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Estimation of Parameters of a Multifractal Process

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Abstract

Multifractal functions are widely used to model irregular signals such as turbulence, data stream or road traffic. Here, we consider multifractal functions defined as lacunar wavelet series observed in a white noise model. These random functions are statistically characterized by two parameters. The first parameter governs the intensity of the wavelet coefficients while the second one governs its sparsity. We construct estimators of these two parameters and discuss statistical properties of this important model: the rate of the Fisher information and a testing procedure to check the multifractal feature of an observed noisy signal.

Key Words: Multifractal analysis, wavelet bases, empirical estimators

AMS subject classification: 62F03, 62A05.

1 Introduction

In the last decade much emphasis has been placed on multifractal models. Roughly speaking, a multifractal function is a function whose local Hölder regularity index is not constant. That means that the function may be very regular in some areas while it is very irregular in others. Such functions, with rapid changes of regularity, have been first introduced to model physical phenomena such as turbulence by Bacry et al. (1991), or road traffic or data traffic on networks by Riedi et al. (1999). Arneodo et al. (1999), Aubry and Jaffard (2002) or Roueff (2001) and Roueff (2003) have recently shown that some lacunary random series built on wavelets have multifractal properties.

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In this paper we address parametric estimation problems in a mixture model arising in random tal functions. To begin with, let us briefly describe the model we will work with. Given a fixed wavelet ψ, for $j_1 \in \mathbb{N}$, let $n = 2^{j_1}$ be the number of observations. Further, consider the triangular array $\mathbf{d}_n = (d_{jk})_{1 \le j \le j_1, 0 \le k \le 2^j - 1}$ of independent random variables. For any j, k the distribution of d_{jk} is the Gaussian mixture:

$$d_{jk} \sim 2^{(\eta_0 - 1)j} \mathcal{N}(2^{-\alpha_0 j}, \frac{\sigma^2}{n}) + (1 - 2^{(\eta_0 - 1)j}) \mathcal{N}(0, \frac{\sigma^2}{n}), \tag{1.1}$$

where $(\alpha_0, \eta_0)^T \in (0,1)^2$ is the unknown parameter vector, $\sigma > 0$ is known and, as usual, $\mathcal{N}(m, \xi^2)$ denotes the Gaussian distribution with mean m and standard deviation ξ. This statistical model comes from random lacunar wavelet series used to model multifractal functions. It can be obtained in the classical white noise model, drawing independently wavelet coefficients with a rescaled Bernoulli distribution. As a matter of fact, if at all levels $j \in [1, j_1]$ the random wavelet coefficient w_{jk}, $k = 0, \ldots, 2^j - 1$ has the rescaled Bernoulli distribution:

$$2^{(\eta_0 - 1)j} \delta_{2^{-\alpha_0 j}} + (1 - 2^{(\eta_0 - 1)j}) \delta_0, \tag{1.2}$$

then the corresponding random function $f_n = \sum_{j=1}^{\log_2 n} \sum_{k=0}^{2^j - 1} w_{jk} \psi_{jk}$ converges to a random function having multifractal properties. The multifractal properties of general sparse random series have been studied in Aubry and Jaffard (2002), Roueff (2001), Jaffard (2000), Arneodo et al. (1998). In Section 2, we recall the multifractal properties of sparse random series. Moreover, we explain how our observation model d_n may be interpreted in this context. In the Bayesian white noise model, when the prior on wavelet coefficients is given by (1.2), the observed wavelet coefficients $(d_{jk})_{1 \le j \le j_1 - 1, 0 \le k \le 2^j - 1}$ are independent and distributed as in (1.1). In a previous work Gamboa and Loubes (2005), we studied nonparametric estimation in this Bayesian setting. Roughly speaking, we have shown that the Bayesian nonparametric posterior estimate is built on a ranked thresholding procedure. We also have studied the rates of convergence which appeared to be quite different than those usually found in the non paradigm. We focus here on empirical estimation of the hyperparameters $(\alpha_0, \eta_0)^T \in (0,1)^2$. As statistical inference based on likelihood is hard to achieve, we propose and study empirical estimators of $(\alpha_0, \eta_0)^T$. Two kinds of empirical estimators are studied. In Section 3.1, we build empirical moment estimators and show that these estimates are convergent and also satisfy a central limit theorem. In Section 3.2 we repeat the study for a level counting estimate, while in Section 3.3 we present an alternative numerical method to estimate the parameters using an EM-type algorithm. Section 3.4 is devoted to the asymptotic study of the Fisher information of the model. Moreover, we compare the rates of convergence of the empirical estimators to the optimal asymptotic rate. We show that the normalizing rate of convergence does not correspond in general to the rate of convergence of Fisher's information. In Section 4, we give some applications of the asymptotic results obtained in Section 3.1 and 3.2. All the proofs are postponed to Section 5.

2 Multifractal wavelet models

In this paper, we will always work with functions on $[0,1]$. To begin with, let us first introduce some useful definitions around multifractal functions. The Haussdorf dimension $d_H(A)$ of a set A is defined as follows. Let $C(A,\delta)$ be the set of all $\delta-$covering (C_i) of A with open sets C_i of diameter $|C_i|\leq\delta$. Let also

$$H_{s,\delta}(A) = \inf_{(C_i)_i\in C(A,\delta)} \sum_i |C_i|^s$$
$$H_s(A) = \lim_{\delta\to 0} H_{s,\delta}(A)$$
$$d_H(A) = \inf\{s\ : H_s(A)=0\}$$

Definition 2.1. *Let f be a function on $[0,1]$.*

1) Let $x_0\in[0,1]$ and $h\geq 0$, the set $C^h(x_0)$ is the set of all functions f on $[0,1]$ such that there exist a polynomial P of degree less than h and a neighborhood V of x_0 satisfying

$$|f(x)-P(x)| = O\left(x-x_0\right)^h \quad (x\in V).$$

2) Let $h_f(x_0)=\sup\{h\geq 0, f\in C^h(x_0)\}$ and

$$E_h = \{x\in[0,1],\ h_f(x)=h\}\ \ (h\geq 0).$$

The spectrum of singularity d_f of f is the function on $\mathbb{R}^+$ which associates to each $h\geq 0$ the Haussdorf dimension of the set E_h.

Multifractal analysis of a function was first introduced in a physical framework in Frisch and Parisi (1985). Given a function f, one of the main goal of this analysis is the computation of the spectrum of singularities d_f. When d_f does not vanish in at least two points we say that f is multifractal. The spectrum of singularities of a function f is a relevant quantity to describe the smoothness variation of f. Multifractal functions can be constructed using their decomposition into an appropriate wavelet basis as described in Arneodo et al. (1999) and Jaffard (2000). Since we restrict attention to functions on $[0,1]$, we will only consider periodized wavelets in the Schwartz class. This implies that all moments of the wavelets vanish. In an equivalent way, we could have used compactly supported wavelets as it is stated in Jaffard (2000) but the results are heavier to state.

Following the construction provided in Meyer (1990) we define a wavelet $\tilde{\psi}$ in the Schwartz class, and construct the periodized wavelet $\psi(x) = \sum_{l\in\mathbb{Z}} \tilde{\psi}(x-l)$. The functions $\psi_{jk} = \psi(2^j x - l), \quad \forall j \in \mathbb{N},\ k \in [0, 2^j - 1]$ are obtained from the first wavelet by dilatation and translation. Then $(2^{j/2}\psi_{jk})_{(j,k)}$ provides an orthonormal basis of the Hilbert space $L^2([0,1])$ (observe the presence of a normalizing factor $2^{j/2}$). Let $f \in L^2([0,1])$ on one hand, its wavelet coefficients may be computed as

$$w_{jk} = 2^j \int_0^1 f(t)\psi_{jk}(t)dt \ \ (j \in \mathbb{N},\ k \in [0, 2^j - 1]).$$

On the other hand, f may be reconstructed using its wavelet coefficients

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} w_{jk}\psi_{jk} \tag{2.1}$$

Using this wavelet representation, we now turn on the construction of random functions exhibiting multifractal properties. This will be done considering sparse random wavelet series. Let ρ_j, $j \in \mathbb{N}^*$ be a repartition function on $\mathbb{R}$. Further, let $Z_j = (Z_{jk})_{k=0,\ldots,2^j-1}$ be 2^j independent random vectors having common distribution ρ_j. Now, build a random function F using the reconstruction formula (2.1) where for any $j \in \mathbb{N}^*, k = 0, \ldots, 2^j - 1$ $|w_{jk}| = 2^{-jZ_{jk}}$. To study the multifractal properties of the random function F, Aubry and Jaffard (2002), Jaffard (2000) introduced the following functions:

$$\tilde{\rho}(\alpha,\epsilon) = \limsup_{j\to\infty} \frac{\log_2(2^j \rho_j[\alpha-\epsilon,\alpha+\epsilon])}{j}$$
$$= 1 + \limsup_{j\to\infty} \frac{\log \mathbf{P}(Z_{jk} \in [\alpha-\epsilon,\alpha+\epsilon])}{j}$$
$$\tilde{\rho}(\alpha) = \inf_{\epsilon>0} \tilde{\rho}(\alpha,\epsilon)$$

Under some assumptions on $(\rho_j)_{j\in\mathbb{N}^*}$, which can be found in Jaffard (2000), or Aubry and Jaffard (2002), Jaffard et al. prove that the spectrum of singularity of F can be calculated. Indeed, they show that, for all $h > 0$

$$d_F(h) = h \sup_{\alpha\in(0,h]} \frac{\rho(\alpha)}{\alpha} \text{ (a.s.).} \tag{2.2}$$

