Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity

Mihai MARIŞ

Université de Franche-Comté
Département de Mathématiques UMR 6623
16, Route de Gray
25030 Besançon, France
e-mail: mihai.maris@univ-fcomte.fr

Dedicated to Jean-Claude Saut,
who gave me water to cross the desert.

Abstract

For a large class of nonlinear Schrödinger equations with nonzero conditions at infinity and for any speed c less than the sound velocity, we prove the existence of finite energy traveling waves moving with speed c in any space dimension $N \geq 3$. Our results are valid as well for the Gross-Pitaevskii equation and for NLS with cubic-quintic nonlinearity.

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1 Introduction

We consider the nonlinear Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + F(|\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N, \tag{1.1}$$

where $\Phi : \mathbf{R}^N \longrightarrow \mathbf{C}$ satisfies the "boundary condition" $|\Phi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, $r_0 > 0$ and F is a real-valued function on $\mathbf{R}_+$ satisfying $F(r_0^2) = 0$.

Equations of the form (1.1), with the considered non-zero conditions at infinity, arise in a large variety of physical problems such as superconductivity, superfluidity in Helium II, phase transitions and Bose-Einstein condensate ([2], [3], [4], [12], [20], [22], [23], [24], [25]). In nonlinear optics, they appear in the context of dark solitons ([27], [28]). Two important particular cases of (1.1) have been extensively studied both by physicists and by mathematicians: the Gross-Pitaevskii equation (where $F(s) = 1 - s$) and the so-called "cubic-quintic" Schrödinger equation (where $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, $\alpha_1, \alpha_3, \alpha_5$ are positive and F has two positive roots).

The boundary condition $|\Phi| \longrightarrow r_0 > 0$ at infinity makes the structure of solutions of (1.1) much more complicated than in the usual case of zero boundary conditions (when the associated dynamics is essentially governed by dispersion and scattering).

Using the Madelung transformation $\Phi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$ (which is well-defined whenever $\Phi \neq 0$), equation (1.1) is equivalent to a system of Euler's equations for a compressible inviscid fluid of density ρ and velocity $2\nabla\theta$. In this context it has been shown that, if F is C^1 near r_0^2 and $F'(r_0^2) < 0$, the sound velocity at infinity associated to (1.1) is $v_s = r_0\sqrt{-2F'(r_0^2)}$ (see the introduction of [33]).

Equation (1.1) is Hamiltonian: denoting $V(s) = \int_s^{r_0^2} F(\tau)\, d\tau$, it is easy to see that, at least formally, the "energy"

$$(1.2) \qquad E(\Phi) = \int_{\mathbf{R}^N} |\nabla \Phi|^2 \, dx + \int_{\mathbf{R}^N} V(|\Phi|^2)\, dx$$

is a conserved quantity.

In a series of papers (see, e.g., [2], [3], [20], [24], [25]), particular attention has been paid to a special class of solutions of (1.1), namely the traveling waves. These are solutions of the form $\Phi(x,t) = \psi(x - cty)$, where $y \in S^{N-1}$ is the direction of propagation and $c \in \mathbf{R}^*$ is the speed of the traveling wave. We say that ψ has finite energy if $\nabla\psi \in L^2(\mathbf{R}^N)$ and $V(|\psi|^2) \in L^1(\mathbf{R}^N)$). These solutions are supposed to play an important role in the dynamics of (1.1). In view of formal computations and numerical experiments, a list of conjectures, often referred to as *the Roberts programme,* has been formulated about the existence, the stability and the qualitative properties of traveling waves. The first of these conjectures asserts that finite energy traveling waves of speed c exist if and only if $|c| < v_s$.

Let ψ be a finite energy traveling-wave of (1.1) moving with speed c. Without loss of generality we may assume that $y = (1, 0, \dots, 0)$. If $N \geq 3$, it follows that $\psi - z_0 \in L^{2^*}(\mathbf{R}^N)$ for some constant $z_0 \in \mathbf{C}$, where $2^* = \frac{2N}{N-2}$ (see, e.g., Lemma 7 and Remark 4.2 pp. 774-775 in [17]). Since $|\psi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, necessarily $|z_0| = r_0$. If Φ is a solution of (1.1) and $\alpha \in \mathbf{R}$, then $e^{i\alpha}\Phi$ is also a solution; hence we may assume that $z_0 = r_0$, thus $\psi - r_0 \in L^{2^*}(\mathbf{R}^N)$. Denoting $u = r_0 - \psi$, we see that u satisfies the equation

$$(1.3) \qquad ic\frac{\partial u}{\partial x_1} - \Delta u + F(|r_0 - u|^2)(r_0 - u) = 0 \qquad \text{in } \mathbf{R}^N.$$

It is obvious that a function u satisfies (1.3) for some velocity c if and only if $u(-x_1, x')$ satisfies (1.3) with c replaced by $-c$. Hence it suffices to consider the case $c > 0$. This assumption will be made throughout the paper.

In space dimension $N = 1$, in many interesting applications equation (1.3) can be integrated explicitly and one obtains traveling waves for all subsonic speeds. The nonexistence of such solutions for supersonic speeds has also been proved under general conditions (cf. Theorem 5.1, p. 1099 in [33]).

Despite of many attempts, a rigorous proof of the existence of traveling waves in higher dimensions has been a long lasting problem. In the particular case of the Gross-Pitaevskii (GP) equation, this problem was considered in a series of papers. In space dimension $N = 2$, the existence of traveling waves has been proved in [7] for all speeds in some interval $(0, \varepsilon)$, where ε is small. In space dimension $N \geq 3$, the existence has been proved in [6] for a sequence of speeds $c_n \longrightarrow 0$ by using constrained minimization; a similar result has been established in [11] for all sufficiently small speeds by using a mountain-pass argument. In a recent paper [5], the existence of traveling waves for (GP) has been proved in space dimension $N = 2$ and $N = 3$ for any speed in a set $A \subset (0, v_s)$. If $N = 2$, A contains points arbitrarily close to 0 and to v_s (although it is not clear that $A = (0, v_s)$), while in dimension $N = 3$ we have $A \subset (0, v_0)$, where $v_0 < v_s$ and 0, v_0 are limit points of A. The traveling waves are obtained in [5] by minimizing the energy at fixed momentum (see the next section for the definition of the momentum) and the propagation speed is the Lagrange multiplier associated to minimizers. In the case of cubic-quintic type nonlinearities, it has been proved in [31] that traveling waves exist for any sufficiently small speed if $N \geq 4$. To our knowledge, even for specific nonlinearities there are no existence results in the literature that cover the whole range $(0, v_s)$ of possible speeds.

The nonexistence of traveling waves for supersonic speeds ($c > v_s$) has been proved in [21] in the case of the Gross-Pitaevskii equation, respectively in [33] for a large class of nonlinearities.

The aim of this paper is to prove the existence of finite energy traveling waves of (1.1) in space dimension $N \geq 3$, under general conditions on the nonlinearity F and for any speed $c \in (-v_s, v_s)$.

We will consider the following set of assumptions:

A1. The function F is continuous on $[0, \infty)$, C^1 in a neighborhood of r_0^2, $F(r_0^2) = 0$ and $F'(r_0^2) < 0$.

A2. There exist $C > 0$ and $p_0 < \frac{2}{N-2}$ such that $|F(s)| \leq C(1 + s^{p_0})$ for any $s \geq 0$.

A3. There exist C, $\alpha_0 > 0$ and $r_* > r_0$ such that $F(s) \leq -Cs^{\alpha_0}$ for any $s \geq r_*$.

If (A1) is satisfied, we denote $V(s) = \displaystyle\int_s^{r_0^2} F(\tau)\, d\tau$ and $a = \sqrt{-\frac{1}{2}F'(r_0^2)}$. Then the sound velocity at infinity associated to (1.1) is $v_s = 2ar_0$ and using Taylor's formula for s in a neighborhood of r_0^2 we have

$$(1.4) \qquad V(s) = \frac{1}{2}V''(r_0^2)(s - r_0^2)^2 + (s - r_0^2)^2\varepsilon(s - r_0^2) = a^2(s - r_0^2)^2 + (s - r_0^2)^2\varepsilon(s - r_0^2),$$

where $\varepsilon(t) \longrightarrow 0$ as $t \longrightarrow 0$. Hence for $|\psi|$ close to r_0, $V(|\psi|^2)$ can be approximated by $a^2(|\psi|^2 - r_0^2)^2$.

We fix an odd function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(s) = s$ for $s \in [0, 2r_0]$, $0 \leq \varphi' \leq 1$ on $\mathbf{R}$ and $\varphi(s) = 3r_0$ for $s \geq 4r_0$. We denote $W(s) = V(s) - V(\varphi^2(\sqrt{s}))$, so that $W(s) = 0$ for $s \in [0, 4r_0^2]$. If assumptions (A1) and (A2) are satisfied, it is not hard to see that there exist C_1, C_2, $C_3 > 0$ such that

$$(1.5) \qquad \begin{array}{l} |V(s)| \leq C_1(s - r_0^2)^2 \quad \text{for any } s \leq 9r_0^2; \\ \text{in particular, } |V(\varphi^2(\tau))| \leq C_1(\varphi^2(\tau) - r_0^2)^2 \text{ for any } \tau; \end{array}$$

$$(1.6) \qquad |V(b) - V(a)| \leq C_2|b - a| \max(a^{p_0}, b^{p_0}) \qquad \text{for any } a,\ b \geq 2r_0^2;$$

$$(1.7) \qquad |W(b^2) - W(a^2)| \leq C_3|b - a| \left(a^{2p_0+1}\mathbb{1}_{\{a>2r_0\}} + b^{2p_0+1}\mathbb{1}_{\{b>2r_0\}}\right) \quad \text{for any } a, b \geq 0.$$

Given $u \in H^1_{loc}(\mathbf{R}^N)$ and Ω an open set in $\mathbf{R}^N$, the modified Ginzburg-Landau energy of u in Ω is defined by

$$(1.8) \qquad E_{GL}^{\Omega}(u) = \int_\Omega |\nabla u|^2\, dx + a^2 \int_\Omega \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\, dx.$$

We simply write $E_{GL}(u)$ instead of $E_{GL}^{\mathbf{R}^N}(u)$. The modified Ginzburg-Landau energy will play a central role in our analysis. We consider the function space

$$(1.9) \qquad \begin{array}{rcl} \mathcal{X} & = & \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N) \mid \varphi^2(|r_0 - u|) - r_0^2 \in L^2(\mathbf{R}^N)\} \\ & = & \{u \in \dot{H}^1(\mathbf{R}^N) \mid u \in L^{2^*}(\mathbf{R}^N),\ E_{GL}(u) < \infty\}, \end{array}$$

where $\mathcal{D}^{1,2}(\mathbf{R}^N)$ is the completion of C_c^∞ for the norm $||v|| = ||\nabla v||_{L^2}$. If $N \geq 3$ and (A1), (A2) are satisfied, it is not hard to see that a function u has finite energy if and only if $u \in \mathcal{X}$ (see Lemma 4.1 below). Note that for $N = 3$, $\mathcal{X}$ is *not* a vector space. However, in any space dimension we have $H^1(\mathbf{R}^N) \subset \mathcal{X}$. If $u \in \mathcal{X}$, it is easy to see that for any $w \in H^1(\mathbf{R}^N)$ with compact support we have $u + w \in \mathcal{X}$. For $N = 3, 4$ it can be proved that $u \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ belongs to $\mathcal{X}$ if and only if $|r_0 - u|^2 - r_0^2 \in L^2(\mathbf{R}^N)$, and consequently $\mathcal{X}$ coincides with the space F_{r_0} introduced by P. Gérard in [17], section 4. It has been proved in [17] that the Cauchy problem for the Gross-Pitaevskii equation is globally well-posed in $\mathcal{X}$ in dimension $N = 3$, respectively it is globally well-posed for small initial data if $N = 4$.

Our main results can be summarized as follows:

Theorem 1.1 *Assume that $N \geq 3$, $0 < c < v_s$, (A1) and one of the conditions (A2) or (A3) are satisfied. Then equation (1.3) admits a nontrivial solution $u \in \mathcal{X}$. Moreover, $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$ and, after a translation, u is axially symmetric with respect to Ox_1.*

At least formally, solutions of (1.3) are critical points of the functional

$$E_c(u) = \int_{\mathbf{R}^N} |\nabla u|^2 \, dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx,$$

where Q is the momentum with respect to the x_1-direction (the functional Q will be defined in the next section). If the assumptins (A1) and (A2) above are satisfied, it can be proved (see Proposition 4.1 p. 1091-1092 in [33]) that any traveling wave $u \in \mathcal{X}$ of (1.1) must satisfy a Pohozaev-type identity $P_c(u) = 0$, where

$$P_c(u) = \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_1} \right|^2 + \frac{N-3}{N-1} \sum_{k=2}^{N} \left| \frac{\partial u}{\partial x_k} \right|^2 dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx.$$

We will prove the existence of traveling waves by showing that the problem of minimizing E_c in the set $\{u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0\}$ admits solutions. Then we show that any minimizer satisfies (1.3) if $N \geq 4$, respectively any minimizer satisfies (1.3) after a scaling in the last two variables if $N = 3$.

In space dimension $N = 2$, the situation is different: if (A1) is true and (A2) holds for some $p_0 < \infty$, any solution $u \in \mathcal{X}$ of (1.3) still satisfies the identity $P_c(u) = 0$, but it can be proved that there are no minimizers of E_c subject to the constraint $P_c = 0$ (in fact, we have $\inf\{E_c(u) \mid u \in \mathcal{X}, u \neq 0, P_c(u) = 0\} = 0$). However, using a different aproach it is still possible to show the existence of traveling waves in the case $N = 2$, at least for a set of speeds that contains elements arbitrarily close to zero and to v_s (and this will be done in a forthcoming paper). Although some of the results in sections $2-4$ are still valid in space dimension $N = 2$ (with straightforward modifications in proofs), for simplicity we assume throughout that $N \geq 3$.

It is easy to see that it suffices to prove Theorem 1.1 only in the case where (A1) and (A2) are satisfied. Indeed, suppose that Theorem 1.1 holds if (A1) and (A2) are true. Assume that (A1) and (A3) are satisfied. Let C, r_*, α_0 be as in (A3). There exist $\beta \in (0, \frac{2}{N-1})$, $\tilde{r} > r_*$, and $C_1 > 0$ such that

$$Cs^{2\alpha_0} - \frac{v_s^2}{4} \geq C_1(s - \tilde{r})^{2\beta} \qquad \text{for any } s \geq \tilde{r}.$$

Let $\tilde{F}$ be a function with the following properties: $F = \tilde{F}$ on $[0, 4\tilde{r}^2]$, $\tilde{F}(s) = -C_2 s^\beta$ for s sufficiently large, and $\tilde{F}(s^2) + \frac{v_s^2}{4} \leq -C_3(s - \tilde{r})^{2\beta}$ for any $s \geq \tilde{r}$, where C_2, C_3 are some positive constants. Then $\tilde{F}$ satisfies (A1), (A2), (A3) and from Theorem 1.1 it follows that equation (1.3) with $\tilde{F}$ instead of F has nontrivial solutions $u \in \mathcal{X}$. From the proof of Proposition 2.2 (i) p. 1079-1080 in [33] it follows that any such solution satisfies $|r_0 - u|^2 \leq 2\tilde{r}^2$, and consequently $F(|r_0 - u|^2) = \tilde{F}(|r_0 - u|^2)$. Thus u satisfies (1.3). Of course, if (A1) and (A3) are satisfied but (A2) does not hold, we do not claim that the solutions of (1.3) obtained as above are still minimizers of E_c subject to the constraint $P_c = 0$ (in fact, only assumptions (A1) and (A3) do not imply that E_c and P_c are well-defined on $\mathcal{X}$ and that the minimization problem makes sense).

In particular, for $F(s) = 1 - s$ the conditions (A1) and (A3) are satisfied and it follows that the Gross-Pitaevskii equation admits traveling waves of finite energy in any space dimension

$N \geq 3$ and for any speed $c \in (0, v_s)$ (although (A2) is not true for $N > 3$: the (GP) equation is critical if $N = 4$, and supercritical if $N \geq 5$). A similar result holds for the cubic-quintic NLS.

We have to mention that, according to the properties of F, for $c = 0$ equation (1.3) may or not have finite energy solutions. For instance, it is an easy consequence of the Pohozaev identities that all finite energy stationary solutions of the Gross-Pitaevskii equation are constant. On the contrary, for nonlinearities of cubic-quintic type the existence of finite energy stationary solutions has been proved in [13] under fairly general assumptions on F. In the case $c = 0$, our proofs imply that E_0 has a minimizer in the set $\{u \in \mathcal{X} \mid u \neq 0, P_0(u) = 0\}$ whenever this set is not empty. Then it is not hard to prove that minimizers satisfy (1.3) for $c = 0$ (modulo a scale change if $N = 3$). However, for simplicity we assume throughout (unless the contrary is explicitly mentioned) that $0 < c < v_s$.

This paper is organized as follows. In the next section we give a convenient definition of the momentum and we study the properties of this functional.

In section 3 we introduce a regularization procedure for functions in $\mathcal{X}$ which will be a key tool for all the variational machinery developed later.

In section 4 we describe the variational framework. In particular, we prove that the set $\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0, \, P_c(u) = 0\}$ is not empty and we have $\inf\{E_c(u) \mid u \in \mathcal{C}\} > 0$.

In section 5 we consider the case $N \geq 4$ and we prove that the functional E_c has minimizers in $\mathcal{C}$ and these minimizers are solutions of (1.3). To show the existence of minimizers we use the concentration-compactness principle and the regularization procedure developed in section 3. Then we use the Pohozaev identities to control the Lagrange multiplier associated to the minimization problem.

Although the results in space dimension $N = 3$ are similar to those in higher dimensions (with one exception: not all minimizers of E_c in $\mathcal{C}$ are solutions of (1.3), as one can easily see by scaling), it turns out that the proofs are quite different. We treat the case $N = 3$ in section 6.

Finally, we prove that traveling waves found by minimization in sections 5 and 6 are axially symmetric (as one would expect from physical considerations, see [24]).

Throughout the paper, $\mathcal{L}^N$ is the Lebesgue measure on $\mathbf{R}^N$. For $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$, we denote $x' = (x_2, \ldots, x_N) \in \mathbf{R}^{N-1}$. We write $\langle z_1, z_2 \rangle$ for the scalar product of two complex numbers z_1, z_2. Given a function f defined on $\mathbf{R}^N$ and λ, $\sigma > 0$, we denote by

$$f_{\lambda,\sigma} = f\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right) \tag{1.10}$$

the dilations of f. The behavior of functions and of functionals with respect to dilations in $\mathbf{R}^N$ will be very important. For $1 \leq p < N$, we denote by p^* the Sobolev exponent associated to p, that is $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

2 The momentum

A good definition of the momentum is essential in any attempt to find solutions of (1.3) by using a variational approach. Roughly speaking, the momentum (with respect to the x_1-direction) should be a functional with derivative $2iu_{x_1}$. Various definitions have been given in the literature (see [7], [5], [6], [31]), any of them having its advantages and its inconvenients. Unfortunately, none of them is valid for all functions in $\mathcal{X}$. We propose a new and more general definition in this section.

It is clear that for functions $u \in H^1(\mathbf{R}^N)$, the momentum should be given by

$$Q_1(u) = \int_{\mathbf{R}^N} \langle iu_{x_1}, u \rangle \, dx, \tag{2.1}$$

and this is indeed a nice functional on $H^1(\mathbf{R}^N)$. The problem is that there are functions $u \in \mathcal{X} \setminus H^1(\mathbf{R}^N)$ such that $\langle iu_{x_1}, u \rangle \notin L^1(\mathbf{R}^N)$.

If $u \in \mathcal{X}$ is such that $r_0 - u$ admits a lifting $r_0 - u = \rho e^{i\theta}$, a formal computation gives

$$\int_{\mathbf{R}^N} \langle iu_{x_1}, u \rangle \, dx = -\int_{\mathbf{R}^N} \rho^2 \theta_{x_1} \, dx = -\int_{\mathbf{R}^N} (\rho^2 - r_0^2)\theta_{x_1} \, dx. \tag{2.2}$$

It is not hard to see that if $u \in \mathcal{X}$ is as above, then $(\rho^2 - r_0^2)\theta_{x_1} \in L^1(\mathbf{R}^N)$. However, there are many "interesting" functions $u \in \mathcal{X}$ such that $r_0 - u$ does not admit a lifting.

Our aim is to define the momentum on $\mathcal{X}$ in such a way that it agrees with (2.1) for functions in $H^1(\mathbf{R}^N)$ and with (2.2) when a lifting as above exists.

Lemma 2.1 *Let $u \in \mathcal{X}$ be such that $m \le |r_0 - u(x)| \le 2r_0$ a.e. on $\mathbf{R}^N$, where $m > 0$. There exist two real-valued functions ρ, θ such that $\rho - r_0 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $r_0 - u = \rho e^{i\theta}$ a.e. on $\mathbf{R}^N$ and*

$$\langle iu_{x_1}, u \rangle = -r_0 \frac{\partial}{\partial x_1}(\mathrm{Im}(u) + r_0\theta) - (\rho^2 - r_0^2)\frac{\partial \theta}{\partial x_1} \qquad a.e. \text{ on } \mathbf{R}^N. \tag{2.3}$$

Moreover, we have $\displaystyle\int_{\mathbf{R}^N} \left|(\rho^2 - r_0^2)\theta_{x_1}\right| dx \le \frac{1}{2am} E_{GL}(u)$.

Proof. Since $r_0 - u \in H^1_{loc}(\mathbf{R}^N)$, the fact that there exist $\rho, \theta \in H^1_{loc}(\mathbf{R}^N)$ such that $r_0 - u = \rho e^{i\theta}$ a.e. is standard and follows from Theorem 3 p. 38 in [9]. We have

$$\left|\frac{\partial u}{\partial x_j}\right|^2 = \left|\frac{\partial \rho}{\partial x_j}\right|^2 + \rho^2 \left|\frac{\partial \theta}{\partial x_j}\right|^2 \qquad \text{a.e. on } \mathbf{R}^N \text{ for } j = 1, \dots, N. \tag{2.4}$$

Since $\rho = |r_0 - u| \ge m$ a.e., it follows that $\nabla\rho, \nabla\theta \in L^2(\mathbf{R}^N)$. If $N \ge 3$, we infer that there exist $\rho_0, \theta_0 \in \mathbf{R}$ such that $\rho - \rho_0$ and $\theta - \theta_0$ belong to $L^{2^*}(\mathbf{R}^N)$. Then it is not hard to see that $\rho_0 = r_0$ and $\theta_0 = 2k_0\pi$, where $k_0 \in \mathbf{Z}$. Replacing θ by $\theta - 2k_0\pi$, we have $\rho - r_0, \theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. Since $\rho \le 2r_0$ a.e., we have $\rho^2 - r_0^2 = \varphi(|r_0 - u|^2) - r_0^2 \in L^2(\mathbf{R}^N)$ because $u \in \mathcal{X}$. Clearly $|\rho - r_0| = \frac{|\rho^2 - r_0^2|}{\rho + r_0} \le \frac{1}{r_0}|\rho^2 - r_0^2|$, hence $\rho - r_0 \in L^2(\mathbf{R}^N)$.

A straightforward computation gives

$$\langle iu_{x_1}, u \rangle = \langle iu_{x_1}, r_0 \rangle - \rho^2\theta_{x_1} = -r_0 \frac{\partial}{\partial x_1}(\mathrm{Im}(u) + r_0\theta) - (\rho^2 - r_0^2)\frac{\partial \theta}{\partial x_1}.$$

By (2.4) we have $\left|\frac{\partial \theta}{\partial x_j}\right| \le \frac{1}{\rho}\left|\frac{\partial u}{\partial x_j}\right| \le \frac{1}{m}\left|\frac{\partial u}{\partial x_j}\right|$ and the Cauchy-Schwarz inequality gives

$$\int_{\mathbf{R}^N} \left|(\rho^2 - r_0^2)\theta_{x_1}\right| dx \le ||\rho^2 - r_0^2||_{L^2} ||\theta_{x_1}||_{L^2} \le \frac{1}{m}||\rho^2 - r_0^2||_{L^2}||u_{x_1}||_{L^2} \le \frac{1}{2am} E_{GL}(u).$$

□

Lemma 2.2 *Let $\chi \in C_c^\infty(\mathbf{C}, \mathbf{R})$ be a function such that $\chi = 1$ on $B(0, \frac{r_0}{4})$, $0 \le \chi \le 1$ and $\mathrm{supp}(\chi) \subset B(0, \frac{r_0}{2})$. For an arbitrary $u \in \mathcal{X}$, denote $u_1 = \chi(u)u$ and $u_2 = (1 - \chi(u))u$. Then $u_1 \in \mathcal{X}$, $u_2 \in H^1(\mathbf{R}^N)$ and the following estimates hold:*

$$|\nabla u_i| \le C|\nabla u| \qquad a.e. \text{ on } \mathbf{R}^N, i = 1, 2, \text{ wehere } C \text{ depends only on } \chi, \tag{2.5}$$

$$||u_2||_{L^2(\mathbf{R}^N)} \le C_1 ||\nabla u||_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}} \quad \text{and} \quad ||(1 - \chi^2(u))u||_{L^2(\mathbf{R}^N)} \le C_1 ||\nabla u||_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}}, \tag{2.6}$$

$$(2.7)\qquad \int_{\mathbf{R}^N} \left(\varphi^2(|r_0-u_1|)-r_0^2\right)^2\,dx \le \int_{\mathbf{R}^N} \left(\varphi^2(|r_0-u|)-r_0^2\right)^2\,dx + C_2||\nabla u||^{2^*}_{L^2(\mathbf{R}^N)},$$

$$(2.8)\qquad \int_{\mathbf{R}^N} \left(\varphi^2(|r_0-u_2|)-r_0^2\right)^2\,dx \le C_2||\nabla u||^{2^*}_{L^2(\mathbf{R}^N)}.$$

Let $r_0-u_1=\rho e^{i\theta}$ be the lifting of r_0-u_1, as given by Lemma 2.1. Then we have

$$(2.9)\qquad \langle iu_{x_1},u\rangle = (1-\chi^2(u))\langle iu_{x_1},u\rangle - (\rho^2-r_0^2)\frac{\partial\theta}{\partial x_1} - r_0\frac{\partial}{\partial x_1}(\mathrm{Im}(u_1)+r_0\theta)$$

a.e. on $\mathbf{R}^N$.

Proof. Since $|u_i|\le|u|$, we have $u_i\in L^{2^*}(\mathbf{R}^N)$, $i=1,2$. It is standard to prove that $u_i\in H^1_{loc}(\mathbf{R}^N)$ (see, e.g., Lemma C1 p. 66 in [9]) and we have

$$(2.10)\qquad \frac{\partial u_1}{\partial x_j} = \left(\partial_1\chi(u)\frac{\partial(\mathrm{Re}(u))}{\partial x_j} + \partial_2\chi(u)\frac{\partial(\mathrm{Im}(u))}{\partial x_j}\right)u + \chi(u)\frac{\partial u}{\partial x_j}.$$

A similar formula holds for u_2. Since the functions $z\longmapsto \partial_i\chi(z)z$, $i=1,2$, are bounded on $\mathbf{C}$, (2.5) follows immediately from (2.10).

Using the Sobolev embedding we have

$$||u_2||^2_{L^2} \le \int_{\mathbf{R}^N}|u|^2\mathbb{1}_{\{|u|>\frac{r_0}{4}\}}(x)\,dx \le \left(\frac{4}{r_0}\right)^{2^*-2}\int_{\mathbf{R}^N}|u|^{2^*}\mathbb{1}_{\{|u|>\frac{r_0}{4}\}}(x)\,dx \le C_1||\nabla u||^{2^*}_{L^2}.$$

This gives the first estimate in (2.6); the second one is similar.

For $|u|\le\frac{r_0}{4}$ we have $u_1(x)=u(x)$, hence

$$\int_{\{|u|\le\frac{r_0}{4}\}}\left(\varphi^2(|r_0-u_1|)-r_0^2\right)^2\,dx = \int_{\{|u|\le\frac{r_0}{4}\}}\left(\varphi^2(|r_0-u|)-r_0^2\right)^2\,dx.$$

There exists $C'>0$ such that $\left(\varphi^2(|r_0-z|)-r_0^2\right)^2\le C'|z|^2$ if $|z|\ge\frac{r_0}{4}$. Proceeding as in the proof of (2.6) we have for $i=1,2$

$$\int_{\{|u|>\frac{r_0}{4}\}}\left(\varphi^2(|r_0-u_i|)-r_0^2\right)^2\,dx \le C'\int_{\{|u|>\frac{r_0}{4}\}}|u_i|^2\,dx \le C_2||\nabla u||^{2^*}_{L^2}.$$

This clearly implies (2.7) and (2.8).

Since $\partial_1\chi(u)\frac{\partial(\mathrm{Re}(u))}{\partial x_j}+\partial_2\chi(u)\frac{\partial(\mathrm{Im}(u))}{\partial x_j}\in\mathbf{R}$, using (2.10) we see that $\langle i\frac{\partial u_1}{\partial x_1},u_1\rangle=\chi^2(u)\langle iu_{x_1},u\rangle$ a.e. on $\mathbf{R}$. Then (2.9) follows from Lemma 2.1. □

We consider the space $\mathcal{Y}=\{\partial_{x_1}\phi \mid \phi\in\mathcal{D}^{1,2}(\mathbf{R}^N)\}$. It is clear that $\phi_1,\phi_2\in\mathcal{D}^{1,2}(\mathbf{R}^N)$ and $\partial_{x_1}\phi_1=\partial_{x_1}\phi_2$ imply $\phi_1=\phi_2$. Defining

$$||\partial_{x_1}\phi||_{\mathcal{Y}} = ||\phi||_{\mathcal{D}^{1,2}} = ||\nabla\phi||_{L^2(\mathbf{R}^N)},$$

it is easy to see that $||\cdot||_{\mathcal{Y}}$ is a norm on $\mathcal{Y}$ and $(\mathcal{Y},||\cdot||_{\mathcal{Y}})$ is a Banach space. The following holds.

Lemma 2.3 *For any $v\in L^1(\mathbf{R}^N)\cap\mathcal{Y}$ we have $\int_{\mathbf{R}^N}v(x)\,dx=0$.*

Proof. Let $\phi \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ be such that $v = \partial_{x_1}\phi$. Then $\phi \in \mathcal{S}'(\mathbf{R}^N)$ and $|\xi|\widehat{\phi} \in L^2(\mathbf{R}^N)$. Hence $\widehat{\phi} \in L^1_{loc}(\mathbf{R}^N \setminus \{0\})$. On the other hand we have $v = \partial_{x_1}\phi \in L^1 \cap L^2(\mathbf{R}^N)$ by hypothesis, hence $\widehat{v} = i\xi_1\widehat{\phi} \in L^2 \cap C^0_b(\mathbf{R}^N)$.

We prove that $\widehat{v}(0) = 0$. We argue by contradiction and assume that $\widehat{v}(0) \neq 0$. By continuity, there exists $m > 0$ and $\varepsilon > 0$ such that $|\widehat{v}(\xi)| \geq m$ for $|\xi| \leq \varepsilon$. For $j = 2, \dots N$ we get

$$|i\xi_j\widehat{\phi}(\xi)| \geq \frac{|\xi_j|}{|\xi_1|}|\widehat{v}(\xi)| \geq m\frac{|\xi_j|}{|\xi_1|} \qquad \text{for a.e. } \xi \in B(0,\varepsilon).$$

But this contradicts the fact that $i\xi_j\widehat{\phi} \in L^2(\mathbf{R}^N)$. Thus necessarily $\widehat{v}(0) = 0$ and this is exactly the conclusion of Lemma 2.3. □

It is obvious that $L_1(v) = \displaystyle\int_{\mathbf{R}^N} v(x)\,dx$ and $L_2(w) = 0$ are continuous linear forms on $L^1(\mathbf{R}^N)$ and on $\mathcal{Y}$, respectively. Moreover, by Lemma 2.3 we have $L_1 = L_2$ on $L^1(\mathbf{R}^N) \cap \mathcal{Y}$. Putting

$$L(v+w) = L_1(v) + L_2(w) = \int_{\mathbf{R}^N} v(x)\,dx \qquad \text{for } v \in L^1(\mathbf{R}^N) \text{ and } w \in \mathcal{Y}, \tag{2.11}$$

we see that L is well-defined and is a continuous linear form on $L^1(\mathbf{R}^N) + \mathcal{Y}$.

