$ is a local minimum of E (for example, it minimizes E on the open set $\{u \in H^1(\mathbf{R}) \mid \inf_{x\in\mathbf{R}} u(x) > a_2(v)\}$). The second solution, $r_{a_2(v)}$, is a critical point of mountain-pass type of E. Indeed, for each continuous path $\gamma : [0,1] \longrightarrow H^1(\mathbf{R})$ such that $\gamma(0) = r_{-1+\frac{v}{\sqrt{2}}}$ and $\gamma(1) = r_{a_1(v)}$, there exists $t \in [0,1]$ such that $E(\gamma(t)) \geq E(r_{a_2(v)}) > \max(E(r_{-1+\frac{v}{\sqrt{2}}}), E(r_{a_1(v)}))$ (when E is suitably extended to $H^1(\mathbf{R})$).

For a general measure U, we do not know the shape of the curve $a \longmapsto h(a)$. However, it will be shown in the next two sections that quite a similar phenomenon takes place.

4 A local minimizer of E

We keep the notation introduced previously. The main result of this section is

Theorem 4.1 *Assume that U is a positive Borel measure and $||U||$ is finite. Then:*

i) There exists $\eta > 0$ such that $h(a) < 0$ for all $a \in (-\eta, 0)$.

ii) Suppose that there exists $a \in [-1+\frac{v}{\sqrt{2}}, 0)$ such that $h(a) \geq 0$. Let $a_0 = \sup\{a \in [-1+\frac{v}{\sqrt{2}}, 0) \mid h(a) \geq 0\}$. Then E has a minimum on the open set

$$V_0 = \{u \in H^1(\mathbf{R}) \mid \inf_{x\in\mathbf{R}} u(x) > a_0\}.$$

Proof. *i*) We have for any $T > 0$

$$\begin{aligned} h(a) \leq E(r_a) &= 4F(a) + \int_{\mathbf{R}} r_a(r_a+2)U(x)dx \\ &\leq 4F(a) + r_a(T)(r_a(T)+2)||\chi_{[-T,T]}U||. \end{aligned} \tag{4.1}$$

Let us denote by $\phi_T(a)$ the right hand side of the above inequality. Clearly ϕ_T is differentiable and

$$\phi_T'(a) = 4f(a) + 2(r_a(T)+1)||\chi_{[-T,T]}U||\frac{d}{da}(r_a(T)).$$

But $r_a(T) = r_{-1+\frac{v}{\sqrt{2}}}(T - c(a))$ and so

$$\frac{d}{da}r_a(T) = -r'_{-1+\frac{v}{\sqrt{2}}}(T-c(a))c'(a) = f(r_{-1+\frac{v}{\sqrt{2}}}(T-c(a)))c'(a). \tag{4.2}$$

Since $r_{-1+\frac{v}{\sqrt{2}}}(c(a)) = r_a(0) = a$, we get $1 = r'_{-1+\frac{v}{\sqrt{2}}}(c(a))c'(a)$. Remember that $c(a) < 0$ and $r'_{-1+\frac{v}{\sqrt{2}}}(c(a)) = f(r_{-1+\frac{v}{\sqrt{2}}}(c(a))) = f(a)$. Therefore $c'(a) = \frac{1}{f(a)}$. Combining this with (4.2), one obtains $\frac{d}{da}(r_a(T)) = \frac{f(r_{-1+\frac{v}{\sqrt{2}}}(T-c(a)))}{f(a)}$. After a straightforward computation, we get

$$\lim_{a\uparrow 0} \frac{d}{da}(r_a(T)) = e^{-\sqrt{2-v^2}T}. \tag{4.3}$$

Thus $\lim_{a\uparrow 0} \phi_T'(a) = 2e^{-\sqrt{2-v^2}T}||\chi_{[-T,T]}U||$. Now fix T such that $||\chi_{[-T,T]}U|| > 0$. Then ϕ_T is continuous, $\phi_T(0) = 0$ and $\lim_{a\uparrow 0} \phi_T'(a) > 0$, so there exists $\eta > 0$ such that $\phi_T(a) < 0$, $\forall a \in (-\eta, 0)$. This clearly implies *i*).

ii) Obviously E is bounded from below on V_0 by $\min_{a\in[a_0,0]} h(a)$. Let $(r_n)_{n\in\mathbf{N}}$ be a minimizing sequence for E on V_0. We may suppose that $a_0 < r_n(s) \leq 0$, $\forall s \in \mathbf{R}$ and $E(r_n) < 0$. Then we have

$$G(r_n) < -\int_{\mathbf{R}} r_n(r_n + 2)U(s)ds \leq -a_0(a_0 + 2)||U||. \tag{4.4}$$

Observe that the function $a \longmapsto 4F(a) + a(a + 2)||U||$ is increasing on an interval $(-1 + \frac{v}{\sqrt{2}}, -1 + \frac{v}{\sqrt{2}} + \delta)$ for some $\delta > 0$. In view of Lemma 2.3 *i*) and *iii*), it follows that $h(-1 + \frac{v}{\sqrt{2}}) < h(a)$, $\forall a \in (-1 + \frac{v}{\sqrt{2}}, -1 + \frac{v}{\sqrt{2}} + \delta)$. Consequently we have $a_0 > -1 + \frac{v}{\sqrt{2}}$ and there exists $C_0 > 0$ such that $f^2(x) \geq C_0 x^2$, $\forall x \in [a_0, 0]$. From (4.4) we infer that (r_n) is bounded in $H^1(\mathbf{R})$. Hence there exists a subsequence (still denoted (r_n)) and $r \in H^1(\mathbf{R})$ such that

$$r_n \rightharpoonup r \text{ weakly in } H^1(\mathbf{R}) \text{ and}$$
$$r_n \to r \text{ a.e. as } n \longrightarrow \infty.$$

By lower semicontinuity we have

$$\int_{\mathbf{R}} |r'|^2 dx \leq \liminf_{n\to\infty} \int_{\mathbf{R}} |r_n'|^2 dx. \tag{4.5}$$

Using Fatou's lemma one has

$$\int_{\mathbf{R}} f^2(r) dx \leq \liminf_{n\to\infty} \int_{\mathbf{R}} f^2(r_n) dx. \tag{4.6}$$

Clearly $|r_n(s)(r_n(s) + 2)| \leq |a_0|(2 + a_0)$ for all $s \in \mathbf{R}$ and $n \in \mathbf{N}$. Since $||U|| < \infty$, Lebesgue's dominated convergence theorem can be applied and gives

$$\int_{\mathbf{R}} r(r + 2)U(x)dx = \lim_{n\to\infty} \int_{\mathbf{R}} r_n(r_n + 2)U(x)dx. \tag{4.7}$$

From (4.5), (4.6) and (4.7) we infer that

$$E(r) \leq \liminf_{n\to\infty} E(r_n) < 0. \tag{4.8}$$

Obviously $r \in \overline{V_0}$ since $r_n \longrightarrow r$ a.e. We cannot have $\inf_{x\in\mathbf{R}} r(x) = a_0$ because in this case we would have $E(r) \geq h(a_0) \geq 0$, which contradicts (4.8). Hence $r \in V_0$ and r is a minimizer of E on V_0. $\Box$

Remark 4.2 The assumption of Theorem 4.1, part *ii*) is clearly satisfied if, for example, $h(-1 + \frac{v}{\sqrt{2}}) \geq 0$, that is if $||U|| \leq \frac{8F(-1+\frac{v}{\sqrt{2}})}{2-v^2}$. Let $\varphi_1(v) = \frac{8F(-1+\frac{v}{\sqrt{2}})}{2-v^2}$. One can see that φ_1 is smooth and positive on $[0, \sqrt{2})$ and $\varphi_1(0) = \frac{4\sqrt{2}}{3}$, $\varphi_1(0) = 0$. $\lim_{v\uparrow\sqrt{2}} \frac{\varphi_1(v)}{\sqrt{2-v^2}} = \frac{5}{12}$. If $||U|| \leq \varphi_1(v)$, then necessarily E has a critical point which is a local minimizer.