In this paper, we focus on the simplest statistical model derived from the ones described in the last paragraph. Let $(X_{jk})_{j\in\mathbb{N}^*,\, k=0,\ldots,2^j-1}$ be a triangular array of independent Bernoulli random variables: for $\eta_0 \in (0,1)$

$$\mathbf{P}(X_{jk}=1) = 1 - \mathbf{P}(X_{jk}=0) = 2^{(\eta_0-1)j}.$$

Further, take for $j \in \mathbb{N}^*$ and $k = 0,\ldots,2^j-1$ random wavelet coefficients $w_{jk} = 2^{-\alpha_0 j} X_{jk}$ for $\alpha_0 \in (0,1)$. So we get

$$w_{jk} \sim 2^{(\eta_0-1)j}\delta_{2^{-\alpha_0 j}} + (1-2^{(\eta_0-1)j})\delta_0 \tag{2.3}$$

The corresponding function f is then defined by its wavelet decomposition into the basis ψ_{jk} by $f = \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} w_{jk}\psi_{jk}$. So, this simple multifractal model is characterized by two parameters η_0 and α_0 in $(0,1)$. On one hand η_0 describes the lacunarity of the wavelet series (that is its sparsity). On the other hand the coefficient α_0 is inversely proportional to the intensity of the value of the wavelet coefficients. These parameters completely characterizes the spectrum of singularity of the random functions involved. Generating a function with this method may seem too restrictive. However, such processes appear naturally when studying multifractal processes. As a matter of fact, Jaffard et al. in their work (Aubry and Jaffard, 2002; Arneodo et al., 1998; Jaffard, 2000) use such modelization and they show in Aubry and Jaffard (2002) that the spectrum of singularity of the function f is almost surely

$$d_f(h) = \frac{1-\eta_0}{\alpha_0} h,\ \forall h \in \left[\alpha_0, \frac{\alpha_0}{1-\eta_0}\right]. \tag{2.4}$$

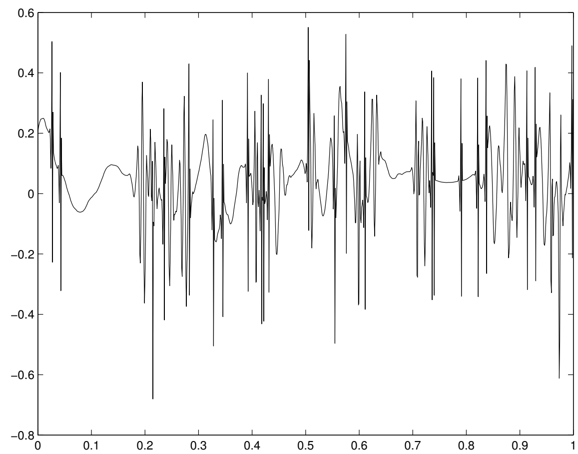

Figure 1: *Multifractal function*

In Figure 1 we plot a realization of such a multifractal function. The lacunarity coefficient is $\eta_0 = 0.4$ while $\alpha_0 = 0.3$. In this paper, we will build and study estimators of these two parameters when the observation is a wavelet series observed in a white noise model.

3 Parametric estimation of lacunarity wavelet series

We aim at estimating the parameters η_0 and α_0 of a multifractal function f observed with some measurement errors. We will not address the nonparametric estimation problem since it has already been tackled by the authors in a previous work (Gamboa and Loubes, 2005). Throughout the paper, we make the assumption that the wavelet basis, in which the function has the decomposition (2.3), is known.

In the white noise model, we observe the wavelet coefficients w_{jk} of the function f together with a Gaussian white noise ϵ_{jk} having variance $\frac{\sigma^2}{n}$ where n is the number of observations. We assume that the observations are dyadic and $n = 2^{j_1}$, $(j_1 > 0)$. Recall that the wavelet coefficients are obtained from discrete regression model

$$Y_i = f(i/2^{j_1}) + W_i,\ i = 1, \dots, n \tag{3.1}$$

by performing the Discrete Wavelet Transform (DWT). Such transform is performed by Mallat's fast algorithm (Mallat, 1989) that requires only $O(n)$ operations. Hence, the observations are drawn from the following model:

$$d_{jk} = w_{jk} + \epsilon_{jk},\ j = 1, \dots, j_1,\ k = 0, \dots, 2^j - 1$$

As a result, the law of the observed coefficients d_{jk} is determined by the prior given by the model (2.3):

$$d_{jk} \sim 2^{(\eta_0-1)j}\mathcal{N}\left(2^{-\alpha_0 j}, \frac{\sigma^2}{n}\right) + (1 - 2^{(\eta_0-1)j})\mathcal{N}\left(0, \frac{\sigma^2}{n}\right) \tag{3.2}$$

We could have also considered an extension of the previous model, allowing the coefficients w_{jk} to differ slightly from the two values 0 and $2^{-\alpha_0 j}$. For this, we may take, using the same notations with $1 \le j \le j_1$, $k = 0, \dots, 2^j - 1$:

$$w_{jk} \sim 2^{(\eta_0-1)j}\mathcal{N}(2^{-\alpha_0 j}, \Delta_j^2) + (1 - 2^{(\eta_0-1)j})\mathcal{N}(0, \Delta_j^2), \tag{3.3}$$

where Δ_j^2 is a strictly positive sequence satisfying the following condition

$$\Delta_j^2 = O(2^{-j}). \tag{3.4}$$

In this case, assumptions (3.4) imply that both models share the same asymptotic properties.

3.1 Moment estimators

A first natural way to build empirical estimates of (η_0, α_0) is to use the moment method. To begin with, observe that we obviously have

$$\mathbf{E}d_{jk} = 2^{(\eta_0-1-\alpha_0)j}, \quad \mathbf{E}d_{jk}^2 = \frac{\sigma^2}{n} + 2^{(\eta_0-1-2\alpha_0)j}.$$

This leads to the following empirical moment estimates of (η_0, α_0): for a function $f_n = \sum_{j=1}^{j_1}\sum_k w_{jk}\psi_{jk}$ whose coefficients are drawn according the prior defined in Section 2, observed in a white noise model, define:

$$\hat{\alpha}_n = \frac{1}{j_1 \log 2}\left(\log\left[\frac{\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} d_{jk}}{\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} d_{jk}^2 - \sigma^2}\right]\right) \tag{3.5}$$

$$\hat{\eta}_n = \hat{\alpha}_n + \frac{1}{j_1 \log 2}\log\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} d_{jk} \tag{3.6}$$

The following theorem describes the asymptotic behavior of the moment estimators.

Theorem 3.1. *Assume that* $\eta_0 - 2\alpha_0 > 0$ *and set*

$$\kappa = -\sum_{j=1}^{j_1} 2^{2(\eta_0-1-\alpha_0)j} - 2^{(\eta_0-1-2\alpha_0)j} = \frac{2^{\eta_0-1-2\alpha_0} - 2^{2(\eta_0-1-\alpha_0)}}{(1-2^{2(\eta_0-1-\alpha_0)})(1-2^{(\eta_0-1-2\alpha_0)})} \tag{3.7}$$

Then, $\hat{\alpha}_n$ *and* $\hat{\eta}_n$ *are consistent estimators of* α_0 *and* η_0*. Further, we have:*

$$\log(n)\sqrt{n^{\eta_0}}(\hat{\alpha}_n - \alpha_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \tag{3.8}$$

$$\log(n)n^{\eta_0-\alpha_0}\left(\hat{\eta}_n - \eta_0\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \kappa + \sigma^2\right) \tag{3.9}$$

Remark 3.1. *If* α_0 *is known, the asymptotic result (3.9) is true in the less restrictive case* $\eta_0 > \alpha_0$*. For the remaining subcase* $\eta_0 - \alpha_0 \leq 0$*, the mean of* d_{jk} *goes to zero. Hence, an estimator directly based on the mean could not be consistent.*

The proof of the previous theorem will be tackled using the two following technical lemmas on asymptotic normality for sums of coefficients d_{jk} and d_{jk}^2. Δ-method (see in van der Vaart, 1998) enables to get the final result. The proof is postponed to the Appendix.

Lemma 3.1. *For* $\eta_0 > \alpha_0$*, set* $C_{j_1} = 2^{(\alpha_0-\eta_0)j_1} = n^{\alpha_0-\eta_0} \to 0$*. Then,*

$$C_{j_1}^{-1}\left(C_{j_1}\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} d_{jk} - 1\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2 + \kappa\right) \tag{3.10}$$

Lemma 3.2. *For* $\eta_0 > 2\alpha_0$*, set* $D_{j_1} = 2^{(2\alpha_0-\eta_0)j_1} = n^{2\alpha_0-\eta_0}$*, and observe that* D_{j_1} *vanishes as* n *goes to infinity. Then we get:*

$$\sqrt{n^{\eta_0}}\left[D_{j_1}\left(\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} d_{jk}^2 - \sigma^2\right) - 1\right] \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \tag{3.11}$$

The two conditions $\eta_0 > \alpha_0$ and $\eta_0 > 2\alpha_0$ of Lemma 3.1 and 3.2 mean that the true signal f must contain enough energy in order to estimate the parameters.

The first estimator defined in (3.6) estimates in fact $\eta_0 - \alpha_0$ while the second estimates the quantity $\eta_0 - 2\alpha_0$. Only a computational trick enables us to estimate the parameters separately, but the relations between these parameters are worth a closer attention. Maybe a change in the representation of the model should be appropriate.