It follows from (2.9) and Lemmas 2.1 and 2.2 that for any $u \in \mathcal{X}$ we have $\langle iu_{x_1}, u\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$. This enables us to give the following

Definition 2.4 *Given $u \in \mathcal{X}$, the momentum of u (with respect to the x_1-direction) is*

$$Q(u) = L(\langle iu_{x_1}, u\rangle).$$

If $u \in \mathcal{X}$ and $\chi, u_1, u_2, \rho, \theta$ are as in Lemma 2.2, from (2.9) we get

$$Q(u) = \int_{\mathbf{R}^N} (1-\chi^2(u))\langle iu_{x_1}, u\rangle - (\rho^2 - r_0^2)\theta_{x_1}\,dx. \tag{2.12}$$

It is easy to check that the right-hand side of (2.12) does not depend on the choice of the cut-off function χ, provided that χ is as in Lemma 2.2.

It follows directly from (2.12) that the functional Q has a nice behavior with respect to dilations in $\mathbf{R}^N$: for any $u \in \mathcal{X}$ and λ, $\sigma > 0$ we have

$$Q(u_{\lambda,\sigma}) = \sigma^{N-1}Q(u). \tag{2.13}$$

The next lemma will enable us to perform "integrations by parts".

Lemma 2.5 *For any $u \in \mathcal{X}$ and $w \in H^1(\mathbf{R}^N)$ we have $\langle iu_{x_1}, w\rangle \in L^1(\mathbf{R}^N)$, $\langle iu, w_{x_1}\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$ and*

$$L(\langle iu_{x_1}, w\rangle + \langle iu, w_{x_1}\rangle) = 0. \tag{2.14}$$

Proof.

Since $w, u_{x_1} \in L^2(\mathbf{R}^N)$, the Cauchy-Schwarz inequality implies $\langle iu_{x_1}, w\rangle \in L^1(\mathbf{R}^N)$.

Let χ, u_1, u_2 be as in Lemma 2.2 and denote $w_1 = \chi(w)w$, $w_2 = (1-\chi(w))w$. Then $u = u_1 + u_2$, $w = w_1 + w_2$ and it follows from Lemma 2.2 that $u_1 \in \mathcal{X} \cap L^\infty(\mathbf{R}^N)$ and u_2, w_1, $w_2 \in H^1(\mathbf{R}^N)$.

As above we have $\langle i\frac{\partial u_2}{\partial x_1}, w\rangle$, $\langle iu_2, \frac{\partial w}{\partial x_1}\rangle \in L^1(\mathbf{R}^N)$ by the Cauchy-Schwarz inequality. The standard integration by parts formula for functions in $H^1(\mathbf{R}^N)$ (see, e.g., [8], p. 197) gives

$$\int_{\mathbf{R}^N} \langle i\frac{\partial u_2}{\partial x_1}, w\rangle + \langle iu_2, \frac{\partial w}{\partial x_1}\rangle \, dx = 0. \tag{2.15}$$

Since $u_1 \in \mathcal{D}^{1,2} \cap L^\infty(\mathbf{R}^N)$ and $w_1 \in H^1 \cap L^\infty(\mathbf{R}^N)$, it is standard to prove that $\langle iu_1, w_1\rangle \in \mathcal{D}^{1,2} \cap L^\infty(\mathbf{R}^N)$ and

$$\langle i\frac{\partial u_1}{\partial x_1}, w_1\rangle + \langle iu_1, \frac{\partial w_1}{\partial x_1}\rangle = \frac{\partial}{\partial x_1}\langle iu_1, w_1\rangle \qquad \text{a.e. on } \mathbf{R}^N. \tag{2.16}$$

Let $A_w = \{x \in \mathbf{R}^N \mid |w(x)| \geq \frac{r_0}{4}\}$. We have $\left(\frac{r_0}{4}\right)^2 \mathcal{L}^N(A_w) \leq \int_{A_w} |w|^2 \, dx \leq ||w||_{L^2}^2$, and consequently A_w has finite measure. It is clear that $w_2 = 0$ and $\nabla w_2 = 0$ a.e. on $\mathbf{R}^N \setminus A_w$. Since $w_2 \in L^{2^*}(\mathbf{R}^N)$ and $\nabla w_2 \in L^2(\mathbf{R}^N)$, we infer that $w_2 \in L^1 \cap L^{2^*}(\mathbf{R}^N)$ and $\nabla w_2 \in L^1 \cap L^2(\mathbf{R}^N)$. Together with the fact that $u_1 \in L^{2^*} \cap L^\infty(\mathbf{R}^N)$ and $\nabla u_1 \in L^2(\mathbf{R}^N)$, this gives $\langle iu_1, w_2\rangle \in L^1 \cap L^{2^*}(\mathbf{R}^N)$ and

$$\langle i\frac{\partial u_1}{\partial x_j}, w_2\rangle \in L^1 \cap L^{\frac{N}{N-1}}(\mathbf{R}^N), \qquad \langle iu_1, \frac{\partial w_2}{\partial x_j}\rangle \in L^1 \cap L^2(\mathbf{R}^N) \qquad \text{for } j = 1, \dots, N.$$

It is easy to see that $\frac{\partial}{\partial x_j}\langle iu_1, w_2\rangle = \langle i\frac{\partial u_1}{\partial x_j}, w_2\rangle + \langle iu_1, \frac{\partial w_2}{\partial x_j}\rangle$ in $\mathcal{D}'(\mathbf{R}^N)$. From the above we infer that $\langle iu_1, w_2\rangle \in W^{1,1}(\mathbf{R}^N)$. It is obvious that $\int_{\mathbf{R}^N} \frac{\partial \psi}{\partial x_j} \, dx = 0$ for any $\psi \in W^{1,1}(\mathbf{R}^N)$ (indeed, let $(\psi_n)_{n\geq 1} \subset C_c^\infty(\mathbf{R}^N)$ be a sequence such that $\psi_n \longrightarrow \psi$ in $W^{1,1}(\mathbf{R}^N)$ as $n \longrightarrow \infty$; then $\int_{\mathbf{R}^N} \frac{\partial \psi_n}{\partial x_j} \, dx = 0$ for each n and $\int_{\mathbf{R}^N} \frac{\partial \psi_n}{\partial x_j} \, dx \longrightarrow \int_{\mathbf{R}^N} \frac{\partial \psi}{\partial x_j} \, dx$ as $n \longrightarrow \infty$). Thus we have $\langle i\frac{\partial u_1}{\partial x_1}, w_2\rangle$, $\langle iu_1, \frac{\partial w_2}{\partial x_1}\rangle \in L^1(\mathbf{R}^N)$ and

$$\int_{\mathbf{R}^N} \langle i\frac{\partial u_1}{\partial x_1}, w_2\rangle + \langle iu_1, \frac{\partial w_2}{\partial x_1}\rangle \, dx = \int_{\mathbf{R}^N} \frac{\partial}{\partial x_1}\langle iu_1, w_2\rangle \, dx = 0. \tag{2.17}$$

Now (2.14) follows from (2.15), (2.16), (2.17) and Lemma 2.5 is proved. □

Corollary 2.6 *Let u, $v \in \mathcal{X}$ be such that $u - v \in L^2(\mathbf{R}^N)$. Then*

$$|Q(u) - Q(v)| \leq ||u - v||_{L^2(\mathbf{R}^N)} \left(\left|\left|\frac{\partial u}{\partial x_1}\right|\right|_{L^2(\mathbf{R}^N)} + \left|\left|\frac{\partial v}{\partial x_1}\right|\right|_{L^2(\mathbf{R}^N)} \right) \tag{2.18}$$

Proof. It is clear that $w = u - v \in H^1(\mathbf{R}^N)$ and using (2.14) we get

$$\begin{aligned} Q(u) - Q(v) &= L(\langle i(u-v)_{x_1}, u\rangle + \langle iv_{x_1}, u - v\rangle) \\ &= L(\langle iu_{x_1}, u - v\rangle + \langle iv_{x_1}, u - v\rangle) \\ &= \int_{\mathbf{R}^N} \langle iu_{x_1} + iv_{x_1}, u - v\rangle \, dx. \end{aligned} \tag{2.19}$$

Then (2.18) follows from (2.19) and the Cauchy-Schwarz inequality. □

The next result will be useful to estimate the contribution to the momentum of a domain where the modified Ginzburg-Landau energy is small.

Lemma 2.7 *Let $M > 0$ and let Ω be an open subset of $\mathbf{R}^N$. Assume that $u \in \mathcal{X}$ satisfies $E_{GL}(u) \leq M$ and let χ, ρ, θ be as in Lemma 2.2. Then we have*

$$\int_\Omega \left|(1 - \chi^2(u))\langle iu_{x_1}, u\rangle - (\rho^2 - r_0^2)\theta_{x_1}\right| dx \leq C(M^{\frac{1}{2}} + M^{\frac{2^*}{4}}) \left(E_{GL}^\Omega(u)\right)^{\frac{1}{2}}. \tag{2.20}$$

Proof. Using (2.6) and the Cauchy-Schwarz inequality we get

$$(2.21)\qquad \begin{aligned}\int_\Omega \left|(1-\chi^2(u))\langle iu_{x_1}, u\rangle\right| dx &\le ||u_{x_1}||_{L^2(\Omega)}||(1-\chi^2(u))u||_{L^2(\Omega)} \\ &\le C_1||u_{x_1}||_{L^2(\Omega)}||\nabla u||_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}}.\end{aligned}$$

We have $|u_1| \le \frac{r_0}{2}$, hence $|r_0 - u_1| \le \frac{3r_0}{2}$ and $\varphi(|r_0-u_1|) = |r_0-u_1| = \rho$. Then (2.7) gives

$$(2.22)\qquad ||\rho^2 - r_0^2||_{L^2(\mathbf{R}^N)} \le C'(E_{GL}(u) + E_{GL}(u)^{\frac{2^*}{2}}) \le C'(M + M^{\frac{2^*}{2}}).$$

From (2.4) and (2.5) we have $\left|\frac{\partial\theta}{\partial x_j}\right| \le \frac{1}{\rho}\left|\frac{\partial u_1}{\partial x_j}\right| \le C''\left|\frac{\partial u}{\partial x_j}\right|$ a.e. on $\mathbf{R}^N$. Therefore

$$(2.23)\qquad \begin{aligned}&\int_\Omega \left|(\rho^2 - r_0^2)\theta_{x_1}\right| dx \le ||\rho^2 - r_0^2||_{L^2(\Omega)}||\theta_{x_1}||_{L^2(\Omega)} \\ &\le C''||\rho^2 - r_0^2||_{L^2(\mathbf{R}^N)}||u_{x_1}||_{L^2(\Omega)} \le C'''\left(M + M^{\frac{2^*}{2}}\right)^{\frac{1}{2}}\left(E_{GL}^\Omega(u)\right)^{\frac{1}{2}}.\end{aligned}$$

Then (2.20) follows from (2.21) and (2.23). □

3 A regularization procedure

Given a function $u \in \mathcal{X}$ and a region $\Omega \subset \mathbf{R}^N$ such that $E_{GL}^\Omega(u)$ is small, we would like to get a fine estimate of the contribution of Ω to the momentum of u. To do this, we will use a kind of "regularization" procedure for arbitrary functions in $\mathcal{X}$. A similar device has been introduced in [1] to get rid of small-scale topological defects of functions; variants of it have been used for various purposes in [7], [6], [5].

Throughout this section, Ω is an open set in $\mathbf{R}^N$. We do not assume Ω bounded, nor connected. If $\partial\Omega \ne \emptyset$, we assume that $\partial\Omega$ is C^2. Let φ be as in the introduction. Let $u \in \mathcal{X}$ and let $h > 0$. We consider the functional

$$G_{h,\Omega}^u(v) = E_{GL}^\Omega(v) + \frac{1}{h^2}\int_\Omega \varphi\left(\frac{|v-u|^2}{32r_0}\right) dx.$$

Note that $G_{h,\Omega}^u(v)$ may equal ∞ for some $v \in \mathcal{X}$; however, $G_{h,\Omega}^u(v)$ is finite whenever $v \in \mathcal{X}$ and $v - u \in L^2(\Omega)$. We denote $H_0^1(\Omega) = \{u \in H^1(\mathbf{R}^N) \mid u = 0 \text{ on } \mathbf{R}^N \setminus \Omega\}$ and

$$H_u^1(\Omega) = \{v \in \mathcal{X} \mid v - u \in H_0^1(\Omega)\}.$$

The next lemma gives the properties of functions that minimize $G_{h,\Omega}^u$ in the space $H_u^1(\Omega)$.

Lemma 3.1 *i) The functional $G_{h,\Omega}^u$ has a minimizer in $H_u^1(\Omega)$.*

ii) Let v_h be a minimizer of $G_{h,\Omega}^u$ in $H_u^1(\Omega)$. There exist constants C_1, C_2, $C_3 > 0$, depending only on N, a and r_0 such that v_h satisfies:

$$(3.1)\qquad E_{GL}^\Omega(v_h) \le E_{GL}^\Omega(u);$$

$$(3.2)\qquad ||v_h - u||_{L^2(\Omega)}^2 \le 32r_0h^2E_{GL}^\Omega(u) + C_1\left(E_{GL}^\Omega(u)\right)^{1+\frac{2}{N}} h^{\frac{4}{N}};$$

$$(3.3)\qquad \int_\Omega \left| \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - v_h|) - r_0^2\right)^2 \right| dx \le C_2hE_{GL}^\Omega(u);$$

$$(3.4)\qquad |Q(u)-Q(v_h)| \le C_3\left(h^2 + \left(E_{GL}^{\Omega}(u)\right)^{\frac{2}{N}} h^{\frac{4}{N}}\right)^{\frac{1}{2}} E_{GL}^{\Omega}(u).$$

iii) For $z \in \mathbf{C}$, denote $H(z) = \left(\varphi^2(|z-r_0|) - r_0^2\right)\varphi(|z-r_0|)\varphi'(|z-r_0|)\frac{z-r_0}{|z-r_0|}$ if $z \ne r_0$ and $H(r_0) = 0$. Then any minimizer v_h of $G_{h,\Omega}^u$ in $H_u^1(\Omega)$ satisfies the equation

$$(3.5)\qquad -\Delta v_h + 2a^2 H(v_h) + \frac{1}{32 r_0 h^2}\varphi'\left(\frac{|v_h-u|^2}{32 r_0}\right)(v_h - u) = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

Moreover, for any $\omega \subset\subset \Omega$ we have $v_h \in W^{2,p}(\omega)$ for $p \in [1,\infty)$; thus, in particular, $v_h \in C^{1,\alpha}(\omega)$ for $\alpha \in [0,1)$.

iv) For any $h > 0$, $\delta > 0$ and $R > 0$ there exists a constant $K = K(a, r_0, N, h, \delta, R) > 0$ such that for any $u \in \mathcal{X}$ with $E_{GL}^{\Omega}(u) \le K$ and for any minimizer v_h of $G_{h,\Omega}^u$ in $H_u^1(\Omega)$ we have

$$(3.6)\qquad r_0 - \delta < |r_0 - v_h(x)| < r_0 + \delta \qquad \text{whenever } x \in \Omega \text{ and } dist(x, \partial\Omega) \ge 4R.$$

Proof. i) It is obviuos that $u \in H_u^1(\Omega)$. Let $(v_n)_{n\ge1}$ be a minimizing sequence for $G_{h,\Omega}^u$ in $H_u^1(\Omega)$. We may assume that $G_{h,\Omega}^u(v_n) \le G_{h,\Omega}^u(u) = E_{GL}^{\Omega}(u)$ and this implies $\int_\Omega |\nabla v_n|^2\,dx \le E_{GL}^{\Omega}(u)$. It is clear that

$$(3.7)\qquad \int_{\Omega\cap\{|v_n-u|\le 8r_0\}} |v_n-u|^2\,dx \le 32 r_0 \int_\Omega \varphi\left(\frac{|v_n-u|^2}{32 r_0}\right)dx \le 32 r_0 h^2 E_{GL}^{\Omega}(u).$$

Since $v_n - u \in H_0^1(\Omega) \subset H^1(\mathbf{R}^N)$, by the Sobolev embedding we have $||v_n - u||_{L^{2^*}(\mathbf{R}^N)} \le C_S||\nabla v_n - \nabla u||_{L^2(\mathbf{R}^N)}$, where C_S depends only on N. Therefore

$$(3.8)\qquad \begin{aligned} &\int_{\{|v_n-u|>8r_0\}} |v_n-u|^2\,dx \le (8r_0)^{2-2^*}\int_{\{|v_n-u|>8r_0\}} |v_n-u|^{2^*}\,dx \\ &\le (8r_0)^{2-2^*}\,||v_n-u||_{L^{2^*}(\mathbf{R}^N)}^{2^*} \le C'||\nabla v_n - \nabla u||_{L^2(\mathbf{R}^N)}^{2^*} \le C\left(E_{GL}^{\Omega}(u)\right)^{\frac{2^*}{2}}. \end{aligned}$$

It follows from (3.7) and (3.8) that $||v_n - u||_{L^2(\Omega)}$ is bounded, hence $v_n - u$ is bounded in $H_0^1(\Omega)$. We infer that there exists a sequence (still denoted $(v_n)_{n\ge1}$) and there is $w \in H_0^1(\Omega)$ such that $v_n - u \rightharpoonup w$ weakly in $H_0^1(\Omega)$, $v_n - u \longrightarrow w$ a.e. and $v_n - u \longrightarrow w$ in $L^p_{loc}(\Omega)$ for $1 \le p < 2^*$. Let $v = u + w$. Then $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^2(\mathbf{R}^N)$ and this implies

$$\int_\Omega |\nabla v|^2\,dx \le \liminf_{n\to\infty}\int_\Omega |\nabla v_n|^2\,dx.$$

Using the a.e. convergence and Fatou's Lemma we infer that

$$\int_\Omega \left(\varphi^2(|r_0 - v|) - r_0^2\right)^2 dx \le \liminf_{n\to\infty}\int_\Omega \left(\varphi^2(|r_0 - v_n|) - r_0^2\right)^2 dx \qquad \text{and}$$

$$\int_\Omega \varphi\left(\frac{|v-u|^2}{32r_0}\right)dx \le \liminf_{n\to\infty}\int_\Omega \varphi\left(\frac{|v_n-u|^2}{32r_0}\right)dx.$$

Therefore $G_{h,\Omega}^u(v) \le \liminf_{n\to\infty} G_{h,\Omega}^u(v_n)$ and consequently v is a minimizer of $G_{h,\Omega}^u$ in $H_u^1(\Omega)$.

ii) Since $u \in H_u^1(\Omega)$, we have $E_{GL}^{\Omega}(v_h) \le G_{h,\Omega}^u(v_h) \le E_{GL}^{\Omega}(u)$; hence (3.1) holds. It is clear that $\varphi\left(\frac{|v_h-u|^2}{32r_0}\right) \ge 2r_0$ if $|v_h - u| \ge 8r_0$, thus

$$2r_0\mathcal{L}^N(\{|v_h - u| \ge 8r_0\}) \le \int_{\mathbf{R}^N}\varphi\left(\frac{|v_h-u|^2}{32r_0}\right)dx \le h^2 G_{h,\Omega}^u(v_h) \le h^2 E_{GL}^{\Omega}(u).$$

Using Hölder's inequality, the above estimate and the Sobolev inequality we get

$$
\begin{aligned}
&\int_{\{|v_h-u|\geq 8r_0\}} |v_h-u|^2\,dx \\
&\leq ||v_h-u||^2_{L^{2^*}(\{|v_h-u|\geq 8r_0\})} \left(\mathcal{L}^N(\{|v_h-u|\geq 8r_0\})\right)^{1-\frac{2}{2^*}} \\
&\leq ||v_h-u||^2_{L^{2^*}(\mathbf{R}^N)} \left(\mathcal{L}^N(\{|v_h-u|\geq 8r_0\})\right)^{1-\frac{2}{2^*}} \\
&\leq C_S||\nabla v_h-\nabla u||^2_{L^2(\mathbf{R}^N)} \left(\frac{h^2}{2r_0}E^{\Omega}_{GL}(u)\right)^{1-\frac{2}{2^*}} \leq C_1 h^{\frac{4}{N}}\left(E^{\Omega}_{GL}(u)\right)^{1+\frac{2}{N}}.
\end{aligned}
\tag{3.9}
$$

It is clear that (3.7) holds with v_h instead of v_n and then (3.2) follows from (3.7) and (3.9).

We claim that

$$
\left|\varphi(|r_0-z|)-\varphi(|r_0-\zeta|)\right| \leq \left[32r_0\varphi\left(\frac{|z-\zeta|^2}{32r_0}\right)\right]^{\frac{1}{2}} \qquad \text{for any } z,\ \zeta\in\mathbf{C}. \tag{3.10}
$$

Indeed, if $|z-r_0|\leq 4r_0$ and $|\zeta-r_0|\leq 4r_0$, then $|z-\zeta|\leq 8r_0$, $\varphi\left(\frac{|z-\zeta|^2}{32r_0}\right)=\frac{|z-\zeta|^2}{32r_0}$ and $\left|\varphi(|r_0-z|)-\varphi(|r_0-\zeta|)\right|\leq\left||r_0-z|-|r_0-\zeta|\right|\leq|z-\zeta|$, hence (3.10) holds.

If $|z-r_0|\leq 4r_0$ and $|\zeta-r_0|>4r_0$, there exists $t\in[0,1)$ such that $w=(1-t)z+t\zeta$ satisfies $|r_0-w|=4r_0$ and

$$
\begin{aligned}
&\left|\varphi(|r_0-z|)-\varphi(|r_0-\zeta|)\right| = \left|\varphi(|r_0-z|)-\varphi(|r_0-w|)\right| \\
&\leq \left[32r_0\varphi\left(\tfrac{|z-w|^2}{32r_0}\right)\right]^{\frac{1}{2}} \leq \left[32r_0\varphi\left(\tfrac{|z-\zeta|^2}{32r_0}\right)\right]^{\frac{1}{2}}.
\end{aligned}
$$

We argue similarly if $|z-r_0|>4r_0$ and $|\zeta-r_0|\leq 4r_0$. Finally, in the case $|z-r_0|>4r_0$ and $|\zeta-r_0|>4r_0$ we have $\varphi(|r_0-z|)=\varphi(|r_0-\zeta|)=3r_0$ and (3.10) trivially holds.

It is obvious that

$$
\begin{aligned}
&\left|\left(\varphi^2(|r_0-u|)-r_0^2\right)^2-\left(\varphi^2(|r_0-v_h|)-r_0^2\right)^2\right| \\
&\leq 6r_0\left|\varphi(|r_0-u|)-\varphi(|r_0-v_h|)\right|\cdot\left|\varphi^2(|r_0-u|)+\varphi^2(|r_0-v_h|)-2r_0^2\right|.
\end{aligned}
\tag{3.11}
$$

Using (3.11), the Cauchy-Schwarz inequality and (3.10) we get

$$
\begin{aligned}
&\int_\Omega\left|\left(\varphi^2(|r_0-u|)-r_0^2\right)^2-\left(\varphi^2(|r_0-v_h|)-r_0^2\right)^2\right|dx \\
&\leq 6r_0\left(\int_\Omega\left|\varphi(|r_0-u|)-\varphi(|r_0-v_h|)\right|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega\left|\varphi^2(|r_0-u|)+\varphi^2(|r_0-v_h|)-2r_0^2\right|^2dx\right)^{\frac{1}{2}} \\
&\leq 6r_0\left(\int_\Omega 32r_0\varphi\left(\tfrac{|v_h-u|^2}{32r_0}\right)dx\right)^{\frac{1}{2}}\left(2\int_\Omega\left(\varphi^2(|r_0-u|)-r_0^2\right)^2+\left(\varphi^2(|r_0-v_h|)-r_0^2\right)^2dx\right)^{\frac{1}{2}} \\
&\leq 48r_0^{\frac{3}{2}}\left(h^2G^u_{h,\Omega}(v_h)\right)^{\frac{1}{2}}\left(\tfrac{1}{a^2}E^{\Omega}_{GL}(u)+\tfrac{1}{a^2}E^{\Omega}_{GL}(v_h)\right)^{\frac{1}{2}} \leq \tfrac{48\sqrt{2}}{a}r_0^{\frac{3}{2}}hE^{\Omega}_{GL}(u)
\end{aligned}
$$

and (3.3) is proved. Finally, (3.4) follows directly from (3.1), (3.2) and Corollary 2.6.

iii) The proof of (3.5) is standard. For any $\psi\in C_c^\infty(\Omega)$ we have $v+\psi\in H^1_u(\Omega)$ and the function $t\longmapsto G^u_{h,\Omega}(v+t\psi)$ achieves its minumum at $t=0$. Hence $\frac{d}{dt}\big|_{t=0}\left(G^u_{h,\Omega}(v+t\psi)\right)=0$ for any $\psi\in C_c^\infty(\Omega)$ and this is precisely (3.5).

For any $z \in \mathbf{C}$ we have

$$|H(z)| \le 3r_0|\varphi^2(|z-r_0|)-r_0^2| \le 24r_0^3. \tag{3.12}$$

Since $v_h \in \mathcal{X}$, we have $\varphi^2(|r_0-v_h|)-r_0^2 \in L^2(\mathbf{R}^N)$ and (3.12) gives $H(v_h) \in L^2 \cap L^\infty(\mathbf{R}^N)$. We also have $\left|\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right| \le |v_h-u|$ and $\left|\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right| \le \sup_{s\ge 0}\varphi'\left(\frac{s^2}{32r_0}\right)s < \infty$. Since $v_h-u \in L^2(\mathbf{R}^N)$, we get $\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u) \in L^2\cap L^\infty(\mathbf{R}^N)$. Using (3.5) we infer that $\Delta v_h \in L^2 \cap L^\infty(\Omega)$. Then (iii) follows from standard elliptic estimates (see, e.g., Theorem 9.11 p. 235 in [19]) and a straightforward bootstrap argument.

iv) Using (3.12) we get

$$\int_\Omega |H(v_h)|^2\,dx \le 9r_0^2\int_\Omega \left(\varphi^2(|r_0-v_h|)-r_0^2\right)^2\,dx \le \frac{9r_0^2}{a^2}E_{GL}^\Omega(v_h) \le \frac{9r_0^2}{a^2}E_{GL}^\Omega(u),$$

hence $||H(v_h)||_{L^2(\Omega)} \le C'\left(E_{GL}^\Omega(u)\right)^{\frac12}$. By interpolation we find for any $p\in[2,\infty]$,

$$||H(v_h)||_{L^p(\Omega)} \le ||H(v_h)||_{L^\infty(\Omega)}^{\frac{p-2}{p}}||H(v_h)||_{L^2(\Omega)}^{\frac{2}{p}} \le C\left(E_{GL}^\Omega(u)\right)^{\frac1p}. \tag{3.13}$$

There exist $m_1,\, m_2>0$ such that $\left|\varphi'\left(\frac{s^2}{32r_0}\right)s\right|^2 \le m_1\varphi\left(\frac{s^2}{32r_0}\right)$ and $\left|\varphi'\left(\frac{s^2}{32r_0}\right)s\right| \le m_2$ for any $s\ge 0$. Then we have

$$\int_\Omega\left|\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|^2dx \le m_1\int_\Omega\varphi\left(\frac{|v_h-u|^2}{32r_0}\right)dx \le m_1h^2E_{GL}^\Omega(u),$$

thus $\left|\left|\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|\right|_{L^2(\Omega)} \le h\left(m_1E_{GL}^\Omega(u)\right)^{\frac12}$. By interpolation we get

$$\begin{aligned}
&\left|\left|\varphi'\left(\tfrac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|\right|_{L^p(\Omega)}\\
&\le \left|\left|\varphi'\left(\tfrac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|\right|_{L^\infty(\Omega)}^{\frac{p-2}{p}}\left|\left|\varphi'\left(\tfrac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|\right|_{L^2(\Omega)}^{\frac2p}\\
&\le Ch^{\frac2p}\left(E_{GL}^\Omega(u)\right)^{\frac1p}
\end{aligned} \tag{3.14}$$

for any $p\in[2,\infty]$. From (3.5), (3.13) and (3.14) we obtain

$$||\Delta v_h||_{L^p(\Omega)} \le C(1+h^{\frac2p-2})\left(E_{GL}^\Omega(u)\right)^{\frac1p} \qquad \text{for any } p\ge 2. \tag{3.15}$$

For a measurable set $\omega\subset\mathbf{R}^N$ with $\mathcal{L}^N(\omega)<\infty$ and for any $f\in L^1(\omega)$, we denote by $m(f,\omega)=\frac{1}{\mathcal{L}^N(\omega)}\displaystyle\int_\omega f(x)\,dx$ the mean value of f on ω.

Let x_0 be such that $B(x_0,4R)\subset\Omega$. Using the Poincaré inequality and (3.1) we have

$$||v_h-m(v_h,B(x_0,4R))||_{L^2(B(x_0,4R))} \le C_PR||\nabla v_h||_{L^2(B(x_0,4R))} \le C_PR\left(E_{GL}^\Omega(u)\right)^{\frac12}. \tag{3.16}$$

We claim that there exist $k\in\mathbf{N}$, depeding only on N, and $C_*=C_*(a,r_0,N,h,R)$ such that

$$||v_h-m(v_h,B(x_0,4R))||_{W^{2,N}(B(x_0,\frac{R}{2^{k-2}}))} \le C_*\left(\left(E_{GL}^\Omega(u)\right)^{\frac12}+\left(E_{GL}^\Omega(u)\right)^{\frac1N}\right). \tag{3.17}$$

It is well-known (see Theorem 9.11 p. 235 in [19]) that for $p\in(1,\infty)$ there exists $C=C(N,r,p)>0$ such that for any $w\in W^{2,p}(B(a,2r))$ we have

$$||w||_{W^{2,p}(B(a,r))} \le C\left(||w||_{L^p(B(a,2r))}+||\Delta w||_{L^p(B(a,2r))}\right). \tag{3.18}$$

From (3.15), (3.16) and (3.18) we infer that

$$||v_h - m(v_h, B(x_0, 4R))||_{W^{2,2}(B(x_0,2R))} \leq C(a, r_0, N, h, R) \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}. \tag{3.19}$$

If $\frac{1}{2} - \frac{2}{N} \leq \frac{1}{N}$, from (3.19) and the Sobolev embedding we find

$$||v_h - m(v_h, B(x_0, 4R))||_{L^N(B(x_0,2R))} \leq C(a, r_0, N, h, R) \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}. \tag{3.20}$$

Then using (3.15) (for $p = N$), (3.20) and (3.18) we infer that (3.17) holds for $k = 2$.