5 A second critical point of E

It is proved below, under certain hypothesis on U, that the functional E has a second critical point of "mountain-pass" type.

We suppose throughout this section that the assumptions of Theorem 4.1 are satisfied. Moreover, we suppose that U has compact support. Let $[x, y]$ be the smallest closed interval containing $supp(U)$.

We use the following mountain-pass theorem due to Ghoussoub and Preiss [3], based on Ekeland's variational principle:

Theorem 5.1 ([3]) *Let X be a Banach space and $\Phi : X \longrightarrow \mathbf{R}$ a C^1 functional. Let $u, v \in X$ and consider the set $\Gamma_{u,v}$ of continuous paths joining u and v, i.e.*

$$\Gamma_{u,v} = \{\gamma \in C^0([0,1], X) \mid \gamma(0) = u, \gamma(1) = v\}.$$

Define $c = \inf_{\gamma \in \Gamma_{u,v}} (\max_{s \in [0,1]} \Phi(\gamma(s)))$. Assume that there exists a closed subset M of X such that $M^c = M \cap \{x \in X \mid \Phi(x) \geq c\}$ separates u and v, i.e. u and v belong to two disjoint connected components of $X \setminus M^c$. Then there exists a sequence $(x_n)_{n \in \mathbf{N}}$ in X such that

i) $\lim_{n \to \infty} dist(x_n, M) = 0,$

ii) $\lim_{n \to \infty} \Phi(x_n) = c,$

iii) $\lim_{n \to \infty} ||\Phi'(x_n)||_{X^\star} = 0.$

A sequence satisfying *ii)* and *iii)* is called a Palais-Smale sequence. Note that the usual mountain-pass theorem corresponds to the case $M = X$.

In order to apply Theorem 5.1, we extend E to $H^1(\mathbf{R})$. Fix $d \in (-1, -1 + \frac{v}{\sqrt{2}})$ and consider a function $\tilde{f} : \mathbf{R} \longrightarrow \mathbf{R}$ such that $\tilde{f} \in C^1(\mathbf{R})$, $\tilde{f} \equiv f$ on $[d, \infty)$ and $\tilde{f}$ is bounded on $(-\infty, d]$. Define $\tilde{E} : H^1(\mathbf{R}) \longrightarrow \mathbf{R}$ by

$$\tilde{E}(u) = \int_{\mathbf{R}} |u'|^2 + \tilde{f}^2(u)dx + H(u).$$

Then $\tilde{E}$ is a C^1 functional on $H^1(\mathbf{R})$ and $\tilde{E} \equiv E$ on a neighbourhood of $V_\star = \{u \in H^1(\mathbf{R}) \mid \inf_{s \in \mathbf{R}} u(s) > -1 + \frac{v}{\sqrt{2}}\}$. We are going to find a critical point $r_1 \in V_\star$ of $\tilde{E}$. Clearly r_1 will be also a critical point of E.

Set

$$w(s) = \begin{cases} r_{-1+\frac{v}{\sqrt{2}}}(s-x) & \text{if} \quad s < x \\ -1 + \frac{v}{\sqrt{2}} & \text{if} \quad x \leq s \leq y \\ r_{-1+\frac{v}{\sqrt{2}}}(s-y) & \text{if} \quad s > y \end{cases} \tag{5.1}$$

so that $w \in H^1(\mathbf{R})$, $\inf_{s \in \mathbf{R}} w(s) = -1 + \frac{v}{\sqrt{2}}$ and $E(w) = h(-1 + \frac{v}{\sqrt{2}})$. Let

$$\Gamma_{r,w} = \{\gamma \in C^0([0,1], H^1(\mathbf{R})) \mid \gamma(0) = r, \gamma(1) = w\}$$

where r is a minimizer of E on V_0 (as in Section 4), and $c = \inf_{\gamma\in\Gamma_{r,w}} (\max_{s\in[0,1]} \tilde{E}(\gamma(s)))$.

We study first the convexity of f^2 on $[-1+\frac{v}{\sqrt{2}}, 0]$. One has

$$(f^2)''(x) = 2\Big(3(x+1)^2 - 1 - \frac{v^2}{4} - \frac{3v^2}{4}\frac{1}{(x+1)^4}\Big).$$

So $(f^2)''$ is strictly increasing on $[-1+\frac{v}{\sqrt{2}}, 0]$, $(f^2)''(-1+\frac{v}{\sqrt{2}}) = (\frac{5}{2}+\frac{3}{v^2})(v^2-2) < 0$, $(f^2)''(0) = 2(2-v^2) > 0$ and f^2 is concave on $[-1+\frac{v}{\sqrt{2}}, -1+\sqrt{\alpha(v)}]$ and convex on $[-1+\sqrt{\alpha(v)}, 0]$, where $\alpha(v)$ is the unique root of the equation $3y^3 - (1+\frac{v^2}{4})y^2 - \frac{3v^2}{4} = 0$ in the interval $[\frac{v^2}{2}, 1]$. It is also easily seen that there exists $\beta(v) \in (-1+\frac{v}{\sqrt{2}}, -1+\sqrt{\alpha(v)})$ such that $(f^2)'$ is positive (and decreasing) on $(-1+\frac{v}{\sqrt{2}}, \beta(v))$ (hence f^2 is concave, increasing and positive on $(-1+\frac{v}{\sqrt{2}}, \beta(v))$) and $(f^2)'$ is negative on $(\beta(v), 0)$. In other words, $\beta(v)$ is the maximum point of f^2 on $(-1, 0]$.

Next, we introduce the following supplementary hypothesis:

H1 $-1+\sqrt{\alpha(v)} \le a_0$ (recall that $a_0 = \sup\{a \in [-1+\frac{v}{\sqrt{2}}, 0) \mid h(a) \ge 0\}$).

H2 There exists $\varepsilon > 0$ such that for any interval $I \subset [x, y]$ we have $\int_I U(x)dx \ge \varepsilon |I|$, where $|I|$ is the length of I.

Proposition 5.2 *Assume that* **H1** *and* **H2** *are satisfied. For $\delta > 0$, δ small, consider the closed subset of $H^1(\mathbf{R})$*

$$M_\delta = \{u \in H^1(\mathbf{R}) \mid -1 + \frac{v}{\sqrt{2}} + \delta \le \inf_{s\in\mathbf{R}} u(s) \le a_0\}.$$

Then there exists $\delta > 0$ such that M_δ^c separates r and w.