Remark 3.2. *The estimation problem can be linked with the estimation of the index of a multifractional process. Such issue is tackled by Benassi et al. (1998). Indeed, both rates of convergence depend on the value of the true parameter, as in a nonparametric framework. Moreover, the technics used in the proofs are quite similar, since in both settings, one deals with a rescaling of a quadratic variation.*

3.2 Level counting estimator

The main drawback of the previous mean estimator is that the estimation of the lacunarity parameter η_0 is linked with the intensity parameter α_0. To overcome this link, we build another estimator for η_0.
For this, recall that the distribution of the coefficients is given by the mixture,

$$d_{jk} \sim 2^{(\eta_0-1)j}\mathcal{N}\left(2^{-\alpha_0 j}, \frac{\sigma^2}{n}\right) + (1 - 2^{(\eta_0-1)j})\mathcal{N}\left(0, \frac{\sigma^2}{n}\right).$$

As a consequence if we rescale the coefficients by $\sqrt{n}$ we get the following distribution

$$\sqrt{n}d_{jk} \sim 2^{(\eta_0-1)j}\mathcal{N}(m_j, \sigma^2) + (1 - 2^{(\eta_0-1)j})\mathcal{N}(0, \sigma^2).$$

with $m_j = 2^{j_1/2-\alpha_0 j}$, $j = 1, \ldots, j_1$. Under the hypothesis of Theorem 3.1 we have that m_j goes to infinity with j. As a result, the two components of the rescaled mixture

$$(1) = \mathcal{N}(m_j, \sigma^2), \quad (2) = \mathcal{N}(0, \sigma^2)$$

are asymptotically well separated. So, the two kinds of wavelet coefficients can be efficiently separated using a thresholding procedure. We will use this idea to build an estimator of the lacunarity parameter η_0.
Let l_n be an increasing sequence of positive real numbers and set

$$S_n = \frac{1}{n}\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} 1_{\sqrt{n}d_{jk} \geq l_n}.$$

Define the following estimator:

$$\tilde{\eta}_n = 1 + \frac{1}{\log_2 n} \log_2 (S_n) \tag{3.12}$$

Since the two groups of random variables are well separated when the level of resolution j increases, the number of rescaled coefficients $\sqrt{n} d_{jk}$ above a fixed level l_n can be used to estimate the proportion of coefficients which belong to the first group (1).

Theorem 3.2. *Assume as previously that $\alpha_0 < \frac{1}{2}$ and $\eta_0 - 2\alpha_0 > 0$ Take $l_n = \log_2 n$, then $\hat{\eta}_n$ is a consistent estimator of the lacunarity parameter of the mixture, η_0, and*

$$n^{\frac{\eta_0}{2}} \log(n) \ (\tilde{\eta}_n - \eta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1). \tag{3.13}$$

Lemma 3.3. *Assume that $\alpha_0 < \frac{1}{2}$ and $\eta_0 - 2\alpha_0 > 0$. Set also $d_{j_1} = 2^{\frac{\eta_0}{2} j_1} = n^{\frac{\eta_0}{2}}$, and $c_{j_1} = 2^{(1-\eta_0) j_1} = n^{1-\eta_0}$. Set also*

$$F(l_n) = \mathbf{P}(\mathcal{N}(0,1) \geq l_n).$$

If l_n satisfies the three following conditions:

$$l_n \to +\infty, \ \ 2^{(\alpha - \frac{1}{2}) j_1} - l_n \to +\infty, \ \ c_{j_1} F(l_n) \leq M,$$

for a given positive constant M. Then as n goes to infinity:

$$d_{j_1} \left(c_{j_1} [S_n - F(l_n)] \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1 + \kappa_2), \tag{3.14}$$

where $\kappa_2 = \lim_{j_1 \to \infty} (c_{j_1} F(l_n) - c_{j_1} F^2(l_n)) < \infty$ on the choice of the growing sequence (l_n).

We point out that such estimator is more accurate than the estimator $\hat{\eta}_n$ defined in Section 3.1. Indeed, the parameter η_0 only depends on the number of non zero whereas the mean based estimator artificially links the position and the intensity of the wavelet coefficients. As a result, the estimator $\tilde{\eta}_n$ counts the number of significant observed coefficients and is more directly concerned with the proportion of the mixture η_0 without any consideration about the values taken by the coefficients.
Hence $n^{\frac{\eta_0}{2}} \log n$ is the common rate of convergence for both estimator $\hat{\alpha}_n$ and $\tilde{\eta}_n$ and is a characteristic of the multifractal model under the restriction that the intensity parameter satisfies $\alpha_0 < \frac{\eta_0}{2}$.

For applications, we first can see that the estimation of the intensity coefficient α_0 relies on the prior knowledge of the variance of the observation noise σ. Hence, this estimation issue is of great interest in practice. A way of estimating this quantity is given by the following procedure: first, compute $\tilde{\eta}$ whose expression (3.12) is independent from the variance. Then, using (3.6), we can deduce an estimation of α_0, when an estimator of η_0 is known:

$$\tilde{\alpha}_n = \tilde{\eta}_n - \frac{1}{j_1 \log 2} \log \sum_{j=1}^{j_1} \sum_{k=0}^{2^j} d_{jk}.$$

Then, using (3.5), we can construct a moment estimator of the variance:

$$\hat{\sigma}_n^2 = \sum_{j=1}^{j_1} \sum_{k=0}^{2^j-1} d_{jk}^2 - 2^{-\tilde{\alpha}_n j_1(n)} \sum_{j=1}^{j_1} \sum_{k=0}^{2^j-1} d_{jk}. \tag{3.15}$$

This estimator can be studied in a similar way as the study of the moment estimator defined by (3.5).

Remark 3.3. *As from Theorem 3.2,* $n^{\frac{\tilde{\eta}_n - \eta_0}{2}} = \exp\left(\frac{\log n}{2}(\tilde{\eta}_n - \eta_0)\right)$ *goes to* 1 *in probability, we have that* $\log n n^{\frac{\tilde{\eta}_n}{2}}(\hat{\alpha}_n - \alpha_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$ *and* $\log n n^{\frac{\tilde{\eta}_n}{2}}(\tilde{\eta}_n - \eta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. *Thus, we may construct asymptotic confidence interval for* α_0 *and* η_0 *and test on the values of these parameters.*

3.3 Estimation of model parameters with EM algorithm

In this part, we show that the parameters of the multifractal signal (intensity α_0, lacunarity η_0 and level of noise σ) can be estimated using an EM algorithm well suited for such a mixture framework. Indeed, the EM algorithm is a recursive algorithm used to maximize log-likelihood when the variables are not directly observed. A direct application is the classification problem in mixture settings (see for instance McLeish and Small, 1986). Let us illustrate this algorithm on a single Gaussian mixture model. Let $Y_1, \ldots, Y_n$ be an i.i.d sample of a random vector Y having density: $f(y, \Psi_0) = \pi_0 \phi\left(y; \mu_1^0, \sigma_0\right) + (1 - \pi_0)\phi\left(y; \mu_2^0, \sigma_0\right)$, where $\phi(y; \mu_i, \sigma)$ is the Gaussian density function with mean μ_i and variance σ^2, for $i \in \{1, 2\}$. The parameter of interest is $\Psi_0 = (\pi_0, \theta^T)^T$, where $\theta = (\mu_1^0, \mu_2^0, \sigma_0)^t$. The log-likelihood is:

$$L(\Psi_0) = \sum_{j=1}^{n} \log\left(\pi_0\phi\left(Y_j;\mu_1^0,\sigma_0\right) + (1-\pi_1^0)\phi\left(Y_j;\mu_2^0,\sigma_0\right)\right).$$

To apply the EM-algorithm, we transform this model into a missing observation model. For $j \in \{1,\ldots,n\}$, let Z_j, be a random variable equal to 1 if Y_j comes from the first component, i.e with law $\mathcal{N}(\mu_1^0,\sigma_0)$, and $\mu_2^0 = 0$ otherwise. The complete data are $X_c = (X_1^T,...,X_n^T)$, with $X_1 = (Y_1,Z_1)^T,...,X_n = (Y_n,Z_n)^T$. In the complete model the log-likelihood is:

$$L_c(\Psi_0) = \sum_{j=1}^{n} z_j \log\left[\pi_0\phi\left(y_j;\mu_1^0,\sigma_0\right)\right] + (1-z_j)\log\left[(1-\pi_0)\phi\left(y_j;0,\sigma_0\right)\right]. \tag{3.16}$$

Set y_{obs} the values of the data $(Y_1,\ldots,Y_n)'$. From the theory of EM algorithm, we know that maximizing in the parameter of interest Ψ the log-likelihood is equivalent to maximizing in a recursive way, the following quantity, where all the estimated quantities are taken at the k-th step:

$$\begin{aligned} Q\left(\Psi,\Psi^{(k)}\right) &= \mathbf{E}\left(L_c(\Psi)/y_{obs};\Psi^{(k)}\right) \\ &= \sum_{j=1}^{n} \mathbf{E}\left(Z_j/y_{obs};\Psi^{(k)}\right)\log\left[\pi^{(k)}\phi\left(y_j;\mu_1^{(k)},\sigma^{(k)}\right)\right] \\ &+ \mathbf{E}\left((1-Z_j)/y_{obs};\Psi^{(k)}\right)\log\left[(1-\pi^{(k)})\phi\left(y_j;0,\sigma^{(k)}\right)\right]. \end{aligned}$$