If $\frac{1}{2} - \frac{2}{N} > \frac{1}{N}$, (3.19) and the Sobolev embedding imply

$$||v_h - m(v_h, B(x_0, 4R))||_{L^{p_1}(B(x_0,2R))} \leq C(a, r_0, N, h, R) \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}, \tag{3.21}$$

where $\frac{1}{p_1} = \frac{1}{2} - \frac{2}{N}$. Then (3.21), (3.15) and (3.18) give

$$||v_h - m(v_h, B(x_0, 4R))||_{W^{2,p_1}(B(x_0,R))} \leq C(a, r_0, N, h, R) \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}} \right). \tag{3.22}$$

If $\frac{1}{p_1} - \frac{2}{N} \leq \frac{1}{N}$, using (3.22), the Sobolev embedding, (3.15) and (3.18) we get

$$||v_h - m(v_h, B(x_0, 4R))||_{W^{2,N}(B(x_0,\frac{R}{2}))} \leq C(a, r_0, N, h, R) \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}} \right);$$

otherwise we repeat the process. After a finite number of steps we find $k \in \mathbf{N}$ such that (3.17) holds.

We will use the following variant of the Gagliardo-Nirenberg inequality:

$$||w - m(w, B(a, r))||_{L^p(B(a,r))} \leq C(p, q, N, r) ||w||_{L^q(B(a,2r))}^{\frac{q}{p}} ||\nabla w||_{L^N(B(a,2r))}^{1-\frac{q}{p}} \tag{3.23}$$

for any $w \in W^{1,N}(B(a, 2r))$, where $1 \leq q \leq p < \infty$ (see, e.g., [26] p. 78).

Using (3.23) with $w = \nabla v_h$ and (3.17) we find

$$\begin{aligned} &||\nabla v_h - m(\nabla v_h, B(x_0, \tfrac{R}{2^{k-1}}))||_{L^p(B(x_0,\frac{R}{2^{k-1}}))} \\ &\leq C ||\nabla v_h||_{L^2(B(x_0,\frac{R}{2^{k-2}}))}^{\frac{2}{p}} ||\nabla^2 v_h||_{L^N(B(x_0,\frac{R}{2^{k-2}}))}^{1-\frac{2}{p}} \\ &\leq C \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{p}} \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}} \right)^{1-\frac{2}{p}} \end{aligned} \tag{3.24}$$

for any $p \in [2, \infty)$, where the constants depend only on a, r_0, N, p, h, R.

Using the Cauchy-Schwarz inequality and (3.1) we have

$$\left| m(\nabla v_h, B(x_0, \frac{R}{2^{k-1}})) \right| \leq \mathcal{L}^N(B(x_0, \frac{R}{2^{k-1}}))^{-\frac{1}{2}} ||\nabla v_h||_{L^2(B(x_0,\frac{R}{2^{k-1}}))} \leq C \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}$$

and we infer that for any $p \in [1, \infty]$ we have the estimate

$$\begin{aligned} &||m(\nabla v_h, B(x_0, \tfrac{R}{2^{k-1}}))||_{L^p(B(x_0,\frac{R}{2^{k-1}}))} \\ &\leq \left| m(\nabla v_h, B(x_0, \tfrac{R}{2^{k-1}})) \right| \left(\mathcal{L}^N(B(x_0, \tfrac{R}{2^{k-1}})) \right)^{\frac{1}{p}} \leq C(N, p, R) \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}. \end{aligned} \tag{3.25}$$

From (3.24) and (3.25) we obtain for any $p \in [2, \infty)$,

$$||\nabla v_h||_{L^p(B(x_0,\frac{R}{2^{k-1}}))} \leq C(a, r_0, N, p, h, R) \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{p}+\frac{1}{N}(1-\frac{2}{p})} \right). \tag{3.26}$$

We will use the Morrey inequality which asserts that, for any $w \in C^0 \cap W^{1,p}(B(x_0, r))$ with $p > N$ we have

$$(3.27) \qquad |w(x) - w(y)| \leq C(p, N)|x - y|^{1-\frac{N}{p}} ||\nabla w||_{L^p(B(x_0,r))} \qquad \text{for any } x, y \in B(x_0, r))$$

(see, e.g., the proof of Theorem IX.12 p. 166 in [8]). Using (3.26) and the Morrey's inequality (3.27) for $p = 2N$ we get

$$(3.28) \qquad |v_h(x) - v_h(y)| \leq C(a, r_0, N, h, R)|x - y|^{\frac{1}{2}} \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}(1+\frac{1}{2^*})} \right)$$

for any $x, y \in B(x_0, \frac{R}{2^{k-1}}))$.

Let $\delta > 0$ and assume that there exists $x_0 \in \Omega$ such that $|\,|v_h(x_0) - r_0| - r_0| \geq \delta$ and $B(x_0, 4R) \subset \Omega$. Since $\big|\,|\,|v_h(x) - r_0| - r_0| - |\,|v_h(y) - r_0| - r_0|\,\big| \leq |v_h(x) - v_h(y)|$, from (3.28) we infer that

$$|\,|v_h(x) - r_0| - r_0| \geq \frac{\delta}{2} \qquad \text{for any } x \in B(x_0, r_\delta),$$

where

$$(3.29) \qquad r_\delta = \min\left(\frac{R}{2^{k-1}}, \left(\frac{\delta}{2C(a, r_0, N, h, R)} \right)^2 \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}(1+\frac{1}{2^*})} \right)^{-2} \right).$$

Let

$$(3.30) \qquad \eta(s) = \inf\{(\varphi^2(\tau) - r_0^2)^2 \mid \tau \in (-\infty, r_0 - s] \cup [r_0 + s, \infty)\}.$$

It is clear that η is nondecreasing and positive on $(0, \infty)$. We have:

$$(3.31) \qquad \begin{aligned} E_{GL}^{\Omega}(u) &\geq E_{GL}^{\Omega}(v_h) \geq a^2 \int_{B(x_0,r_\delta)} \left(\varphi^2(|r_0 - v_h|) - r_0^2\right)^2 \, dx \\ &\geq a^2 \int_{B(x_0,r_\delta)} \eta(\tfrac{\delta}{2}) \, dx = \mathcal{L}^N(B(0, 1)) a^2 \eta(\tfrac{\delta}{2}) r_\delta^N, \end{aligned}$$

where r_δ is given by (3.29). It is obvious that there exists a constant $K > 0$, depending only on a, r_0, N, h, R, δ such that (3.31) cannot hold for $E_{GL}^{\Omega}(u) \leq K$. We infer that $|\,|v_h(x_0) - r_0| - r_0| < \delta$ if $B(x_0, 4R) \subset \Omega$ and $E_{GL}^{\Omega}(u) \leq K$. This completes the proof of Lemma 3.1. ☐

Lemma 3.2 *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence of functions satisfying:*

a) $E_{GL}(u_n)$ is bounded and

b) $\lim\limits_{n\to\infty} \Big(\sup\limits_{y\in\mathbf{R}^N} E_{GL}^{B(y,1)}(u_n)\Big) = 0$.

There exists a sequence $h_n \longrightarrow 0$ such that for any minimizer v_n of $G_{h_n,\mathbf{R}^N}^{u_n}$ in $H_{u_n}^1(\mathbf{R}^N)$ we have $||\,|v_n - r_0| - r_0||_{L^\infty(\mathbf{R}^N)} \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. Let $M = \sup\limits_{n\geq 1} E_{GL}(u_n)$. For $n \geq 1$ and $x \in \mathbf{R}^N$ we denote

$$m_n(x) = m(u_n, B(x, 1)) = \frac{1}{\mathcal{L}^N(B(0, 1))} \int_{B(x,1)} u_n(y) \, dy.$$

By the Poincaré inequality, there exists $C_0 > 0$ such that

$$\int_{B(x,1)} |u_n(y) - m_n(x)|^2 \, dy \;\leq C_0 \int_{B(x,1)} ||\nabla u_n(y)||^2 \, dy.$$

From (b) it follows that

$$\sup_{x\in\mathbf{R}^N} ||u_n - m_n(x)||_{L^2(B(x,1))} \longrightarrow 0 \qquad \text{as } x \longrightarrow \infty. \tag{3.32}$$

Let H be as in Lemma 3.1 (iii). From (3.12) and (b) we get

$$\sup_{x\in\mathbf{R}^N} ||H(u_n)||^2_{L^2(B(x,1))} \le \sup_{x\in\mathbf{R}^N} 9r_0^2 \int_{B(x,1)} \left(\varphi^2(|r_0 - u_n(y)|) - r_0^2\right)^2 dy \longrightarrow 0 \tag{3.33}$$

as $n \longrightarrow \infty$. It is obvious that H is Lipschitz on $\mathbf{C}$. Using (3.32) we find

$$\sup_{x\in\mathbf{R}^N} ||H(u_n) - H(m_n(x))||_{L^2(B(x,1))} \le C_1 \sup_{x\in\mathbf{R}^N} ||u_n - m_n(x)||_{L^2(B(x,1))} \longrightarrow 0 \tag{3.34}$$

as $n \longrightarrow \infty$. From (3.33) and (3.34) we infer that $\sup_{x\in\mathbf{R}^N} ||H(m_n(x))||_{L^2(B(x,1))} \longrightarrow 0$ as $n \longrightarrow \infty$. Since $||H(m_n(x))||_{L^2(B(x,1))} = \mathcal{L}^N(B(0,1)|H(m_n(x))|$, we have proved that

$$\lim_{n\to\infty} \sup_{x\in\mathbf{R}^N} |H(m_n(x))| = 0. \tag{3.35}$$

Let

$$h_n = \max\left(\left(\sup_{x\in\mathbf{R}^N} ||u_n - m_n(x)||_{L^2(B(x,1))}\right)^{\frac{1}{N+2}}, \left(\sup_{x\in\mathbf{R}^N} |H(m_n(x))|\right)^{\frac{1}{N}}\right). \tag{3.36}$$

From (3.32) and (3.35) it follows that $h_n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus we may assume that $0 < h_n < 1$ for any n (if $h_n = 0$, we see that u_n is constant a.e. and there is nothing to prove). Let v_n be a minimizer of $G^{u_n}_{h_n,\mathbf{R}^N}$ (such minimizers exist by Lemma 3.1 (i)). It follows from Lemma 3.1 (iii) that v_n satisfies (3.5). We will prove that there exist $R_N > 0$ and $C > 0$, independent on n, such that

$$||\Delta v_n||_{L^N(B(x,R_N))} \le C \qquad \text{for any } x \in \mathbf{R}^N \text{ and } n \in \mathbf{N}^*. \tag{3.37}$$

Clearly, it suffices to prove (3.37) for $x = 0$. We denote $m_n = m_n(0)$ and $\tilde{\varphi}(s) = \varphi(\frac{s}{32r_0})$. Then (3.5) can be written as

$$-\Delta v_n + \frac{1}{h_n^2}\tilde{\varphi}'(|v_n - m_n|^2)(v_n - m_n) = f_n, \tag{3.38}$$

where

$$\begin{aligned} f_n &= -2a^2\left(H(v_n) - H(m_n)\right) - 2a^2 H(m_n) \\ &\quad + \tfrac{1}{h_n^2}\left(\tilde{\varphi}'(|v_n - m_n|^2)(v_n - m_n) - \tilde{\varphi}'(|v_n - u_n|^2)(v_n - u_n)\right). \end{aligned} \tag{3.39}$$

In view of Lemma 3.1 (iii), equality (3.38) holds in $L^p_{loc}(\mathbf{R}^N)$ (and not only in $\mathcal{D}'(\mathbf{R}^N)$).

The function $z \longmapsto \tilde{\varphi}'(|z|^2)z$ belongs to $C_c^\infty(\mathbf{C})$ and consequently it is Lipschitz. Using (3.36), we see that there exists $C_2 > 0$ such that

$$\begin{aligned} &||\tilde{\varphi}'(|v_n - m_n|^2)(v_n - m_n) - \tilde{\varphi}'(|v_n - u_n|^2)(v_n - u_n)||_{L^2(B(0,1))} \\ &\le C_2||u_n - m_n||_{L^2(B(0,1))} \le C_2 h_n^{N+2}. \end{aligned} \tag{3.40}$$

By (3.36) we have also $||H(m_n)||_{L^2(B(0,1))} = \left(\mathcal{L}^N(B(0,1)\right)^{\frac{1}{2}} |H(m_n)| \le \left(\mathcal{L}^N(B(0,1)\right)^{\frac{1}{2}} h_n^N$. From this estimate, (3.39), (3.40) and the fact that H is Lipschitz we get

$$||f_n||_{L^2(B(0,R))} \le C_3||v_n - m_n||_{L^2(B(0,R))} + C_4 h_n^N \qquad \text{for any } R \in (0,1]. \tag{3.41}$$

Let $\chi \in C_c^\infty(\mathbf{R}^N, \mathbf{R})$. Taking the scalar product (in $\mathbf{C}$) of (3.38) by $\chi(x)(v_n(x) - m_n)$ and integrating by parts we find

$$\begin{aligned}
&\int_{\mathbf{R}^N} \chi |\nabla v_n|^2 \, dx + \frac{1}{h_n^2} \int_{\mathbf{R}^N} \chi \tilde{\varphi}'(|v_n - m_n|^2) |v_n - m_n|^2 \, dx \\
&= \frac{1}{2} \int_{\mathbf{R}^N} (\Delta \chi) |v_n - m_n|^2 \, dx + \int_{\mathbf{R}^N} \langle f_n(x), v_n(x) - m_n \rangle \chi(x) \, dx.
\end{aligned} \tag{3.42}$$

From (3.2) we have $||v_n - u_n||_{L^2(\mathbf{R}^N)} \le C_5 h_n^{\frac{2}{N}}$, thus

$$||v_n - m_n||_{L^2(B(0,1))} \le ||v_n - u_n||_{L^2(B(0,1))} + ||u_n - m_n||_{L^2(B(0,1))} \le K_0 h_n^{\frac{2}{N}}. \tag{3.43}$$

We prove that

$$||v_n - m_n||_{L^2(B(0, \frac{1}{2^{j-1}}))} \le K_j h_n^{\frac{2j}{N}} \qquad \text{for } 1 \le j \le \left[\frac{N^2}{2} \right] + 1, \tag{3.44}$$

where K_j does not depend on n. We proceed by induction. From (3.43) it follows that (3.44) is true for $j = 1$.

Assume that (3.44) holds for some $j \in \mathbf{N}^*$, $j \le \left[\frac{N^2}{2} \right]$. Let $\chi_j \in C_c^\infty(\mathbf{R}^N)$ be a real-valued function such that $0 \le \chi_j \le 1$, $\text{supp}(\chi_j) \subset B(0, \frac{1}{2^{j-1}})$ and $\chi_j = 1$ on $B(0, \frac{1}{2^j})$. Replacing χ by χ_j in (3.42), then using the Cauchy-Schwarz inequality and (3.41) we find

$$\begin{aligned}
&\int_{B(0, \frac{1}{2^j})} |\nabla v_n|^2 \, dx + \frac{1}{h_n^2} \int_{B(0, \frac{1}{2^j})} \tilde{\varphi}'(|v_n - m_n|^2) |v_n - m_n|^2 \, dx \\
&\le \tfrac{1}{2} ||\Delta \chi_j||_{L^\infty(\mathbf{R}^N)} ||v_n - m_n||^2_{L^2(B(0, \frac{1}{2^{j-1}}))} + ||f_n||_{L^2(B(0, \frac{1}{2^{j-1}}))} ||v_n - m_n||_{L^2(B(0, \frac{1}{2^{j-1}}))} \\
&\le A_j ||v_n - m_n||^2_{L^2(B(0, \frac{1}{2^{j-1}}))} + C_4 h_n^N ||v_n - m_n||_{L^2(B(0, \frac{1}{2^{j-1}}))} \le A_j' h_n^{\frac{4j}{N}}.
\end{aligned} \tag{3.45}$$

From (3.44) and (3.45) we infer that $||v_n - m_n||_{H^1 B(0, \frac{1}{2^j})} \le B_j h_n^{\frac{2j}{N}}$. Then the Sobolev embedding implies

$$||v_n - m_n||_{L^{2^*} B(0, \frac{1}{2^j})} \le D_j h_n^{\frac{2j}{N}}. \tag{3.46}$$

The function $z \longmapsto \tilde{\varphi}(|z|^2)$ is clearly Lipschitz on $\mathbf{C}$, thus we have

$$\begin{aligned}
&\int_{B(0,1)} |\tilde{\varphi}(|v_n - u_n|^2) - \tilde{\varphi}(|v_n - m_n|^2)| \, dx \le C_6' \int_{B(0,1)} |u_n - m_n| \, dx \\
&\le C_6 ||u_n - m_n||_{L^2(B(0,1))} \le C_6 h_n^{N+2}.
\end{aligned}$$

It is clear that $\int_{B(0,1)} \tilde{\varphi}(|v_n - u_n|^2) \, dx \le h_n^2 G^{u_n}_{h_n, \mathbf{R}^N}(v_n) \le h_n^2 E_{GL}(u_n) \le h_n^2 M$ and we obtain

$$\int_{B(0,1)} \tilde{\varphi}(|v_n - m_n|^2) \, dx \le C_7 h_n^2. \tag{3.47}$$

If $|v_n(x) - m_n| \ge 8r_0$ we have $\tilde{\varphi}(|v_n(x) - m_n|^2) = \varphi\left(\frac{|v_n(x) - m_n|^2}{32 r_0} \right) \ge 2r_0$, hence

$$2r_0 \mathcal{L}^N(\{x \in B(0,1) \mid |v_n(x) - m_n| \ge 8r_0\}) \le \int_{B(0,1)} \tilde{\varphi}\left(|v_n - m_n|^2\right) \, dx \le C_7 h_n^2. \tag{3.48}$$

By Hölder's inequality, (3.46) and (3.48) we have

$$\begin{aligned}(3.49)\qquad &\int_{\{|v_n-m_n|\geq 8r_0\}\cap B(0,\frac{1}{2^j})} |v_n-m_n|^2\,dx\\ &\leq ||v_n-m_n||^2_{L^{2^*}B(0,\frac{1}{2^j})}\left(\mathcal{L}^N(\{x\in B(0,1)\mid |v_n(x)-m_n|\geq 8r_0\})\right)^{1-\frac{2}{2^*}}\\ &\leq \left(D_j h_n^{\frac{2j}{N}}\right)^2\left(\frac{C_7}{2r_0}h_n^2\right)^{1-\frac{2}{2^*}}\leq E_j h_n^{\frac{4j+4}{N}}.\end{aligned}$$

From (3.45) it follows that

$$\begin{aligned}(3.50)\qquad &\int_{\{|v_n-m_n|<8r_0\}\cap B(0,\frac{1}{2^j})}|v_n-m_n|^2\,dx\leq \int_{B(0,\frac{1}{2^j})}\tilde{\varphi}'(|v_n-m_n|^2)|v_n-m_n|^2\,dx\\ &\leq A_j' h_n^{2+\frac{4j}{N}}\leq A_j' h_n^{\frac{4j+4}{N}}.\end{aligned}$$

Then (3.49) and (3.50) imply that (3.44) holds for $j+1$ and the induction is complete. Thus (3.44) is established. Denoting $j_N=\left[\frac{N^2}{2}\right]+1$ and $R_N=\frac{1}{2^{j_N-1}}$, we have proved that

$$(3.51)\qquad ||v_n-m_n||_{L^2(B(0,R_N))}\leq K_{j_N}h_n^{\frac{2j_N}{N}}\leq K_{j_N}h_n^N.$$

It follows that

$$\begin{aligned}(3.52)\qquad &\int_{B(0,R_N)}\left|\frac{1}{h_n^2}\tilde{\varphi}'(|v_n-m_n|^2)(v_n-m_n)\right|^N dx\\ &\leq \frac{1}{h_n^{2N}}\sup_{z\in\mathbf{C}}\left|\tilde{\varphi}'\left(|z|^2\right)z\right|^{N-2}\int_{B(0,R_N)}|v_n-m_n|^2\,dx\leq C_8.\end{aligned}$$

Arguing as in (3.40) and using (3.36) we get

$$\begin{aligned}(3.53)\qquad &||\tilde{\varphi}'(|v_n-m_n|^2)(v_n-m_n)-\tilde{\varphi}'(|v_n-u_n|^2)(v_n-u_n)||^N_{L^N(B(0,1))}\\ &\leq C_9\sup_{z\in\mathbf{C}}\left|\tilde{\varphi}'\left(|z|^2\right)z\right|^{N-2}||u_n-m_n||^2_{L^2(B(0,1)}\leq C_{10}h_n^{2N+4}.\end{aligned}$$

From (3.39), (3.53) and the fact that H is bounded on $\mathbf{C}$ it follows that $||f_n||_{L^N(B(0,R_N))}\leq C_{11}$, where C_{11} does not depend on n. Using this estimate, (3.52) and (3.38), we infer that (3.37) holds.

Since any ball of radius 1 can be covered by a finite number of balls of radius R_N, it follows that there exists $C>0$ such that

$$(3.54)\qquad ||\Delta v_n||_{L^N(B(x,1))}\leq C\qquad \text{for any } x\in\mathbf{R}^N \text{ and } n\in\mathbf{N}^*.$$

We will use (3.18) and (3.54) to prove that there exist $\tilde{R}_N\in(0,1]$ and $C>0$ such that

$$(3.55)\qquad ||v_n-m_n(x)||_{W^{2,N}(B(x,\tilde{R}_N))}\leq C\qquad \text{for any } x\in\mathbf{R}^N \text{ and } n\in\mathbf{N}^*.$$

As previously, it suffices to prove (3.55) for $x_0=0$. From (3.54) and Hölder's inequality it follows that for $1\leq p\leq N$ we have

$$(3.56)\qquad ||\Delta v_n||_{L^p(B(x,1))}\leq \left(\mathcal{L}^N(B(0,1))\right)^{1-\frac{p}{N}}||\Delta v_n||^{\frac{p}{N}}_{L^N(B(x,1))}\leq C(p).$$

Using (3.43), (3.54) and (3.18) we obtain

$$(3.57)\qquad ||v_n-m_n(0)||_{W^{2,2}(B(x,\frac{1}{2}))}\leq C.$$

If $\frac{1}{2} - \frac{2}{N} \leq \frac{1}{N}$, (3.57) and the Sobolev embedding give

$$||v_n - m_n(0)||_{L^N(B(x,\frac{1}{2}))} \leq C,$$

and this estimate together with (3.54) and (3.18) imply that (3.55) holds for $\tilde{R}_N = \frac{1}{4}$.

If $\frac{1}{2} - \frac{2}{N} > \frac{1}{N}$, from (3.57) and the Sobolev embedding we find $||v_n - m_n(0)||_{L^{p_1}(B(x,\frac{1}{2}))} \leq C$, where $\frac{1}{p_1} = \frac{1}{2} - \frac{2}{N}$. This estimate, (3.56) and (3.18) imply $||v_n - m_n(0)||_{W^{2,p_1}(B(x,\frac{1}{4}))} \leq C$. If $\frac{1}{p_1} - \frac{2}{N} \leq \frac{1}{N}$, from the Sobolev embedding we obtain $||v_n - m_n(0)||_{L^N(B(x,\frac{1}{4}))} \leq C$, and then using (3.54) and (3.18) we infer that (3.55) holds for $\tilde{R}_N = \frac{1}{8}$. Otherwise we repeat the above argument. After a finite number of steps we see that (3.55) holds.

Next we proceed as in the proof of Lemma 3.1 (iv). By (3.23) and (3.55) we have for $p \in [2, \infty)$ and any $x_0 \in \mathbf{R}^N$,

$$\begin{aligned} &||\nabla v_n - m(\nabla v_n, B(x_0, \tfrac{1}{2}\tilde{R}_N))||_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))} \\ &\leq C||\nabla v_n||^{\frac{2}{p}}_{L^2(B(x_0,\tilde{R}_N))} ||\nabla^2 v_n||^{1-\frac{2}{p}}_{L^N(B(x_0,\tilde{R}_N))} \leq C_1(p). \end{aligned} \tag{3.58}$$

Arguing as in (3.25) we see that $||m(\nabla v_n, B(x_0, \frac{1}{2}\tilde{R}_N))||_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))}$ is bounded independently on n and hence

$$||\nabla v_n||_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))} \leq C_2(p) \qquad \text{for any } n \in \mathbf{N}^* \text{ and } x_0 \in \mathbf{R}^N.$$

Using this estimate for $p = 2N$ together with the Morrey inequality (3.27), we see that there exists $C_* > 0$ such that for any $x, y \in \mathbf{R}^N$ with $|x - y| \leq \frac{\tilde{R}_N}{2}$ and any $n \in \mathbf{N}^*$ we have

$$|v_n(x) - v_n(y)| \leq C_*|x - y|^{\frac{1}{2}}. \tag{3.59}$$

Let $\delta_n = ||\,|v_n - r_0| - r_0||_{L^\infty(\mathbf{R}^N)}$ and choose $x_n \in \mathbf{R}^N$ such that $|\,|v_n(x_n) - r_0| - r_0| \geq \frac{\delta_n}{2}$. From (3.59) it follows that $|\,|v_n(x) - r_0| - r_0| \geq \frac{\delta_n}{4}$ for any $x \in B(x_n, r_n)$, where

$$r_n = \min\left(\frac{\tilde{R}_N}{2}, \left(\frac{\delta_n}{4C_*}\right)^2\right).$$

Then we have

$$\begin{aligned} &\int_{B(x_n,1)} \left(\varphi^2(|r_0 - v_n(y)|) - r_0^2\right)^2 \, dy \geq \int_{B(x_n,r_n)} \left(\varphi^2(|r_0 - v_n(y)|) - r_0^2\right)^2 \, dy \\ &\geq \int_{B(x_n,r_n)} \eta\left(\tfrac{\delta_n}{4}\right) \, dy = \mathcal{L}^N(B(0,1)\eta\left(\tfrac{\delta_n}{4}\right) r_n^N, \end{aligned} \tag{3.60}$$

where η is as in (3.30).

On the other hand, the function $z \longmapsto \left(\varphi^2(|r_0 - z|) - r_0^2\right)^2$ is Lipschitz on $\mathbf{C}$. Using this fact, the Cauchy-Schwarz inequality, (3.2) and assumption (a) we get

$$\begin{aligned} &\int_{B(x,1)} \left| \left(\varphi^2(|r_0 - v_n(y)|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - u_n(y)|) - r_0^2\right)^2 \right| dy \\ &\leq C \int_{B(x,1)} |v_n(y) - u_n(y)| \, dy \leq C'||v_n - u_n||_{L^2(B(x,1))} \leq C'||v_n - u_n||_{L^2(\mathbf{R}^N)} \leq C'' h_n^{\frac{2}{N}}. \end{aligned}$$

Then using assumption (b) we infer that

$$\sup_{x \in \mathbf{R}^N} \int_{B(x,1)} \left(\varphi^2(|r_0 - v_n(y)|) - r_0^2\right)^2 \, dy \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.61}$$

From (3.60) and (3.61) we get $\lim_{n\to\infty} \eta\left(\frac{\delta_n}{4}\right) r_n^N = 0$ and this clearly implies $\lim_{n\to\infty} \delta_n = 0$. Lemma 3.2 is thus proved. $\Box$

The next result is based on Lemma 3.1 and will be very useful in the next sections to prove the "concentration" of minimizing sequences. For $0 < R_1 < R_2$ we denote $\Omega_{R_1,R_2} = B(0,R_2) \setminus \overline{B}(0,R_1)$.

Lemma 3.3 *Let $A > A_3 > A_2 > 1$. There exist $\varepsilon_0 = \varepsilon_0(a, r_0, N, A, A_2, A_3) > 0$ and $C_i = C_i(a, r_0, N, A, A_2, A_3) > 0$ such that for any $R \geq 1$, $\varepsilon \in (0, \varepsilon_0)$ and $u \in \mathcal{X}$ verifying $E_{GL}^{\Omega_{AR,R}}(u) \leq \varepsilon$, there exist two functions u_1, $u_2 \in \mathcal{X}$ and a constant $\theta_0 \in [0, 2\pi)$ satisfying the following properties:*

i) $\text{supp}(u_1) \subset B(0, A_2R)$ *and* $r_0 - u_1 = e^{-i\theta_0}(r_0 - u)$ *on* $B(0,R)$,

ii) $u_2 = u$ *on* $\mathbf{R}^N \setminus B(0, AR)$ *and* $r_0 - u_2 = r_0 e^{i\theta_0} =$ *constant on* $B(0, A_3R)$,

iii) $\displaystyle\int_{\mathbf{R}^N} \left| \left|\frac{\partial u}{\partial x_j}\right|^2 - \left|\frac{\partial u_1}{\partial x_j}\right|^2 - \left|\frac{\partial u_2}{\partial x_j}\right|^2 \right| dx \leq C_1\varepsilon$ *for* $j = 1, \dots, N$,

iv) $\displaystyle\int_{\mathbf{R}^N} \left| \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - u_1|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - u_2|) - r_0^2\right)^2 \right| dx \leq C_2\varepsilon$,

v) $|Q(u) - Q(u_1) - Q(u_2)| \leq C_3\varepsilon$,

vi) If assumptions (A1) and (A2) in the introduction hold, then

$$\int_{\mathbf{R}^N} \left|V(|r_0 - u|^2) - V(|r_0 - u_1|^2) - V(|r_0 - u_2|^2)\right| dx \leq C_4\varepsilon + C_5\sqrt{\varepsilon}\left(E_{GL}(u)\right)^{\frac{2^*-1}{2}}.$$

Proof. Fix $k > 0$, A_1 and A_4 such that $1 + 4k < A_1 < A_2 < A_3 < A_4 < A - 4k$. Let $h = 1$ and $\delta = \frac{r_0}{2}$. We will prove that Lemma 3.3 holds for $\varepsilon_0 = K(a, r_0, N, h = 1, \delta = \frac{r_0}{2}, k)$, where $K(a, r_0, N, h, \delta, R)$ is as in Lemma 3.1 (iv).