Remark 5.3 We assume that **H1** holds only for technical reasons (the convexity of f^2 in a neighbourhood of $[\inf_{s\in\mathbf{R}} r(s), 0]$ is used in proofs). Using only the assumptions of Theorem 4.1, hypothesis **H2** and the fact that U has compact support, the proofs given below still work and it can be deduced, for example, that the set $\{u \in H^1(\mathbf{R}) \mid -1+\frac{v}{\sqrt{2}} + \delta \le \inf_{s\in\mathbf{R}} u(s) \le -\eta\} \cap \{u \in H^1(\mathbf{R}) \mid \tilde{E}(u) \ge c'\}$ separates 0 and w, where $c' = \inf_{\gamma\in\Gamma_{0,w}} (\max_{s\in[0,1]} \tilde{E}(\gamma(s)))$. We still get a critical point of E. However, we are not able to prove that this critical point is different from r.

In view of Lemma 2.3, *i*), a sufficient (but not necessary) condition for **H1** to be satisfied is that $4F(-1+\sqrt{\alpha(v)}) + (\alpha(v)-1)||U|| \ge 0$. Therefore, if U has compact support and $||U|| \le \varphi_2(v)$, where $\varphi_2(v) = \frac{4F(-1+\sqrt{\alpha(v)})}{1-\alpha(v)}$, then (1.9) has a second solution r_1. Moreover, it will be seen that $\inf_{s\in\mathbf{R}} r_1(s) < \inf_{s\in\mathbf{R}} r(s)$. Note that φ_2 is continuous and positive on $[0, \sqrt{2})$ and $\varphi_2(0) = 2\sqrt{2} - \frac{8}{9}\sqrt{6}$.

The proof of Proposition 5.2 is based on the following three lemmas:

Lemma 5.4 *Let $u \in H^1(\mathbf{R})$ be such that $a = \inf_{s\in\mathbf{R}} u(s) \ge -1+\frac{v}{\sqrt{2}}$. There exists a continuous path $\psi : [0,1] \longrightarrow H^1(\mathbf{R})$ with the following properties:*

i) $\psi(0) = u$;

ii) $\inf_{s\in\mathbf{R}} \psi(t)(s) \geq a$ *and* $E(\psi(t)) \leq E(u)$, $\forall t \in [0,1]$;

iii) *there exists* $z \in [x,y]$ *such that* $\psi(1)(z) = a$ *and* $\psi(1)(s) \leq r_a(s-z)$ *for all* $s \in \mathbf{R}$.

Proof. For $t \in [0,1]$ set $v_t = -u^- + tu^+$, where u^+ and u^- are the positive, respectively the negative part of u. Clearly the map $t \longmapsto v_t$ is continuous from $[0,1]$ to $H^1(\mathbf{R})$, $v_1 = u$ and $v_0 = -u^-$, $a \leq v_0 \leq 0$. Since the functions $s \longmapsto f^2(s)$ and $s \longmapsto s(s+2)$ are increasing on $[0,\infty)$, we have $E(v_t) \leq E(u)$, $\forall t \in [0,1]$.

For $t \in [0,\infty)$ define

$$u_t(s) = \begin{cases} r_{v_0(x-t)}(s-x+t) & \text{if} \quad s < x-t \\ v_0(s) & \text{if} \quad x-t \leq s \leq y+t \\ r_{v_0(y+t)}(s-y-t) & \text{if} \quad s > y+t \end{cases} \tag{5.2}$$

It is easy to check that $t \longmapsto u_t$ is a continuous map from $[0,\infty)$ to $H^1(\mathbf{R})$, $a \leq u_t(s) \leq 0$, $\forall s \in \mathbf{R}, \forall t \in [0,\infty)$ and $u_t \longrightarrow v_0$ in $H^1(\mathbf{R})$ as $t \longrightarrow \infty$.

By Lemma 2.2 *vii)* and Remark 2.1, we have for all $t \in [0,\infty)$

$$\begin{aligned} E(v_0) &= \Big(\int_{-\infty}^{x-t} + \int_{x-t}^{y+t} + \int_{y+t}^{\infty}\Big)(|v_0'|^2(s) + f^2(v_0(s)))ds \\ &\quad + \int_x^y v_0(v_0+2)U(s)ds \\ &\geq 2F(v_0(x-t)) + \int_{x-t}^{y+t} |v_0'|^2(s) + f^2(v_0(s))ds \\ &\quad + 2F(v_0(y+t)) + \int_x^y v_0(v_0+2)U(s)ds = E(u_t). \end{aligned} \tag{5.3}$$

Since u_0 is decreasing on $(-\infty, x]$ and increasing on $[y,\infty)$, there exists $z \in [x,y]$ such that $u_0(z) = b = \inf_{s\in\mathbf{R}} u_0(s)$. Clearly $b \geq a$.

If $b > a$, there exists $z_1 \in \mathbf{R} \setminus [x,y]$ such that $v_0(z_1) = a$. Suppose that $z_1 < x$. Using Remark 2.1 we have

$$\begin{aligned} E(v_0) - E(u_0) &= \Big(\int_{-\infty}^{x} + \int_{y}^{\infty}\Big)(|v_0'|^2 + f^2(v_0))ds \\ &\quad - \Big(\int_{-\infty}^{x} + \int_{y}^{\infty}\Big)(|u_0'|^2 + f^2(u_0))ds \\ &= \Big(\int_{-\infty}^{z_1} + \int_{z_1}^{x} + \int_{y}^{\infty}\Big)(|v_0'|^2 + f^2(v_0))ds - 2F(v_0(x)) - 2F(v_0(y)) \\ &\geq 2F(v_0(z_1)) + 2(F(v_0(z_1)) - F(v_0(x))) + 2F(v_0(y)) \\ &\qquad -2F(v_0(x)) - 2F(v_0(y)) \\ &= 4F(a) - 4F(v_0(x)) \geq 4F(a) - 4F(b). \end{aligned} \tag{5.4}$$

Obviously the same is true if $z_1 > y$.

For $t \in [a,0]$ set $u_t(s) = \min(u_0(s), r_t(s-z))$ (note that this definition is not ambiguous for $t = 0$). Since the mapping $t \longmapsto r_t(\cdot - z)$ is continuous from $[a,0]$ to $H^1(\mathbf{R})$, we infer that the mapping $t \longmapsto u_t$ is also continuous.

Let us show that $E(u_t) \leq E(v_0)$, $\forall t \in [a, 0]$. Fix t. Since $a \leq u_t \leq u_0 \leq 0$ we have

$$\int_x^y u_t(u_t+2)U(s)ds \leq \int_x^y u_0(u_0+2)U(s)ds = \int_x^y v_0(v_0+2)U(s)ds. \tag{5.5}$$

The set $O_t = \{s \in \mathbf{R} \mid u_0(s) > r_t(s-z)\}$ is open, hence there exists a family at most countable of disjoint open intervals $((x_i, y_i))_{i\in I}$ such that $O_t = \cup_{i\in I}(x_i, y_i)$. For each $i \in I$ we have

- either $x_i = -\infty$ or $u_0(x_i) = r_t(x_i - z)$
- either $y_i = \infty$ or $u_0(y_i) = r_t(y_i - z)$.