We now may apply this general algorithm to our wavelet model. Write $\mu_1^0 = 2^{-\alpha_0 j}$, $\mu_2 = 0$ and $\pi_0 = 2^{(\eta_0-1)j}$. At a fixed level j, the augmented likelihood is

$$\begin{aligned} &L(d_{jk}^*,m,\pi) \\ &= \left(\log\frac{\pi}{1-\pi};m^2;m\right)\left(\sum_k z_{jk};-\frac{n}{2(\sigma^{(k)})^2}\sum_k z_{jk};\frac{n}{(\sigma^{(k)})^2}\sum_k d_{jk}^* z_{jk}\right)' \\ &+ 2^j\log(1-\pi) = a(\theta)'b(X) + c(\theta) + d(X), \end{aligned}$$

with $\theta = (\alpha_0\ \eta_0\ \sigma_0)'$. We recognize an exponential family. Then, the EM algorithm can be written at the $i+1$-step:

- E step: $\mathbf{E}(b(X)|d^*,\theta^i) = (\sum_k \hat{z}_{jk}^{(i)}; -\frac{n}{2(\sigma^{(k)})^2}\sum_k \hat{z}_{jk}^{(i)}; \frac{n}{(\sigma^{(k)})^2}\sum_k d^*_{jk}\hat{z}_{jk}^{(i)})$ where $\hat{z}_{jk}^{(i)} = \mathbf{P}(z_{jk} = 1|d^*, \theta^{(i)})$.
- M step: in order to maximize the functions:

$$\begin{cases} f(\pi) & = \log\left(\frac{\pi}{1-\pi}\right)\sum_k z_{jk} + 2^j \log(1-\pi) \\ g(m) & = -\frac{n}{2(\sigma^{(k)})^2} m^2 \sum_k z_{jk} + \frac{nm}{(\sigma^{(k)})^2}\sum_k d^*_{jk} z_{jk} \end{cases}$$

write the first order condition and this gives raise to the three estimated parameters:

$$\hat{m}^{(i+1)} = \frac{\sum_k d^*_{jk}\hat{z}_{jk}^{(i)}}{\sum_k \hat{z}_{jk}^{(i)}} \quad \hat{\pi}^{(i+1)} = \frac{1}{2^j}\sum_k \hat{z}_{jk}^{(i)},$$

$$(\hat{\sigma}^{(i+1)})^2 = \frac{\sum_k \hat{z}_{jk}^{(i)}(d^*_{jk} - \hat{m}^{(i+1)})}{\sum_k \hat{z}_{jk}^{(i)}}.$$

The starting point of each iteration is the estimator obtained in the previous step.
An application of the previous estimators is the reconstruction of a multifractal signal observed in a white noise model, as it is done in Gamboa and Loubes (2005).

3.4 On the efficiency of the estimation procedure

We have constructed different estimators for the parameters of the lacunarity wavelet series that defines the multifractal function. We can see that such prior model creates asymptotic behaviour which are very different from the one encountered in regular models (see for instance van der Vaart, 1998). In particular, the rate of convergence of the estimators depends on the values of both the intensity parameter α_0 and the lacunarity parameter η_0.
Define $\mathcal{L}_n(\eta,\alpha)$ the log-likelihood of the model for the parameters η and α in $(0,1)$.

$$\mathcal{L}_n(\eta,\alpha) = \sum_{j=1}^{\log_2 n}\sum_{k=0}^{2^j-1}\log\left(2^{(\eta-1)j}\phi_{2^{-\alpha j},\frac{\sigma^2}{n}}(d_{jk}) + (1-2^{(\eta-1)j})\phi_{0,\frac{\sigma^2}{n}}(d_{jk})\right),$$

where $\phi_{m,\sigma^2}(x)$ is the density of a Gaussian variable with mean m and variance σ^2. The following propositions give its asymptotic expansion

Theorem 3.3. *Write* $I_n(\eta_0) = \mathbf{E}\left(-\frac{d^2}{d\eta^2}\big|_{(\alpha_0,\eta_0)} \mathcal{L}_n(\eta,\alpha)\right)$ *the Fisher's information in* η *for the model* (2.3). *We have the following asymptotic expansion:*

$$I_n(\eta_0) = \log(n) n^{2\eta_0 - 1} + o(1).$$

This result must be compared with the rate of convergence of the estimator of the lacunarity parameter. We recall that we proved the following Central Limit Theorem: $\log(n) n^{\frac{\eta_0}{2}}(\tilde{\eta}_n - \eta_0) \to \mathcal{N}(0,1)$. As a result, if we compare the rate of convergence with the order of Fisher's information of the model, we can see that in the range $\frac{1}{2} < \eta_0 < \frac{2}{3}$, the estimator $\tilde{\eta}_n$ is asymptotically super efficient in the Cramer Rao sense.

Theorem 3.4. *Set* $J_n(\alpha_0) = \mathbf{E}\left(-\frac{d^2}{d\alpha^2}\big|_{(\alpha_0,\eta_0)} \mathcal{L}_n(\eta,\alpha)\right)$ *the Fisher's information in* α *for the model* (2.3). *The following asymptotic order holds:*

$$J_n(\alpha_0) = \log^2(n) n^{1+\eta_0 - 2\alpha_0} + o(1).$$

In this case, we recall that $\log(n) n^{\eta_0 - \alpha_0}(\hat{\alpha}_n - \alpha_0) \to \mathcal{N}(0, \kappa + \sigma^2)$. As a consequence, the normalizing rate of convergence for the Fisher's information never corresponds to the rate of convergence of the Central Limit Theorem.

4 Testing for a multifractal structure against white noise

Our goal is to construct a procedure to check if an observed data correspond to the model (2.3), where both α_0 and η_0 are known. Due to the complexity of the mixture, with both mean and variance depending on n (the number of observations), we only focus on the last resolution level $j_1(n) = \log_2 n$: for all $k = 0, \dots, 2^j - 1$,

$$\sqrt{n} w_{j_1 k} = X_k \sim n^{\eta_0 - 1}\mathcal{N}(m_n, 1) + (1 - n^{\eta_0 - 1})\mathcal{N}(0,1),$$

where $m_n = n^{\frac{1}{2} - \alpha_0}$.
We construct a goodness of fit test by the Neyman-Pearson procedure. We test

$$\mathbf{H_0} : \ w_0, \dots, w_{n-1} \sim h_0 = n^{\eta_0 - 1}\phi_{m_n,1}(x) + (1 - n^{\eta_0 - 1})\phi_{0,1}(x)$$

against

$$\mathbf{H_1}: \; w_0, \ldots, w_{n-1} \sim h_1 = \phi_{0,1}(x).$$

We have used the following notations:
h_0 is the density of the rescaled coefficients under the null hypothesis H_0 while h_1 is the density under the hypothesis H_1.
Going back to the white noise model (3.1), under H_1 the observed signal $(Y_i)_{i=1,\ldots,n}$ only contains white noise ($f = 0$).
Hence, we are testing the multifractal model (α_0, η_0) in white noise, against pure white noise. The so-called Neyman-Pearson procedure rejects H_0 when the log-likelihood ratio statistic is too small. Hence, setting

$$S_n(X) = \frac{1}{n} \sum_{i=0}^{n-1} \log \left[\frac{h_0}{h_1}(X_i) \right].$$

We reject H_0 when this statistic is less than some threshold depending on the required level. It is well known, that this is the most powerful way to decide between H_0 and H_1.
We will now study the asymptotic behaviour of this testing procedure.
The following lemma studied the rate of $\mathbf{E}(S_n(X))$ under H_0 and H_1.

Lemma 4.1.

$$\mathbf{E}_{H_0}(S_n(X)) = \frac{1}{2} n^{\eta_0 - 2\alpha_0}(1 + o(1)) \tag{4.1}$$

$$\mathbf{E}_{H_1}(S_n(X)) = n^{\eta_0 - 1}(1 + o(1)). \tag{4.2}$$

Unusually, under H_1, this expectation goes to zero, while it tends to infinity under H_0. Hence, we will be able to construct a convergent test.

Theorem 4.1. *Under the conditions*

$$\alpha_0 < \frac{1}{2}, \;\; \eta_0 - 2\alpha_0 > 0, \tag{4.3}$$

we have the following asymptotics:

$$2n^{-1-\frac{\eta_0}{2}+2\alpha_0} \sum_{k=0}^{n-1} \left[\log\left(\frac{h_0}{h_1}(X_k) \right) - \frac{1}{2} n^{\eta_0 - 2\alpha_0} \right] \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \tag{4.4}$$

under the null hypothesis H_0.

Let for $\xi \in]0,1[$, N_ξ be the upper ξ-quantile of the Gaussian distribution. As a consequence of the two previous results, the test:

$$\text{decide } H_0 \text{ if } 2n^{-1-\frac{\eta_0}{2}+2\alpha_0} \sum_{k=0}^{n-1} \left[\log\left(\frac{h_0}{h_1}(X_k)\right) - \frac{1}{2} n^{\eta_0 - 2\alpha_0} \right] \geq -N_\xi$$
$$H_1 \text{ if } 2n^{-1-\frac{\eta_0}{2}+2\alpha_0} \sum_{k=0}^{n-1} \left[\log\left(\frac{h_0}{h_1}(X_k)\right) - \frac{1}{2} n^{\eta_0 - 2\alpha_0} \right] < -N_\xi,$$

has asymptotic level ξ and is convergent. In the following figure, we study the power of these tests when the parameters are not known but replaced by their estimates found in Section 3. Indeed, we compute the probability under assumption H_1 to accept the hypothesis H_1 when ξ increases. For $n = 1024$ observations, we set

$$T_n = 2n^{-1-\tilde{\eta}_n/2+2\hat{\alpha}_n} \sum_{k=0}^{n-1} \left[\frac{\hat{h}_0}{h_1}(x_k) - \frac{1}{2} n^{\hat{\eta}_n - 2\hat{\alpha}_n} \right],$$

where $\hat{h}_0$ is the density of the mixture with parameters $\hat{\alpha}_n$ and $\tilde{\eta}_n$. In Figure 2, we have drawn the power of the test when the level increases for different values of the true parameters (η_0, α_0). We point out that the power of the test is very high, even when we use the estimated coefficients $\tilde{\eta}_n$ and $\hat{\alpha}_n$. The reason of this result is that the estimation error does not change the structure of the test and therefore, the two assumptions are still very well separated. Moreover, as expected, the power is higher when the lacunarity coefficient η_0 increases and when the intensity coefficient α_0 decreases.