Consider $\eta_1, \eta_2 \in C^\infty(\mathbf{R})$ satisfying the following properties:

$$\eta_1 = 1 \text{ on } (-\infty, A_1], \quad \eta_1 = 0 \text{ on } [A_2, \infty), \quad \eta_1 \text{ is nonincreasing,}$$
$$\eta_2 = 0 \text{ on } (-\infty, A_3], \quad \eta_2 = 1 \text{ on } [A_4, \infty), \quad \eta_2 \text{ is nondecreasing.}$$

Let $\varepsilon < \varepsilon_0$ and let $u \in \mathcal{X}$ be such that $E_{GL}^{\Omega_{R,AR}}(u) \leq \varepsilon$. Let v_1 be a minimizer of $G_{1,\Omega_{R,AR}}^u$ in the space $H_u^1(\Omega_{R,AR})$. The existence of v_1 is guaranteed by Lemma 3.1. We also know that $v_1 \in W_{loc}^{2,p}(\Omega_{R,AR})$ for any $p \in [1, \infty)$. Moreover, since $E_{GL}^{\Omega_{R,AR}}(u) \leq K(a, r_0, N, 1, \frac{r_0}{2}, k)$, Lemma 3.1 (iv) implies that

$$\frac{r_0}{2} < |r_0 - v_1(x)| < \frac{3r_0}{2} \qquad \text{if } R + 4k \leq |x| \leq AR - 4k. \tag{3.62}$$

Since $N \geq 3$, Ω_{A_1R,A_4R} is simply connected and it follows directly from Theorem 3 p. 38 in [9] that there exist two real-valued functions ρ, $\theta \in W^{2,p}(\Omega_{A_1R,A_4R})$, $1 \leq p < \infty$, such that

$$r_0 - v_1(x) = \rho(x)e^{i\theta(x)} \qquad \text{on } \Omega_{A_1R,A_4R}. \tag{3.63}$$

For $j = 1, \dots, N$ we have

$$\frac{\partial v_1}{\partial x_j} = \left(-\frac{\partial \rho}{\partial x_j} - i\rho\frac{\partial \theta}{\partial x_j}\right)e^{i\theta} \quad \text{and} \quad \left|\frac{\partial v_1}{\partial x_j}\right|^2 = \left|\frac{\partial \rho}{\partial x_j}\right|^2 + \rho^2\left|\frac{\partial \theta}{\partial x_j}\right|^2 \quad \text{a.e. on } \Omega_{A_1R,A_4R}. \tag{3.64}$$

Thus we get the following estimates:

$$\int_{\Omega_{A_1R,\,A_4R}} |\nabla\rho|^2\,dx \leq \int_{\Omega_{A_1R,\,A_4R}} |\nabla v_1|^2\,dx \leq \varepsilon, \tag{3.65}$$

$$(3.66)\qquad a^2\int_{\Omega_{A_1R,\,A_4R}}\left(\rho^2-r_0^2\right)^2\,dx\le E_{GL}^{\Omega_{A_1R,\,A_4R}}(v_1)\le\varepsilon,$$

$$(3.67)\qquad \int_{\Omega_{A_1R,\,A_4R}}|\nabla\theta|^2\,dx\le\frac{4}{r_0^2}\int_{\Omega_{A_1R,\,A_4R}}|\nabla v_1|^2\,dx\le\frac{4}{r_0^2}\varepsilon.$$

The Poincaré inequality and a scaling argument imply that

$$(3.68)\qquad \int_{\Omega_{A_1R,\,A_4R}}|f-m(f,\Omega_{A_1R,A_4R})|^2\,dx\le C(N,A_1,A_4)R^2\int_{\Omega_{A_1R,\,A_4R}}|\nabla f|^2\,dx$$

for any $f\in H^1(\Omega_{A_1R,\,A_4R})$, where $C(N,A_1,A_4)$ does not depend on R. Let $\theta_0=m(\theta,\Omega_{A_1R,A_4R})$. We may assume that $\theta_0\in[0,2\pi)$ (otherwise we replace θ by $\theta-2\pi\left[\frac{\theta}{2\pi}\right]$). Using (3.67) and (3.68) we get

$$(3.69)\qquad \int_{\Omega_{A_1R,\,A_4R}}|\theta-\theta_0|^2\,dx\le C(r_0,N,A_1,A_4)R^2\int_{\Omega_{A_1R,\,A_4R}}|\nabla v_1|^2\,dx\le C(r_0,N,A_1,A_4)R^2\varepsilon.$$

We define $\tilde u_1$ and u_2 by

$$(3.70)\qquad r_0-\tilde u_1(x)=\begin{cases} r_0-u(x) & \text{if } x\in\overline{B}(0,R),\\ r_0-v_1(x) & \text{if } x\in B(0,A_1R)\setminus\overline{B}(0,R),\\ \left(r_0+\eta_1(\frac{|x|}{R})(\rho(x)-r_0)\right)e^{i\left(\theta_0+\eta_1(\frac{|x|}{R})(\theta(x)-\theta_0)\right)} & \\ \qquad\qquad \text{if } x\in B(0,A_4R)\setminus B(0,A_1R), & \\ r_0e^{i\theta_0} & \text{if } x\in\mathbf{R}^N\setminus B(0,A_4R),\end{cases}$$

$$(3.71)\qquad r_0-u_2(x)=\begin{cases} r_0e^{i\theta_0} & \text{if } x\in\overline{B}(0,A_1R),\\ \left(r_0+\eta_2(\frac{|x|}{R})(\rho(x)-r_0)\right)e^{i\left(\theta_0+\eta_2(\frac{|x|}{R})(\theta(x)-\theta_0)\right)} & \\ \qquad\qquad \text{if } x\in B(0,A_4R)\setminus\overline{B}(0,A_1R), & \\ r_0-v_1(x) & \text{if } x\in B(0,AR)\setminus B(0,A_4R),\\ r_0-u(x) & \text{if } x\in\mathbf{R}^N\setminus B(0,AR),\end{cases}$$

then we define u_1 in such a way that $r_0-u_1=e^{-i\theta_0}(r_0-\tilde u_1)$. Since $u\in\mathcal{X}$ and $u-v_1\in H_0^1(\Omega_{R,AR})$, it is clear that $u_1\in H^1(\mathbf{R}^N)$, $u_2\in\mathcal{X}$ and (i), (ii) hold.

Since $\rho+r_0\ge\frac{3}{2}r_0$ on Ω_{A_1R,A_4R}, from (3.66) we get

$$(3.72)\qquad ||\rho-r_0||^2_{L^2(\Omega_{A_1R,\,A_4R})}\le\frac{4}{9r_0^2a^2}\varepsilon.$$

Obviously, $\nabla\left(r_0+\eta_i(\frac{|x|}{R})(\rho(x)-r_0)\right)=\frac{1}{R}\eta_i'(\frac{|x|}{R})(\rho(x)-r_0)\frac{x}{|x|}+\eta_i(\frac{|x|}{R})\nabla\rho$ and using (3.65), (3.72) and the fact that $R\ge 1$ we get

$$(3.73)\qquad \begin{aligned}&||\nabla\left(r_0+\eta_i(\tfrac{|x|}{R})(\rho(x)-r_0)\right)||_{L^2(\Omega_{A_1R,\,A_4R})}\\ &\le\tfrac{1}{R}\sup|\eta_i'|\cdot||\rho-r_0||_{L^2(\Omega_{A_1R,\,A_4R})}+||\eta_i(\tfrac{|\cdot|}{R})\nabla\rho||_{L^2(\Omega_{A_1R,\,A_4R})}\le C\sqrt{\varepsilon}.\end{aligned}$$

Similarly, using (3.67) and (3.69) we find

$$(3.74)\qquad \begin{aligned}&||\nabla\left(\theta_0+\eta_i(\tfrac{|x|}{R})(\theta(x)-\theta_0)\right)||_{L^2(\Omega_{A_1R,\,A_4R})}\\ &\le\tfrac{1}{R}\sup|\eta_i'|\cdot||\theta-\theta_0||_{L^2(\Omega_{A_1R,\,A_4R})}+||\eta_i(\tfrac{|\cdot|}{R})\nabla\theta||_{L^2(\Omega_{A_1R,\,A_4R})}\le C\sqrt{\varepsilon}.\end{aligned}$$

From (3.73), (3.74) and the definition of u_1, u_2 it follows that $||\nabla u_i||_{L^2(\Omega_{A_1R,\,A_4R})} \le C\sqrt{\varepsilon}$, $i = 1, 2$. Therefore

$$\int_{\mathbf{R}^N} \left| \left|\frac{\partial u}{\partial x_j}\right|^2 - \left|\frac{\partial u_1}{\partial x_j}\right|^2 - \left|\frac{\partial u_2}{\partial x_j}\right|^2 \right| dx = \int_{\Omega_{R,AR}} \left| \left|\frac{\partial u}{\partial x_j}\right|^2 - \left|\frac{\partial u_1}{\partial x_j}\right|^2 - \left|\frac{\partial u_2}{\partial x_j}\right|^2 \right| dx$$

$$\le \int_{\Omega_{R,A_1R}\cup\Omega_{A_4R,AR}} \left|\frac{\partial u}{\partial x_j}\right|^2 + \left|\frac{\partial v_1}{\partial x_j}\right|^2 dx + \int_{\Omega_{A_1R,A_4R}} \left|\frac{\partial u}{\partial x_j}\right|^2 + \left|\frac{\partial u_1}{\partial x_j}\right|^2 + \left|\frac{\partial u_2}{\partial x_j}\right|^2 dx \le C_1\varepsilon$$

and (iii) is proved. On Ω_{A_1R,A_4R} we have $\rho \in [\frac{r_0}{2}, \frac{3r_0}{2}]$, hence $\varphi\left(r_0 + \eta_i(\frac{|x|}{R})(\rho(x) - r_0)\right) = r_0 + \eta_i(\frac{|x|}{R})(\rho(x) - r_0)$ and

$$\begin{aligned}(3.75)\qquad &\left(\varphi^2\left(r_0 + \eta_i(\tfrac{|x|}{R})(\rho(x) - r_0)\right) - r_0^2\right)^2 = (\rho - r_0)^2\eta_i^2(\tfrac{|x|}{R})\left(2r_0 + \eta_i(\tfrac{|x|}{R})(\rho - r_0)\right)^2\\ &\le \left(\tfrac{5}{2}r_0\right)^2(\rho - r_0)^2.\end{aligned}$$

From (3.70)–(3.72) and (3.75) it follows that $||\varphi^2(|r_0 - u_i|) - r_0^2||_{L^2(\Omega_{A_1R,\,A_4R})} \le C\sqrt{\varepsilon}$. As above, we get

$$\int_{\mathbf{R}^N} \left| \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - u_1|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - u_2|) - r_0^2\right)^2 \right|$$

$$\le \int_{\Omega_{R,A_1R}\cup\Omega_{A_4R,AR}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 + \left(\varphi^2(|r_0 - v_1|) - r_0^2\right)^2 dx$$

$$+ \int_{\Omega_{A_1R,A_4R}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 + \left(\varphi^2(|r_0 - u_1|) - r_0^2\right)^2 + \left(\varphi^2(|r_0 - u_2|) - r_0^2\right)^2 dx \le C_2\varepsilon.$$

This proves (iv).

Next we prove (v). Since $\langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle$ has compact support, a simple computation gives

$$(3.76)\quad Q(u_1) = L(\langle i\frac{\partial u_1}{\partial x_1}, u_1\rangle) = L(\langle ie^{-i\theta_0}\frac{\partial \tilde{u}_1}{\partial x_1}, r_0 - e^{-i\theta_0}r_0 + e^{-i\theta_0}\tilde{u}_1\rangle) = \int_{\mathbf{R}^N} \langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle\, dx.$$

From the definition of $\tilde{u}_1$ and u_2 and the fact that $u = v_1$ on $\mathbf{R}^N \setminus \Omega_{R,AR}$ we get $\langle i\frac{\partial v_1}{\partial x_1}, v_1\rangle - \langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle - \langle i\frac{\partial u_2}{\partial x_1}, u_2\rangle = 0$ a.e. on $\mathbf{R}^N \setminus \Omega_{A_1R,\,A_4R}$. Using this identity, Definition 2.4, (3.76), then (2.3) and (3.70), (3.71) we obtain

$$\begin{aligned}
&Q(v_1) - Q(u_1) - Q(u_2) = \int_{\Omega_{A_1R,\,A_4R}} \langle i\frac{\partial v_1}{\partial x_1}, v_1\rangle - \langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle - \langle i\frac{\partial u_2}{\partial x_1}, u_2\rangle\, dx\\
&= \int_{\Omega_{A_1R,\,A_4R}} \langle i\frac{\partial v_1}{\partial x_1} - \frac{\partial \tilde{u}_1}{\partial x_1} - \frac{\partial u_2}{\partial x_1}, r_0\rangle\, dx - \int_{\Omega_{A_1R,\,A_4R}} (\rho^2 - r_0^2)\frac{\partial \theta}{\partial x_1}\, dx\\
(3.77)\qquad &+ \int_{\Omega_{A_1R,\,A_4R}} \sum_{i=1}^{2} \left(\left(r_0 + \eta_i(\frac{|x|}{R})(\rho - r_0)\right)^2 - r_0^2\right) \frac{\partial}{\partial x_1}\left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta - \theta_0)\right) dx\\
&- \int_{\Omega_{A_1R,\,A_4R}} r_0^2 \left(\frac{\partial \theta}{\partial x_1} - \sum_{i=1}^{2} \frac{\partial}{\partial x_1}\left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta(x) - \theta_0)\right)\right) dx.
\end{aligned}$$

The functions $v_1 - \tilde{u}_1 - u_2$ and $\theta^* = \theta - \sum_{i=1}^2 \Big(\theta_0 + \eta_i(\frac{|x|}{R})(\theta(x) - \theta_0)\Big)$ belong to $C^1(\Omega_{R,AR})$ and $v_1 - \tilde{u}_1 - u_2 = r_0(e^{i\theta_0} - 1) = const.$, $\theta^* = -\theta_0 = const.$ on $\Omega_{R,AR} \setminus \Omega_{A_1R,A_4R}$. Therefore

$$\text{(3.78)} \qquad \int_{\Omega_{A_1R,\,A_4R}} \langle i\frac{\partial}{\partial x_1}(v_1 - \tilde{u_1} - u_2), r_0\rangle \, dx = 0 \qquad \text{and} \qquad \int_{\Omega_{A_1R,\,A_4R}} \frac{\partial \theta^*}{\partial x_1}\, dx = 0.$$

Using (3.66), (3.67) and the Cauchy-Schwarz inequality we have

$$\text{(3.79)} \qquad \Big| \int_{\Omega_{A_1R,\,A_4R}} (\rho^2 - r_0^2)\frac{\partial \theta}{\partial x_1}\, dx \Big| \le C\varepsilon.$$

Similarly, from (3.72), (3.74), (3.75) and the Cauchy-Schwarz inequality we get

$$\text{(3.80)} \qquad \Big| \int_{\Omega_{A_1R,\,A_4R}} \left(\Big(r_0 + \eta_i(\frac{|x|}{R})(\rho - r_0)\Big)^2 - r_0^2\right) \frac{\partial}{\partial x_1}\Big(\theta_0 + \eta_i(\frac{|x|}{R})(\theta - \theta_0)\Big)\, dx \Big| \le C\varepsilon.$$

From (3.77)–(3.80) we obtain $|Q(v_1) - Q(u_1) - Q(u_2)| \le C\varepsilon$ and (3.4) gives $|Q(u) - Q(v_1)| \le CE_{GL}^{\Omega_{R,AR}}(u) \le C\varepsilon$. These estimates clearly imply (v).

It remains to prove (vi). Assume that (A1) and (A2) are satisfied and let W be as in the introduction. Using (1.5) and (1.7), then Hölder's inequality we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^N} \Big| V(|r_0 - u|^2) - V(|r_0 - v_1|^2)\Big|\, dx \\
&\le \int_{\Omega_{R,\,AR}} \Big| V(\varphi^2(|r_0 - u|)) - V(\varphi^2(|r_0 - v_1|))\Big| + \Big| W(|r_0 - u|^2) - W(|r_0 - v_1|^2)\Big| dx \\
&\le C \int_{\Omega_{R,\,AR}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 + \left(\varphi^2(|r_0 - v_1|) - r_0^2\right)^2 \, dx \\
\text{(3.81)} \qquad & \quad + C\int_{\Omega_{R,\,AR}} \Big| \, |r_0 - u| - |r_0 - v_1| \, \Big| \, \big(|r_0 - u|^{2p_0+1} \mathbb{1}_{\{|r_0 - u| > 2r_0\}} \\
& \qquad\qquad + |r_0 - v_1|^{2p_0+1} \mathbb{1}_{\{|r_0 - v_1| > 2r_0\}} \big) \, dx \\
&\le C'\varepsilon + C' \int_{\Omega_{R,\,AR}} |u - v_1| \, \Big(|r_0 - u|^{2^* - 1} \mathbb{1}_{\{|r_0 - u| > 2r_0\}} + |r_0 - v_1|^{2^* - 1} \mathbb{1}_{\{|r_0 - v_1| > 2r_0\}} \Big) dx \\
&\le C'\varepsilon + C' ||u - v_1||_{L^{2^*}(\Omega_{R,\,AR})} \Big(|| \, |r_0 - u| \mathbb{1}_{\{|r_0 - u| > 2r_0\}} ||_{L^{2^*}(\Omega_{R,\,AR})}^{2^* - 1} \\
& \qquad + || \, |r_0 - v_1| \mathbb{1}_{\{|r_0 - v_1| > 2r_0\}} ||_{L^{2^*}(\Omega_{R,\,AR})}^{2^* - 1} \Big).
\end{aligned}$$

From the Sobolev embedding we have

$$\text{(3.82)} \qquad \begin{aligned} &||u - v_1||_{L^{2^*}(\mathbf{R}^N)} \le C_S ||\nabla(u - v_1)||_{L^2(\mathbf{R}^N)} \\ &\le C_S(||\nabla u||_{L^2(\Omega_{R,\,AR})} + ||\nabla v_1||_{L^2(\Omega_{R,\,AR})}) \le 2C_S\sqrt{\varepsilon}. \end{aligned}$$

It is clear that $|r_0 - u| > 2r_0$ implies $|u| > r_0$ and $|r_0 - u| < 2|u|$, hence

$$\text{(3.83)} \qquad \begin{aligned} &|| \, |r_0 - u| \mathbb{1}_{\{|r_0 - u| > 2r_0\}} ||_{L^{2^*}(\Omega_{R,\,AR})} \\ &\le 2||u||_{L^{2^*}(\mathbf{R}^N)} \le 2C_S ||\nabla u||_{L^2(\mathbf{R}^N)} \le 2C_S \left(E_{GL}(u)\right)^{\frac{1}{2}}. \end{aligned}$$

Obviously, a similar estimate holds for v_1. Combining (3.81), (3.82) and (3.83) we find

$$\int_{\Omega_{R,\,AR}} \Big| V(|r_0 - u|^2) - V(|r_0 - v_1|^2) \Big| \, dx \le C'\varepsilon + C''\sqrt{\varepsilon}\,(E_{GL}(u))^{\frac{2^*-1}{2}} . \tag{3.84}$$

From (3.70) and (3.71) it follows that $V(|r_0 - v_1|^2) - V(|r_0 - u_1|^2) - V(|r_0 - u_2|^2) = 0$ on $\mathbf{R}^N \setminus \Omega_{A_1R,\,A_4R}$ and $|r_0 - v_1|$, $|r_0 - u_1|$, $|r_0 - u_2| \in \left[\frac{r_0}{2}, \frac{3r_0}{2}\right]$ on $\Omega_{A_1R,\,A_4R}$. Then using (1.5), (3.66), (3.75) and (3.72) we get

$$\int_{\Omega_{A_1R,\,A_4R}} |V(|r_0 - v_1|^2)| \, dx \le C \int_{\Omega_{A_1R,\,A_4R}} (\rho^2 - r_0^2)^2 \, dx \le C\varepsilon, \qquad \text{respectively} \tag{3.85}$$

$$\int_{\Omega_{A_1R,\,A_4R}} |V(|r_0 - u_i|^2)| \, dx \le C \int_{\Omega_{A_1R,\,A_4R}} \left(\left(r_0 + \eta_i(\tfrac{|x|}{R})(\rho - r_0) \right)^2 - r_0^2 \right)^2 dx \le C\varepsilon. \tag{3.86}$$

Therefore

$$\begin{aligned} &\int_{\mathbf{R}^N} \Big| V(|r_0 - v_1|^2) - V(|r_0 - u_1|^2) - V(|r_0 - u_2|^2) \Big| \, dx \\ &\le \int_{\Omega_{A_1R,\,A_4R}} |V(|r_0 - v_1|^2)| + |V(|r_0 - u_1|^2)| + |V(|r_0 - u_2|^2)| \, dx \le C\varepsilon. \end{aligned} \tag{3.87}$$

Then (iv) follows from (3.84) and (3.87) and Lemma 3.3 is proved. □

4 Variational formulation

We assume throughout that assumptions (A1) and (A2) in the introduction are satisfied. We introduce the following functionals:

$$E_c(u) = \int_{\mathbf{R}^N} |\nabla u|^2 \, dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx,$$

$$A(u) = \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left| \frac{\partial u}{\partial x_j} \right|^2 dx,$$

$$B_c(u) = \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx,$$

$$P_c(u) = \tfrac{N-3}{N-1} A(u) + B_c(u).$$

It is clear that $E_c(u) = A(u) + B_c(u) = \frac{2}{N-1} A(u) + P_c(u)$. Let

$$\mathcal{C} = \{ u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0 \}.$$

The aim of this section is to study the properties of the above functionals. In particular, we will prove that $\mathcal{C} \neq \emptyset$ and $\inf\{E_c(u) \mid u \in \mathcal{C}\} > 0$. This will be done in a sequence of lemmas. In the next sections we show that E_c admits a minimizer in $\mathcal{C}$ and this minimizer is a solution of (1.3).

We begin by proving that the above functionals are well-defined on $\mathcal{X}$. Since we have already seen in section 2 that Q is well-defined on $\mathcal{X}$, all we have to do is to prove that $V(|r_0 - u|^2) \in L^1(\mathbf{R}^N)$ for any $u \in \mathcal{X}$. This will be done in the next lemma.

Lemma 4.1 *For any $u \in \mathcal{X}$ we have $V(|r_0 - u|^2) \in L^1(\mathbf{R}^N)$. Moreover, for any $\delta > 0$ there exist $C_1(\delta)$, $C_2(\delta) > 0$ such that for any $u \in \mathcal{X}$ we have*

$$(1-\delta)a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 dx - C_1(\delta)||\nabla u||^{2^*}_{L^2(\mathbf{R}^N)}$$

$$\leq \int_{\mathbf{R}^N} V(|r_0 - u|^2)\, dx \tag{4.1}$$

$$\leq (1+\delta)a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 dx + C_2(\delta)||\nabla u||^{2^*}_{L^2(\mathbf{R}^N)}.$$

Proof. Fix $\delta > 0$. Using (1.4) we see that there exists $\beta = \beta(\delta) \in (0, r_0]$ such that

$$(1-\delta)a^2(s - r_0^2)^2 \leq V(s) \leq (1+\delta)a^2(s - r_0^2)^2 \quad \text{for any } s \in ((r_0-\beta)^2, (r_0+\beta)^2). \tag{4.2}$$

Let $u \in \mathcal{X}$. If $|u(x)| < \beta$ we have $|r_0 - u(x)|^2 \in ((r_0-\beta)^2, (r_0+\beta)^2)$ and it follows from (4.2) that $V(|r_0 - u|^2)\mathbb{1}_{\{|u|<\beta\}} \in L^1(\mathbf{R}^N)$ and

$$(1-\delta)a^2 \int_{\{|u|<\beta\}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 dx \leq \int_{\{|u|<\beta\}} V(|r_0 - u|^2)\, dx$$

$$\leq (1+\delta)a^2 \int_{\{|u|<\beta\}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 dx. \tag{4.3}$$

Assumption (A2) implies that there exists $C_1'(\delta) > 0$ such that

$$\left|V(|r_0 - z|^2) - (1-\delta)a^2(\varphi^2(|r_0 - z|) - r_0^2)^2\right| \leq C_1'(\delta)|z|^{2p_0+2} \leq C_1''(\delta)|z|^{2^*}$$

for any $z \in \mathbf{C}$ satisfying $|z| \geq \beta$. Using the Sobolev embedding we obtain

$$\int_{\{|u|\geq\beta\}} \left|V(|r_0 - u|^2) - (1-\delta)a^2(\varphi^2(|r_0 - u|) - r_0^2)^2\right| dx$$

$$\leq C_1''(\delta) \int_{\{|u|\geq\beta\}} |u|^{2^*} dx \leq C_1''(\delta) \int_{\mathbf{R}^N} |u|^{2^*} dx \leq C_1(\delta)||\nabla u||^{2^*}_{L^2(\mathbf{R}^N)}. \tag{4.4}$$

Consequently $V(|r_0 - u|^2)\mathbb{1}_{\{|u|\geq\beta\}} \in L^1(\mathbf{R}^N)$ and it follows from (4.3) and (4.4) that the first inequality in (4.1) holds; the proof of the second inequality is similar. □

Lemma 4.2 *Let $\delta \in (0, r_0)$ and let $u \in \mathcal{X}$ be such that $r_0 - \delta \leq |r_0 - u| \leq r_0 + \delta$ a.e. on $\mathbf{R}^N$. Then*

$$|Q(u)| \leq \frac{1}{2a(r_0 - \delta)} E_{GL}(u).$$

Proof. From Lemma 2.1 we know that there are two real-valued functions ρ, θ such that $\rho - r_0 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ and $r_0 - u = \rho e^{i\theta}$ a.e. on $\mathbf{R}^N$. Moreover, from (2.3) and Definition 2.4 we infer that

$$Q(u) = -\int_{\mathbf{R}^N} (\rho^2 - r_0^2)\theta_{x_1}\, dx.$$

Using the Cauchy-Schwarz inequality we obtain

$$2a(r_0 - \delta)|Q(u)| \leq 2a(r_0 - \delta)||\theta_{x_1}||_{L^2(\mathbf{R}^N)}||\rho^2 - r_0^2||_{L^2(\mathbf{R}^N)}$$

$$\leq (r_0 - \delta)^2 \int_{\mathbf{R}^N} |\theta_{x_1}|^2\, dx + a^2 \int_{\mathbf{R}^N} \left(\rho^2 - r_0^2\right)^2 dx$$

$$\leq \int_{\mathbf{R}^N} \rho^2|\nabla\theta|^2 + a^2\left(\rho^2 - r_0^2\right)^2 dx \leq E_{GL}(u).$$

□

Lemma 4.3 *Assume that $0 \le c < v_s$ and let $\varepsilon \in (0, 1 - \frac{c}{v_s})$. There exists a constant $K_1 = K_1(F, N, c, \varepsilon) > 0$ such that for any $u \in \mathcal{X}$ satisfying $E_{GL}(u) < K_1$ we have*

$$\int_{\mathbf{R}^N} |\nabla u|^2 \, dx + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx - c|Q(u)| \ge \varepsilon E_{GL}(u).$$

Proof. Fix ε_1 such that $\varepsilon < \varepsilon_1 < 1 - \frac{c}{v_s}$. Then fix $\delta_1 \in (0, \varepsilon_1 - \varepsilon)$. By Lemma 4.1, there exists $C_1(\delta_1) > 0$ such that for any $u \in \mathcal{X}$ we have

$$\int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx \ge (1 - \delta_1) a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 \, dx - C_1(\delta_1) \left(E_{GL}(u)\right)^{\frac{2^*}{2}}. \tag{4.5}$$

Using (3.4) we see that there exists $A > 0$ such that for any $w \in \mathcal{X}$ with $E_{GL}(w) \le 1$, for any $h \in (0, 1]$ and for any minimizer v_h of $G^w_{h,\mathbf{R}^N}$ in $H^1_w(\mathbf{R}^N)$ we have

$$|Q(w) - Q(v_h)| \le A h^{\frac{2}{N}} E_{GL}(w). \tag{4.6}$$

Choose $h \in (0, 1]$ such that $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} > \varepsilon$ (this choice is possible because $\varepsilon_1 - \delta_1 - \varepsilon > 0$). Then fix $\delta > 0$ such that $\frac{c}{2a(r_0 - \delta)} < 1 - \varepsilon_1$ (such δ exist because $\varepsilon_1 < 1 - \frac{c}{v_s} = 1 - \frac{c}{2ar_0}$).

Let $K = K(a, r_0, N, h, \delta, 1)$ be as in Lemma 3.1 (iv).

Consider $u \in \mathcal{X}$ such that $E_{GL}(u) \le \min(K, 1)$. Let v_h be a minimizer of $G^u_{h,\mathbf{R}^N}$ in $H^1_u(\mathbf{R}^N)$. The existence of v_h follows from Lemma 3.1 (i). By Lemma 3.1 (iv) we have $r_0 - \delta < |r_0 - v_h| < r_0 + \delta$ a.e. on $\mathbf{R}^N$ and then Lemma 4.2 implies

$$c|Q(v_h)| \le \frac{c}{2a(r_0 - \delta)} E_{GL}(v_h) \le (1 - \varepsilon_1) E_{GL}(v_h) \le (1 - \varepsilon_1) E_{GL}(u). \tag{4.7}$$

We have:

$$\begin{aligned}
&\int_{\mathbf{R}^N} |\nabla u|^2 \, dx + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx - c|Q(u)| \\
&\ge (1 - \delta_1) E_{GL}(u) - C_1(\delta_1) \left(E_{GL}(u)\right)^{\frac{2^*}{2}} - c|Q(u)| \qquad \text{by (4.5)} \\
&\ge (1 - \delta_1) E_{GL}(u) - C_1(\delta_1) \left(E_{GL}(u)\right)^{\frac{2^*}{2}} - c|Q(u) - Q(v_h)| - c|Q(v_h)| \\
&\ge (1 - \delta_1) E_{GL}(u) - C_1(\delta_1) \left(E_{GL}(u)\right)^{\frac{2^*}{2}} - cAh^{\frac{2}{N}} E_{GL}(u) - (1 - \varepsilon_1) E_{GL}(u) \\
&\qquad\qquad \text{by (4.6) and (4.7)} \\
&= \left(\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} - C_1(\delta_1) \left(E_{GL}(u)\right)^{\frac{2^*}{2} - 1}\right) E_{GL}(u).
\end{aligned} \tag{4.8}$$

Note that (4.8) holds for any $u \in \mathcal{X}$ with $E_{GL}(u) \le \min(K, 1)$. Since $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} > \varepsilon$, it is obvious that $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} - C_1(\delta_1) \left(E_{GL}(u)\right)^{\frac{2^*}{2} - 1} > \varepsilon$ if $E_{GL}(u)$ is sufficiently small and the conclusion of Lemma 4.3 follows. □

An obvious consequence of Lemma 4.3 is that $E_c(u) > 0$ if $u \in \mathcal{X} \setminus \{0\}$ and $E_{GL}(u)$ is sufficiently small. An easy corollary of the next lemma is that there are functions $v \in \mathcal{X}$ such that $E_c(v) < 0$.

Lemma 4.4 *Let $N \ge 2$. Let $D = \{(R, \varepsilon) \in \mathbf{R}^2 \mid R > 0, \ 0 < \varepsilon < \frac{R}{2}\}$. There exists a continuous map from D to $H^1(\mathbf{R}^N)$, $(R, \varepsilon) \longmapsto v^{R,\varepsilon}$ such that $v^{R,\varepsilon} \in C_c(\mathbf{R}^N)$ for any $(R, \varepsilon) \in D$ and the following estimates hold:*

i) $\displaystyle\int_{\mathbf{R}^N} |\nabla v^{R,\varepsilon}|^2 \, dx \le C_1 R^{N-2} + C_2 R^{N-2} \ln \frac{R}{\varepsilon}$, *ii)* $\left| \int_{\mathbf{R}^N} V(|r_0 - v^{R,\varepsilon}|^2)\, dx \right| \le C_3 \varepsilon^2 R^{N-2}$,

iii) $\left| \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - v^{R,\varepsilon}|) - r_0^2\right)^2 \, dx \right| \le C_4 \varepsilon^2 R^{N-2}$,

iv) $-2\pi r_0^2 \omega_{N-1} R^{N-1} \le Q(v^{R,\varepsilon}) \le -2\pi r_0^2 \omega_{N-1} (R - 2\varepsilon)^{N-1}$,

where the constants $C_1 - C_4$ depend only on N and $\omega_{N-1} = \mathcal{L}^{N-1}(B_{\mathbf{R}^{N-1}}(0,1))$.

Proof. Let $A > 0$ and

$$T_{A,R} = \{x \in \mathbf{R}^N \mid 0 \le |x'| \le R, \;\; -\tfrac{A(R-|x'|)}{R} < x_1 < \tfrac{A(R-|x'|)}{R}\}.$$

We define $\theta^{A,R} : \mathbf{R}^N \longrightarrow \mathbf{R}$ in the following way: if $|x'| \ge R$ we put $\theta^{A,R}(x) = 0$ and if $|x'| < R$ we define

$$\theta^{A,R}(x) = \begin{cases} 0 & \text{if } x_1 \le -\frac{A(R-|x'|)}{R}, \\ \frac{\pi R}{A(R-|x'|)} x_1 + \pi & \text{if } x \in T_{A,R}, \\ 2\pi & \text{if } x_1 \ge \frac{A(R-|x'|)}{R}. \end{cases} \tag{4.9}$$

It is easy to see that $x \longmapsto e^{i\theta^{A,R}(x)}$ is continuous on $\mathbf{R}^N \setminus \{x \mid x_1 = 0, \; |x'| = R\}$ and equals 1 on $\mathbf{R}^N \setminus T_{A,R}$.