Then

$$\begin{aligned} E(u_t) - E(u_0) \leq \sum_{i\in I} \Big(&\int_{x_i}^{y_i} |r_t'(s-z)|^2 + f^2(r_t(s-z))ds \\ &- \int_{x_i}^{y_i} |u_0'(s)|^2 + f^2(u_0(s))ds \Big). \end{aligned} \tag{5.6}$$

If $(x_i, y_i) \subset (-\infty, z)$ or $(x_i, y_i) \subset (z, \infty)$ then

$$\int_{x_i}^{y_i} |r_t'(s-z)|^2 + f^2(r_t(s-z))ds \leq \int_{x_i}^{y_i} |u_0'(s)|^2 + f^2(u_0(s))ds \tag{5.7}$$

by Lemma 2.2, part *vii*). Note that if $t \geq b$, we have $(x_i, y_i) \subset (-\infty, z)$ or $(x_i, y_i) \subset (z, \infty)$ for all $i \in I$. If $t < b$, there exists exactly one $i_0 \in I$ such that $z \in (x_0, y_0)$ and $(x_i, y_i) \subset ((-\infty, z) \cup (z, \infty))$ for all other $i \in I$. For i_0 we have

$$\begin{aligned} &\int_{x_{i_0}}^{y_{i_0}} |r_t'(s-z)|^2 + f^2(r_t(s-z))ds \\ &= \Big(\int_{x_{i_0}}^{z} + \int_{z}^{y_{i_0}} \Big) |r_t'(s-z)|^2 + f^2(r_t(s-z))ds \\ &= (2F(t) - 2F(u_0(x_{i_0}))) + (2F(t) - 2F(u_0(y_{i_0}))) \end{aligned}$$

and by Remark 2.1,

$$\int_{x_{i_0}}^{y_{i_0}} |u_0'(s)|^2 + f^2(u_0(s))ds \geq 2|F(u_0(y_{i_0})) - F(u_0(x_{i_0}))|.$$

Therefore

$$\begin{aligned} &\int_{x_{i_0}}^{y_{i_0}} |r_t'(s-z)|^2 + f^2(r_t(s-z))ds - \int_{x_{i_0}}^{y_{i_0}} |u_0'(s)|^2 + f^2(u_0(s))ds \\ &\leq 4F(t) - 4\max(F(u_0(x_{i_0})), F(u_0(y_{i_0}))) \leq 4F(t) - 4F(b). \end{aligned} \tag{5.8}$$

From (5.6), (5.7), (5.8) and (5.4) we infer that $E(u_t) - E(u_0) \leq 4F(t) - 4F(b) \leq 4F(a) - 4F(b) \leq E(v_0) - E(u_0)$. Hence $E(u_t) \leq E(v_0) \leq E(u)$ for all $t \in [a, 0]$.

Finally, define $\psi : [0,1] \longrightarrow H^1(\mathbf{R})$ by

$$\psi(t) = \begin{cases} v_{1-2t} & \text{if} \quad t \in [0, \frac{1}{2}] \\ u_{a\frac{(4t-3)}{2t-1}} & \text{if} \quad t \in (\frac{1}{2}, 1]. \end{cases}$$

It is easy to check that ψ is continuous and satisfies Lemma 5.4. $\square$

Lemma 5.5 *Suppose that the hypothesis* **H1** *and* **H2** *are satisfied. There exists* $\delta > 0$ *(depending on* ε*) such that for each* $u \in H^1(\mathbf{R})$ *verifying*

$$b = \inf_{s\in\mathbf{R}} u(s) \in [-1 + \frac{v}{\sqrt{2}}, -1 + \frac{v}{\sqrt{2}} + \delta]$$

there exists a continuous path $\lambda : [0,1] \longrightarrow \overline{V_\star}$ *such that*

i) $\lambda(0) = u$*;*

ii) $E(\lambda(t)) \le E(u),\ \forall u \in [0,1]$*;*

iii) $\lambda(1)(s) \le 0,\ \forall s \in \mathbf{R}$ *and there exists* $z \in [x,y]$ *such that* $\lambda(1)(z) = -1 + \frac{v}{\sqrt{2}}$*.*

Proof. Fix $a \in (-1+\frac{v}{\sqrt{2}}, \beta(v))$ sufficiently close to $-1+\frac{v}{\sqrt{2}}$ so that $c(a) > x-y$. (The value of a will be chosen later). Recall that $\beta(v)$ is the maximum point of f^2 on $(-1,0]$ and f^2 is concave and increasing on $[-1+\frac{v}{\sqrt{2}}, \beta(v)]$.

Let $u \in H^1(\mathbf{R})$ be such that $b = \inf_{s\in\mathbf{R}} u(s) \in [-1+\frac{v}{\sqrt{2}}, a)$.

Consider the path ψ given by Lemma 5.4 and denote $u_1 = \psi(1)$. There exists $z \in [x,y]$ such that $u_1(z) = b$ and $u_1(s) \le r_b(s-z),\ \forall s \in \mathbf{R}$.

Let

$$x_a = \inf\{t < z \mid u_1(s) < a \text{ on } (t,z]\},$$
$$y_a = \sup\{t > z \mid u_1(s) < a \text{ on } [z,t)\}.$$

Clearly $u_1(x_a) = u_1(y_a) = a$. Since $u_1(s) \le r_b(s-z)$ we have $x_a \le z - (c(b)-c(a))$ and $y_a \ge z + (c(b)-c(a))$. For $t \in [a,0]$ define

$$\lambda_1(t)(s) = \begin{cases} \min(u_1(s), r_t(s-x_a)) & \text{if} \quad s \in (-\infty, x_a] \\ u_1(s) & \text{if} \quad s \in (x_a, y_a) \\ \min(u_1(s), r_t(s-y_a) & \text{if} \quad s \in [y_a, \infty). \end{cases}$$

Then λ_1 is continuous from $[a,0]$ to $H^1(\mathbf{R})$, $b \le \lambda_1(t)(s) \le 0$ for all $t,\ s$ and $\lambda_1(0) = u_1$. As in the proof of Lemma 5.4 one shows that $E(\lambda_1(t)) \le E(u_1),\ \forall t \in [a,0]$. Denote $u_2 = \lambda_1(a)$. We have $u_2(s) \le r_a(s-x_a)$ on $(-\infty, x_a]$, $u_2(s) = u_1(s)$ on (x_a, y_a) and $u_2(s) \le r_a(s-y_a)$ on $[y_a, \infty)$.