5 Proofs

Throughout the proofs, we will use the following notations.

The maximum number of observations available is linked with the maximal resolution level by the relationship $2^{j_1} = n$.

The densities of the Gaussian mixture are written $f_\alpha = \Phi_{2^{-\alpha j}, \sigma^2/n}$ and $f_0 = \Phi_{0,\sigma^2/n}$, while $\sigma_n^2 = \sigma^2/n$ denotes the variance of the observation noise. $m_{j_1} = 2^{j_1(1/2-\alpha_0)} = m_n$ is the mean of the first component of the rescaled mixture.

Also recall the important sequences $C_{j_1} = 2^{(\alpha_0-\eta_0)j_1}$, $\quad D_{j_1} = 2^{(2\alpha_0-\eta_0)j_1}$,

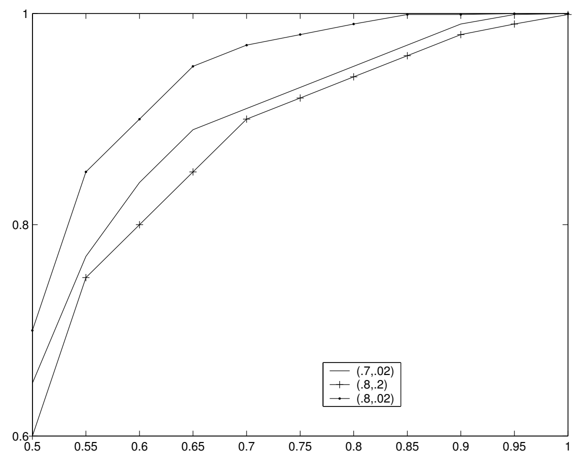

Figure 2: Power of the test for different (η, α)

and $c_{j_1} = 2^{(1-\eta_0)j_1}$, $d_{j_1} = 2^{j_1\eta_0/2}$. Their asymptotic behaviour depends on the range of the parameters α_0 and η_0, which will be made precise at the beginning of each proof.

Proof of Lemma 3.1. To begin with, we point out that the parameters are such that $\eta_0 - \alpha_0 > 0$, hence C_{j_1} decreases to zero. For $\eta_0 \in (0,1)$, define $L_n(\eta_0) = C_{j_1} \sum_{j=1}^{j_1} \sum_{k=0}^{2^j-1} d_{jk}$.
By obvious calculations we can see that the normalizing constant C_{j_1} is such that:

$$\mathbf{E}L_n(\eta_0) = C_{j_1} \sum_{j=1}^{j_1} 2^{(\eta_0-\alpha_0)j} \stackrel{j_1\to\infty}{\longrightarrow} 1 \tag{5.1}$$

Taking into account the independence of the wavelet coefficients, the variance of $L_n(\eta_0)$ is given by:

$$\mathrm{Var}(L_n(\eta_0)) = C_{j_1}^2 \sigma^2 + C_{j_1}^2 \sum_{j=1}^{j_1} 2^{(\eta_0-2\alpha_0)j}(1 - 2^{(\eta_0-1)j}) \tag{5.2}$$

As a result, Chebychev's inequality, as well as the calculations (5.1) and

(5.2) leads to: $L_n(\eta_0) \xrightarrow{\mathbf{P}} 1$. So consistency in probability of the empirical mean estimator $\hat{\eta}_n$ is proved.

Using the Δ-method (see for instance in van der Vaart, 1998), we now turn on the asymptotic distribution of $L_n(\eta_0)$. By straightforward calculations we obtain that for all $t \in \mathbb{R}$, the characteristic function of the random variable $L_n(\eta_0)$, using Taylor's expansion of order 2:

$$\begin{aligned} \mathbf{E} \exp\left(itL_n(\eta_0)\right) &= \mathbf{E}\exp\left(itC_{j_1}\sum_{j=1}^{j_1}\sum_{k} d_{jk}\right) \\ &= \exp(it)\exp\left(-C_{j_1}^2\frac{t^2}{2}(\sigma^2+\kappa)\right) + O(t^2 C_{j_1}^2) \qquad (5.3) \end{aligned}$$

where κ is given in (3.7). As a consequence, by a continuity argument, using Lévy's theorem, Lemma 3.1 is proved. □

Proof of Lemma 3.2. Here, we assume that $\eta_0 > 2\alpha_0$, hence we have $D_{j_1} \to 0$. Our aim is to prove a Central Limit Theorem for the quantity

$$D_{j_1}\left[\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} d_{jk}^2 - \sigma^2\right] - 1,$$

with a proper scaling coefficients.

Using a Taylors's expansion up to the second order in t we find the following expansions:

$$\left(1 - 2it\frac{\sigma^2}{n}\right)^{-\frac{n}{2}} = \exp\left(it\sigma^2 - \frac{t^2\sigma^4}{n}\right) + o(t^2/n)$$

$$\exp\left(\frac{i2^{-2\alpha_0 j}t}{1-2it\frac{\sigma^2}{n}}\right) = 1 + i2^{-2\alpha_0 j}t - \left(2\ \cdot 2^{-2\alpha_0 j}\frac{\sigma^2}{n} + \frac{2^{-4\alpha_0 j}}{2}\right)t^2 + o(1)$$

$$\begin{aligned} &\left[1 + 2^{(\eta_0-1)j}\left(\exp\left(\frac{i2^{-2\alpha_0 j}t}{1-2it\frac{\sigma^2}{n}}\right) - 1\right)\right]^{2^j} \\ &= \exp\left(2^{(\eta_0-2\alpha_0)j}t - t^2\frac{n^{(\eta_0-4\alpha_0)j}}{2}\right) + o(1). \end{aligned}$$

As a result we have the following asymptotic expansion of the characteristic function

$$\mathbf{E}\exp\left(it\sqrt{n^{\eta_0}}\left[D_{j_1}\left(\sum_{jk} d_{jk}^2 - 1\right) - 1\right]\right) = \exp\left(-\frac{t^2}{2}\right) + o(1) \quad (5.4)$$

Using (5.4), a continuity argument and Levy's theorem, we may conclude that

$$\sqrt{n^{\eta_0}}\left[D_{j_1}\left(\sum_{jk} d_{jk}^2 - \sigma^2\right) - 1\right] \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \quad (5.5)$$

which proves the statement of Lemma 3.2. □

Proof of Theorem 3.1. We recall that $\eta_0 - 2\alpha_0 > 0$.
The proof falls into two parts: the first one deals with the asymptotic distribution of $\hat{\alpha}_n$ while the other is the asymptotic normality of $\hat{\eta}_n$.

1. Define the quantity T_n as

$$T_n = \frac{1}{j_1}\log_2 C_{j_1}\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} d_{jk} = \frac{1}{j_1}\log_2 C_{j_1} + \alpha_0 - \hat{\eta}_n = \eta_0 - \hat{\eta}_n.$$

Since (5.3) shows that

$$C_{j_1}^{-1}\left(C_{j_1}\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1} d_{jk} - 1\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2 + \kappa\right) \quad (5.6)$$

so the Δ-method together gives the following limit:

$$j_1 C_{j_1}^{-1} T_n \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2 + \kappa\right) \quad (5.7)$$

Indeed, if Φ is Hadamard differentiable at a point θ, provided there exists a sequence r_n such that $r_n(X_n - \theta) \xrightarrow{\mathcal{L}} X$, the following convergence holds $r_n(\Phi(X_n) - \Phi(\theta)) \xrightarrow{\mathcal{L}} \Phi'(\theta)X$. Applying the last result with $\theta = 0$ and $\Phi(x) = \log(1+x)$ proves (5.7). Finally, the form of the estimator together with the result (5.7) proves the statement of Theorem 3.1. We must keep in mind that the data are dyadic with the correspondence $2^{j_1} = n$. This gives a rate of convergence in $\log(n)n^{\alpha_0 - \eta_0}$.