Fix $\psi \in C^\infty(\mathbf{R})$ such that $\psi = 0$ on $(-\infty, 1]$, $\psi = 1$ on $[2, \infty)$ and $0 \le \psi' \le 2$. Let

$$\psi^{R,\varepsilon}(x) = \psi\Big(\frac{1}{\varepsilon}\sqrt{x_1^2 + (|x'| - R)^2}\Big) \quad \text{and} \quad w_{A,R,\varepsilon}(x) = r_0 \left(1 - \psi^{R,\varepsilon}(x) e^{i\theta^{A,R}(x)}\right). \tag{4.10}$$

It is obvious that $w_{A,R,\varepsilon} \in C_c(\mathbf{R}^N)$ (in fact, $w_{A,R,\varepsilon}$ is C^∞ on $\mathbf{R}^N \setminus B$, where $B = \partial T_{A,R} \cup \{(x_1, 0, \dots, 0) \mid x_1 \in [-A, A]\}$). On $\mathbf{R}^N \setminus B$ we have

$$\frac{\partial \theta^{A,R}}{\partial x_1} = \begin{cases} \frac{\pi R}{A(R-|x'|)} & \text{if } x \in T_{A,R}, \\ 0 & \text{otherwise}, \end{cases} \qquad \frac{\partial \theta^{A,R}}{\partial x_j} = \begin{cases} \frac{\pi R x_1}{A(R-|x'|)^2} \frac{x_j}{|x'|} & \text{if } x \in T_{A,R}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.11}$$

$$\frac{\partial \psi^{R,\varepsilon}}{\partial x_1}(x) = \frac{1}{\varepsilon} \psi' \left(\frac{\sqrt{x_1^2 + (|x'| - R)^2}}{\varepsilon} \right) \frac{x_1}{\sqrt{x_1^2 + (|x'| - R)^2}}, \tag{4.12}$$

$$\frac{\partial \psi^{R,\varepsilon}}{\partial x_j}(x) = \frac{1}{\varepsilon} \psi' \left(\frac{\sqrt{x_1^2 + (|x'| - R)^2}}{\varepsilon} \right) \frac{|x'| - R}{\sqrt{x_1^2 + (|x'| - R)^2}} \frac{x_j}{|x'|} \quad \text{for } j \ge 2. \tag{4.13}$$

Then a simple computation gives $\langle i \frac{\partial w_{A,R,\varepsilon}}{\partial x_1}, w_{A,R,\varepsilon} \rangle = -r_0^2 (\psi^{R,\varepsilon})^2 \frac{\partial \theta^{A,R}}{\partial x_1} + r_0^2 \frac{\partial}{\partial x_1} \left(\psi^{R,\varepsilon} \sin(\theta^{A,R}) \right)$ on $\mathbf{R}^N \setminus B$. Thus we have

$$Q(w_{A,R,\varepsilon}) = -r_0^2 \int_{\mathbf{R}^N} (\psi^{R,\varepsilon})^2 \frac{\partial \theta^{A,R}}{\partial x_1} \, dx.$$

It is obvious that

$$\int_{-\infty}^{\infty} \frac{\partial \theta^{A,R}}{\partial x_1} \, dx_1 = 0 \quad \text{if } |x'| > R \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\partial \theta^{A,R}}{\partial x_1} \, dx_1 = 2\pi \quad \text{if } 0 < |x'| < R. \tag{4.14}$$

Since $\frac{\partial \theta^{A,R}}{\partial x_1} \ge 0$ a.e. on $\mathbf{R}^N$ and $0 \le \psi^{R,\varepsilon} \le 1$, we get

$$\int_{\{|R-|x'|| \ge 2\varepsilon\}} \frac{\partial \theta^{A,R}}{\partial x_1} \, dx \le \int_{\mathbf{R}^N} \left(\psi^{R,\varepsilon}\right)^2 \frac{\partial \theta^{A,R}}{\partial x_1} \, dx_1 \le \int_{\mathbf{R}^N} \frac{\partial \theta^{A,R}}{\partial x_1} \, dx_1,$$

and using Fubini's theorem and (4.14) we obtain that $w_{A,R,\varepsilon}$ satisfies (iv).

Using cylindrical coordinates (x_1, r, ζ) in $\mathbf{R}^N$, where $r = |x'|$ and $\zeta = \frac{x'}{|x'|} \in S^{N-2}$, we get

$$\int_{\mathbf{R}^N} V(|r_0 - w_{A,R,\varepsilon}|^2)\, dx = |S^{N-2}| \int_{-\infty}^{\infty} \int_0^{\infty} V\left(r_0^2 \psi^2 \left(\frac{\sqrt{x_1^2+(r-R)^2}}{\varepsilon}\right)\right) r^{N-2}\, dr\, dx_1. \tag{4.15}$$

Next we use polar coordinates in the (x_1, r) plane, that is we write $x_1 = \tau\cos\alpha$, $r = R + \tau\sin\alpha$ (thus $\tau = \sqrt{x_1^2 + (R-r)^2}$). Since $V(r_0^2\psi^2(s)) = 0$ for $s \geq 2$, we get

(4.16)

$$\int_{-\infty}^{\infty} \int_0^{\infty} V\left(r_0^2 \psi^2 \left(\frac{\sqrt{x_1^2+(r-R)^2}}{\varepsilon}\right)\right) r^{N-2}\, dr\, dx_1 = \int_0^{2\varepsilon} \int_0^{2\pi} V(r_0^2\psi^2(\tfrac{\tau}{\varepsilon}))(R + \tau\sin\alpha)^{N-2}\, d\alpha\, \tau\, d\tau$$

$$= \varepsilon^2 \int_0^2 \int_0^{2\pi} V(r_0^2\psi^2(s))(R + \varepsilon s\sin\alpha)^{N-2}\, d\alpha\, s\, ds.$$

It is obvious that $\left| \int_0^{2\pi} (R + \varepsilon s \sin\alpha)^{N-2} d\alpha \right| \leq 2\pi(R + 2\varepsilon)^{N-2}$ for any $s \in [0,2]$, and then using (4.15) and (4.16) we infer that $w_{A,R,\varepsilon}$ satisfies (ii). The proof of (iii) is similar.

It is clear that on $\mathbf{R}^N \setminus B$ we have

$$|\nabla w_{A,R,\varepsilon}| = r_0^2 |\nabla\psi^{R,\varepsilon}|^2 + r_0^2 |\psi^{R,\varepsilon}|^2 |\nabla\theta^{A,R}|^2. \tag{4.17}$$

From (4.12) and (4.13) we see that $|\nabla\psi^{R,\varepsilon}(x)|^2 = \frac{1}{\varepsilon^2}\left|\psi'\left(\frac{\sqrt{x_1^2+(|x'|-R)^2}}{\varepsilon}\right)\right|^2$. Proceeding as above and using cylindrical coordinates (x_1, r, ζ) in $\mathbf{R}^N$, then passing to polar coordinates $x_1 = \tau\cos\alpha$, $r = R + \tau\sin\alpha$, we obtain

$$\int_{\mathbf{R}^N} \left|\psi'\left(\frac{\sqrt{x_1^2 + (|x'| - R)^2}}{\varepsilon}\right)\right|^2 dx \leq 2\pi |S^{N-2}| \varepsilon^2 (R + 2\varepsilon)^{N-2} \int_0^2 s|\psi'(s)|^2\, ds. \tag{4.18}$$

It is easily seen from (4.11) that $|\nabla\theta^{A,R}(x)|^2 = \frac{\pi^2 R^2}{A^2(R-|x'|)^2}\left(1 + \frac{x_1^2}{(R-|x'|)^2}\right)$ if $x \in T_{A,R}$, $|x'| \neq 0$, and $\nabla\theta^{A,R}(x) = 0$ a.e. on $\mathbf{R}^N \setminus \overline{T}_{A,R}$. Moreover, if $(x_1, x') \in T_{A,R}$ and $|x'| \geq R - \frac{R\varepsilon}{\sqrt{A^2+R^2}}$, we have $\psi^{R,\varepsilon}(x_1, x') = 0$. Therefore

$$\int_{\mathbf{R}^N} |\psi^{R,\varepsilon}|^2 |\nabla\theta^{A,R}|^2\, dx \leq \int_{T_{A,R} \cap \{|x'| < R - \frac{R\varepsilon}{\sqrt{A^2+R^2}}\}} |\nabla\theta^{A,R}|^2\, dx$$

$$= \int_{\{|x'| < R - \frac{R\varepsilon}{\sqrt{A^2+R^2}}\}} \int_{-\frac{A(R-|x'|)}{R}}^{\frac{A(R-|x'|)}{R}} |\nabla\theta^{A,R}|^2\, dx_1\, dx'$$

$$= \int_{\{|x'| < R - \frac{R\varepsilon}{\sqrt{A^2+R^2}}\}} \frac{2\pi^2 R}{A(R - |x'|)} + \frac{2\pi^2}{3} \frac{A}{R} \frac{1}{R - |x'|}\, dx' \tag{4.19}$$

$$= 2\pi^2 \left(\frac{R}{A} + \frac{3A}{R}\right) |S^{N-2}| \int_0^{R - \frac{R\varepsilon}{\sqrt{A^2+R^2}}} \frac{r^{N-2}}{R - r}\, dr$$

$$= 2\pi^2 \left(\frac{R}{A} + \frac{3A}{R}\right) |S^{N-2}| R^{N-2} \left(\sum_{k=1}^{N-2} \frac{1}{k}\left(1 - \frac{\varepsilon}{\sqrt{A^2+R^2}}\right)^k + \ln\left(\frac{\sqrt{A^2+R^2}}{\varepsilon}\right)\right).$$

Now it suffices to take $v^{R,\varepsilon} = w_{R,R,\varepsilon}$. From (4.17), (4.18) and (4.19) it follows that $v^{R,\varepsilon}$ satisfies (i). It is not hard to see that the mapping $(R,\varepsilon) \longmapsto v^{R,\varepsilon}$ is continuous from D to $H^1(\mathbf{R}^N)$ and Lemma 4.4 is proved. $\Box$

Lemma 4.5 *For any $k > 0$, the functional Q is bounded on the set*

$$\{u \in \mathcal{X} \mid E_{GL}(u) \le k\}.$$

Proof. Let $c \in (0, v_s)$ and let $\varepsilon \in (0, 1 - \frac{c}{v_s})$. From Lemmas 4.1 and 4.3 it follows that there exist two positive constants $C_2(\frac{\varepsilon}{2})$ and K_1 such that for any $u \in \mathcal{X}$ satisfying $E_{GL}(u) < K_1$ we have

$$(1 + \tfrac{\varepsilon}{2})E_{GL}(u) + C_2(\tfrac{\varepsilon}{2})\,(E_{GL}(u))^{\frac{2^*}{2}} - c|Q(u)|$$

$$\ge \int_{\mathbf{R}^N} |\nabla u|^2\,dx + \int_{\mathbf{R}^N} V(|r_0 - u|^2)\,dx - c|Q(u)| \ge \varepsilon E_{GL}(u).$$

This inequality implies that there exists $K_2 \le K_1$ such that for any $u \in \mathcal{X}$ satisfying $E_{GL}(u) \le K_2$ we have

$$c|Q(u)| \le E_{GL}(u). \tag{4.20}$$

Hence Lemma 4.5 is proved if $k \le K_2$.

Now let $u \in \mathcal{X}$ be such that $E_{GL}(u) > K_2$. Using the notation (1.10), it is clear that for $\sigma > 0$ we have $Q(u_{\sigma,\sigma}) = \sigma^{N-1}Q(u)$ (see (2.14) and

$$E_{GL}(u_{\sigma,\sigma}) = \sigma^{N-2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx + \sigma^N a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\,dx.$$

Let $\sigma_0 = \left(\frac{K_2}{E_{GL}(u)}\right)^{\frac{1}{N-2}}$. Then $\sigma_0 \in (0,1)$ and we have $E_{GL}(u_{\sigma_0,\sigma_0}) \le \sigma_0^{N-2}E_{GL}(u) = K_2$. Using (4.20) we infer that $c|Q(u_{\sigma_0,\sigma_0})| \le E_{GL}(u_{\sigma_0,\sigma_0})$, and this implies $c\sigma_0^{N-1}|Q(u)| \le \sigma_0^{N-2}E_{GL}(u)$, or equivalently

$$|Q(u)| \le \frac{1}{c\sigma_0}E_{GL}(u) = \frac{1}{c}K_2^{-\frac{1}{N-2}}\,(E_{GL}(u))^{\frac{N-1}{N-2}}. \tag{4.21}$$

Since (4.21) holds for any $u \in \mathcal{X}$ with $E_{GL}(u) > K_2$, Lemma 4.5 is proved. $\Box$

From Lemma 4.1 and Lemma 4.5 it follows that for any $k > 0$, the functional E_c is bounded on the set $\{u \in \mathcal{X} \mid E_{GL}(u) = k\}$. For $k > 0$ we define

$$E_{c,\,min}(k) = \inf\{E_c(u) \mid u \in \mathcal{X},\ E_{GL}(u) = k\}.$$

Clearly, the function $E_{c,\,min}$ is bounded on any bounded interval in $\mathbf{R}$. The next result will be important for our variational argument.

Lemma 4.6 *Assume that $N \ge 3$ and $0 < c < v_s$. The function $E_{c,\,min}$ has the following properties:*

i) There exists $k_0 > 0$ such that $E_{c,\,min}(k) > 0$ for any $k \in (0, k_0)$.

ii) We have $\lim_{k\to\infty} E_{c,\,min}(k) = -\infty$.

iii) For any $k > 0$ we have $E_{c,\,min}(k) < k$.

Proof. (i) is an easy consequence of Lemma 4.3.

(ii) It is obvious that $H^1(\mathbf{R}^N) \subset \mathcal{X}$ and the functionals E_{GL}, E_c and Q are continuous on $H^1(\mathbf{R}^N)$. For $\varepsilon = 1$ and $R > 2$, consider the functions $v^{R,1}$ constructed in Lemma 4.4. Clearly, $R \longmapsto v^{R,1}$ is a continuous curve in $H^1(\mathbf{R}^N)$. Lemma 4.4 implies $E_c(v^{R,1}) \longrightarrow -\infty$ as $R \longrightarrow \infty$. From Lemma 4.5 we infer that $E_{GL}(v^{R,1}) \longrightarrow \infty$ as $R \longrightarrow \infty$ and then it is not hard to see that (ii) holds.

(iii) Fix $k > 0$. Let $v^{R,1}$ be as above and let $u = v^{R,1}$ for some R sufficiently large, so that

$$E_{GL}(u) > k, \quad Q(u) < 0 \quad \text{ and } E_c(u) < 0.$$

In particular, we have

$$E_c(u) - E_{GL}(u) = cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) - a^2 \left(\varphi^2(|r_0 - u|^2) - r_0^2\right)^2 dx < 0.$$

It is obvious that $E_{GL}(u_{\sigma,\sigma}) \longrightarrow 0$ as $\sigma \longrightarrow 0$, hence there exists $\sigma_0 \in (0,1)$ such that $E_{GL}(u_{\sigma_0,\sigma_0}) = k$. We have

$$\begin{aligned}
&E_c(u_{\sigma_0,\sigma_0}) - E_{GL}(u_{\sigma_0,\sigma_0}) \\
&= \sigma_0^{N-1} cQ(u) + \sigma_0^N \int_{\mathbf{R}^N} V(|r_0 - u|^2) - a^2 \left(\varphi^2(|r_0 - u|^2) - r_0^2\right)^2 dx \\
&= (\sigma_0^{N-1} - \sigma_0^N) cQ(u) + \sigma_0^N (E_c(u) - E_{GL}(u)) < 0.
\end{aligned}$$

Thus $E_c(u_{\sigma_0,\sigma_0}) < E_{GL}(u_{\sigma_0,\sigma_0})$. Since $E_{GL}(u_{\sigma_0,\sigma_0}) = k$, we have necessarily $E_{c,\,min}(k) \leq E_c(u_{\sigma_0,\sigma_0}) < k$. ☐

From Lemma 4.6 (i) and (ii) it follows that

$$0 < S_c := \sup\{E_{c,min}(k) \mid k > 0\} < \infty. \tag{4.22}$$

Lemma 4.7 *The set $\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0,\ P_c(u) = 0\}$ is not empty and we have*

$$T_c := \inf\{E_c(u) \mid u \in \mathcal{C}\} \geq S_c > 0.$$

Proof. Let $u \in \mathcal{X} \setminus \{0\}$ be such that $E_c(w) < 0$ (we have seen in the proof of Lemma 4.6 that such functions exist). It is obvious that $A(w) > 0$ and $\displaystyle\int_{\mathbf{R}^N} \left|\frac{\partial w}{\partial x_1}\right|^2 dx > 0$; therefore $B_c(w) = E_c(w) - A(w) < 0$ and $P_c(w) = E_c(w) - \frac{2}{N-1} A(w) < 0$. Clearly,

$$P_c(w_{\sigma,1}) = \frac{1}{\sigma} \int_{\mathbf{R}^N} \left|\frac{\partial w}{\partial x_1}\right|^2 dx + \frac{N-3}{N-1} \sigma A(w) + cQ(w) + \sigma \int_{\mathbf{R}^3} V(|r_0 - w|^2)\, dx. \tag{4.23}$$

Since $P_c(w_{1,1}) = P_c(w) < 0$ and $\lim\limits_{\sigma \to 0} P_c(w_{\sigma,1}) = \infty$, there exists $\sigma_0 \in (0,1)$ such that $P_c(w_{\sigma_0,1}) = 0$, that is $w_{\sigma_0,1} \in \mathcal{C}$. Thus $\mathcal{C} \neq \emptyset$.

To prove the second part of Lemma 4.7, consider first the case $N \geq 4$. Let $u \in \mathcal{C}$. It is clear that $A(u) > 0$, $B_c(u) = -\frac{N-3}{N-1} A(u) < 0$ and for any $\sigma > 0$ we have $E_c(u_{1,\sigma}) = A(u_{1,\sigma}) + B_c(u_{1,\sigma}) = \sigma^{N-3} A(u) + \sigma^{N-1} B_c(u)$, hence

$$\frac{d}{d\sigma}(E_c(u_{1,\sigma})) = (N-3)\sigma^{N-4} A(u) + (N-1)\sigma^{N-2} B_c(u)$$

is positive on $(0,1)$ and negative on $(1,\infty)$. Consequently the function $\sigma \longmapsto E_c(u_{1,\sigma})$ achieves its maximum at $\sigma = 1$.

On the other hand, we have

$$E_{GL}(u_{1,\sigma}) = \sigma^{N-3}A(u) + \sigma^{N-1}\left(\int_{\mathbf{R}^N}\left|\frac{\partial u}{\partial x_1}\right|^2 + a^2\left(\varphi^2(|r_0-u|) - r_0^2\right)^2\,dx\right).$$

It is easy to see that the mapping $\sigma \longmapsto E_{GL}(u_{1,\sigma})$ is strictly increasing and one-to-one from $(0,\infty)$ to $(0,\infty)$. Hence for any $k>0$, there is a unique $\sigma(k,u)>0$ such that $E_{GL}(u_{1,\sigma(k,u)}) = k$. Then we have

$$E_{c,min}(k) \le E_c(u_{1,\sigma(k,u)}) \le E_c(u_{1,1}) = E_c(u).$$

Since this is true for any $k>0$ and any $u\in\mathcal{C}$, the conclusion follows.

Next we consider the case $N=3$. Let $u\in\mathcal{C}$. We have $P_c(u) = B_c(u) = 0$ and $E_c(u) = A(u) > 0$. For $\sigma>0$ we get

$$E_c(u_{1,\sigma}) = A(u) + \sigma^2 B_c(u) = A(u) \qquad \text{and}$$
$$E_{GL}(u_{1,\sigma}) = A(u) + \sigma^2\left(\int_{\mathbf{R}^3}\left|\frac{\partial u}{\partial x_1}\right|^2 + a^2\left(\varphi^2(|r_0-u|) - r_0^2\right)^2\,dx\right).$$

Clearly, $\sigma \longmapsto E_{GL}(u_{1,\sigma})$ is increasing on $(0,\infty)$ and is one-to-one from $(0,\infty)$ to $(A(u),\infty)$.

Let $\varepsilon>0$. Let $k_\varepsilon>0$ be such that $E_{c,min}(k_\varepsilon) > S_c-\varepsilon$. If $A(u)\ge k_\varepsilon$, from Lemma 4.6 (iii) we have

$$E_c(u) = A(u) \ge k_\varepsilon > E_{c,min}(k_\varepsilon) > S_c - \varepsilon.$$

If $A(u)<k_\varepsilon$, there exists $\sigma(k_\varepsilon,u)>0$ such that $E_{GL}(u_{1,\sigma(k_\varepsilon,u)}) = k_\varepsilon$. Then we get

$$E_c(u) = A(u) = E_c(u_{1,\sigma(k_\varepsilon,u)}) \ge E_{c,min}(k_\varepsilon) > S_c-\varepsilon.$$

So far we have proved that for any $u\in\mathcal{C}$ and any $\varepsilon>0$ we have $E_c(u) > S_c-\varepsilon$. The conclusion follows letting $\varepsilon\longrightarrow 0$, then taking the infimum for $u\in\mathcal{C}$. $\square$

In Lemma 4.7, we do not know whether $T_c = S_c$.

Lemma 4.8 *Let T_c be as in Lemma 4.7. The following assertions hold.*

i) For any $u\in\mathcal{X}$ with $P_c(u)<0$ we have $A(u) > \frac{N-1}{2}T_c$.

ii) Let $(u_n)_{n\ge1}\subset\mathcal{X}$ be a sequence such that $(E_{GL}(u_n))_{n\ge1}$ is bounded and $\lim\limits_{n\to\infty} P_c(u_n) = \mu < 0$. Then $\liminf\limits_{n\to\infty} A(u_n) > \frac{N-1}{2}T_c$.

Proof. i) Since $P_c(u)<0$, it is clear that $u\ne0$ and $\int_{\mathbf{R}^N}\left|\frac{\partial u}{\partial x_1}\right|^2 dx > 0$. As in the proof of Lemma 4.7, we have $P_c(u_{1,1}) = P_c(u) < 0$ and (4.23) implies that $\lim\limits_{\sigma\to0} P_c(u_{\sigma,1}) = \infty$, hence there exists $\sigma_0\in(0,1)$ such that $P_c(u_{\sigma_0,1}) = 0$. From Lemma 4.7 we get $E_c(u_{\sigma_0,1})\ge T_c$ and this implies $E_c(u_{\sigma_0,1}) - P_c(u_{\sigma_0,1}) \ge T_c$, that is $\frac{2}{N-1}A(u_{\sigma_0,1})\ge T_c$. From the last inequality we find

$$A(u) \ge \frac{N-1}{2}\frac{1}{\sigma_0}T_c > \frac{N-1}{2}T_c. \tag{4.24}$$

ii) For n sufficiently large (so that $P_c(u_n)<0$) we have $u_n\ne0$ and $\int_{\mathbf{R}^N}\left|\frac{\partial u_n}{\partial x_1}\right|^2 dx > 0$. As in the proof of part (i), using (4.23) we see that for each n sufficiently big there exists $\sigma_n\in(0,1)$ such that

$$P_c((u_n)_{\sigma_n,1}) = 0 \tag{4.25}$$

and we infer that $A(u_n) \geq \frac{N-1}{2}\frac{1}{\sigma_n}T_c$. We claim that

$$\limsup_{n\to\infty} \sigma_n < 1. \tag{4.26}$$

Notice that if (4.26) holds, we have $\liminf_{n\to\infty} A(u_n) \geq \frac{N-1}{2}\frac{1}{\limsup_{n\to\infty}\sigma_n}T_c > \frac{N-1}{2}T_c$ and Lemma 4.8 is proved.

To prove (4.26) we argue by contradition and assume that there is a subsequence $(\sigma_{n_k})_{k\geq 1}$ such that $\sigma_{n_k} \longrightarrow 1$ as $k \longrightarrow \infty$. Since $(E_{GL}(u_n))_{n\geq 1}$ is bounded, using Lemmas 4.1 and 4.5 we infer that $\left(\int_{\mathbf{R}^N}\left|\frac{\partial u_n}{\partial x_1}\right|^2 dx\right)_{n\geq 1}$, $\left(\int_{\mathbf{R}^N} V(|r_0-u_n|^2)\,dx\right)_{n\geq 1}$, $(A(u_n))_{n\geq 1}$, and $(Q(u_n))_{n\geq 1}$ are bounded. Consequently there is a subsequence $(n_{k_\ell})_{\ell\geq 1}$ and there are $\alpha_1, \alpha_2, \beta, \gamma \in \mathbf{R}$ such that

$$\int_{\mathbf{R}^N}\left|\frac{\partial u_{n_{k_\ell}}}{\partial x_1}\right|^2 dx \longrightarrow \alpha_1, \qquad \int_{\mathbf{R}^N} V(|r_0-u_{n_{k_\ell}}|^2)\,dx \longrightarrow \gamma$$

$$A(u_{n_{k_\ell}}) \longrightarrow \alpha_2, \qquad Q(u_{n_{k_\ell}}) \longrightarrow \beta \quad \text{as } \ell \longrightarrow \infty.$$

Writing (4.25) and (4.23) (with $(u_{n_{k_\ell}})_{\sigma_{n_{k_\ell}},1}$ instead of $(u_n)_{\sigma_n,1}$ and $w_{\sigma,1}$, respectively) then passing to the limit as $\ell \longrightarrow \infty$ and using the fact that $\sigma_{n_k} \longrightarrow 1$ we find $\alpha_1 + \frac{N-3}{N-1}\alpha_2 + c\beta + \gamma = 0$. On the other hand we have $\lim_{\ell\to\infty} P_c(u_{n_{k_\ell}}) = \mu < 0$ and this gives $\alpha_1 + \frac{N-3}{N-1}\alpha_2 + c\beta + \gamma = \mu < 0$. This contradiction proves that (4.26) holds and the proof of Lemma 4.8 is complete. □

5 The case $N \geq 4$

Throughout this section we assume that $N \geq 4$, $0 < c < v_s$ and the assumptions (A1) and (A2) are satisfied. Most of the results below do *not* hold for $c > v_s$. Some of them may not hold for $c = 0$ and some particular nonlinearities F.

Lemma 5.1 *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that $(E_c(u_n))_{n\geq 1}$ is bounded and $P_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Then $(E_{GL}(u_n))_{n\geq 1}$ is bounded.

Proof. We have $\frac{2}{N-1}A(u_n) = E_c(u_n) - P_c(u_n)$, hence $(A(u_n))_{n\geq 1}$ is bounded. It remains to prove that $\int_{\mathbf{R}^N}\left|\frac{\partial u_n}{\partial x_1}\right| + a^2\left(\varphi^2(|r_0-u_n|) - r_0^2\right)^2 dx$ is bounded. We argue by contradiction and we assume that there is a subsequence, still denoted $(u_n)_{n\geq 1}$, such that

$$\int_{\mathbf{R}^N}\left|\frac{\partial u_n}{\partial x_1}\right| + a^2\left(\varphi^2(|r_0-u_n|) - r_0^2\right)^2 dx \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty. \tag{5.1}$$

Fix $k_0 > 0$ such that $E_{c,min}(k_0) > 0$. Arguing as in the proof of Lemma 4.7, it is easy to see that there exists a sequence $(\sigma_n)_{n\geq 1}$ such that

$$E_{GL}((u_n)_{1,\sigma_n}) = \sigma_n^{N-3}A(u_n) + \sigma_n^{N-1}\int_{\mathbf{R}^N}\left|\frac{\partial u_n}{\partial x_1}\right| + a^2\left(\varphi^2(|r_0-u_n|) - r_0^2\right)^2 dx = k_0. \tag{5.2}$$

From (5.1) and (5.2) we have $\sigma_n \longrightarrow 0$ as $n \longrightarrow \infty$. Since $B_c(u_n) = -\frac{N-3}{N-1}A(u_n) + P_c(u_n)$, it is clear that $(B_c(u_n))_{n\geq 1}$ is bounded and we obtain

$$E_c((u_n)_{1,\sigma_n}) = \sigma_n^{N-3}A(u_n) + \sigma_n^{N-1}B_c(u_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

But this contradicts the fact that $E_{c,min}(k_0) > 0$ and Lemma 5.1 is proved. □

Lemma 5.2 *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence satisfying the following properties:*
a) There exist C_1, $C_2 > 0$ such that $C_1 \leq E_{GL}(u_n)$ and $A(u_n) \leq C_2$ for any $n \geq 1$.
b) $P_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.
Then $\liminf_{n\to\infty} E_c(u_n) \geq T_c$, where T_c is as in Lemma 4.7.

Note that in Lemma 5.2 the assumption $E_{GL}(u_n) \geq C_1 > 0$ is necessary. To see this, consider a sequence $(u_n)_{n\geq 1} \subset H^1(\mathbf{R}^N)$ such that $u_n \neq 0$ and $u_n \longrightarrow 0$ as $n \longrightarrow \infty$. It is clear that $P_c(u_n) \longrightarrow 0$ and $E_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. First we prove that

$$C_3 := \liminf_{n\to\infty} A(u_n) > 0. \tag{5.3}$$

To see this, fix $k_0 > 0$ such that $E_{c,min}(k_0) > 0$. Exactly as in the proof of Lemma 4.7, it is easy to see that for each n there exists a unique $\sigma_n > 0$ such that (5.2) holds. Since $k_0 = E_{GL}((u_n)_{1,\sigma_n}) \geq \min(\sigma_n^{N-3}, \sigma_n^{N-1}) E_{GL}((u_n)) \geq \min(\sigma_n^{N-3}, \sigma_n^{N-1}) C_1$, it follows that $(\sigma_n)_{n\geq 1}$ is bounded. On the other hand, we have $E_c((u_n)_{1,\sigma_n}) = \sigma_n^{N-3} A(u_n) + \sigma_n^{N-1} B_c(u_n) \geq E_{c,min}(k_0) > 0$, that is

$$\sigma_n^{N-3} A(u_n) + \sigma_n^{N-1} \left(P_c(u_n) - \frac{N-3}{N-1} A(u_n) \right) \geq E_{c,min}(k_0) > 0. \tag{5.4}$$

If there is a subsequence $(u_{n_k})_{k\geq 1}$ such that $A(u_{n_k}) \longrightarrow 0$, putting u_{n_k} in (5.4) and letting $k \longrightarrow \infty$ we would get $0 \geq E_{c,min}(k_0) > 0$, a contradiction. Thus (5.3) is proved.

We have $B_c(u_n) = P_c(u_n) - \frac{N-3}{N-1} A(u_n)$ and using (b) and (5.3) we obtain

$$\limsup_{n\to\infty} B_c(u_n) \leq -\frac{N-3}{N-1} C_3 < 0. \tag{5.5}$$

Clearly, for any $\sigma > 0$ we have

$$P_c((u_n)_{1,\sigma}) = \sigma^{N-3} \frac{N-3}{N-1} A(u_n) + \sigma^{N-1} B_c(u_n) = \sigma^{N-3} \left(\frac{N-3}{N-1} A(u_n) + \sigma^2 B_c(u_n) \right).$$

For n sufficiently big (so that $B_c(u_n) < 0$), let $\tilde{\sigma}_n = \left(\frac{\frac{N-3}{N-1} A(u_n)}{-B_c(u_n)} \right)^{\frac{1}{2}}$. Then $P_c((u_n)_{1,\tilde{\sigma}_n}) = 0$, or equivalently $(u_n)_{1,\tilde{\sigma}_n} \in \mathcal{C}$. From Lemma 4.7 we obtain $E_c((u_n)_{1,\tilde{\sigma}_n}) = \tilde{\sigma}_n^{N-3} \frac{N-3}{N-1} A(u_n) + \tilde{\sigma}_n^{N-1} B_c(u_n) \geq T_c$, that is

$$E_c(u_n) + \left(\tilde{\sigma}_n^{N-3} - 1\right) A(u_n) + \left(\tilde{\sigma}_n^{N-1} - 1\right) \left(P_c(u_n) - \frac{N-3}{N-1} A(u_n) \right) \geq T_c. \tag{5.6}$$

Clearly, $\tilde{\sigma}_n$ can be written as $\tilde{\sigma}_n = \left(\frac{P_c(u_n)}{-B_c(u_n)} + 1 \right)^{\frac{1}{2}}$ and using (b) and (5.5) it follows that $\lim_{n\to\infty} \tilde{\sigma}_n = 1$. Then passing to the limit as $n \longrightarrow \infty$ in (5.6) and using the fact that $(A(u_n))_{n\geq 1}$ and $(P_c(u_n))_{n\geq 1}$ are bounded, we obtain $\liminf_{n\to\infty} E_c(u_n) \geq T_c$. □

We can now state the main result of this section.