For $t \in [0, b+1-\frac{v}{\sqrt{2}}]$ define

$$\lambda_2(t)(s) = \begin{cases} \min(u_2(s), r_{a-t}(s-x_a)) & \text{if} \quad s \in (-\infty, x_a] \\ u_1(s) - t & \text{if} \quad s \in (x_a, y_a) \\ \min(u_2(s), r_{a-t}(s-y_a) & \text{if} \quad s \in [y_a, \infty). \end{cases}$$

One easily checks that the map $t \longmapsto \lambda_2(t)$ is continuous for the norm of $H^1(\mathbf{R})$. As in the proof of Lemma 5.4 we obtain

$$\int_{\mathbf{R}\setminus[x_a,y_a]} |\lambda_2(t)'(s)|^2 + f^2(\lambda_2(t)(s))ds$$
$$\le \int_{\mathbf{R}\setminus[x_a,y_a]} |u_2'(s)|^2 + f^2(u_2(s))ds + 4F(a-t) - 4F(a).$$

We have

$$\int_{\mathbf{R}\setminus[x_a,y_a]} \lambda_2(t)(\lambda_2(t)+2)U(s)ds - \int_{\mathbf{R}\setminus[x_a,y_a]} u_2(u_2+2)U(s)ds \le 0$$

because $-1+\frac{v}{\sqrt{2}} \le \lambda_2(t)(s) \le u_2(s)$, $\forall s \in \mathbf{R}$. Obviously $\lambda_2(t)'(s) = u_1'(s)$ for $s \in (x_a, y_a)$. Therefore

$$\begin{aligned} &E(\lambda_2(t)) - E(u_2) \\ &\le 4F(a-t) + 4F(a) + \int_{x_a}^{y_a} f^2(u_1(s)-t) - f^2(u_1(s))ds \\ &\quad + \int_{x_a}^{y_a} \Big(-2t(u_1(s)+1) + t^2\Big)U(s)ds. \end{aligned} \tag{5.9}$$

We have $f^2(u_1(s)-t) - f^2(u_1(s)) \le -2tff'(u_1(s)) \le -2tff'(a)$ for $s \in [x_a, y_a]$ by the concavity of f^2. Since $u_1(s)+1 \ge b+1 \ge \frac{v}{\sqrt{2}}$ we obtain

$$\begin{aligned} &E(\lambda_2(t)) - E(u_2) \\ &\le 4F(a-t) + 4F(a) - 2tff'(a)(y_a - x_a) + (t^2 - \sqrt{2}vt)\int_{x_a}^{y_a} U(s)ds. \end{aligned} \tag{5.10}$$

Using the fact that $y_a - z \ge c(b) - c(a)$, $z - x_a \ge c(b) - c(a)$, $z \in [x, y]$, $-c(a) \le y - x$ and hypothesis **H2** (note that this is the only point in the proof of Proposition 5.2 where this hypothesis is needed), we get $\int_{x_a}^{y_a} U(s)ds \ge \varepsilon|[x,y] \cap [x_a, y_a]| \ge \varepsilon(c(b) - c(a))$. Hence

$$\begin{aligned} &E(\lambda_2(t)) - E(u_2) \le 4F(a-t) + 4F(a) \\ &-4t(c(b)-c(a))ff'(a) + \varepsilon(t^2 - \sqrt{2}vt)(c(b)-c(a)). \end{aligned} \tag{5.11}$$

Recall that f is negative and decreasing on $[-1+\frac{v}{\sqrt{2}}, a]$, so $F(a-t) - F(a) \le -tf(a)$. For $t \le \frac{\sqrt{2}v}{2}$ we have

$$\begin{aligned} &E(\lambda_2(t)) - E(u_2) \\ &\le -4tf(a) - 4t(c(b)-c(a))ff'(a) - \tfrac{\varepsilon\sqrt{2}v}{2}t(c(b)-c(a)) \\ &= \Big[\Big(-4f(a) + 4ff'(a)c(a) + \tfrac{\varepsilon\sqrt{2}v}{2}\cdot c(a)\Big) - \Big(4ff'(a) + \tfrac{\varepsilon\sqrt{2}v}{2}\Big)c(b)\Big]t. \end{aligned} \tag{5.12}$$

By a straightforward computation one has

$$\begin{aligned} \lim_{a\downarrow -1+\frac{v}{\sqrt{2}}} \frac{f(a)}{\sqrt{2(a+1)^2 - v^2}} &= \frac{v^2-2}{2\sqrt{2}v}; \\ \lim_{a\downarrow -1+\frac{v}{\sqrt{2}}} \frac{c(a)}{\sqrt{2(a+1)^2 - v^2}} &= -\frac{2}{2-v^2}; \\ \lim_{a\downarrow -1+\frac{v}{\sqrt{2}}} ff'(a) &= \frac{1}{v\sqrt{2}}(\frac{v^2}{2} - 1)^2. \end{aligned}$$

Consequently, we find that

$$\lim_{a\downarrow -1+\frac{v}{\sqrt{2}}} \frac{-4f(a) + 4ff'(a)c(a) + \frac{\varepsilon\sqrt{2}v}{2}\cdot c(a)}{\sqrt{2(a+1)^2 - v^2}} = -\frac{\sqrt{2}v\varepsilon}{2-v^2} < 0. \tag{5.13}$$

Hence $-4f(a)+4ff'(a)c(a)+\frac{\varepsilon\sqrt{2}v}{2}c(a)<0$ if a is "sufficiently close" to $-1+\frac{v}{\sqrt{2}}$.

Now choose $a\in(-1+\frac{v}{\sqrt{2}},\beta(v))$ such that $-c(a)\le y-x$ and $-4f(a')+4ff'(a')c(a')+\frac{\varepsilon\sqrt{2}v}{2}\cdot c(a')<0$ for all $a'\in[-1+\frac{v}{\sqrt{2}},a]$. In view of (5.13), this is possible.

Next, choose $\delta\in(0,\frac{v\sqrt{2}}{2})$ such that

$$-4f(a)+4ff'(a)c(a)+\frac{\varepsilon\sqrt{2}v}{2}c(a)-\Big(4ff'(a)+\frac{\varepsilon\sqrt{2}v}{2}\Big)c(b)<0 \tag{5.14}$$

for all $b\in[-1+\frac{v}{\sqrt{2}},-1+\frac{v}{\sqrt{2}}+\delta]$. This is also possible because $\lim\limits_{a\downarrow-1+\frac{v}{\sqrt{2}}}c(b)=0$.

Let $u\in H^1(\mathbf{R})$ be such that $b=\inf\limits_{s\in\mathbf{R}}u(s)\in[-1+\frac{v}{\sqrt{2}},-1+\frac{v}{\sqrt{2}}+\delta]$. Let ψ be the path given by Lemma 5.4 and let $u_1=\psi(1)$. Define λ_1 as before. It is clear that $t\longmapsto\lambda_1(-t)$, $t\in[0,-a]$ is a continuous path in $\overline{V_\star}$ joining u_1 and $u_2=\lambda_1(a)$. Next, define λ_2 as previously for $t\in J=[0,b+1-\frac{v}{\sqrt{2}}]$. Then the estimates (5.9) - (5.11) hold. We see that $b+1-\frac{v}{\sqrt{2}}<\frac{v\sqrt{2}}{2}$, hence (5.12) is true for all $t\in J$. From (5.12) and (5.14) we infer that $E(\lambda_2(t))\le E(u_2)\le E(u)$, $\forall t\in J$. Let $u_3=\lambda_2(b+1-\frac{v}{\sqrt{2}})$. It is easy to see that $\inf\limits_{s\in\mathbf{R}}u_3(s)=u_3(z)=-1+\frac{v}{\sqrt{2}}$ and λ_2 is a continuous path in $\overline{V_\star}$ joining u_2 and u_3. It suffices to add the paths ψ, $\lambda_1(-\cdot)$ and λ_2 to obtain a continuous path $\lambda:[0,1]\longrightarrow\overline{V_\star}=\{u\in H^1(\mathbf{R})\mid\inf\limits_{s\in\mathbf{R}}u(s)\ge-1+\frac{v}{\sqrt{2}}\}$ such that $\lambda(0)=u$, $\lambda(1)=u_3$ and $E(\lambda(t))\le E(u)$, $\forall t\in[0,1]$. This proves Lemma 5.5. □

Lemma 5.6 *Let $u\in H^1(\mathbf{R})$ be such that $-1+\frac{v}{\sqrt{2}}\le u\le0$ and there exists $z\in[x,y]$ such that $u(z)=-1+\frac{v}{\sqrt{2}}$. Then there exists a continuous path $\mu:[0,1]\longrightarrow\overline{V_\star}$ satisfying:*

i) $\mu(0)=u$, $\mu(1)=w$, where w is given by (5.1);

ii) $\mu(t)(z)=-1+\frac{v}{\sqrt{2}}$, $\forall t\in[0,1]$;

iii) $E(\mu(t))\le E(u)$ for all $t\in[0,1]$.