2. As in the previous proof, the Δ-method, with $\Phi(x) = \log(1+x)$ gives the asymptotic distribution of the estimator.

$$\sqrt{n^{\eta_0}} \log \left[D_{j_1} \left(\sum_{j=1}^{j_1} \sum_{k=0}^{2^j-1} d_{jk}^2 - \sigma^2 \right) \right] \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \tag{5.8}$$

As a result, we may split the estimator of α_0 as follows:

$$\begin{aligned} \hat{\alpha}_n &= \frac{1}{j_1 \log 2} \left(\log \left[\frac{\sum_{jk} d_{jk}}{\sum_{jk} d_{jk}^2 - \sigma^2} \right] \right) \\ &= \frac{1}{j_1 \log 2} \left(\hat{\eta}_n - \eta_0 + \log \left(d_{j_1} \left(\sum_{jk} d_{jk}^2 - \sigma^2 \right) \right) \right) + \alpha_0 \end{aligned}$$

This implies that

$$\begin{aligned} &\log(n) \sqrt{n^{\eta_0}} \hat{\alpha}_n - \alpha_0 \\ &= \frac{\sqrt{n^{\eta_0}}}{n^{\eta_0 - \alpha_0}} n^{\eta_0 - \alpha_0} (\hat{\eta}_n - \eta_0) + \sqrt{n^{\eta_0}} \log \left(D_{j_1} \left(\sum_{jk} d_{jk}^2 - \sigma^2 \right) \right) \end{aligned}$$

But $\eta_0 > 2\alpha_0$, so the first term in the previous sum goes to zero. Indeed $\frac{\sqrt{n^{\eta_0}}}{n^{\eta_0 - \alpha_0}} \to 0$. So the asymptotic distribution is given by the second term, which proves the statement (3.8) of the theorem.

□

From now on, we recall that the parameters of the mixture are such that: $\alpha_0 < \frac{1}{2}$ and $\eta_0 - 2\alpha_0 > 0$.

Proof of Lemma 3.3. First, note that $nS_n = \sum_{j=1}^{j_1} \sum_{k=0}^{2^j-1} 1_{X_{jk} \geq l_n}$, and $Z_j = \sum_{k=0}^{2^j-1} 1_{X_{jk} \geq l_n} \sim \mathcal{B}(2^j, \mathbf{P}(X_{jk} \geq l_n))$, where $\mathcal{B}(N, \theta)$ denotes the Binomial distribution with parameters N and θ. Under the assumptions:

$$l_n \to +\infty, \; 2^{(\alpha - \frac{1}{2})j_1} - l_n \to +\infty, \; c_{j_1} F(l_n) \leq M,$$

for a given positive constant M, we get the following asymptotic expansions.

$$\mathbf{E}(c_{j_1} S_n) = 1 - F(l_n) + o(1)$$
$$\mathrm{Var}(d_{j_1}[c_{j_1}(S_n - F(l_n))]) = 1 + c_{j_1} F(l_n) + o(1)$$

As a matter of fact:

$$\mathbf{E}(c_{j_1} S_n) = c_{j_1}\left(F(l_n) + 2^{(\eta_0-1)j_1}(1 - F(l_n))\right) + o(1)$$
$$\mathrm{Var}[c_{j_1}(S_n - F(l_n))] = 2^{-\eta_0 j_1}\left(1 + c_{j_1}F(l_n) - F(l_n)\right).$$

which proves the previous result. Now, see that

$$\begin{aligned}&\mathbf{E}\left[\exp(itd_{j_1}\left(d_{j_1}[c_{j_1}(S_n - F(l_n))] - 1\right))\right]\\ &= \exp(-itd_{j_1})\exp(-itc_{j_1}d_{j_1}F(l_n))\underbrace{\mathbf{E}\left[\exp(itd_{j_1}c_{j_1}S_n)\right]}_{T_n}.\end{aligned}$$

$$\begin{aligned}T_n &= \prod_{j=1}^{j_1}\exp\left[2^j\log\left(1 + 2^{(\eta_0-1)j}\left[e^{i\frac{td_{j_1}c_{j_1}}{n}} - 1\right]\right)\right]\\ &= \exp(itd_{j_1})\exp(itc_{j_1}d_{j_1}F(l_n))\\ &\quad\cdot\exp\left(-\frac{t^2}{2}(1 + c_{j_1}F(l_n) - c_{j_1}F^2(l_n))\right) + o(1)\end{aligned}$$

As a result we get:

$$d_{j_1}\left[c_{j_1}(S_n - F(l_n))\right] \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 1 + \lim(c_{j_1}F(l_n) - c_{j_1}F^2(l_n))\right)$$

as soon as $l_n \to +\infty$, with $F(l_n) = O\left(\frac{1}{c_{j_1}}\right)$, and $2^{(\alpha_0 - \frac{1}{2})j_1} > l_n$. □

Proof of Theorem 3.2. Using the fact that

$$\underbrace{\mathbf{E}\left(\frac{1}{n}\sum_{j=1}^{j_1}\sum_{k=0}^{2^j-1}1_{\{X_{jk}\geq l_n\}}\right)}_{S_n} = F(l_n) + 2^{(\eta_0-1)j_1}(1 - F(l_n)) + o(1)$$

we get the following asymptotic equivalence: $\frac{\mathbf{E}(S_n) - F(l_n)}{1 - F(l_n)} \sim 2^{(\eta_0-1)j_1}$. As a result, consider for estimator of η_0

$$\tilde{\eta}_n = 1 + \frac{1}{j_1}\log_2\left(\frac{S_n - F(l_n)}{1 - F(l_n)}\right) \tag{5.9}$$

Since $F(l_n) \to 0$, we get that $j_1(\tilde{\eta}_n - \eta_0) = \log(c_{j_1} S_n)$. Using the previous lemma 3.3, we know the asymptotic distribution of $d_{j_1}(c_{j_1} S_n - 1)$. As a consequence, the Δ-method, as in the previous proofs, enables us to conclude that

$$\log(n) n^{\frac{\eta_0}{2}} (\tilde{\eta}_n - \eta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

which concludes the proof of the theorem. □

Proof of Theorem 3.3. The log-likelihood of the mixture is given for (η, α) by

$$\mathcal{L}_n(\eta,\alpha) = \sum_{j=1}^{\log_2 n} \sum_{k=0}^{2^j-1} \log\left(2^{(\eta-1)j} f_\alpha(d_{jk}) + (1 - 2^{(\eta-1)j}) f_0(d_{jk}) \right).$$

$$\frac{d}{d\eta}\mathcal{L}_n(\eta,\alpha) = \sum_{j=1}^{\log_2 n} \sum_{k=0}^{2^j-1} \frac{j\log(2) 2^{(\eta-1)j}(f_\alpha(d_{jk}) - f_0(d_{jk}))}{2^{(\eta-1)j}(f_\alpha(d_{jk}) - f_0(d_{jk})) + f_0(d_{jk})}$$

$$\frac{d^2}{d\eta^2}\mathcal{L}_n(\eta,\alpha) = \sum_{j=1}^{\log_2 n} \sum_{k=0}^{2^j-1} \frac{(j\log 2)^2 2^{(\eta-1)j}(f_\alpha(d_{jk}) - f_0(d_{jk}))f_0(d_{jk})}{[2^{(\eta-1)j}(f_\alpha(d_{jk}) - f_0(d_{jk})) + f_0(d_{jk})]^2}$$

Fisher's information is given by

$$\begin{aligned} I(\eta_0) &= \mathbf{E}\left(-\frac{d^2}{d\eta^2}\big|_{(\eta_0,\alpha_0)} \mathcal{L}_n(\eta,\alpha) \right) \\ &= -\sum_{j=1}^{\log_2 n} (j\log 2)^2 2^j \underbrace{\int f_0(x) \frac{2^{(\eta_0-1)j}(f_{\alpha_0}(x) - f_0(x))}{2^{(\eta_0-1)j}(f_{\alpha_0}(x) - f_0(x)) + f_0(x)} dx}_{u_j} \end{aligned}$$

The proof relies on asymptotic expansions in the both cases, $x > m_n/2$ and $x < m_n/2$ and an extensive use of the equality

$$\frac{1}{1-x} - (1+x) = \frac{x^2}{1-x}.$$

Since $\alpha_0 < \frac{1}{2}$, $m_{j_1} = 2^{j_1(1-2\alpha_0)} \to \infty$ and we get:

$$\begin{aligned}
u_j &= \int 2^{(\eta_0-1)j}(f_{\alpha_0}(x) - f_0(x)) \frac{dx}{1 + 2^{(\eta_0-1)j}\dfrac{f_{\alpha_0}(x) - f_0(x)}{f_0(x)}} \\
&= \int_{2x \le 2^{-\alpha_0 j}} 2^{(\eta_0-1)j} \frac{(f_{\alpha_0}(x) - f_0(x))dx}{1 + 2^{(\eta_0-1)j}\left[\exp\left(-\frac{m_j(1-2x2^{\alpha_0 j})}{2}\right) - 1\right]} \\
&\quad + \int_{2x > 2^{-\alpha_0 j}} \frac{2^{(\eta_0-1)j} f_0(x)dx}{2^{(\eta_0-1)j} + \dfrac{f_0(x)}{f_{\alpha_0}(x) - f_0(x)}} \\
&= (I) + (II)
\end{aligned}$$