Theorem 5.3 *Let $(u_n)_{n\geq 1} \subset \mathcal{X} \setminus \{0\}$ be a sequence such that*

$$P_c(u_n) \longrightarrow 0 \qquad \text{and} \qquad E_c(u_n) \longrightarrow T_c \qquad \text{as } n \longrightarrow \infty.$$

There exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^N$ and $u \in \mathcal{C}$ such that

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad \text{and} \quad \varphi^2(|r_0 - u_{n_k}(\cdot + x_k)|) - r_0^2 \longrightarrow \varphi^2(|r_0 - u|) - r_0^2 \quad \text{in } L^2(\mathbf{R}^N).$$

Moreover, we have $E_c(u) = T_c$, that is u minimizes E_c in $\mathcal{C}$.

Proof. From Lemma 5.1 we know that $E_{GL}(u_n)$ is bounded. We have $\frac{2}{N-1}A(u_n) = E_c(u_n) - P_c(u_n) \longrightarrow T_c$ as $n \longrightarrow \infty$. Therefore

$$(5.7) \qquad \lim_{n\to\infty} A(u_n) = \frac{N-1}{2}T_c \text{ and } \liminf_{n\to\infty} E_{GL}(u_n) \geq \lim_{n\to\infty} A(u_n) = \frac{N-1}{2}T_c.$$

Passing to a subsequence if necessary, we may assume that there exists $\alpha_0 \geq \frac{N-1}{2}T_c$ such that

$$(5.8) \qquad E_{GL}(u_n) \longrightarrow \alpha_0 \qquad \text{as } n \longrightarrow \infty.$$

We will use the concentration-compactness principle ([30]). We denote by $q_n(t)$ the concentration function of $E_{GL}(u_n)$, that is

$$(5.9) \qquad q_n(t) = \sup_{y\in\mathbf{R}^N} \int_{B(y,t)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx.$$

As in [30], it follows that there exists a subsequence of $((u_n, q_n))_{n\geq 1}$, still denoted $((u_n, q_n))_{n\geq 1}$, there exists a nondecreasing function $q : [0, \infty) \longrightarrow \mathbf{R}$ and there is $\alpha \in [0, \alpha_0]$ such that

$$(5.10) \qquad q_n(t) \longrightarrow q(t) \text{ a.e on } [0,\infty) \text{ as } n \longrightarrow \infty \qquad \text{and} \qquad q(t) \longrightarrow \alpha \text{ as } t \longrightarrow \infty.$$

We claim that

$$(5.11) \qquad \text{there is a nondecreasing sequnce } t_n \longrightarrow \infty \text{ such that } \lim_{n\to\infty} q_n(t_n) = \alpha.$$

To prove the claim, fix an increasing sequence $x_k \longrightarrow \infty$ such that $q_n(x_k) \longrightarrow q(x_k)$ as $n \longrightarrow \infty$ for any k. Then there exists $n_k \in \mathbf{N}$ such that $|q_n(x_k) - q(x_k)| < \frac{1}{k}$ for any $n \geq n_k$; clearly, we may assume that $n_k < n_{k+1}$ for all k. If $n_k \leq n < n_{k+1}$, put $t_n = x_k$. Then for $n_k \leq n < n_{k+1}$ we have

$$|q_n(t_n) - \alpha| = |q_n(x_k) - \alpha| \leq |q_n(x_k) - q(x_k)| + |q(x_k) - \alpha| \leq \frac{1}{k} + |q(x_k) - \alpha| \longrightarrow 0$$

as $k \longrightarrow \infty$ and (5.11) is proved.

Next we claim that

$$(5.12) \qquad q_n(t_n) - q_n\left(\frac{t_n}{2}\right) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

To see this, fix $\varepsilon > 0$. Take $y > 0$ such that $q(y) > \alpha - \frac{\varepsilon}{4}$ and $q_n(y) \longrightarrow q(y)$ as $n \longrightarrow \infty$. There is some $\tilde{n} \geq 1$ such that $q_n(y) > \alpha - \frac{\varepsilon}{2}$ for $n \geq \tilde{n}$. Then we can find $n_* \geq \tilde{n}$ such that $t_n > 2y$ for $n \geq n_*$, and consequently we have $q_n(\frac{t_n}{2}) \geq q_n(y) > \alpha - \frac{\varepsilon}{2}$. Therefore $\limsup_{n\to\infty} \left(q_n(t_n) - q_n(\frac{t_n}{2})\right) = \lim_{n\to\infty} q_n(t_n) - \liminf_{n\to\infty} q_n(\frac{t_n}{2}) < \varepsilon$. Since ε was arbitrary, (5.12) follows.

Our aim is to show that $\alpha = \alpha_0$ in (5.10). It follows from the next lemma that $\alpha > 0$.

Lemma 5.4 *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence satisfying*

a) $M_1 \leq E_{GL}(u_n) \leq M_2$ for some positive constants M_1, M_2.

b) $\lim_{n\to\infty} P_c(u_n) = 0$.

There exists $k > 0$ such that $\sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \geq k$ for all sufficiently large n.

Proof. We argue by contradiction and we suppose that the conclusion is false. Then there exists a subsequence (still denoted $(u_n)_{n\geq 1}$) such that

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \, dx = 0. \tag{5.13}$$

We will prove that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \left| V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx = 0. \tag{5.14}$$

Fix $\varepsilon > 0$. Assumptions (A1) and (A2) imply that there exists $\delta(\varepsilon) > 0$ such that

$$\left|V(|r_0 - z|^2) - a^2 \left(\varphi^2(|r_0 - z|) - r_0^2\right)^2 \right| \leq \varepsilon a^2 \left(\varphi^2(|r_0 - z|) - r_0^2\right)^2 \tag{5.15}$$

for any $z \in \mathbf{C}$ satisfying $\big| |r_0 - z| - r_0 \big| \leq \delta(\varepsilon)$ (see (4.2)). Therefore

$$\begin{aligned} &\int_{\{|\,|r_0-u_n|-r_0|\leq\delta(\varepsilon)\}} \left| V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx \\ &\leq \varepsilon a^2 \int_{\{|\,|r_0-u_n|-r_0|\leq\delta(\varepsilon)\}} \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \, dx \leq \varepsilon M_2. \end{aligned} \tag{5.16}$$

Assumption (A2) implies that there exists $C(\varepsilon) > 0$ such that

$$\left| V(|r_0 - z|^2) - a^2 \left(\varphi^2(|r_0 - z|) - r_0^2\right)^2 \right| \leq C(\varepsilon) \big| |r_0 - z| - r_0 \big|^{2p_0+2} \tag{5.17}$$

for any $z \in \mathbf{C}$ verifying $\big| |r_0 - z| - r_0 \big| \geq \delta(\varepsilon)$.

Let $w_n = \big| |r_0 - u_n| - r_0 \big|$. It is clear that $|w_n| \leq |u_n|$. Using the inequality $|\nabla |v|\,| \leq |\nabla v|$ a.e. for $v \in H^1_{loc}(\mathbf{R}^N)$, we infer that $w_n \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ and

$$\int_{\mathbf{R}^N} |\nabla w_n|^2 \, dx \leq M_2 \qquad \text{for any } n. \tag{5.18}$$

Using (5.17), Hölder's inequality, the Sobolev embedding and (5.18) we find

$$\begin{aligned} &\int_{\{|r_0-u_n|-r_0|>\delta(\varepsilon)\}} \left| V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx \\ &\leq C(\varepsilon) \int_{\{w_n>\delta(\varepsilon)\}} |w_n|^{2p_0+2} \, dx \\ &\leq C(\varepsilon) \left(\int_{\{w_n>\delta(\varepsilon)\}} |w_n|^{2^*} \, dx \right)^{\frac{2p_0+2}{2^*}} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\})\right)^{1-\frac{2p_0+2}{2^*}} \\ &\leq C(\varepsilon) C_S^{2p_0+2} ||\nabla w_n||_{L^2(\mathbf{R}^N)}^{2p_0+2} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\})\right)^{1-\frac{2p_0+2}{2^*}} \\ &\leq C(\varepsilon) C_S^{2p_0+2} M_2^{p_0+1} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\})\right)^{1-\frac{2p_0+2}{2^*}}. \end{aligned} \tag{5.19}$$

We claim that for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathcal{L}^N(\{w_n > \delta(\varepsilon)\}) = 0. \tag{5.20}$$

To prove the claim, we argue by contradiction and assume that there exist $\varepsilon_0 > 0$, a subsequence $(w_{n_k})_k \geq 1$ and $\gamma > 0$ such that $\mathcal{L}^N\left(\{w_{n_k} > \delta(\varepsilon_0)\}\right) \geq \gamma > 0$ for any $k \geq 1$. Since $||\nabla w_n||_{L^2(\mathbf{R}^N)}$ is bounded, using Lieb's lemma (see Lemma 6 p. 447 in [29] or Lemma 2.2 p. 101 in [10]), we infer that there exists $\beta > 0$ and $y_k \in \mathbf{R}^N$ such that $\mathcal{L}^N\left(\{w_{n_k} > \frac{\delta(\varepsilon_0)}{2}\} \cap B(y_k, 1)\right) \geq \beta$. Let η be as in (3.30). Then $w_{n_k}(x) \geq \frac{\delta(\varepsilon_0)}{2}$ implies $\left(\varphi^2(|r_0 - u_{n_k}(x)|) - r_0^2\right)^2 \geq \eta\left(\frac{\delta(\varepsilon_0)}{2}\right) > 0$. Therefore

$$\int_{B(y_k,1)} \left(\varphi^2(|r_0 - u_{n_k}(x)|) - r_0^2\right)^2 dx \geq \eta\left(\frac{\delta(\varepsilon_0)}{2}\right)\beta > 0$$

for any $k \geq 1$, and this clearly contradicts (5.13). Thus we have proved that (5.20) holds.

From (5.16), (5.19) and (5.20) it follows that

$$\int_{\mathbf{R}^N} \left| V(|r_0 - u_n|^2) - a^2\left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx \leq 2\varepsilon M_2$$

for all sufficiently large n. Thus (5.14) is proved.

From Lemma 5.2 we know that $\liminf_{n\to\infty} E_c(u_n) \geq T_c$. Combined with (b), this implies $\liminf_{n\to\infty} \frac{2}{N-1} A(u_n) \geq T_c$. Let $\sigma_0 = \sqrt{\frac{2(N-1)}{N-3}}$ and let $\tilde{u}_n = (u_n)_{1,\sigma_0}$. It is obvious that

$$\liminf_{n\to\infty} A(\tilde{u}_n) = \sigma_0^{N-3} \liminf_{n\to\infty} A(u_n) \geq \frac{N-1}{2}\sigma_0^{N-3} T_c. \tag{5.21}$$

Using assumption (a), (5.13) and (5.14) it is easy to see that

(5.22) there exist $\tilde{M}_1$, $\tilde{M}_2 > 0$ such that $\tilde{M}_1 \leq E_{GL}(\tilde{u}_n) \leq \tilde{M}_2$ for any n,

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |\nabla \tilde{u}_n|^2 + a^2\left(\varphi^2(|r_0 - \tilde{u}_n|) - r_0^2\right)^2 dx = 0 \quad \text{and} \tag{5.23}$$

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \left| V(|r_0 - \tilde{u}_n|^2) - a^2\left(\varphi^2(|r_0 - \tilde{u}_n|) - r_0^2\right)^2 \right| dx = 0. \tag{5.24}$$

It is clear that $P_c(u_n) = \frac{N-3}{N-1}\sigma_0^{3-N} A(\tilde{u}_n) + \sigma_0^{1-N} B_c(\tilde{u}_n)$ and then assumption (b) implies

$$\lim_{n\to\infty}\left(\frac{N-3}{N-1}\sigma_0^2 A(\tilde{u}_n) + B_c(\tilde{u}_n)\right) = \lim_{n\to\infty}\left(A(\tilde{u}_n) + E_c(\tilde{u}_n)\right) = 0. \tag{5.25}$$

Using (5.22), (5.23) and Lemma 3.2 we infer that there exists a sequence $h_n \longrightarrow 0$ and for each n there exists a minimizer v_n of $G^{\tilde{u}_n}_{h_n,\mathbf{R}^N}$ in $H^1_{\tilde{u}_n}(\mathbf{R}^N)$ such that $\delta_n := ||\,|v_n - r_0| - r_0||_{L^\infty(\mathbf{R}^N)} \longrightarrow 0$ as $n \longrightarrow \infty$. Then using Lemma 4.2 and the fact that $|c| < v_s = 2ar_0$ we obtain

$$E_{GL}(v_n) + cQ(v_n) \geq 0 \qquad \text{for all sufficiently large } n. \tag{5.26}$$

From (5.22) and (3.4) we obtain

$$|Q(\tilde{u}_n) - Q(v_n)| \leq \left(h_n^2 + h_n^{\frac{4}{N}} \tilde{M}_2^{\frac{2}{N}}\right)^{\frac{1}{2}} \tilde{M}_2 \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{5.27}$$

Since $E_{GL}(v_n) \leq E_{GL}(\tilde{u}_n)$, it is clear that

$$\begin{aligned} E_c(\tilde{u}_n) &= E_{GL}(\tilde{u}_n) + cQ(\tilde{u}_n) + \int_{\mathbf{R}^N} V(|r_0 - \tilde{u}_n|^2) - a^2\left(\varphi^2(|r_0 - \tilde{u}_n|) - r_0^2\right)^2 dx \\ &\geq E_{GL}(v_n) + cQ(v_n) + c(Q(\tilde{u}_n) - Q(v_n)) \\ &\qquad - \int_{\mathbf{R}^N} \left| V(|r_0 - \tilde{u}_n|^2) - a^2\left(\varphi^2(|r_0 - \tilde{u}_n|) - r_0^2\right)^2 \right| dx \end{aligned}$$

Using the last inequality and (5.24), (5.26), (5.27) we infer that $\liminf_{n\to\infty} E_c(\tilde{u}_n) \geq 0$. Combined with (5.25), this gives $\limsup_{n\to\infty} A(\tilde{u}_n) \leq 0$, which clearly contradicts (5.21). This completes the proof of Lemma 5.4. □

Next we prove that we cannot have $\alpha \in (0, \alpha_0)$. To do this we argue again by contradiction and we assume that $0 < \alpha < \alpha_0$. Let t_n be as in (5.11) and let $R_n = \frac{t_n}{2}$. For each $n \geq 1$, fix $y_n \in \mathbf{R}^N$ such that $E_{GL}^{B(y_n,R_n)}(u_n) \geq q_n(R_n) - \frac{1}{n}$. Using (5.12), we have

$$(5.28) \qquad \begin{aligned} \varepsilon_n &:= \int_{B(y_n,2R_n)\setminus B(y_n,R_n)} |\nabla u_n|^2 + a^2\left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \\ &\leq q_n(2R_n) - \left(q_n(R_n) - \tfrac{1}{n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

After a translation, we may assume that $y_n = 0$. Using Lemma 3.3 with $A = 2$, $R = R_n$, $\varepsilon = \varepsilon_n$, we infer that for all n sufficiently large there exist two functions $u_{n,1}$, $u_{n,2}$ having the properties (i)-(vi) in Lemma 3.3.

From Lemma 3.3 (iii) and (iv) we get $|E_{GL}(u_n) - E_{GL}(u_{n,1}) - E_{GL}(u_{n,2})| \leq C\varepsilon_n$, while Lemma 3.3 (i) and (ii) implies $E_{GL}(u_{n,1}) \geq E_{GL}^{B(0,R_n)}(u_n) > q_n(R_n) - \frac{1}{n}$, respectively $E_{GL}(u_{n,2}) \geq E_{GL}^{\mathbf{R}^N\setminus B(0,2R_n)}(u_n) \geq E_{GL}(u_n) - q_n(2R_n)$. Taking into account (5.11), (5.12) and (5.28), we infer that

$$(5.29) \qquad E_{GL}(u_{n,1}) \longrightarrow \alpha \qquad \text{and} \qquad E_{GL}(u_{n,2}) \longrightarrow \alpha_0 - \alpha \qquad \text{as } n \longrightarrow \infty.$$

By (5.28) and Lemma 3.3 (iii)−(vi) we obtain

$$(5.30) \qquad |A(u_n) - A(u_{n,1}) - A(u_{n,2})| \longrightarrow 0,$$

$$(5.31) \qquad |E_c(u_n) - E_c(u_{n,1}) - E_c(u_{n,2})| \longrightarrow 0, \qquad \text{and}$$

$$(5.32) \qquad |P_c(u_n) - P_c(u_{n,1}) - P_c(u_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

From (5.32) and the fact that $P_c(u_n) \longrightarrow 0$ we infer that $P_c(u_{n,1}) + P_c(u_{n,2}) \longrightarrow 0$ as $n \longrightarrow \infty$. Moreover, Lemmas 4.1 and 4.5 imply that the sequences $(P_c(u_{n,i}))_{n\geq 1}$ and $(E_c(u_{n,i}))_{n\geq 1}$ are bounded, $i = 1, 2$. Passing again to a subsequence (still denoted $(u_n)_{n\geq 1}$), we may assume that $\lim_{n\to\infty} P_c(u_{n,1}) = p_1$ and $\lim_{n\to\infty} P_c(u_{n,2}) = p_2$ where p_1, $p_2 \in \mathbf{R}$ and $p_1 + p_2 = 0$. There are only two possibilities: either $p_1 = p_2 = 0$, or one element of $\{p_1, p_2\}$ is negative.

If $p_1 = p_2 = 0$, then (5.29) and Lemma 5.2 imply that $\liminf_{n\to\infty} E_c(u_{n,i}) \geq T_c$, $i = 1, 2$. Using (5.31), we obtain $\liminf_{n\to\infty} E_c(u_n) \geq 2T_c$ and this clearly contradicts the assumption $E_c(u_n) \longrightarrow T_c$ in Theorem 5.3.

If $p_i < 0$, it follows from (5.29) and Lemma 4.8 (ii) that $\liminf_{n\to\infty} A(u_{n,i}) > \frac{N-1}{2}T_c$. Using (5.30) and the fact that $A \geq 0$, we obtain $\liminf_{n\to\infty} A(u_n) > \frac{N-1}{2}T_c$, which is in contradiction with (5.7).

We conclude that we cannot have $\alpha \in (0, \alpha_0)$.

So far we have proved that $\lim_{t \to \infty} q(t) = \alpha_0$. Proceeding as in [30], it follows that for each $n \geq 1$ there exists $x_n \in \mathbf{R}^N$ such that for any $\varepsilon > 0$ there is $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ satisfying

$$E_{GL}^{B(x_n, R_\varepsilon)}(u_n) > \alpha_0 - \varepsilon \qquad \text{for any } n \geq n_\varepsilon. \tag{5.33}$$

Let $\tilde{u}_n = u_n(\cdot + x_n)$, so that $\tilde{u}_n$ satisfies (5.33) with $B(0, R_\varepsilon)$ instead of $B(x_n, R_\varepsilon)$. Let $\chi \in C_c^\infty(\mathbf{C}, \mathbf{R})$ be as in Lemma 2.2 and denote $\tilde{u}_{n,1} = \chi(\tilde{u}_n)\tilde{u}_n$, $\tilde{u}_{n,1} = (1 - \chi(\tilde{u}_n))\tilde{u}_n$. Since $E_{GL}(\tilde{u}_n) = E_{GL}(u_n)$ is bounded, we infer from Lemma 2.2 that $(\tilde{u}_{n,1})_{n \geq 1}$ is bounded in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, $(\tilde{u}_{n,2})_{n \geq 1}$ is bounded in $H^1(\mathbf{R}^N)$ and $(E_{GL}(\tilde{u}_{n,i}))_{n \geq 1}$ is bounded, $i = 1, 2$.

Using Lemma 2.1 we may write $r_0 - \tilde{u}_{n,1} = \rho_n e^{i\theta_n}$, where $\frac{1}{2} r_0 \leq \rho_n \leq \frac{3}{2} r_0$ and $\theta_n \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. From (2.4) and (2.7) we find that $(\rho_n - r_0)_{n \geq 1}$ is bounded in $H^1(\mathbf{R}^N)$ and $(\theta_n)_{n \geq 1}$ is bounded in $\mathcal{D}^{1,2}(\mathbf{R}^N)$.

We infer that there exists a subsequence $(n_k)_{k \geq 1}$ and there are functions $u_1 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $u_2 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $\rho \in r_0 + H^1(\mathbf{R}^N)$ such that

$$\tilde{u}_{n_k,1} \rightharpoonup u_1 \qquad \text{and} \qquad \theta_{n_k} \rightharpoonup \theta \qquad \text{weakly in } \mathcal{D}^{1,2}(\mathbf{R}^N),$$

$$\tilde{u}_{n_k,2} \rightharpoonup u_2 \qquad \text{and} \qquad \rho_{n_k} - r_0 \rightharpoonup \rho - r_0 \qquad \text{weakly in } H^1(\mathbf{R}^N),$$

$$\tilde{u}_{n_k,1} \longrightarrow u_1, \qquad \tilde{u}_{n_k,2} \longrightarrow u_2, \qquad \theta_{n_k} \longrightarrow \theta, \qquad \rho_{n_k} - r_0 \longrightarrow \rho - r_0$$

strongly in $L^p(K)$, $1 \leq p < 2^*$ for any compact set $K \subset \mathbf{R}^N$ and almost everywhere on $\mathbf{R}^N$. Since $\tilde{u}_{n_k,1} = r_0 - \rho_{n_k} e^{i\theta_{n_k}} \longrightarrow r_0 - \rho e^{i\theta}$ a.e., we have $r_0 - u_1 = \rho e^{i\theta}$ a.e. on $\mathbf{R}^N$.

Denoting $u = u_1 + u_2$, we see that $\tilde{u}_{n_k} \rightharpoonup u$ weakly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, $\tilde{u}_{n_k} \longrightarrow u$ a.e. on $\mathbf{R}^N$ and strongly in $L^p(K)$, $1 \leq p < 2^*$ for any compact set $K \subset \mathbf{R}^N$.

Since $E_{GL}(\tilde{u}_n)$ is bounded, it is clear that $\left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)_{k \geq 1}$ is bounded in $L^2(\mathbf{R}^N)$ and converges a.e. on $\mathbf{R}^N$ to $\varphi^2(|r_0 - u|) - r_0^2$. From Lemma 4.8 p. 11 in [26] it follows that

$$\left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right) \rightharpoonup \varphi^2(|r_0 - u|) - r_0^2 \quad \text{weakly in } L^2(\mathbf{R}^N). \tag{5.34}$$

The weak convergence $\tilde{u}_{n_k} \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ implies

$$\int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_j} \right|^2 dx \leq \liminf_{k \to \infty} \int_{\mathbf{R}^N} \left| \frac{\partial \tilde{u}_{n_k}}{\partial x_j} \right|^2 dx < \infty \qquad \text{for } j = 1, \ldots, N. \tag{5.35}$$

Using the a.e. convergence and Fatou's lemma we obtain

$$\int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 dx \leq \liminf_{k \to \infty} \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2 dx \tag{5.36}$$

From (5.35) and(5.36) it follows that $u \in \mathcal{X}$ and $E_{GL}(u) \leq \liminf_{k \to \infty} E_{GL}(\tilde{u}_{n_k})$.

We will prove that

$$\lim_{k \to \infty} \int_{\mathbf{R}^N} V(|r_0 - \tilde{u}_{n_k}|^2) \, dx = \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx \tag{5.37}$$

and

$$\lim_{k \to \infty} Q(\tilde{u}_{n_k}) = Q(u). \tag{5.38}$$

Fix $\varepsilon > 0$. Let R_ε be as in (5.33). Since $E_{GL}(\tilde{u}_{n_k}) \longrightarrow \alpha_0$ as $k \longrightarrow \infty$, it follows from (5.33) that there exists $k_\varepsilon \geq 1$ such that

$$E_{GL}^{\mathbf{R}^N \setminus B(0, R_\varepsilon)}(\tilde{u}_{n_k}) < 2\varepsilon \qquad \text{for any } k \geq k_\varepsilon. \tag{5.39}$$

As in (5.35)−(5.36), the weak convergence $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^N \setminus B(0, R_\varepsilon))$ implies

$$\int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |\nabla u|^2 \, dx \le \liminf_{k\to\infty} \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |\nabla \tilde{u}_{n_k}|^2 \, dx,$$

while the fact that $\tilde{u}_{n_k} \longrightarrow u$ a.e. on $\mathbf{R}^N$ and Fatou's lemma imply

$$\int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 \, dx \le \liminf_{k\to\infty} \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2 \, dx.$$

Therefore

$$E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(u) \le \liminf_{k\to\infty} E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(\tilde{u}_{n_k}) \le 2\varepsilon. \tag{5.40}$$

Let $v \in \mathcal{X}$ be a function satisfying $E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(v) \le 2\varepsilon$. As in the introduction, we write $V(s) = V(\varphi^2(\sqrt{s})) + W(s)$. Using (1.5) we find

$$\begin{aligned} \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |V(\varphi^2(|r_0 - v|))| \, dx &\le C_1 \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|r_0 - v|) - r_0^2\right)^2 \, dx \\ &\le \tfrac{C_1}{a^2} E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(v) \le \tfrac{2C_1}{a^2}\varepsilon. \end{aligned} \tag{5.41}$$

It is clear that $W(|r_0 - v(x)|^2) = 0$ if $|r_0 - v(x)| \le 2r_0$. On the other hand, $|r_0 - v(x)| > 2r_0$ implies $\left(\varphi^2(|r_0 - v(x)|) - r_0^2\right)^2 > 9r_0^4$, consequently

$$9r_0^4 \mathcal{L}^N(\{x \in \mathbf{R}^N \setminus B(0,R_\varepsilon) \mid |r_0 - v(x)| > 2r_0\}) \le \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|r_0 - v|) - r_0^2\right)^2 \, dx \le \frac{2\varepsilon}{a^2}.$$

Using (1.7), Hölder's inequality, the above estimate and the Sobolev embedding we find

$$\begin{aligned} \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |W(|r_0 - v|^2)| \, dx &\le C \int_{(\mathbf{R}^N \setminus B(0,R_\varepsilon)) \cap \{|r_0 - v| > 2r_0\}} |v|^{2p_0+2} \, dx \\ &\le C \left(\int_{\mathbf{R}^N} |v|^{2^*} \, dx \right)^{\frac{2p_0+2}{2^*}} \left(\mathcal{L}^N(\{x \in \mathbf{R}^N \setminus B(0,R_\varepsilon) \mid |r_0 - v(x)| > 2r_0\}) \right)^{1-\frac{2p_0+2}{2^*}} \\ &\le C' ||\nabla v||_{L^2(\mathbf{R}^N)}^{2p_0+2} \varepsilon^{1-\frac{2p_0+2}{2^*}} \le C' \left(E_{GL}(v)\right)^{p_0+1} \varepsilon^{1-\frac{2p_0+2}{2^*}}. \end{aligned} \tag{5.42}$$

It is obvious that u and $\tilde{u}_{n_k}$ (with $k \ge k_\varepsilon$) satisfy (5.41) and (5.42). If $M > 0$ is such that $E_{GL}(u_n) \le M$ for any n, from (5.41) and (5.42) we infer that

$$\begin{aligned} &\int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |V(|r_0 - \tilde{u}_{n_k}|^2) - V(|r_0 - u|^2)| \, dx \\ &\le \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |V(|r_0 - \tilde{u}_{n_k}|^2)| + |V(|r_0 - u|^2)| \, dx \le C\varepsilon + CM^{p_0+1}\varepsilon^{1-\frac{2p_0+2}{2^*}}. \end{aligned} \tag{5.43}$$

Since $z \longmapsto V(|r_0 - z|^2)$ is C^1, $|V(|r_0 - z|^2)| \le C(1 + |z|^{2p_0+2})$ and $\tilde{u}_{n_k} \longrightarrow u$ in $L^{2p_0+2}(B(0,R_\varepsilon))$ and almost everywhere, it follows that $V(|r_0 - \tilde{u}_{n_k}|^2) \longrightarrow V(|r_0 - u|^2)$ in $L^1(B(0,R_\varepsilon))$ (see, e.g., Theorem A2 p. 133 in [36]). Hence

$$\int_{B(0,R_\varepsilon)} |V(|r_0 - \tilde{u}_{n_k}|^2) - V(|r_0 - u|^2)| \, dx \le \varepsilon \qquad \text{if } k \text{ is sufficiently large.} \tag{5.44}$$

Since $\varepsilon > 0$ is arbitrary, (5.37) follows from (5.43) and (5.44).

From (2.6) we obtain $||(1-\chi^2(u_n))u_n||_{L^2(\mathbf{R}^N)} \le C||\nabla u_n||_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}} \le C\left(E_{GL}(u_n)\right)^{\frac{2^*}{4}}$. Using the Cauchy-Schwarz inequality and (5.39) we get

$$\begin{aligned}
(5.45)\qquad & \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \Big|(1-\chi^2(\tilde{u}_{n_k})\langle i\frac{\partial \tilde{u}_{n_k}}{\partial x_1}, \tilde{u}_{n_k}\rangle\Big| dx \\
& \le ||(1-\chi^2(u_n))u_n||_{L^2(\mathbf{R}^N)} ||\tfrac{\partial \tilde{u}_{n_k}}{\partial x_1}||_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \le CM^{\frac{2^*}{4}}\sqrt{\varepsilon} \quad \text{ for any } k \ge k_\varepsilon.
\end{aligned}$$

From (2.7) we infer that

$$||\rho_n^2 - r_0^2||_{L^2(\mathbf{R}^N)} \le C\left(E_{GL}(u_n) + ||\nabla u_n||^{2^*}_{L^2(\mathbf{R}^N)}\right)^{\frac12} \le C\left(M + M^{\frac{2^*}{2}}\right)^{\frac12}.$$

Using (2.4) and (2.5) we obtain $\left|\frac{\partial \theta_n}{\partial x_1}\right| \le \frac{2}{r_0}\left|\frac{\partial(\chi(\tilde{u}_n)\tilde{u}_n)}{\partial x_1}\right| \le C\left|\frac{\partial \tilde{u}_n}{\partial x_1}\right|$ a.e. on $\mathbf{R}^N$ and then (5.39) implies $||\frac{\partial \theta_{n_k}}{\partial x_1}||_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \le C\sqrt{\varepsilon}$ for any $k \ge k_\varepsilon$. Using again the Cauchy-Schwarz inequality we find

$$\begin{aligned}
(5.46)\qquad & \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \Big|\left(\rho_{n_k}^2 - r_0^2\right)\frac{\partial \theta_{n_k}}{\partial x_1}\Big| dx \le ||\rho_{n_k}^2 - r_0^2||_{L^2(\mathbf{R}^N)} \Big|\Big|\frac{\partial \theta_{n_k}}{\partial x_1}\Big|\Big|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \\
& \le C\left(M + M^{\frac{2^*}{2}}\right)^{\frac12}\sqrt{\varepsilon} \quad \text{ for any } k \ge k_\varepsilon.
\end{aligned}$$

It is obvious that the estimates (5.45) and (5.46) also hold with u instead of $\tilde{u}_{n_k}$.