Proof. Let $v(s)=\min(u(s),r_{-1+\frac{v}{\sqrt{2}}}(s-z))$. For $t\in[0,1-\frac{v}{\sqrt{2}}]$ define $\mu_1(t)(s)=\min(u(s),r_{-t}(s-z))$. Then μ_1 is a continuous path joining u and v and one shows as previously that $E(\mu_1(t))\le E(u)$ for all t.

For $k\in[0,\infty)$ set $\mu_2^\star(k)(s)=\min(v(s),u_{-1+\frac{v}{\sqrt{2}},k}(s-z))$, where $u_{-1+\frac{v}{\sqrt{2}},k}$ was defined in (2.8). Then $\mu_2^\star$ is continuous from $[0,\infty)$ to $H^1(\mathbf{R})$ (because $k\longmapsto u_{-1+\frac{v}{\sqrt{2}},k}$ is continuous) and $\mu_2(0)=v$. As in the previous lemmas one proves that $E(\mu_2^\star(k))\le E(v)$, $\forall k\in[0,\infty)$. Since $v(s)\longrightarrow0$ as $s\longrightarrow\infty$, there exists $k_0>0$ such that $supp(U)\subset[z-k_0,z+k_0]$ and $-1+\sqrt{\alpha(v)}<v(s)\le0$ for all $s\in\mathbf{R}\setminus[z-k_0,z+k_0]$. Let $v_1=\mu_2^\star(k_0)$. Then $v_1(s)=u_{-1+\frac{v}{\sqrt{2}},k_0}(s-z)$ if $s\in I_1=[z-k_0+c(-1+\sqrt{\alpha(v)}),z+k_0-c(-1+\sqrt{\alpha(v)})]$ and $-1+\sqrt{\alpha(v)}<v_1(s)\le0$ for $s\in\mathbf{R}\setminus I_1$. Denote by μ_2 the restriction of $\mu_2^\star$ to $[0,k_0]$, so that μ_2 is a continuous path and it joins v and v_1.

Set $\mu_3(t)=(1-t)v_1+tu_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot-z)$, $t\in[0,1]$. Obviously μ_3 is continuous and $\mu_3(t)\equiv u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot-z)$ on I_1, for all $t\in[0,1]$.

Since $v_1(s), u_{-1+\frac{v}{\sqrt{2}},k_0}(s-z) \in (-1+\sqrt{\alpha(v)}, 0]$ if $s \in \mathbf{R} \setminus I_1$, by the convexity of f^2 on $(-1+\sqrt{\alpha(v)}, 0]$ we get $E(\mu_3(t)) \leq (1-t)E(v_1)+tE(u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot-z)) \leq E(u)$, for all $t \in [0,1]$ (note that $E(u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot - z)) = h(-1+\frac{v}{\sqrt{2}}) \leq E(u)$).

For $t \in [z-k_0, x]$ set

$$\mu_4(t)(s) = \begin{cases} r_{-1+\frac{v}{\sqrt{2}}}(s-t) & \text{if} \quad s < t \\ u_{-1+\frac{v}{\sqrt{2}},k_0}(s-z) & \text{if} \quad s \geq t. \end{cases}$$

Denote $\mu_4(x)$ by v_2. Clearly μ_4 is a continuous path joining $u_{-1+\frac{v}{\sqrt{2}},k_0}(\cdot - z)$ and v_2.

Finally, for $t \in [0, z+k_0-y]$ let

$$\mu_5(t)(s) = \begin{cases} v_2(s) & \text{if} \quad s \leq z+k_0-t \\ r_{-1+\frac{v}{\sqrt{2}}}(s-t) & \text{if} \quad s > z+k_0-t. \end{cases}$$

Then μ_5 is a continuous path joining v_2 and w.

Adding the paths μ_i, $1 \leq i \leq 5$, we obtain a continuous path $\mu : [0,1] \longrightarrow \overline{V_\star}$ satisfying Lemma 5.6. $\Box$

Proof of Proposition 5.2. For a given path $\gamma \in \Gamma_{r,w}$, denote $l(t) = \inf_{s \in \mathbf{R}} \gamma(t)(s)$. The function l is continuous, $l(0) = \inf_{s \in \mathbf{R}} r(s) > a_0$ (as seen in Section 4) and $l(1) = -1+\frac{v}{\sqrt{2}}$. If $l(t) \in [-1+\frac{v}{\sqrt{2}}, 0]$ we necessarily have $\tilde{E}(\gamma(t)) = E(\gamma(t)) \geq h(l(t))$, therefore

$$\max_{t \in [0,1]} \tilde{E}(\gamma(t)) \geq \max_{a \in [-1+\frac{v}{\sqrt{2}}, a_0]} h(a).$$

Consequently, we have $c \geq \max_{a \in [-1+\frac{v}{\sqrt{2}}, a_0]} h(a)$. In particular, $c > E(r)$ and using Lemma 2.3 we infer that $c > E(w) = h(-1+\frac{v}{\sqrt{2}})$.

Fix $\delta = \delta(\varepsilon)$ as given by Lemma 5.5. We show that Proposition 5.2 holds for this choice of δ.

We reason by contradiction. Suppose that M_δ^c does not separate r and w, i.e. there exists a continuous path $\gamma : [0,1] \longrightarrow (H^1(\mathbf{R}) \setminus M_\delta) \cup \{u \in M_\delta \mid \tilde{E}(u) < c\}$ such that $\gamma(0) = r$ and $\gamma(1) = w$. As before, set $l(t) = \inf_{s \in \mathbf{R}} \gamma(t)(s)$. Let

$$t_0 = \sup\{t \in [0,1] \mid l(t) = a_0\} \text{ and}$$
$$t_1 = \inf\{t \in [t_0,1] \mid l(t) = -1+\frac{v}{\sqrt{2}}+\delta\}.$$

Then $0 < t_0 < t_1 < 1$ and for $t \in [t_0, t_1]$ we have $-1+\frac{v}{\sqrt{2}}+\delta \leq l(t) \leq a_0$, hence $\gamma(t) \in M_\delta$. By our assumption, $E(\gamma(t)) < c$ for all $t \in [t_0, t_1]$. Let $u_0 = \gamma(t_0)$, $u_1 = \gamma(t_1)$.

Using the convexity of f^2 on $[a_0, \infty)$ we have

$$\tilde{E}((1-t)r+tu_0) = E((1-t)r+tu_0) \leq (1-t)E(r)+tE(u_0) < c, \quad \forall t \in [0,1].$$

Define $\gamma_1 : [0,1] \longrightarrow H^1(\mathbf{R})$, $\gamma_1(t) = (1-t)r+tu_0$.