Hence, we get:

$$\begin{aligned}
(I) &= \int_{2x \le 2^{-\alpha_0 j}} 2^{2(\eta_0-1)j} \frac{e^{-\frac{x^2}{2\sigma_n^2}}}{\sqrt{2\pi}\sigma_n} \\
&\quad \cdot \left(1 - \exp\left(-\frac{m_j(1 - 2x2^{-\alpha_0 j})}{2}\right)\right) dx(1 + o(1)) \\
&= \int_{2x \le 2^{-\alpha_0 j}} 2^{2(\eta_0-1)j} \frac{e^{-\frac{x^2}{2\sigma_n^2}}}{\sqrt{2\pi}\sigma_n} dx(1 + o(1)) \\
&= 2^{2(\eta_0-1)j}(1 + o(1)).
\end{aligned}$$

$$\begin{aligned}
(II) &= \int_{2x > 2^{-\alpha_0 j}} 2^{(\eta_0-1)j} \frac{f_0(x)}{2^{(\eta_0-1)j} + \left(\exp\left(-\frac{m_j(1-2x2^{\alpha_0 j})}{2}\right) - 1\right)^{-1}} dx \\
&= \int_{2x > 2^{-\alpha_0 j}} 2^{(\eta_0-1)j} \frac{f_0(x)}{2^{(\eta_0-1)j} + \dfrac{\exp\left(\frac{m_j(1-2x2^{\alpha_0 j})}{2}\right)}{\left(1 - \exp\left(\frac{m_j(1-2x2^{\alpha_0 j})}{2}\right)\right)}} dx \\
&= \int_{2x > 2^{-\alpha_0 j}} f_0(x)(1 - 2^{(1-\eta_0)j} \exp\left(\frac{m_j(1 - 2x2^{\alpha_0 j})}{2}\right) dx + o(1) \\
&= ce^{-m_j^2}(1 + o(1)),
\end{aligned}$$

for a given positive constant c.
As a result, we get the asymptotic equivalence $u_j = 2^{2(\eta_0-1)j} + ce^{-m_j^2} + o(1)$

which implies the following asymptotic expansion for Fisher's information: $I_n(\eta_0) = \log(n)\, n^{(2\eta_0-1)} + o(1)$. □

Proof of Theorem 3.4. Recall that the loglikelihood is given by

$$\mathcal{L}_n(\eta, \alpha) = \sum_{j=1}^{\log_2 n} \sum_{k=0}^{2^j-1} \log \left[2^{(\eta-1)j} (f_\alpha(d_{jk}) - f_0(d_{jk})) + f_0(d_{jk}) \right].$$

So we get

$$\begin{aligned}
J(\alpha_0) &= \mathbf{E}\left[-\frac{d^2}{d\alpha^2} |_{\alpha=\alpha_0} \mathcal{L}_n(\eta, \alpha) \right] \\
&= \sum_{j=1}^{\log_2 n} \frac{2^{\eta_0 j}(j \log 2)^2}{\sigma_n^2} \int f_{\alpha_0}(x)(x - 2 \cdot 2^{-2\alpha_0 j})dx \\
&\quad + \sum_{j=1}^{\log_2 n} \left(\frac{j \log 2}{\sigma_n} \right)^2 2^{(\eta_0 - 2\alpha_0)j} (1 - 2^{(\eta_0-1)j}) \\
&\quad \cdot \int \frac{(x - 2^{-\alpha_0 j})^2 f_0(x) f_{\alpha_0}(x)}{2^{\eta_0 - 1}(f_{\alpha_0}(x) - f_0(x)) + f_0(x)} dx \\
&= A \,+\, B
\end{aligned}$$

The first term is analogous to the one studied in the previous proof, so $A = \log(n) n^{\eta_0 - \alpha_0 + 1}(1 + o(1))$. The second term B is such that:

$$B = \sum_j \left(\frac{j \log 2}{\sigma_n} \right)^2 2^{(\eta_0 - 2\alpha_0)j} \underbrace{\int \frac{(x - 2^{-\alpha_0 j})^2 f_0(x) f_{\alpha_0}(x)}{2^{(\eta_0-1)j}(f_{\alpha_0}(x) - f_0(x)) + f_0(x)}}_{u_j}.$$

Using the following decomposition we get:

$$\begin{aligned}
u_j &= \int_{1-2x2^{\alpha_0 j} \geq 0} (x - 2^{-\alpha_0 j})^2 \frac{f_{\alpha_0}(x)}{1 + 2^{(\eta_0-1)j} \dfrac{f_{\alpha_0}(x) - f_0(x)}{f_0(x)}} dx \\
&\quad + \int_{1-2x2^{\alpha_0 j} < 0} (x - 2^{-\alpha_0 j})^2 \frac{f_0(x)}{\frac{f_0(x)}{f_{\alpha_0}(x)} + 2^{(\eta_0-1)j} \left[1 - \frac{f_0(x)}{f_{\alpha_0}(x)} \right]} dx \\
&= \int_{1-2x2^{\alpha_0 j} \geq 0} (x - 2^{-\alpha_0 j})^2 f_{\alpha_0}(x)(1 + 2^{(\eta_0-1)j}) dx (1 + o(1))
\end{aligned}$$

$$+ 2^{(1-\eta_0)j} \int_{1-2x2^{\alpha_0 j}<0} (x - 2^{-\alpha_0 j})^2 f_0(x) dx (1 + o(1))$$
$$\frac{\sigma^2}{n}(1 + 2^{(\eta_0-1)j}) + 2^{(1-\eta_0)j} \left[\frac{\sigma^2}{n} + 2^{-2\alpha_0 j}\right](1 + o(1))$$

Hence,

$$B = \sum_j j^2 n 2^{(\eta_0 - 2\alpha_0)j} \left(2^{(\eta_0-1)j}\frac{1}{n} + 2^{(1-\eta_0)j}\frac{1}{n}\right) + o(1)$$
$$= (\log n)^2 n^{2(\eta_0-\alpha_0)-1}(1 + o(1)).$$

As a result, the following asymptotic expansion holds

$$J_n(\alpha_0) = \log^2(n) n^{1+\eta_0-2\alpha_0}(1 + o(1)).$$

□

Proof of Lemma 4.1. First, we point out that we have:

$$\log \frac{h_0}{h_1}(x) = \log \left[1 - n^{\eta_0-1} + n^{\eta_0-1} \exp\left(m_n\left(x - \frac{m_n}{2}\right)\right)\right].$$

The proof relies on the asymptotic expansion of this log-likelihood depending whether $e^{m_n(x-m_n/2)} \to 0$ or $e^{m_n(x-m_n/2)} \to +\infty$. Hence, observe that for $u < 1/2$

$$|\log(1-u) + u| = -u - \log(1-u) \leq 2u^2 \tag{5.10}$$

and apply this bound either to $\log \left[1 \underbrace{-n^{\eta_0-1} + n^{\eta_0-1}e^{m_n(x-m_n/2)}}_{-u}\right]$ for n large enough, or to

$$\log\left[1 - n^{\eta_0-1} + n^{\eta_0-1}e^{m_n(x-m_n/2)}\right] = (\eta_0 - 1)\log(n) + m_n\left(x - \frac{m_n}{2}\right)$$
$$+ \log\left[1 - \underbrace{(1 - n^{1-\eta_0})e^{-m_n(x-m_n/2)}}_{u}\right].$$

So we obtain

$$\begin{aligned}\mathbf{E}_{H_0}(S_n(X)) &= (1-n^{\eta_0-1})\int \frac{e^{-x^2/2}}{\sqrt{2\pi}}\log\left[1-n^{\eta_0-1}+n^{\eta_0-1}e^{m_n(x-m_n/2)}\right]dx \\ &+ n^{\eta_0-1}\int \frac{e^{-\frac{(x-m_n)^2}{2}}}{\sqrt{2\pi}}\log\left[1-n^{\eta_0-1}+n^{\eta_0-1}e^{m_n(x-m_n/2)}\right]dx \\ &= A+B.\end{aligned}$$

On the one hand we have that $(1-n^{\eta_0-1})^{-1}A$ is equal to:

$$\begin{aligned}&\int_{x<m_n/2}\Big(\log\left[1-n^{\eta_0-1}+n^{\eta_0-1}e^{m_n(x-m_n/2)}\right] \\ &+ n^{\eta_0-1}-n^{\eta_0-1}e^{m_n(x-m_n/2)}\Big)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\ &- \int_{x<m_n/2}\left(n^{\eta_0-1}-n^{\eta_0-1}e^{m_n(x-m_n/2)}\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\ &+ \int_{x>m_n/2}\left[(\eta_0-1)\log(n)+m_n\left(x-\frac{m_n}{2}\right)\right]\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\ &+ \int_{x>m_n/2}\Big(\log\left[1-(1-n^{1-\eta_0})e^{-m_n(x-m_n/2)}\right] \\ &- (1-n^{1-\eta_0})e^{-m_n(x-m_n/2)}\Big)\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\ &+ \int_{x>m_n/2}(1-n^{1-\eta_0})e^{-m_n(x-m_n/2)}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx.\end{aligned}$$

Using inequality (5.10), the first integral is $n^{\eta_0-1}o(1)$ while by Lebesgue theorem, the second one is $n^{\eta_0-1}(-1+o(1))$. Moreover, there exists a constant c_2 such that, for n large enough, the three last integrals are bounded by $\exp(-c_2 m_n)$. Hence, these integrals are $n^{\eta_0-1}o(1)$. Finally, we get

$$A=(1-n^{\eta_0-1})n^{\eta_0-1}(-1+o(1))=n^{\eta_0-1}(-1+o(1)).$$

On the other hand, we may split $n^{1-\eta_0}B$ as:

$$\begin{aligned}&\int_{x<m_n/2}\Big[\log\left[1-n^{\eta_0-1}+n^{\eta_0-1}e^{m_n(x-m_n/2)}\right] \\ &+ \left(n^{\eta_0-1}-n^{\eta_0-1}e^{m_n(x-m_n/2)}\right)\Big]\frac{e^{-\frac{(x-m_n)^2}{2}}}{\sqrt{2\pi}}dx\end{aligned}$$

$$
\begin{aligned}
&- \int_{x<m_n/2} \left(n^{\eta_0-1} - n^{\eta_0-1} e^{m_n(x-m_n/2)}\right) \frac{e^{-\frac{(x-m_n)^2}{2}}}{\sqrt{2\pi}} dx \\
&+ \int_{x>m_n/2} \left[(\eta_0-1)\log(n) + m_n(x - \frac{m_n}{2})\right] \frac{e^{-\frac{(x-m_n)^2}{2}}}{\sqrt{2\pi}} dx \\
&+ \int_{x>m_n/2} \log\left[1-(1-n^{1-\eta_0})e^{-m_n(x-m_n/2)}\right] \frac{e^{-\frac{(x-m_n)^2}{2}}}{\sqrt{2\pi}} dx.
\end{aligned}
$$