Using the fact that $\tilde{u}_{n_k} \longrightarrow u$ and $\rho_{n_k} - r_0 \longrightarrow \rho - r_0$ in $L^2(B(0,R_\varepsilon))$ and a.e. and the dominated convergence theorem we infer that

$$(1-\chi^2(\tilde{u}_{n_k}))\tilde{u}_{n_k} \longrightarrow (1-\chi^2(u))u \qquad \text{and } \rho_{n_k}^2 - r_0^2 \longrightarrow \rho^2 - r_0^2 \qquad \text{in } L^2(B(0,R_\varepsilon)).$$

This information and the fact that $\frac{\partial \tilde{u}_{n_k}}{\partial x_1} \rightharpoonup \frac{\partial u}{\partial x_1}$ and $\frac{\partial \theta_{n_k}}{\partial x_1} \rightharpoonup \frac{\partial \theta}{\partial x_1}$ weakly in $L^2(B(0,R_\varepsilon))$ imply

$$(5.47)\qquad \int_{B(0,R_\varepsilon)} \langle i\frac{\partial \tilde{u}_{n_k}}{\partial x_1}, (1-\chi^2(\tilde{u}_{n_k}))\tilde{u}_{n_k}\rangle\, dx \longrightarrow \int_{B(0,R_\varepsilon)} \langle i\frac{\partial u}{\partial x_1}, (1-\chi^2(u))u\rangle\, dx \quad \text{ and}$$

$$(5.48)\qquad \int_{B(0,R_\varepsilon)} \left(\rho_{n_k}^2 - r_0^2\right)\frac{\partial \theta_{n_k}}{\partial x_1}\, dx \longrightarrow \int_{B(0,R_\varepsilon)} \left(\rho^2 - r_0^2\right)\frac{\partial \theta}{\partial x_1}\, dx.$$

Using (5.45)$-$(5.48) and the representation formula (2.12) we infer that there is some $k_1(\varepsilon) \ge k_\varepsilon$ such that for any $k \ge k_1(\varepsilon)$ we have

$$|Q(\tilde{u}_{n_k}) - Q(u)| \le C\left(M^{\frac12} + M^{\frac{2^*}{4}}\right)\sqrt{\varepsilon},$$

where C does not depend on $k \ge k_1(\varepsilon)$ and ε. Since $\varepsilon > 0$ is arbitrary, (5.38) is proved.

It is obvious that

$$\begin{aligned}
& -cQ(\tilde{u}_{n_k}) - \int_{\mathbf{R}^N} V(|r_0 - \tilde{u}_{n_k}|^2)\, dx \\
& = \frac{N-3}{N-1}A(\tilde{u}_{n_k}) + \int_{\mathbf{R}^N}\Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\Big|^2 dx - P_c(\tilde{u}_{n_k}) \ge \frac{N-3}{N-1}A(\tilde{u}_{n_k}) - P_c(\tilde{u}_{n_k}).
\end{aligned}$$

Passing to the limit as $k \longrightarrow \infty$ in this inequality and using (5.37), (5.38) and the fact that $A(u_n) \longrightarrow \frac{N-1}{2}T_c$, $P_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ we find

$$-cQ(u) - \int_{\mathbf{R}^N} V(|r_0 - u|^2)\,dx \geq \frac{N-3}{2}T_c > 0. \tag{5.49}$$

In particular, (5.49) implies that $u \neq 0$.

From (5.35) we get

$$A(u) \leq \liminf_{k\to\infty} A(\tilde{u}_{n_k}) = \frac{N-1}{2}T_c. \tag{5.50}$$

Using (5.35), (5.37) and (5.38) we find

$$P_c(u) \leq \liminf_{k\to\infty} P_c(\tilde{u}_{n_k}) = 0. \tag{5.51}$$

If $P_c(u) < 0$, from Lemma 4.8 (i) we get $A(u) > \frac{N-1}{2}T_c$, contradicting (5.50). Thus necessarily $P_c(u) = 0$, that is $u \in \mathcal{C}$. Since $A(v) \geq \frac{N-1}{2}T_c$ for any $v \in \mathcal{C}$, we infer from (5.50) that $A(u) = \frac{N-1}{2}T_c$, therefore $E_c(u) = T_c$ and u is a minimizer of E_c in $\mathcal{C}$.

It follows from the above that

$$A(u) = \frac{N-1}{2}T_c = \lim_{k\to\infty} A(\tilde{u}_{n_k}). \tag{5.52}$$

Since $P_c(u) = 0$, $\lim\limits_{k\to\infty} P_c(\tilde{u}_{n_k}) = 0$ and (5.37), (5.38) and (5.52) hold, it is obvious that

$$\int_{\mathbf{R}^N} \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \lim_{k\to\infty} \int_{\mathbf{R}^N} \left|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\right|^2 dx. \tag{5.53}$$

Now (5.52) and (5.53) imply $\lim\limits_{k\to\infty} ||\nabla \tilde{u}_{n_k}||^2_{L^2(\mathbf{R}^N)} = ||\nabla u||^2_{L^2(\mathbf{R}^N)}$. Since $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ weakly in $L^2(\mathbf{R}^N)$, we infer that $\nabla \tilde{u}_{n_k} \longrightarrow \nabla u$ strongly in $L^2(\mathbf{R}^N)$, that is $\tilde{u}_{n_k} \longrightarrow u$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$.

Proceeding as in the proof of (5.37) we show that

$$\lim_{k\to\infty} \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2 dx = \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 dx. \tag{5.54}$$

Together with the weak convergence $\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \rightharpoonup \varphi^2(|r_0 - u|) - r_0^2$ in $L^2(\mathbf{R}^N)$ (see (5.34)), this implies $\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \longrightarrow \varphi^2(|r_0 - u|) - r_0^2$ strongly in $L^2(\mathbf{R}^N)$. The proof of Theorem 5.3 is complete. □

In order to prove that the minimizers provided by Theorem 5.3 solve equation (1.3), we need the following regularity result.

Lemma 5.5 *Let $N \geq 3$. Assume that the conditions (A1) and (A2) in the Introduction hold and that $u \in \mathcal{X}$ satisfies (1.3) in $\mathcal{D}'(\mathbf{R}^N)$. Then $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for $p \in [2, \infty)$, $u \in C^{1,\alpha}(\mathbf{R}^N)$ for $\alpha \in [0, 1)$ and $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.*

Proof. First we prove that for any $R > 0$ and $p \in [2, \infty)$ there exists $C(R, p) > 0$ (depending on u, but not on $x \in \mathbf{R}^N$) such that

$$\|u\|_{W^{2,p}(B(x,R))} \leq C(R, p) \qquad \text{for any } x \in \mathbf{R}^N. \tag{5.55}$$

We write $u = u_1 + u_2$, where u_1 and u_2 are as in Lemma 2.2. Then $|u_1| \leq \frac{r_0}{2}$, $\nabla u_1 \in L^2(\mathbf{R}^N)$ and $u_2 \in H^1(\mathbf{R}^N)$, hence for any $R > 0$ there exists $C(R) > 0$ such that

$$\|u\|_{H^1(B(x,R))} \leq C(R) \qquad \text{for any } x \in \mathbf{R}^N. \tag{5.56}$$

Let $\phi(x) = e^{-\frac{icx_1}{2}}(r_0 - u(x))$. It is easy to see that ϕ satisfies

$$\Delta\phi + \left(F(|\phi|^2) + \frac{c^2}{4}\right)\phi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{5.57}$$

Moreover, (5.56) holds for ϕ instead of u. From (5.56), (5.57), (3.18) and a standard bootstrap argument we infer that ϕ satisfies (5.55). (Note that assumption (A2) is needed for this bootstrap argument.) It is then clear that (5.55) also holds for u.

From (5.55), the Sobolev embeddings and Morrey's inequality (3.27) we find that u and ∇u are continuous and bounded on $\mathbf{R}^N$ and $u \in C^{1,\alpha}(\mathbf{R}^N)$ for $\alpha \in [0,1)$. In particular, u is Lipschitz; since $u \in L^{2^*}(\mathbf{R}^N)$, we have necessarily $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

The boundedness of u implies that there is some $C > 0$ such that $\big|F(|r_0 - u|^2)(r_0 - u)\big| \leq C\big|\varphi^2(|r_0 - u|) - r_0^2\big|$ on $\mathbf{R}^N$. Therefore $F(|r_0-u|^2)(r_0-u) \in L^2 \cap L^\infty(\mathbf{R}^N)$. Since $\nabla u \in L^2(\mathbf{R}^N)$, from (1.3) we find $\Delta u \in L^2(\mathbf{R}^N)$. It is well known that $\Delta u \in L^p(\mathbf{R}^N)$ with $1 < p < \infty$ implies $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbf{R}^N)$ for any i, j (see, e.g., Theorem 3 p. 96 in [34]). Thus we get $\nabla u \in W^{1,2}(\mathbf{R}^N)$. Then the Sobolev embedding implies $\nabla u \in L^p(\mathbf{R}^N)$ for $p \in [2, 2^*]$. Repeating the previous argument, after an easy induction we find $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. □

Proposition 5.6 *Assume that the conditions (A1) and (A2) in the introduction are satisfied. Let $u \in \mathcal{C}$ be a minimizer of E_c in $\mathcal{C}$. Then $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for $p \in [2, \infty)$ and u is a solution of (1.3).*

Proof. It is standard to prove that for any $R > 0$, $J_u(v) = \displaystyle\int_{\mathbf{R}^N} V(|r_0 - u - v|^2)\, dx$ is a C^1 functional on $H_0^1(B(0,R))$ and $J_u'(v).w = 2\displaystyle\int_{\mathbf{R}^N} F(|r_0 - u - v|^2)\langle r_0 - u - v, w\rangle\, dx$ (see, e.g., Lemma 17.1 p. 64 in [26] or the appendix A in [36]). It follows easily that for any $R > 0$, the functionals $\tilde{P}_c(v) = P_c(u + v)$ and $\tilde{E}_c(v) = E_c(u + v)$ are C^1 on $H_0^1(B(0,R))$. We divide the proof of Proposition 5.6 into several steps.

Step 1. There exists a function $w \in C_c^1(\mathbf{R}^N)$ such that $\tilde{P}_c'(0).w \neq 0$.

To prove this, we argue by contradiction and we assume that the above statement is false. Then u satisfies

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{N-3}{N-1}\left(\sum_{k=2}^{N} \frac{\partial^2 u}{\partial x_k^2}\right) + icu_{x_1} + F(|r_0 - u|^2)(r_0 - u) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{5.58}$$

Let $\sigma = \sqrt{\frac{N-1}{N-3}}$. It is not hard to see that $u_{1,\sigma}$ satisfies (1.3) in $\mathcal{D}'(\mathbf{R}^N)$. Hence the conclusion of Lemma 5.5 holds for $u_{1,\sigma}$ (and thus for u). This regularity is enough to prove that u satisfies the Pohozaev identity

$$\int_{\mathbf{R}^N} \Big|\frac{\partial u_{1,\sigma}}{\partial x_1}\Big|^2 dx + \frac{N-3}{N-1}\int_{\mathbf{R}^N} \sum_{k=2}^{N} \Big|\frac{\partial u_{1,\sigma}}{\partial x_k}\Big|^2 dx + cQ(u_{1,\sigma}) + \int_{\mathbf{R}^N} V(|r_0 - u_{1,\sigma}|^2)\, dx = 0. \tag{5.59}$$

To prove (5.59), we multiply (1.3) by $\sum_{k=2}^N \tilde{\chi}(\frac{x}{n})\frac{\partial u_{1,\sigma}}{\partial x_k}$, where $\tilde{\chi} \in C_c^\infty(\mathbf{R}^N)$ is a cut-off function such that $\tilde{\chi} = 1$ on $B(0,1)$ and $\mathrm{supp}(\tilde{\chi}) \subset B(0,2)$, we integrate by parts, then we let $n \longrightarrow \infty$; see the proof of Proposition 4.1 and equation (4.13) in [33] for details.

Since $\sigma = \sqrt{\frac{N-1}{N-3}}$, (5.59) is equivalent to $\left(\frac{N-3}{N-1}\right)^2 A(u) + B_c(u) = 0$. On the other hand we have $P_c(u) = \frac{N-3}{N-1}A(u) + B_c(u) = 0$ and we infer that $A(u) = 0$. But this contradicts the fact that $A(u) = T_c > 0$ and the proof of step 1 is complete.

Step 2. Existence of a Lagrange multiplier.

Let w be as above and let $v \in H^1(\mathbf{R}^N)$ be a function with compact support such that $\tilde{P}_c'(0).v = 0$. For $s, t \in \mathbf{R}$, put $\Phi(t,s) = P_c(u + tv + sw) = \tilde{P}_c(tv + sw)$, so that $\Phi(0,0) = 0$, $\frac{\partial \Phi}{\partial t}(0,0) = \tilde{P}_c'(0).v = 0$ and $\frac{\partial \Phi}{\partial s}(0,0) = \tilde{P}_c'(0).w \neq 0$. The implicit function theorem implies that there exist $\delta > 0$ and a C^1 function $\eta : (-\delta, \delta) \longrightarrow \mathbf{R}$ such that $\eta(0) = 0$, $\eta'(0) = 0$ and $P_c(u + tv + \eta(t)w) = P_c(u) = 0$ for $t \in (-\delta, \delta)$. Since u is a minimizer of A in $\mathcal{C}$, the function $t \longmapsto A(u + tv + \eta(t)w)$ achieves a minimum at $t = 0$. Differentiating at $t = 0$ we get $A'(u).v = 0$.

Hence $A'(u).v = 0$ for any $v \in H^1(\mathbf{R}^N)$ with compact support satisfying $\tilde{P}_c'(0).v = 0$. Taking $\alpha = \frac{A'(u).w}{\tilde{P}_c'(0).w}$ (where w is as in step 1), we see that

$$A'(u).v = \alpha P_c'(u).v \quad \text{for any } v \in H^1(\mathbf{R}^N) \text{ with compact support.} \tag{5.60}$$

Step 3. We have $\alpha < 0$.

To see this, we argue by contradition. Suppose that $\alpha > 0$. Let w be as in step 1. We may assume that $P_c'(u).w > 0$. From (5.60) we obtain $A'(u).w > 0$. Since $A'(u).w = \lim\limits_{t \to 0} \frac{A(u+tw)-A(u)}{t}$ and $P_c'(u).w = \lim\limits_{t \to 0} \frac{P_c(u+tw)-P_c(u)}{t}$, we see that for $t < 0$, t sufficiently close to 0 we have $u + tw \neq 0$, $P_c(u + tw) < P_c(u) = 0$ and $A(u + tw) < A(u) = \frac{N-1}{2} T_c$. But this contradicts Lemma 4.8 (i). Therefore $\alpha \leq 0$.

Assume that $\alpha = 0$. Then (5.60) implies

$$\int_{\mathbf{R}^N} \sum_{k=2}^{N} \langle \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \rangle \, dx = 0 \quad \text{for any } v \in H^1(\mathbf{R}^N) \text{ with compact support.} \tag{5.61}$$

Let $\tilde{\chi} \in C_c^\infty(\mathbf{R}^N)$ be such that $\chi = 1$ on $B(0,1)$ and $\text{supp}(\tilde{\chi}) \subset B(0,2)$. Put $v_n(x) = \chi(\frac{x}{n})u(x)$, so that $\nabla v_n(x) = \frac{1}{n} \nabla \tilde{\chi}(\frac{x}{n})u + \tilde{\chi}(\frac{x}{n})\nabla u$. It is easy to see that $\tilde{\chi}(\frac{\cdot}{n})\nabla u \longrightarrow \nabla u$ in $L^2(\mathbf{R}^N)$ and $\frac{1}{n}\nabla \tilde{\chi}(\frac{\cdot}{n})u \rightharpoonup 0$ weakly in $L^2(\mathbf{R}^N)$. Replacing v by v_n in (5.61) and passing to the limit as $n \longrightarrow \infty$ we get $A(u) = 0$, which contradicts the fact that $A(u) = \frac{N-1}{2} T_c$. Hence we cannot have $\alpha = 0$. Thus necessarily $\alpha < 0$.

Step 4. Conclusion.

Since $\alpha < 0$, it follows from (5.60) that u satisfies

$$-\frac{\partial^2 u}{\partial x_1^2} - \left(\frac{N-3}{N-1} - \frac{1}{\alpha} \right) \sum_{k=2}^{N} \frac{\partial^2 u}{\partial x_k^2} + icu_{x_1} + F(|r_0 - u|^2)(r_0 - u) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{5.62}$$

Let $\sigma_0 = \left(\frac{N-3}{N-1} - \frac{1}{\alpha} \right)^{-\frac{1}{2}}$. It is easy to see that u_{1,σ_0} satisfies (1.3) in $\mathcal{D}'(\mathbf{R}^N)$. Therefore the conclusion of Lemma 5.5 holds for u_{1,σ_0} (and consequently for u). Then Proposition 4.1 in [33] implies that u_{1,σ_0} satisfies the Pohozaev identity $\frac{N-3}{N-1} A(u_{1,\sigma_0}) + B_c(u_{1,\sigma_0}) = 0$, or equivalently $\frac{N-3}{N-1} \sigma_0^{N-3} A(u) + \sigma_0^{N-1} B_c(u) = 0$, which implies

$$\frac{N-3}{N-1} \left(\frac{N-3}{N-1} - \frac{1}{\alpha} \right) A(u) + B_c(u) = 0.$$

On the other hand we have $P_c(u) = \frac{N-3}{N-1} A(u) + B_c(u) = 0$. Since $A(u) > 0$, we get $\frac{N-3}{N-1} - \frac{1}{\alpha} = 1$. Then coming back to (5.62) we see that u satisfies (1.3). □

6 The case $N = 3$

This section is devoted to the proof of Theorem 1.1 in space dimension $N = 3$. We only indicate the differences with respect to the case $N \geq 4$. Clearly, if $N = 3$ we have $P_c = B_c$. For $v \in \mathcal{X}$ we denote

$$D(v) = \int_{\mathbf{R}^3} \left| \frac{\partial v}{\partial x_1} \right|^2 dx + a^2 \int_{\mathbf{R}^3} \left(\varphi^2(|r_0 - v|) - r_0^2 \right)^2 dx.$$

For any $v \in \mathcal{X}$ and $\sigma > 0$ we have

$$(6.1) \qquad A(v_{1,\sigma}) = A(v), \qquad B_c(v_{1,\sigma}) = \sigma^2 B_c(v) \qquad \text{and} \qquad D(v_{1,\sigma}) = \sigma^2 D(v).$$

If $N = 3$ we cannot have a result similar to Lemma 5.1. To see this consider $u \in \mathcal{C}$, so that $B_c(u) = 0$. Using (6.1) we see that $u_{1,\sigma} \in \mathcal{C}$ for any $\sigma > 0$ and we have $E_c(u_{1,\sigma}) = A(u) + \sigma^2 B_c(u) = A(u)$, while $E_{GL}(u_{1,\sigma}) = A(u) + \sigma^2 D(u) \longrightarrow \infty$ as $\sigma \longrightarrow \infty$.

However, for any $u \in \mathcal{C}$ there exists $\sigma > 0$ such that $D(u_{1,\sigma}) = 1$ (and obviously $u_{1,\sigma} \in \mathcal{C}$, $E_c(u_{1,\sigma}) = E_c(u)$). Since $\mathcal{C} \neq \emptyset$ and $T_c = \inf\{E_c(u) \mid u \in \mathcal{C}\}$, we see that there exists a sequence $(u_n)_{n \geq 1} \subset \mathcal{C}$ such that

$$(6.2) \qquad D(u_n) = 1 \qquad \text{and} \qquad E_c(u_n) = A(u_n) \longrightarrow T_c \quad \text{as } n \longrightarrow \infty.$$

In particular, (6.2) implies $E_{GL}(u_n) \longrightarrow T_c + 1$ as $n \longrightarrow \infty$.

The following result is the equivalent of Lemma 5.2 in the case $N = 3$.

Lemma 6.1 *Let $N = 3$ and let $(u_n)_{n \geq 1} \subset \mathcal{X}$ be a sequence satisfying*
a) There exists $C > 0$ such that $D(u_n) \geq C$ for any n, and
b) $B_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.
Then $\liminf_{n \to \infty} E_c(u_n) = \liminf_{n \to \infty} A(u_n) \geq S_c$, where S_c is given by (4.22).

Proof. It suffices to prove that for any $k > 0$ we have

$$(6.3) \qquad \liminf_{n \to \infty} A(u_n) \geq E_{c,min}(k).$$

Fix $k > 0$. Let $n \geq 1$. If $A(u_n) \geq k$, by Lemma 4.6 (iii) we have $A(u_n) \geq k > E_{c,min}(k)$. If $A(u_n) < k$, since $E_{GL}((u_n)_{1,\sigma}) = A(u_n) + \sigma^2 D(u_n)$ we see that there exists $\sigma_n > 0$ such that $E_{GL}((u_n)_{1,\sigma_n}) = k$. Obviously, we have $\sigma_n^2 D(u_n) < k$, hence $\sigma_n^2 \leq \frac{k}{C}$ by (a). It is clear that $E_c((u_n)_{1,\sigma_n}) = A(u_n) + \sigma_n^2 B_c(u_n) \geq E_{c,min}(k)$, therefore $A(u_n) \geq E_{c,min}(k) - \sigma_n^2 |B_c(u_n)| \geq E_{c,min}(k) - \frac{k}{C}|B_c(u_n)|$. Passing to the limit as $n \longrightarrow \infty$ we obtain (6.3). Since $k > 0$ is arbitrary, Lemma 6.1 is proved. □

Let

$$\Lambda_c = \{\lambda \in \mathbf{R} \mid \text{ there exists a sequence } (u_n)_{n \geq 1} \subset \mathcal{X} \text{ such that } D(u_n) \geq 1,\ B_c(u_n) \longrightarrow 0 \text{ and } A(u_n) \longrightarrow \lambda \text{ as } n \longrightarrow \infty\}.$$

Using a scaling argument, we see that

$$\Lambda_c = \{\lambda \in \mathbf{R} \mid \text{ there exist a sequence } (u_n)_{n \geq 1} \subset \mathcal{X} \text{ and } C > 0 \text{ such that } D(u_n) \geq C,\ B_c(u_n) \longrightarrow 0 \text{ and } A(u_n) \longrightarrow \lambda \text{ as } n \longrightarrow \infty\}.$$

Let $\lambda_c = \inf \Lambda_c$. From (6.2) it follows that $T_c \in \Lambda_c$. It is standard to prove that Λ_c is closed in $\mathbf{R}$, hence $\lambda_c \in \Lambda_c$. From Lemma 6.1 we obtain

$$(6.4) \qquad S_c \leq \lambda_c \leq T_c.$$

The main result of this section is as follows.

Theorem 6.2 *Let $N = 3$ and let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that*

$$(6.5) \qquad D(u_n) \longrightarrow 1, \quad B_c(u_n) \longrightarrow 0 \quad \text{and} \quad A(u_n) \longrightarrow \lambda_c \quad \text{as } n \longrightarrow \infty.$$

There exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^3$ and $u \in \mathcal{C}$ such that

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad \text{and} \quad |r_0 - u_{n_k}(\cdot + x_k)|^2 - r_0^2 \longrightarrow |r_0 - u|^2 - r_0^2 \quad \text{in } L^2(\mathbf{R}^3).$$

Moreover, we have $E_c(u) = A(u) = T_c = \lambda_c$ and u minimizes E_c in $\mathcal{C}$.

Proof. By (6.5) we have $E_{GL}(u_n) = A(u_n) + D(u_n) \longrightarrow \lambda_c + 1$ as $n \longrightarrow \infty$. Let $q_n(t)$ be the concentration function of $E_{GL}(u_n)$, as in (5.9). Proceeding as in the proof of Theorem 5.3, we infer that there exist a subsequence of $(u_n, q_n)_{n\geq 1}$, still denoted $(u_n, q_n)_{n\geq 1}$, a nondecreasing function $q : [0, \infty) \longrightarrow [0, \infty)$ and $\alpha \in [0, \lambda_c + 1]$ such that (5.10) holds. We see also that there exists a sequence $t_n \longrightarrow \infty$ satisfying (5.11) and (5.12).

Clearly, our aim is to prove that $\alpha = \lambda_c + 1$. The next result implies that $\alpha > 0$.

Lemma 6.3 *Assume that $N = 3$, $0 \leq c < v_s$ and let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that $D(u_n) \longrightarrow 1$, $B_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sup\limits_{n\geq 1} E_{GL}(u_n) = M < \infty$.*

There exists $k > 0$ such that $\sup\limits_{y\in\mathbf{R}^3} \int_{B(y,1)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \geq k$ for all sufficiently large n.

Proof. We argue by contradiction and assume that the conclusion of Lemma 6.3 is false. Then there exists a subsequence, still denoted $(u_n)_{n\geq 1}$, such that

$$(6.6) \qquad \sup_{y\in\mathbf{R}^3} E_{GL}^{B(y,1)}(u_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

Exactly as in Lemma 5.4 we prove that (5.14) holds, that is

$$(6.7) \qquad \lim_{n\to\infty} \int_{\mathbf{R}^3} \left| V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx = 0.$$

Using (6.7) and the assumptions of Lemma 6.3 we find

$$(6.8) \quad cQ(u_n) = B_c(u_n) - D(u_n) - \int_{\mathbf{R}^3} V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \longrightarrow -1$$

as $n \longrightarrow \infty$. If $c = 0$, (6.8) gives a contradiction and Lemma 6.3 is proved. From now on we assume that $0 < c < v_s$.

Fix $c_1 \in (c, v_s)$, then fix $\sigma > 0$ such that

$$(6.9) \qquad \sigma^2 > \frac{Mc}{c_1 - c}.$$

A simple change of variables shows that $\tilde{M} := \sup\limits_{n\geq 1} E_{GL}((u_n)_{1,\sigma}) < \infty$ and (6.7) holds with $(u_n)_{1,\sigma}$ instead of u_n. It is easy to see that $((u_n)_{1,\sigma})_{n\geq 1}$ also satisfies (6.6). Using Lemma 3.2 we infer that there exists a sequence $h_n \longrightarrow 0$ and for each n there exists a minimizer v_n of $G_{h_n,\mathbf{R}^3}^{(u_n)_{1,\sigma}}$ in $H^1_{(u_n)_{1,\sigma}}(\mathbf{R}^3)$ such that

$$(6.10) \qquad \|\, |v_n - r_0| - r_0 \|_{L^\infty(\mathbf{R}^3)} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

From (3.4) we obtain

$$|Q((u_n)_{1,\sigma}) - Q(v_n)| \leq \left(h_n^2 + h_n^{\frac{4}{3}} \tilde{M}^{\frac{2}{3}}\right)^{\frac{1}{2}} \tilde{M} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{6.11}$$

Using (6.10), the fact that $0 < c_1 < 2ar_0$ and Lemma 4.2 we infer that for all sufficiently large n we have

$$E_{GL}(v_n) + c_1 Q(v_n) \geq 0. \tag{6.12}$$

Since $E_{GL}(v_n) \leq E_{GL}((u_n)_{1,\sigma})$, for large n we have

$$\begin{aligned}
&0 \leq E_{GL}(v_n) + c_1 Q(v_n) \\
&\leq E_{GL}((u_n)_{1,\sigma}) + c_1 Q((u_n)_{1,\sigma}) + c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| \\
&= A(u_n) + B_c((u_n)_{1,\sigma}) + (c_1 - c) Q((u_n)_{1,\sigma}) + c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| \\
&\quad + \int_{\mathbf{R}^3} a^2 \left(\varphi^2(|r_0 - (u_n)_{1,\sigma}|) - r_0^2\right)^2 - V(|r_0 - (u_n)_{1,\sigma}|^2)\, dx \\
&= A(u_n) + \sigma^2 B_c(u_n) + \sigma^2 (c_1 - c) Q(u_n) + a_n \\
&\leq M + \sigma^2 B_c(u_n) + \sigma^2 (c_1 - c) Q(u_n) + a_n,
\end{aligned} \tag{6.13}$$

where

$$a_n = c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| + \int_{\mathbf{R}^3} a^2 \left(\varphi^2(|r_0 - (u_n)_{1,\sigma}|) - r_0^2\right)^2 - V(|r_0 - (u_n)_{1,\sigma}|^2)\, dx.$$

From (6.7) and (6.11) we infer that $\lim\limits_{n\to\infty} a_n = 0$. Then passing to the limit as $n \longrightarrow \infty$ in (6.13), using (6.8) and the fact that $\lim\limits_{n\to\infty} B_c(u_n) = 0$ we find $0 \leq M - \sigma^2 \frac{c_1 - c}{c}$. The last inequality clearly contradicts the choice of σ in (6.9). This contradiction shows that (6.6) cannot hold and Lemma 6.3 is proved. □

Next we show that we cannot have $\alpha \in (0, \lambda_c + 1)$. We argue again by contradiction and we assume that $\alpha \in (0, \lambda_c + 1)$. Proceeding exactly as in the proof of Theorem 5.3 and using Lemma 3.3, we infer that for each n sufficiently large there exist two functions $u_{n,1}$, $u_{n,2}$ having the following properties:

$$E_{GL}(u_{n,1}) \longrightarrow \alpha, \qquad E_{GL}(u_{n,1}) \longrightarrow \lambda_c + 1 - \alpha, \tag{6.14}$$

$$|A(u_n) - A(u_{n,1}) - A(u_{n,2})| \longrightarrow 0, \tag{6.15}$$

$$|B_c(u_n) - B_c(u_{n,1}) - B_c(u_{n,2})| \longrightarrow 0, \tag{6.16}$$

$$|D(u_n) - D(u_{n,1}) - D(u_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{6.17}$$

Since $(E_{GL}(u_{n,i}))_{n\geq 1}$ are bounded, from Lemmas 4.1 and 4.5 we see that $B_c(u_{n,i}))_{n\geq 1}$ are bounded. Moreover, by (6.16) we have $\lim\limits_{n\to\infty} (B_c(u_{n,1}) + B_c(u_{n,2})) = \lim\limits_{n\to\infty} B_c(u_n) = 0$. Similarly, $(D(u_{n,i}))_{n\geq 1}$ are bounded and $\lim\limits_{n\to\infty} (D(u_{n,1}) + D(u_{n,2})) = \lim\limits_{n\to\infty} D(u_n) = 1$. Passing again to a subsequence (still denoted $(u_n)_n \geq 1$), we may assume that

$$\lim_{n\to\infty} B_c(u_{n,1}) = b_1, \qquad \lim_{n\to\infty} B_c(u_{n,2}) = b_2, \qquad \text{where } b_i \in \mathbf{R},\ b_1 + b_2 = 0, \tag{6.18}$$

(6.19) $$\lim_{n\to\infty} D(u_{n,1}) = d_1, \qquad \lim_{n\to\infty} D(u_{n,2}) = d_2, \qquad \text{where } d_i \geq 0,\ d_1 + d_2 = 1.$$

From (6.18) it follows that either $b_1 = b_2 = 0$, or one of b_1 or b_2 is negative.

Case 1. If $b_1 = b_2 = 0$, we distinguish two subcases:

Subcase 1a. We have $d_1 > 0$ and $d_2 > 0$. Let $\sigma_i = \frac{2}{\sqrt{d_i}}$, $i = 1, 2$. Then $D((u_{n,i})_{1,\sigma_i}) = \sigma_i^2 D(u_{n,i}) \longrightarrow 4$ and $B_c((u_{n,i})_{1,\sigma_i}) = \sigma_i^2 B_c(u_{n,i}) \longrightarrow 0$ as $n \longrightarrow \infty$. From (6.1) and the definition of λ_c it follows that $\liminf_{n\to\infty} A(u_{n,i}) = \liminf_{n\to\infty} A((u_{n,i})_{1,\sigma_i}) \geq \lambda_c$, $i = 1, 2$. Then (6.15) implies

$$\liminf_{n\to\infty} A(u_n) \geq \liminf_{n\to\infty} A(u_{n,1}) + \liminf_{n\to\infty} A(u_{n,2}) \geq 2\lambda_c$$

an this is a contradiction because by (6.5) we have $\lim_{n\to\infty} A(u_n) = \lambda_c$.