We have $\inf_{s \in \mathbf{R}} u_1(s) = -1+\frac{v}{\sqrt{2}}+\delta$. Therefore Lemma 5.5 can be applied for u_1 and gives us a path $\lambda : [0,1] \longrightarrow \overline{V_\star}$ such that $\inf_{s \in \mathbf{R}} \lambda(1)(s) = \min_{s \in [x,y]} \lambda(1)(s) = -1+\frac{v}{\sqrt{2}}$

and $\lambda(1) \leq 0$. Next, apply Lemma 5.6 to $\lambda(1)$ in order to obtain a continuous path μ joining $\lambda(1)$ and w. Adding the paths λ and μ we obtain a continuous path $\gamma_2 : [0,1] \longrightarrow \overline{V_\star}$ such that $\gamma_2(0) = u_1$, $\gamma_2(1) = w$ and $E(\gamma_2(t)) \leq E(u_1)$, $\forall t \in [0,1]$.

We define a new path in the following way: we start from r and go to u_0 along the path γ_1; then we go from u_0 to u_1 along the path $t \longmapsto \gamma(t)$, $t \in [t_0, t_1]$; finally we go from u_1 to w along the path γ_2. It suffices to make the corresponding changes of parameter to obtain a continuous path $\gamma_\star \in \Gamma_{r,w}$. Since $\max_{t\in[0,1]} \tilde{E}(\gamma_1(t)) = E(u_0)$ and $\max_{t\in[0,1]} \tilde{E}(\gamma_2(t)) = E(u_1)$, we have

$$\max_{t\in[0,1]} \tilde{E}(\gamma_\star(t)) = \max_{t\in[t_0,t_1]} E(\gamma(t)) < c,$$

which contradicts the definition of c. This proves Proposition 5.2. $\Box$

Proposition 5.7 *Assume that the hypothesis* **H1** *and* **H2** *are satisfied. There exists a solution r_1 of equation (1.9) and $z \in [x, y]$ such that $\inf_{s\in\mathbf{R}} r_1(s) = r_1(z) \in [-1+\frac{v}{\sqrt{2}}+\delta(\varepsilon), a_0]$, where $\delta(\varepsilon)$ is given by Lemma 5.5. Moreover, we have $E(r_1) \leq c$.*

Proof. From Proposition 5.2 and Theorem 5.1 it follows that there exists a sequence $(u_n) \in H^1(\mathbf{R})$ such that

$$\lim_{n\longrightarrow\infty} dist(u_n, M_{\delta(\varepsilon)}) = 0; \tag{5.15}$$

$$\lim_{n\longrightarrow\infty} \tilde{E}(u_n) = c; \tag{5.16}$$

$$\lim_{n\longrightarrow\infty} ||\tilde{E}'(u_n)||_{H^{-1}(\mathbf{R})} = 0. \tag{5.17}$$

Using (5.15) we may suppose that $\inf_{s\in\mathbf{R}} u_n(s) > -1 + \frac{v}{\sqrt{2}} + \frac{1}{2}\delta(\varepsilon)$, $\forall n \in \mathbf{N}$ and so $\tilde{E}(u_n) = E(u_n)$ and $\tilde{E}'(u_n) = E'(u_n)$. Since there exists a constant $C > 0$ such that $f^2(x) \geq Cx^2$ if $x \in [-1 + \frac{v}{\sqrt{2}} + \frac{1}{2}\delta(\varepsilon), \infty)$, (5.16) implies that the sequence (u_n) is bounded in $H^1(\mathbf{R})$.

Let $a_n = \inf_{s\in\mathbf{R}} u_n(s)$. For each n, fix a point $z_n \in \mathbf{R}$ such that $u_n(z_n) = a_n$.

The sequence $u_n(\cdot - z_n)$ is bounded in $H^1(\mathbf{R})$. Passing to a subsequence if necessary, we may suppose that there exists $u \in H^1(\mathbf{R})$ such that

$$u_n(\cdot - z_n) \rightharpoonup u \text{ weakly in } H^1(\mathbf{R}). \tag{5.18}$$

Using Arzela - Ascoli's Theorem and passing again to a subsequence, we may suppose that

$$u_n(\cdot - z_n) \longrightarrow u \text{ uniformly on each compact } K \subset \mathbf{R}. \tag{5.19}$$

It is clear that $\inf_{s\in\mathbf{R}} u(s) = u(0) \in [-1 + \dfrac{v}{\sqrt{2}} + \delta(\varepsilon), a_0]$.

Let $\phi \in \mathcal{S}(\mathbf{R})$. By (5.17), we have

$$E'(u_n)\phi(\cdot + z_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{5.20}$$

But

$$\int_{\mathbf{R}} u_n'(s)\phi'(s+z_n)ds = \int_{\mathbf{R}} u_n'(t-z_n)\phi'(t)dt \longrightarrow \int_{\mathbf{R}} u'(t)\phi'(t)dt$$

by (5.18) and

$$\int_{\mathbf{R}} ff'(u_n(s))\phi(s+z_n)ds \longrightarrow \int_{\mathbf{R}} ff'(u(t))\phi(t)dt \text{ as } n \longrightarrow \infty$$

by (5.19) and Lebesgue's dominated convergence theorem.

If there exists a subsequence (z_{n_k}) which tends to $+\infty$ or to $-\infty$ as $k \longrightarrow \infty$, we would have $\int_{\mathbf{R}} (u_{n_k}(s)+1)\phi(s+z_{n_k})U(s)ds \longrightarrow 0$ as $k \longrightarrow \infty$ by the dominated convergence theorem. From (5.20) we obtain

$$\int_{\mathbf{R}} u'(s)\phi'(s)ds + \int_{\mathbf{R}} ff'(u(s))\phi(s)ds = 0, \quad \forall \phi \in \mathcal{S}(\mathbf{R}),$$

that is u satisfies (1.9) (in the distributional sense) for $U \equiv 0$. On the other hand we have $\inf_{s\in\mathbf{R}} u(s) \in [-1 + \frac{v}{\sqrt{2}} + \delta(\varepsilon), a_0]$. We have seen in Section 3 that this is impossible. Therefore the sequence (z_n) is bounded.

Passing again to a subsequence, we may suppose that $\lim_{n\longrightarrow\infty} z_n = z \in \mathbf{R}$. By (5.19), $u_n(s) \longrightarrow u(s+z)$, $\forall s \in \mathbf{R}$ and

$$\int_{\mathbf{R}} (u_n(s)+1)\phi(s+z_n)U(s)ds \longrightarrow \int_{\mathbf{R}} (u(t)+1)\phi(t)U(t-z)dt.$$

From (5.20) we obtain for all $\phi \in \mathcal{S}(\mathbf{R})$

$$\int_{\mathbf{R}} u'(s)\phi'(s)ds + \int_{\mathbf{R}} ff'(u(s))\phi(s)ds + \int_{\mathbf{R}} (u(s)+1)\phi(s)U(s-z)ds = 0.$$

Therefore u satisfies the equation

$$-u''(s) + ff'(u(s)) + (u(s)+1)U(s-z) = 0$$

or equivalently, $r_1 = u(\cdot + z)$ satisfies (1.9). Furthermore, r_1 achieves its minimum at z and $r_1(z) \in [-1+\frac{v}{\sqrt{2}}+\delta(\varepsilon), a_0]$. From the discussion in Introduction, it follows that $r_1 \leq 0$ and r_1 satisfies (1.12) on $(-\infty, x)\cup(y,\infty)$. Let $a = r_1(x)$ and $b = r_1(y)$. By a standard argument we infer that $r_1 \equiv r_a(\cdot - x)$ on $(-\infty, x)$ and $r_1 \equiv r_b(\cdot - y)$ on (y, ∞) so that necessarily $z \in [x, y]$.