Using the same kinds of argument as those used to expand A we may conclude that the main contribution comes from the last term of the third integral, which can be written after substituting $x - m_n/2$ with u: $\int_{u<-m_n/2} m_n \cdot (u + \frac{m_n}{2}) \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$ of order $m_n^2/2$. Finally, we get that

$$
B = 1/2 n^{\eta_0 - 2\alpha_0}(1 + o(1)).
$$

Thus, the first statement of the lemma is proved.
In a similar way, under the hypothesis H_1, we have:

$$
\begin{aligned}
\mathbf{E}_{H_1}(S_n(X)) &= \int \frac{e^{-x^2/2}}{\sqrt{2\pi}} \log\left[1 - n^{\eta_0-1} + n^{\eta_0-1} e^{m_n(x-\frac{m_n}{2})}\right] dx \\
&= \int \left(\mathbf{1}_{x<\frac{m_n}{2}} + \mathbf{1}_{x\geq\frac{m_n}{2}}\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \\
&\quad \cdot \log\left[1 - n^{\eta_0-1} + n^{\eta_0-1} e^{m_n(x-\frac{m_n}{2})}\right] dx \\
&= n^{\eta_0-1}(1 + o(1)) \to 0.
\end{aligned}
$$

□

Proof of Theorem 4.1. $\forall\, k = 0, \ldots, 2^j - 1$, set

$$
Z_k^n = \log \frac{h_0}{h_1}(X_k) - \mathbf{E}_{H_0}\left(\log \frac{h_0}{h_1}(X_k)\right).
$$

First, let us compute the variance of the variables Z_k^n. Reasonning as previously, we obtain

$$\begin{aligned}
&\mathbf{E}_{H_0}\left(\log\frac{h_0}{h_1}(X_k)\right)^2\\
&=(1-n^{\eta_0-1})\int\frac{e^{-x^2/2}}{\sqrt{2\pi}}\left(\log\left[1-n^{\eta_0-1}+n^{\eta_0-1}e^{m_n(x-\frac{m_n}{2})}\right]\right)^2dx\\
&+n^{\eta_0-1}\int\frac{e^{-\frac{(x-m_n)^2}{2}}}{\sqrt{2\pi}}\left(\log\left[1-n^{\eta_0-1}+n^{\eta_0-1}e^{m_n(x-\frac{m_n}{2})}\right]\right)^2dx\\
&=\frac{1}{4}n^{1+\eta_0-4\alpha_0}(1+o(1)).
\end{aligned}$$

As a result we have, using results of Lemma 4.1:

$$\mathrm{Var}_{H_0}(Z_k^n)=\frac{1}{4}n^{1+\eta_0-4\alpha_0}(1+o(1)).$$

Our aim is to apply Lindeberg theorem, see for instance Billingsley (1995). For this, set $s_n^2=\sum_{k=0}^{n-1}\sigma_{nk}^2$ with $\sigma_{nk}^2=\mathbf{E}(Z_k^n)^2$. We recall that the data $Z_k^n,\ k=0,\dots,n-1$ will satisfy a Central Limit Theorem

$$\frac{\sum_{k=0}^{n-1}Z_k^n}{s_n}\xrightarrow{\mathcal{L}}\mathcal{N}(0,1)$$

whenever the two following conditions are fulfilled

$$s_n>0 \tag{5.11}$$

$$\forall\epsilon>0,\ \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{s_n^2}\int_{|Z_k^n|\geq\epsilon s_n}(Z_k^n)^2d\mathbf{P}_{H_0}=0. \tag{5.12}$$

First of all, $s_n=\frac{1}{2}n^{1+\frac{\eta_0}{2}-2\alpha_0}+o(1)>0$, so condition (5.11) is checked. In our context, condition (5.12) can be written as follows:

$$\lim_{n\to\infty}n^{4\alpha_0-1-\eta_0}\mathbf{E}_{H_0}\left[(Z_k^n)^2\mathbf{1}_{|Z_k^n|\geq\frac{\epsilon}{2}n^{1+\frac{\eta_0}{2}-2\alpha_0}}\right]=0 \tag{5.13}$$

Let us check this assumption. For $\epsilon > 0$, we have:

$$
\begin{aligned}
&\mathbf{E}_{H_0}\left[(Z_k^n)^2 \mathbf{1}_{|Z_k^n| \geq \frac{\epsilon}{2} n^{1+\frac{\eta_0}{2}-2\alpha_0}}\right] \\
\leq& \mathbf{E}_{H_0}\left[(Z_k^n)^2 \mathbf{1}_{\left|\log \frac{h_0}{h_1}(X_k)\right| \geq \frac{\epsilon}{2} n^{1+\frac{\eta_0}{2}-2\alpha_0} - \frac{1}{2} n^{\eta_0 - 2\alpha_0}}\right] \\
\leq& c_1 \mathbf{E}_{H_0}\left[(Z_k^n)^2 \mathbf{1}_{\left|\log \frac{h_0}{h_1}(X_k)\right| \geq \frac{\epsilon}{2} n^{1+\frac{\eta_0}{2}-2\alpha_0}}\right] \\
\leq& 2c_1\left(\frac{1}{4} n^{2\eta_0 - 4\alpha_0} + \mathbf{E}_{H_0}\left[(\log \frac{h_0}{h_1}(X_k))^2 \mathbf{1}_{\left|\log \frac{h_0}{h_1}(X_k)\right| \geq \frac{\epsilon}{2} n^{1+\frac{\eta_0}{2}-2\alpha_0}}\right]\right) \\
\leq& 2c_1(A+B),
\end{aligned}
$$

where c_1 is a positive finite constant.
For the first term A, we have $n^{4\alpha_0-1-\eta_0} A \leq \frac{1}{4} n^{\eta_0-1} \to 0$. For the second term B, first note that $\left|\log \frac{h_0}{h_1}(X)\right| \geq \frac{\epsilon}{2} n^{1+\frac{\eta_0}{2}-2\alpha_0}$ is equivalent, for n large enough, to

$$\exp\left[m_n\left(x - \frac{m_n}{2}\right)\right] \geq 1 - n^{1-\eta_0} + n^{1-\eta_0} e^{\frac{\epsilon}{2} n^{1+\frac{\eta_0}{2}-2\alpha_0}}.$$

So that, setting

$$\Theta_n := \left\{x :\ x \geq \frac{m_n}{2} + m_n \log\left[1 + n^{1-\eta_0}\left(\exp\left[\frac{\epsilon}{2} n^{1+\frac{\eta_0}{2}-2\alpha_0}\right]\right) - 1\right]\right\}$$

we may write, for n large enough,

$$
\begin{aligned}
B &\leq 2\int_{\Theta_n}\left(\log\left[n^{\eta_0-1}\exp\left[m_n\left(x-\frac{m_n}{2}\right)\right]\right]\right)^2 \frac{\exp(-\frac{x^2}{2})}{\sqrt{2\pi}} dx \\
&\leq 2(\eta_0-1)^2 \log^2 n + 2m_n^2 \int_{\Theta_n}\left(x - \frac{m_n}{2}\right)^2 \frac{\exp(-\frac{x^2}{2})}{\sqrt{2\pi}} dx
\end{aligned}
$$

As the last integral may be bounded, for n large enough, by $\exp(-c_2 m_n)$ where c_2 is a positive constant, we get that $n^{4\alpha_0-1-\eta_0} B \to 0$. So condition (5.13) is fulfilled, and Lindeberg theorem holds, so that:

$$2\, n^{2\alpha_0-1-\frac{\eta_0}{2}} \sum_{k=0}^{n-1}\left[\log\left(\frac{h_0}{h_1}(X_k)\right) - \mathbf{E}_{H_0}\log\left(\frac{h_0}{h_1}(X_k)\right)\right] \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

So we can write

$$2n^{-1-\frac{\eta_0}{2}+2\alpha_0}\sum_{k=0}^{n-1}\left[\log\left(\frac{h_0}{h_1}(X_k)\right)-\frac{1}{2}n^{\eta_0-2\alpha_0}\right]$$
$$=2n^{-1-\frac{\eta_0}{2}+2\alpha_0}\sum_{k=0}^{n-1}\left[\log\left(\frac{h_0}{h_1}(X_k)\right)-\mathbf{E}_{H_0}\log\left(\frac{h_0}{h_1}(X_k)\right)\right.$$
$$\left.+\mathbf{E}_{H_0}\log\left(\frac{h_0}{h_1}(X_k)\right)-\frac{1}{2}n^{\eta_0-2\alpha_0}\right]$$

Using the second order error term in the calculation of $\mathbf{E}_{H_0}(S_n)$ in the proof of Lemma 4.1, we get that

$$2\,n^{-1-\frac{\eta_0}{2}+2\alpha_0}\sum_{k=0}^{n-1}\left[\mathbf{E}\log\left(\frac{h_0}{h_1}(X_k)\right)-\frac{1}{2}n^{\eta_0-2\alpha_0}\right]\leq 2c_3n^{-\eta_0/2}\to 0,$$

for c_3 a finite positive constant. As a consequence, Slutsky's theorem enables to prove the result. □

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