Subcase 1b. One of d_i's is zero, say $d_1 = 0$. Then necessarily $d_2 = 1$, that is $\lim_{n\to\infty} D(u_{n,2}) = 1$. Since $E_{GL}(u_{n,2}) = A(u_{n,2}) + D(u_{n,2}) \longrightarrow 1 + \lambda_c - \alpha$ as $n \longrightarrow \infty$, we infer that $\lim_{n\to\infty} A(u_{n,2}) = \lambda_c - \alpha$. Hence $D(u_{n,2}) \longrightarrow 1$, $B_c(u_{n,2}) \longrightarrow 0$ and $A(u_{n,2}) \longrightarrow \lambda_c - \alpha$ as $n \longrightarrow \infty$, which implies $\lambda_c - \alpha \in \Lambda_c$. Since $\alpha > 0$, this contradicts the definition of λ_c.

Case 2. One of b_i's is negative, say $b_1 < 0$. From Lemma 4.8 (ii) we get $\liminf_{n\to\infty} A(u_{n,1}) > T_c \geq \lambda_c$ and then using (6.15) we find $\liminf_{n\to\infty} A(u_n) > \lambda_c$, in contradiction with (6.5).

Consequently in all cases we get a contradiction and this proves that we cannot have $\alpha \in (0, \lambda_c + 1)$.

Up to now we have proved that $\lim_{t\to\infty} q(t) = \lambda_c + 1$, that is "concentration" occurs.

Proceeding as in the case $N \geq 4$, we see that there exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence of points $(x_k)_{k\geq 1} \subset \mathbf{R}^3$ and $u \in \mathcal{X}$ such that, denoting $\tilde{u}_{n_k}(x) = u_{n_k}(x + x_k)$, we have:

(6.20) $$\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u \text{ and } \varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \rightharpoonup \varphi^2(|r_0 - u|) - r_0^2 \text{ weakly in } L^2(\mathbf{R}^3),$$

(6.21) $$\tilde{u}_{n_k} \longrightarrow u \quad \text{in } L^p_{loc}(\mathbf{R}^3) \text{ for } 1 \leq p < 6 \text{ and a.e. on } \mathbf{R}^3,$$

(6.22) $$\int_{\mathbf{R}^3} V(|r_0 - \tilde{u}_{n_k}|^2)\, dx \longrightarrow \int_{\mathbf{R}^3} V(|r_0 - u|^2)\, dx,$$

(6.23) $$\int_{\mathbf{R}^3} \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2\, dx \longrightarrow \int_{\mathbf{R}^3} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\, dx,$$

(6.24) $$Q(\tilde{u}_{n_k}) \longrightarrow Q(u) \qquad \text{as } k \longrightarrow \infty.$$

Passing to the limit as $k \longrightarrow \infty$ in the identity

$$\int_{\mathbf{R}^3} V(|r_0 - \tilde{u}_{n_k}|^2) - a^2 \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2\, dx + cQ(\tilde{u}_{n_k}) = B_c(\tilde{u}_{n_k}) - D(\tilde{u}_{n_k}),$$

using (6.22)–(6.24) and the fact that $B_c(\tilde{u}_{n_k}) \longrightarrow 0$, $D(\tilde{u}_{n_k}) \longrightarrow 1$ we get

$$\int_{\mathbf{R}^3} V(|r_0 - u|^2) - a^2 \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\, dx + cQ(u) = -1.$$

Thus $u \neq 0$.

From the weak convergence $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^3)$ we get

$$(6.25)\qquad \int_{\mathbf{R}^3} \Big|\frac{\partial u}{\partial x_j}\Big|^2 dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^3} \Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_j}\Big|^2 dx \qquad \text{for } j = 1, \dots, N.$$

In particular, we have

$$(6.26)\qquad A(u) \leq \lim_{k\to\infty} A(\tilde{u}_{n_k}) = \lambda_c.$$

From (6.22), (6.24) and (6.25) we obtain

$$(6.27)\qquad B_c(u) \leq \lim_{k\to\infty} B_c(\tilde{u}_{n_k}) = 0.$$

Since $u \neq 0$, (6.27) and Lemma 4.8 (i) imply $A(u) \geq T_c$. Then using (6.26) and the fact that $\lambda_c \leq T_c$, we infer that necessarily

$$(6.28)\qquad A(u) = T_c = \lambda_c = \lim_{k\to\infty} A(\tilde{u}_{n_k}).$$

The fact that $B_c(\tilde{u}_{n_k}) \longrightarrow 0$, (6.22) and (6.24) imply that $\left(\int_{\mathbf{R}^3} \Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\Big|^2 dx\right)_{k\geq 1}$ converges. If $\int_{\mathbf{R}^3} \Big|\frac{\partial u}{\partial x_1}\Big|^2 dx < \lim_{k\to\infty} \int_{\mathbf{R}^3} \Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\Big|^2 dx$, we get $B_c(u) < \lim_{k\to\infty} B_c(\tilde{u}_{n_k}) = 0$ in (6.27) and then Lemma 4.8 (i) implies $A(u) > T_c$, a contradiction. Taking (6.25) into account, we see that necessarily

$$(6.29)\qquad \int_{\mathbf{R}^3} \Big|\frac{\partial u}{\partial x_1}\Big|^2 dx = \lim_{k\to\infty} \int_{\mathbf{R}^3} \Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\Big|^2 dx \qquad \text{and} \qquad B_c(u) = 0.$$

Thus we have proved that $u \in \mathcal{C}$ and $||\nabla u||_{L^2(\mathbf{R}^3)} = \lim_{k\to\infty} ||\nabla \tilde{u}_{n_k}||_{L^2(\mathbf{R}^3)}$. Combined with the weak convergence $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^3)$, this implies the strong convergence $\nabla \tilde{u}_{n_k} \longrightarrow \nabla u$ in $L^2(\mathbf{R}^3)$. Then using the Sobolev embedding we find $\tilde{u}_{n_k} \longrightarrow u$ in $L^6(\mathbf{R}^3)$.

From the second part of (6.20) and (6.23) it follows that

$$(6.30)\qquad \varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \longrightarrow \varphi^2(|r_0 - u|) - r_0^2 \qquad \text{in } L^2(\mathbf{R}^3).$$

Let $G(z) = |r_0 - z|^2 - \varphi^2(|r_0 - z|)$. It is obvious that $G \in C^\infty(\mathbf{C}, \mathbf{R})$ and $|G(z)| \leq C|r_0 - z|^2 \mathbb{1}_{\{|r_0 - z| > 2r_0\}} \leq C'|z|^2 \mathbb{1}_{\{|z|>r_0\}} \leq C''|z|^3 \mathbb{1}_{\{|z|>r_0\}}$. Since $\tilde{u}_{n_k} \longrightarrow u$ in $L^6(\mathbf{R}^3)$, it is easy to see that $G(\tilde{u}_{n_k}) \longrightarrow G(u)$ in $L^2(\mathbf{R}^3)$ (see Theorem A4 p. 134 in [36]). Together with (6.30), this gives $|r_0 - \tilde{u}_{n_k}|^2 - r_0^2 \longrightarrow |r_0 - u|^2 - r_0^2$ in $L^2(\mathbf{R}^3)$ and the proof of Theorem 6.2 is complete. □

To prove that any minimizer provided by Theorem 6.2 satisfies an Euler-Lagrange equation, we will need the next lemma. It is clear that for any $v \in \mathcal{X}$ and any $R > 0$, the functional $\tilde{B}_c^v(w) := B_c(v + w)$ is C^1 on $H_0^1(B(0, R))$. We denote by $(\tilde{B}_c^v)'(0).w = \lim_{t\to 0} \frac{B_c(v+tw) - B_c(v)}{t}$ its derivative at the origin.

Lemma 6.4 *Assume that $N \geq 3$ and the conditions (A1) and (A2) are satisfied. Let $v \in \mathcal{X}$ be such that $(\tilde{B}_c^v)'(0).w = 0$ for any $w \in C_c^1(\mathbf{R}^N)$. Then $v = 0$ almost everywhere on $\mathbf{R}^N$.*

Proof. We denote by v^* be the precise representative of v, that is $v^*(x) = \lim_{r\to 0} m(v, B(x, r))$ if this limit exists, and 0 otherwise. Since $v \in L^1_{loc}(\mathbf{R}^N)$, it is well-known that $v = v^*$ almost everywhere on $\mathbf{R}^N$ (see, e.g., Corollary 1 p. 44 in [14]). Throughout the proof of Lemma 6.4 we replace v by v^*. We proceed in three steps.

Step 1. There exists a set $S \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(S) = 0$ and for any $x' \in \mathbf{R}^{N-1} \setminus S$ the function $v_{x'} := v(\cdot, x')$ belongs to $C^2(\mathbf{R})$ and solves the differential equation

$$(6.31) \qquad -(v_{x'})''(s) + ic(v_{x'})'(s) + F(|r_0 - v_{x'}(s)|^2)(r_0 - v_{x'}(s)) = 0 \qquad \text{for any } s \in \mathbf{R}.$$

Moreover, we have $|v_{x'}(s)| \longrightarrow 0$ as $s \longrightarrow \pm\infty$ and $v_{x'}$ satisfies the following properties:

$$(6.32) \quad v_{x'} \in L^{2^*}(\mathbf{R}), \qquad \varphi^2(|r_0 - v_{x'}|) - r_0^2 \in L^2(\mathbf{R}) \quad \text{and} \quad (v_{x'})' = \frac{\partial v}{\partial x_1}(\cdot, x') \in L^2(\mathbf{R}),$$

$$(6.33) \qquad F(|r_0 - v_{x'}|^2)(r_0 - v_{x'}) \in L^2(\mathbf{R}) + L^{\frac{2^*}{2p_0+1}}(\mathbf{R}).$$

It is easy to see that $F(|r_0 - v|^2)(r_0 - v) \in L^2(\mathbf{R}^N) + L^{\frac{2^*}{2p_0+1}}(\mathbf{R}^N)$. Since $v \in H^1_{loc}(\mathbf{R}^3)$, using Theorem 2 p. 164 in [14] and Fubini's Theorem, respectively, we see that there exists a set $\tilde{S} \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(\tilde{S}) = 0$ and for any $x' \in \mathbf{R}^{N-1} \setminus \tilde{S}$ the function $v_{x'}$ is absolutely continuous, $v_{x'} \in H^1_{loc}(\mathbf{R})$ and (6.32)$-$(6.33) hold.

Given $\phi \in C^1_c(\mathbf{R})$, we denote $\Lambda_\phi(x_1, x') = \langle \frac{\partial v}{\partial x_1}(x_1, x'), \phi'(x_1)\rangle + c\langle i \frac{\partial v}{\partial x_1}(x_1, x'), \phi(x_1)\rangle + \langle F(|r_0 - v|^2)(r_0 - v)(x_1, x'), \phi(x_1)\rangle$. From (6.32) and (6.33) it follows that $\Lambda_\phi(\cdot, x') \in L^1(\mathbf{R})$ for $x' \in \mathbf{R}^{N-1} \setminus \tilde{S}$. For such x' we define $\lambda_\phi(x') = \displaystyle\int_{\mathbf{R}} \Lambda_\phi(x_1, x')dx_1$, then we extend the function λ_ϕ in an arbitrary way to $\mathbf{R}^{N-1}$. Let $\psi \in C^1_c(\mathbf{R}^{N-1})$. It is obvious that the function $(x_1, x') \longmapsto \Lambda_\phi(x_1, x')\psi(x')$ belongs to $L^1(\mathbf{R}^N)$ and using Fubini's Theorem we get $\displaystyle\int_{\mathbf{R}^N} \Lambda_\phi(x_1, x')\psi(x')\, dx = \int_{\mathbf{R}^{N-1}} \lambda_\phi(x')\psi(x')\, dx'$. On the other hand, using the assumption of Lemma 6.4 we obtain $2\displaystyle\int_{\mathbf{R}^N} \Lambda_\phi(x_1, x')\psi(x')\, dx = \left(\tilde{B}^v_c\right)'(0).(\phi(x_1)\psi(x')) = 0$. Hence we have $\displaystyle\int_{\mathbf{R}^{N-1}} \lambda_\phi(x')\psi(x')\, dx' = 0$ for any $\psi \in C^1_c(\mathbf{R}^{N-1})$ and this implies that there exists a set $S_\phi \subset \mathbf{R}^{N-1} \setminus \tilde{S}$ such that $\mathcal{L}^{N-1}(S_\phi) = 0$ and $\lambda_\phi = 0$ on $\mathbf{R}^{N-1} \setminus (\tilde{S} \cup S_\phi)$.

Denote $q_0 = \frac{2^*}{2p_0+1} \in (1, \infty)$. There exists a coutable set $\{\phi_n \in C^1_c(\mathbf{R}) \mid n \in \mathbf{N}\}$ which is dense in $H^1(\mathbf{R}) \cap L^{q_0'}(\mathbf{R})$. For each n consider the set $S_{\phi_n} \subset \mathbf{R}^{N-1}$ as above. Let $S = \tilde{S} \cup \bigcup\limits_{n \in \mathbf{N}} S_{\phi_n}$. It is clear that $\mathcal{L}^{N-1}(S) = 0$.

Let $x' \in \mathbf{R}^{N-1} \setminus S$. Fix $\phi \in C^1_c(\mathbf{R})$. There is a sequence $(\phi_{n_k})_{k \geq 1}$ such that $\phi_{n_k} \longrightarrow \phi$ in $H^1(\mathbf{R})$ and in $L^{q_0'}(\mathbf{R})$. Then $\lambda_{\phi_{n_k}}(x') = 0$ for each k and (6.32)$-$(6.33) imply that $\lambda_{\phi_{n_k}}(x') \longrightarrow \lambda_\phi(x')$. Consequently $\lambda_\phi(x') = 0$ for any $\phi \in C^1_c(\mathbf{R})$ and this implies that $v_{x'}$ satisfies the equation (6.31) in $\mathcal{D}'(\mathbf{R})$. Using (6.31) we infer that $(v_{x'})''$ (the *weak* second derivative of $v_{x'}$) belongs to $L^1_{loc}(\mathbf{R})$ and then it follows that $(v_{x'})'$ is continuous on $\mathbf{R}$ (see, e.g., Lemma VIII.2 p. 123 in [8]). In particular, we have $v_{x'} \in C^1(\mathbf{R})$. Coming back to (6.31) we see that $(v_{x'})''$ is continuous, hence $v_{x'} \in C^2(\mathbf{R})$ and (6.31) holds at each point of $\mathbf{R}$. Finally, we have $|v_{x'}(s_2) - v_{x'}(s_1)| \leq |s_2 - s_1|^{\frac{1}{2}} ||\,(v_{x'})'\,||_{L^2}$; this estimate and the fact that $v_{x'} \in L^{2^*}(\mathbf{R})$ imply that $v_{x'}(s) \longrightarrow 0$ as $s \longrightarrow \pm\infty$.

Step 2. There exist two positive constants k_1, k_2 (depending only on F and c) such that for any $x' \in \mathbf{R}^{N-1} \setminus S$ we have either $v_{x'} = 0$ on $\mathbf{R}$ or there exists an interval $I_{x'} \subset \mathbf{R}$ with $\mathcal{L}^1(I_{x'}) \geq k_1$ and $|\,|r_0 - v_{x'}| - r_0| \geq k_2$ on $I_{x'}$.

To see this, fix $x' \in \mathbf{R}^{N-1} \setminus S$ and denote $g = |r_0 - v_{x'}|^2 - r_0^2$. Then $g \in C^2(\mathbf{R}, \mathbf{R})$ and g tends to zero at $\pm\infty$. Proceeding exactly as in [33], p. 1100-1101 we integrate (6.31) and we see that g satisfies

$$(6.34) \qquad (g')^2(s) + c^2 g^2(s) - 4(g(s) + r_0^2)V(g(s) + r_0^2) = 0 \qquad \text{in } \mathbf{R}.$$

Using (1.4) we have $c^2t^2 - 4(t+r_0^2)V(t+r_0^2) = t^2(c^2 - v_s^2 + \varepsilon_1(t))$, where $\varepsilon_1(t) \longrightarrow 0$ as $t \longrightarrow 0$. In particular, there exists $k_0 > 0$ such that

$$(6.35) \qquad c^2t^2 - 4(t+r_0^2)V(t+r_0^2) < 0 \qquad \text{for } t \in [-2k_0, 0) \cup (0, 2k_0].$$

If $g = 0$ on $\mathbf{R}$ then $|r_0 - v_{x'}| = r_0$ and consequently there exists a lifting $r_0 - v_{x'}(s) = r_0 e^{i\theta(s)}$ with $\theta \in C^2(\mathbf{R}, \mathbf{R})$. Using equation (6.31) and proceeding as in [33] p. 1101 we see that either $r_0 - v_{x'}(s) = r_0 e^{i\theta_0}$ or $r_0 - v_{x'}(s) = r_0 e^{ics+\theta_0}$, where $\theta_0 \in \mathbf{R}$ is a constant. Since $v_{x'} \in L^{2^*}(\mathbf{R})$, we must have $v_{x'} = 0$.

If $g \not\equiv 0$, the function g achieves a negative minimum or a positive maximum at some $s_0 \in \mathbf{R}$. Then $g'(s_0) = 0$ and using (6.34) and (6.35) we infer that $|g(s_0)| > 2k_0$. Let $s_2 = \inf\{s < s_0 \mid |g(s)| \geq 2k_0\}$, $s_1 = \sup\{s < s_2 \mid g(s) \leq k_0\}$, so that $s_1 < s_2$, $|g(s_1)| = k_0$, $|g(s_2)| = 2k_0$ and $k_0 \leq |g(s)| \leq 2k_0$ for $s \in [s_1, s_2]$. Denote $M = \sup\{4(t+r_0^2)V(t+r_0^2) - c^2t^2 \mid t \in [-2k_0, 2k_0]\}$. From (6.34) we obtain $|g'(s)| \leq \sqrt{M}$ if $g(s) \in [-2k_0, 2k_0]$ and we infer that

$$k_0 = |g(s_2)| - |g(s_1)| \leq \Big| \int_{s_1}^{s_2} g'(s)\, ds \Big| \leq \sqrt{M}(s_2 - s_1),$$

hence $s_2 - s_1 \geq \frac{k_0}{\sqrt{M}}$. Obviously, there exists $k_2 > 0$ such that $|\,|r_0 - z|^2 - r_0^2| \geq k_0$ implies $|\,|r_0 - z| - r_0| \geq k_2$. Taking $k_1 = \frac{k_0}{\sqrt{M}}$ and $I_{x'} = [s_1, s_2]$, the proof of step 2 is complete.

Step 3. Conclusion.

Let $K = \{x' \in \mathbf{R}^{N-1} \setminus S \mid v_{x'} \not\equiv 0\}$. It is standard to prove that K is $\mathcal{L}^{N-1}$−measurable. The conclusion of Lemma 6.4 follows if we prove that $\mathcal{L}^{N-1}(K) = 0$. We argue by contradiction and we assume that $\mathcal{L}^{N-1}(K) > 0$.

If $x' \in K$, it follows from step 2 that there exists an interval $I_{x'}$ of length at least k_1 such that $\left(\varphi^2(|r_0 - v_{x'}|) - r_0^2\right)^2 \geq \eta(k_2)$ on $I_{x'}$, where η is as in (3.30). This implies $\displaystyle\int_{\mathbf{R}} \left(\varphi^2(|r_0 - v(x_1, x')|) - r_0^2\right)^2 dx_1 \geq k_1\eta(k_2)$ and using Fubini's theorem we get

$$\int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - v(x)|) - r_0^2\right)^2 dx = \int_K \left(\int_{\mathbf{R}} \left(\varphi^2(|r_0 - v(x_1, x')|) - r_0^2\right)^2 dx_1 \right) dx'$$
$$\geq k_1\eta(k_2)\mathcal{L}^{N-1}(K).$$

Since $v \in \mathcal{X}$, we infer that $\mathcal{L}^{N-1}(K)$ is finite.

It is obvious that there exist $x_1' \in K$ and $x_2' \in \mathbf{R}^{N-1} \setminus (K \cup S)$ arbitrarily close to each other. Then $|v_{x_1'}| \geq k_2$ on an interval $I_{x_1'}$ of length k_1, while $v_{x_2'} \equiv 0$. If we knew that v is uniformly continuous, this would lead to a contradiction. However, the equation (6.31) satisfied by v involves only derivatives with respect to x_1 and does not imply any regularity properties of v with respect to the transverse variables (note that if v is a solution of (6.31), then $v(x_1 + \delta(x'), x')$ is also a solution, even if δ is discontinuous). For instance, for the Gross-Pitaevskii nonlinearity $F(s) = 1 - s$ it is possible to construct bounded, C^∞ functions v such that $v \in L^{2^*}(\mathbf{R}^N)$, (6.31) is satisfied for a.e. x', and the set K constructed as above is a nontrivial ball in $\mathbf{R}^{N-1}$ (of course, these functions do not tend uniformly to zero at infinity, are not uniformly continuous and their gradient is not in $L^2(\mathbf{R}^N)$).

We use that fact that one transverse derivative of v (for instance, $\frac{\partial v}{\partial x_2}$) is in $L^2(\mathbf{R}^N)$ to get a contradiction.

For $x' = (x_2, x_3, \ldots, x_N) \in \mathbf{R}^{N-1}$, we denote $x'' = (x_3, \ldots, x_N)$. Since $v \in H^1_{loc}(\mathbf{R}^N)$, from Theorem 2 p. 164 in [14] it follows that there exists $J \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(J) = 0$ and $u(x_1, \cdot, x'') \in H^1_{loc}(\mathbf{R}^N)$ for any $(x_1, x'') \in \mathbf{R}^{N-1} \setminus J$. Given $x'' \in \mathbf{R}^{N-2}$, we denote

$$K_{x''} = \{x_2 \in \mathbf{R} \mid (x_2, x'') \in K\},$$
$$S_{x''} = \{x_2 \in \mathbf{R} \mid (x_2, x'') \in S\},$$
$$J_{x''} = \{x_1 \in \mathbf{R} \mid (x_1, x'') \in J\}.$$

Fubini's Theorem implies that the sets $K_{x''}$, $S_{x''}$, $J_{x''}$ are $\mathcal{L}^1-$measurable, $\mathcal{L}^1(K_{x''}) < \infty$ and $\mathcal{L}^1(S_{x''}) = \mathcal{L}^1(J_{x''}) = 0$ for $\mathcal{L}^{N-2}-$a.e. $x'' \in \mathbf{R}^{N-2}$. Let

$$\begin{aligned} G \;=\; \{x'' \in \mathbf{R}^{N-2} \mid & \; K_{x''}, S_{x''}, J_{x''} \text{ are } \mathcal{L}^1 \text{ measurable}, \\ & \mathcal{L}^1(S_{x''}) = \mathcal{L}^1(J_{x''}) = 0 \text{ and } 0 < \mathcal{L}^1(K_{x''}) < \infty\}. \end{aligned} \tag{6.36}$$

Clearly, G is $\mathcal{L}^{N-2}-$measurable and $\displaystyle\int_G \mathcal{L}^1(K_{x''})\,dx'' = \mathcal{L}^{N-1}(K) > 0$, thus $\mathcal{L}^{N-2}(G) > 0$. We claim that

$$\int_{\mathbf{R}^2} \Big|\frac{\partial v}{\partial x_2}(x_1, x_2, x'')\Big|^2 dx_1\,dx_2 = \infty \qquad \text{for any } x'' \in G. \tag{6.37}$$

Indeed, let $x'' \in G$. Fix $\varepsilon > 0$. Using (6.36) we infer that there exist s_1, $s_2 \in \mathbf{R}$ such that $(s_1, x'') \in \mathbf{R}^{N-1} \setminus (K \cup S)$, $(s_2, x'') \in K$ and $|s_2 - s_1| < \varepsilon$. Then $v(t, s_1, x'') = 0$ for any $t \in \mathbf{R}$. From step 2 it follows that there exists an interval I with $\mathcal{L}^1(I) \geq k_1$ such that $|v(t, s_2, x'')| \geq |\,|r_0 - v(t, s_2, x'')| - r_0| \geq k_2$ for $t \in I$. Assume $s_1 < s_2$. If $t \in I \setminus J_{x''}$ we have $v(t, \cdot, x'') \in H^1_{loc}(\mathbf{R})$, hence

$$\begin{aligned} k_2 &\leq |v(t, s_2, x'') - v(t, s_1, x'')| = \Big|\int_{s_1}^{s_2} \frac{\partial v}{\partial x_2}(t, \tau, x'')\,d\tau\Big| \\ &\leq (s_2 - s_1)^{\frac{1}{2}} \left(\int_{s_1}^{s_2} \Big|\frac{\partial v}{\partial x_2}(t, \tau, x'')\Big|^2 d\tau\right)^{\frac{1}{2}}. \end{aligned}$$

Clearly, this implies $\displaystyle\int_{s_1}^{s_2} \Big|\frac{\partial v}{\partial x_2}(t, \tau, x'')\Big|^2 d\tau \geq \frac{k_2^2}{\varepsilon}$. Consequently

$$\int_{\mathbf{R}^2} \Big|\frac{\partial v}{\partial x_2}(x_1, x_2, x'')\Big|^2 dx_1\,dx_2 \geq \int_I \int_{s_1}^{s_2} \Big|\frac{\partial v}{\partial x_2}(t, \tau, x'')\Big|^2 d\tau\,dt \geq \frac{k_1 k_2^2}{\varepsilon}.$$

Since the last inequality holds for any $\varepsilon > 0$, (6.37) is proved. Using (6.37), the fact that $\mathcal{L}^{N-2}(G) > 0$ and Fubini's Theorem we get $\displaystyle\int_{\mathbf{R}^N} \Big|\frac{\partial v}{\partial x_2}\Big|^2 dx = \infty$, contradicting the fact that $v \in \mathcal{X}$. Thus necessarily $\mathcal{L}^{N-1}(K) = 0$ and the proof of Lemma 6.4 is complete. ☐

Proposition 6.5 *Assume that $N = 3$ and the conditions (A1) and (A2) are satisfied. Let $u \in \mathcal{C}$ be a minimizer of E_c in $\mathcal{C}$. Then $u \in W^{2,p}_{loc}(\mathbf{R}^3)$ for any $p \in [1, \infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^3)$ for $p \in [2, \infty)$ and there exists $\sigma > 0$ such that $u_{1,\sigma}$ is a solution of (1.3).*

Proof. The proof is very similar to the proof of Proposition 5.6. It is clear that $A(u) = E_c(u) = T_c$ and u is a minimizer of A in $\mathcal{C}$. For any $R > 0$, the functionals $\tilde{B}^u_c$ and $\tilde{A}(v) := A(u + v)$ are C^1 on $H^1_0(B(0, R))$. We proceed in four steps.

Step 1. There exists $w \in C^1_c(\mathbf{R}^3)$ such that $(\tilde{B}^u_c)'(0).w \neq 0$. This follows from Lemma 6.4.

Step 2. There exists a Lagrange multiplier $\alpha \in \mathbf{R}$ such that

$$\tilde{A}'(0).v = \alpha(\tilde{B}^u_c)'(0).v \qquad \text{for any } v \in H^1(\mathbf{R}^3),\ v \text{ with compact support.} \tag{6.38}$$

Step 3. We have $\alpha < 0$.

The proof of steps 2 and 3 is the same as the proof of steps 2 and 3 in Proposition 5.6.

Step 4. Conclusion.

Let $\beta = -\frac{1}{\alpha}$. Then (6.38) implies that u satisfies

$$-\frac{\partial^2 u}{\partial x_1^2} - \beta\left(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) + icu_{x_1} + F(|r_0 - u|^2)(r_0 - u) = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3).$$

For $\sigma^2 = \frac{1}{\beta}$ we see that $u_{1,\sigma}$ satisfies (1.3). It is clear that $u_{1,\sigma} \in \mathcal{C}$ and $u_{1,\sigma}$ minimizes A (respectively E_c) in $\mathcal{C}$. Finally, the regularity of $u_{1,\sigma}$ (thus the regularity of u) follows from Lemma 5.5. ☐

7 Further properties of traveling waves

By Propositions 5.6 and 6.5 we already know that the solutions of (1.3) found there are in $W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$ and in $C^2(\mathbf{R}^N)$. In general, a straightforward boot-strap argument shows that the finite energy traveling waves of (1.1) have the best regularity allowed by the nonlinearity F. For instance, if $F \in C^k([0, \infty))$ for some $k \in \mathbf{N}^*$, it can be proved that all finite energy solutions of (1.3) are in $W^{k+2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$ (see, for instance, Proposition 2.2 (ii) in [33]). If F is analytic, it can be proved that finite energy traveling waves are also analytic. In the case of the Gross-Pitaevskii equation, this has been done in [5].

A lower bound $K(c, N)$ on the energy of traveling waves of speed $c < v_s$ for the Gross-Pitaevskii equation has been found in [35]. The constant $K(c, N)$ is known explicitly and we have $K(c, N) \longrightarrow 0$ as $c \longrightarrow v_s$. In the case of general nonlinearities, we know that *any* finite energy traveling wave u of speed c satisfies the Pohozaev identity $P_c(u) = 0$, that is $u \in \mathcal{C}$. Then it follows from Lemma 4.7 that $A(u) \geq \frac{N-1}{2} T_c > 0$.

Our next result concerns the symmetry of those solutions of (1.3) that minimize E_c in $\mathcal{C}$.

Proposition 7.1 *Assume that $N \geq 3$ and the conditions (A1), (A2) in the introduction hold. Let $u \in \mathcal{C}$ be a minimizer of E_c in $\mathcal{C}$. Then, after a translation in the variables $(x_2, \dots, x_N)$, u is axially symmetric with respect to Ox_1.*

Proof. Let T_c be as in Lemma 4.7. We know that any minimizer u of E_c in $\mathcal{C}$ satisfies $A(u) = \frac{N-1}{2} T_c > 0$. Using Lemma 4.8 (i), it is easy to prove that a function $u \in \mathcal{X}$ is a minimizer of E_c in $\mathcal{C}$ if and only if

$$(7.1) \qquad u \text{ minimizes the functional } -P_c \text{ in the set } \{v \in \mathcal{X} \mid A(v) = \frac{N-1}{2} T_c\}.$$

The minimization problem (7.1) is of the type studied in [32]. All we have to do is to verify that the assumptions made in [32] are satisfied, then to apply the general theory developed there.

Let Π be an affine hyperplane in $\mathbf{R}^N$ parallel to Ox_1. We denote by s_Π the symmetry of $\mathbf{R}^N$ with respect to Π and by Π^+, Π^- the two half-spaces determined by Π. Given a function $v \in \mathcal{X}$, we denote

$$v_{\Pi^+}(x) = \begin{cases} v(x) & \text{if } x \in \Pi^+ \cup \Pi, \\ v(s_\Pi(x)) & \text{if } x \in \Pi^-, \end{cases} \qquad \text{and} \qquad v_{\Pi^-}(x) = \begin{cases} v(x) & \text{if } x \in \Pi^- \cup \Pi, \\ v(s_\Pi(x)) & \text{if } x \in \Pi^+. \end{cases}$$

It is easy to see that $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$. Moreover, for any $v \in \mathcal{X}$ we have

$$A(v_{\Pi^+}) + A(v_{\Pi^-}) = 2A(v) \quad \text{and} \quad P_c(v_{\Pi^+}) + P_c(v_{\Pi^-}) = 2P_c(v).$$

This implies that assumption ($\mathbf{A1}_c$) in [32] is satisfied.

By Propositions 5.6 and 6.5 and Lemma 5.5 we know that any minimizer of (7.1) is C^1 on $\mathbf{R}^N$, hence assumption ($\mathbf{A2}_c$) in [32] holds. Then the axial symmetry of solutions of (7.1) follows directly from Theorem 2' in [32]. ☐

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