As in the proof of Theorem 4.1 one has

$$E(r_1) = E(u(\cdot + z)) \leq \liminf_{n\to\infty} E(u_n) = c. \qquad \square \tag{5.21}$$

In fact, hypothesis **H2** is not necessary for the existence of a second solution of equation (1.9). It can be eliminated using Proposition 5.7 and a simple approximation procedure. This will be seen in the next theorem, which is the main result of this section.

Theorem 5.8 *Let U be a positive Borel measure with supp$(U) \subset [x,y]$. Suppose that $||U|| < \varphi_2(v)$, where φ_2 is the function introduced in Remark 5.3. Then equation (1.9) admits a solution r_1 with $\inf_{s\in\mathbf{R}} r_1(s) \in [-1+\frac{v}{\sqrt{2}}, a_0]$, where $a_0 = \sup\{a \in [-1+\frac{v}{\sqrt{2}}, 0) \mid h(a) \geq 0\}$. Furthermore, $E(r_1) \leq c$.*

Proof. We have seen that if $||U|| < \varphi_2(v)$, then $h(-1+\sqrt{\alpha(v)}) > 0$.

For $\varepsilon > 0$ define $U_\varepsilon = U + \varepsilon\chi_{[x,y]}$. Denote by H_ε, E_ε, h_ε the corresponding quantities for the measure U_ε. It is easily seen that $h_\varepsilon(a) \leq h(a)$, $\forall a \in [-1+\frac{v}{\sqrt{2}}, 0]$ and $h_\varepsilon(a) \longrightarrow h(a)$ as $\varepsilon \longrightarrow 0$, so $h_\varepsilon(-1+\sqrt{\alpha(v)}) > 0$ if ε is sufficiently small, say, if $\varepsilon \in (0, \varepsilon_0)$. For $\varepsilon \in (0,\varepsilon_0)$, define $a_{0,\varepsilon}$ as in Theorem 4.1. Then $a_{0,\varepsilon} \leq a_0$ and E_ε has a minimizer r_ε on the set $V_{0,\varepsilon} = \{u \in H^1(\mathbf{R}) \mid \inf_{s\in\mathbf{R}} u(s) > a_{0,\varepsilon}\}$. Define c_ε as before. It is obvious that $c_\varepsilon \leq c$.

Applying Proposition 5.7 for the measure U_ε, we get a critical point $r_{1,\varepsilon}$ of E_ε and $z_\varepsilon \in [x,y]$ such that $\inf_{s\in\mathbf{R}} r_{1,\varepsilon}(z) = r_{1,\varepsilon}(z_\varepsilon) \in [-1+\frac{v}{\sqrt{2}} + \delta(\varepsilon), a_0]$. Furthermore, we have $E_\varepsilon(r_{1,\varepsilon}) \leq c_\varepsilon \leq c$, which implies that

$$\int_{\mathbf{R}} |r_{1,\varepsilon}'|^2(s)ds \leq E_\varepsilon(r_{1,\varepsilon}) - \int_{\mathbf{R}} r_{1,\varepsilon}(r_{1,\varepsilon}+2)U_\varepsilon(s)ds \leq c + \Big(1 - \frac{v^2}{2}\Big)||U_\varepsilon||.$$

Hence $\int_{\mathbf{R}} |r_{1,\varepsilon}'|^2(s)ds$ is uniformly bounded for $\varepsilon \in (0,\varepsilon_0)$. Let $a_\varepsilon = r_{1,\varepsilon}(x)$ and $b_\varepsilon = r_{1,\varepsilon}(y)$. We know that $r_{1,\varepsilon} = r_{a_\varepsilon}(\cdot - x)$ on $(-\infty, x)$ and $r_{1,\varepsilon} = r_{b_\varepsilon}(\cdot - y)$ on (y,∞). Since $-1+\frac{v}{\sqrt{2}} \leq r_{1,\varepsilon}(s) \leq 0$ for $s \in [x,y]$, we infer that $r_{1,\varepsilon}$ is uniformly bounded in $L^2(\mathbf{R})$, hence $r_{1,\varepsilon}$ is bounded in $H^1(\mathbf{R})$. Consequently, there exists a sequence $\varepsilon_n \longrightarrow 0$ and $r_1 \in H^1(\mathbf{R})$ such that $r_{1,\varepsilon_n} \rightharpoonup r_1$ weakly in $H^1(\mathbf{R})$ as $n \longrightarrow \infty$. Using Arzela - Ascoli's theorem, we may suppose that $r_{1,\varepsilon_n} \longrightarrow r_1$ uniformly on $[x,y]$. In fact, the particular form of r_{1,ε_n} implies that $r_{1,\varepsilon_n} \longrightarrow r_1$ uniformly on $\mathbf{R}$ and $r_1 = r_a(\cdot - x)$ on $(-\infty,x)$, respectively $r_1 = r_b(\cdot - y)$ on (y,∞), where $a = r_1(x)$ and $b = r_1(y)$. Clearly the minimum of r_1 on $\mathbf{R}$ is achieved at a point $z \in [x,y]$. By the uniform convergence, $r_1(z) \in [-1+\frac{v}{\sqrt{2}}, a_0]$.

For each test function ϕ one has

$$\int_{\mathbf{R}} r_{1,\varepsilon_n}'\phi' ds + \int_{\mathbf{R}} ff'(r_{1,\varepsilon_n})\phi ds + \int_{\mathbf{R}} (1+r_{1,\varepsilon_n})\phi U(s)ds + \varepsilon_n \int_x^y (1+r_{1,\varepsilon_n})\phi ds = 0.$$

Passing to the limit as $n \longrightarrow \infty$, we obtain that r_1 is a solution of (1.9).

The weak convergence of r_{1,ε_n} in $H^1(\mathbf{R})$ and the uniform convergence on $\mathbf{R}$ imply $E(r_1) \leq \liminf_{n\longrightarrow\infty} E(r_{1,\varepsilon_n}) \leq c$. $\square$

Coming back to (1.8), we determine the corresponding phases θ and θ_1 for the solutions r, respectively r_1 of (1.9). If U has compact support, θ' and θ_1' are integrable on $\mathbf{R}$ because of the particular form of r and r_1 outside *supp*(U). We impose that $\theta(x) \longrightarrow 0$, $\theta_1(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Then $\theta(x) \longrightarrow \mu$, $\theta_1(x) \longrightarrow \mu_1$ as $x \longrightarrow -\infty$ for some positive constants μ and μ_1. Thus we obtain two solutions A and A_1 of (1.5). Remark that A and A_1 tend exponentially to 1 at ∞ and to $e^{i\mu}$ (respectively to $e^{i\mu_1}$) at $-\infty$. Vortices are replaced in one dimension by a density depression around *supp*(U).

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