Green's function pointwise estimates for spectrally stable discrete shock profiles

Lucas Coeuret

Institut de Mathématiques de Toulouse (IMT)

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Advisors : Jean-François Coulombel and Grégory Faye (IMT)

Conservation law, shocks and finite difference schemes

- We consider a one-dimensional scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R},$$
$$u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R},$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

The result that will be presented also holds for systems of conservations laws.

- Steady Lax shock: We consider $(u^-, u^+) \in \mathbb{R}^2$ such that

$$f(u^-) = f(u^+) \quad \text{(Rankine-Hugoniot condition)}$$

and

$$f'(u^+) < 0 < f'(u^-). \quad \text{(Lax shock)}$$

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.

We introduce a conservative one-step explicit finite difference scheme $\mathcal{N}:\mathbb{R}^{\mathbb{Z}}\to\mathbb{R}^{\mathbb{Z}}$ such that for $u=(u_j)_{j\in\mathbb{Z}}\in\mathbb{R}^{\mathbb{Z}}$ and $j\in\mathbb{Z}$

$$\mathcal{N}(u)_j := u_j - \nu\left(F\left(\nu; u_{j-p+1},\dots,u_{j+q}\right) - F\left(\nu; u_{j-p},\dots,u_{j+q-1}\right)\right),$$

where $p,q\in\mathbb{N}\backslash\{0\}$, the numerical flux $F:]0,+\infty[\times\mathbb{R}^{p+q}\to\mathbb{R}^d$ is a smooth function and we fix $\nu=\frac{\Delta t}{\Delta x}>0$ satisfying a CFL condition.

Assumptions:

- $\forall u\in\mathbb{R},\quad F(\nu;u,\dots,u)=f(u)$ (consistency condition)
- ℓ^2-stability for some constant states
- The scheme introduces numerical diffusion (numerical viscosity) rather than numerical dispersion (at least for the states $u^{\pm}$).

Example : We consider the modified Lax Friedrichs scheme

$$\forall u\in\mathbb{R}^{\mathbb{Z}},\forall j\in\mathbb{Z},\quad \mathcal{N}(u)_j := \frac{u_{j+1}+u_j+u_{j-1}}{3} - \nu\frac{f(u_{j+1})-f(u_{j-1})}{2}.$$

Stationary discrete shock profiles: Definition and existence

We are interested in solutions of

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n), \qquad u^0 \in \mathbb{R}^{\mathbb{Z}}. \tag{1}$$

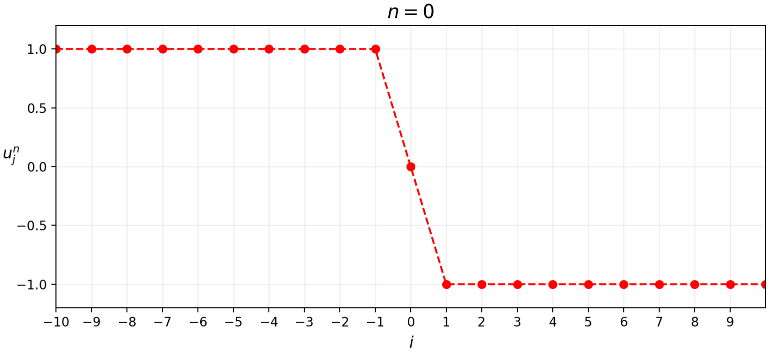
Stationary discrete shock profile (SDSP): We suppose that there exists a sequence $\overline{u} = (\overline{u}_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfies

$$\mathcal{N}(\overline{u}) = \overline{u} \quad \text{and} \quad \overline{u}_j \underset{j\to\pm\infty}{\to} u^{\pm}.$$

For moving shocks, discrete shock profiles are traveling waves solutions of (1) *that link two states $u^{\pm}$. (Difficulties depending on whether the speed of the traveling wave is rational or not).*

Example : We consider the initial condition (mean of the standing shock on each cell $\left[(j-\frac{1}{2})\Delta x, (j+\frac{1}{2})\Delta x\right[$)

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$

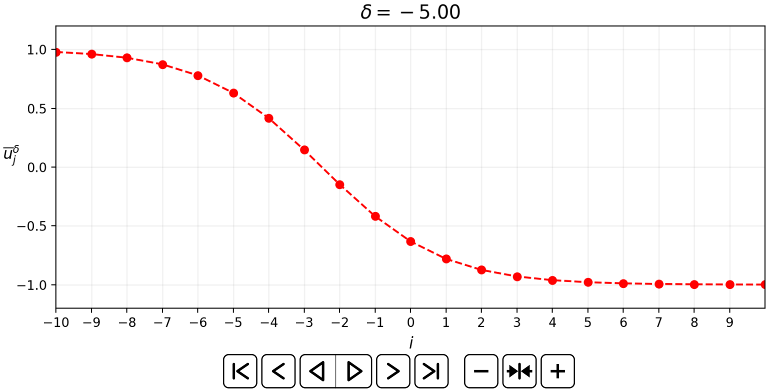


Existence of DSPs

For standing Lax shocks, in some cases, we have the proof of the existence of a continuous one-parameter family $(\overline{u}^{\delta})_{\delta \in I}$ of SDSPs.

- Jennings, *Discrete shocks* (1974)
  - scalar case
  - conservative monotone scheme
- Majda and Ralston, *Discrete Shock Profiles for Systems of Conservation Laws* (1979)
  - system case
  - weak Lax shocks
- Different cases: Smyrlis (1990), Liu-Yu (1999), Serre (2004) etc...

Example : We consider the same initial condition u^0 as before but add a mass δ at $j = 0$. We look at the limit of the solution of the numerical scheme.



We will use the terms "translation of the profile" and "derivative of the profile" even tough we are in a discrete setting.

Stability of discrete shock profiles

The end goal would be to prove a property of nonlinear orbital stability for some SDSPs:

For admissible perturbations h, prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \overline{u} + h$ converges towards the set of translations of the SDSP $\left\{\overline{u}^\delta, \delta \in I\right\}$.

We are going to present a possible first step towards such a result.

- Jennings, *Discrete shocks* (1974)
    - scalar case
    - conservative monotone scheme
- Liu-Xin, L^1*-stability of stationary discrete shocks*, (1993)
    - system case
    - Lax-Friedrichs scheme
    - weak Lax shocks
    - zero mass perturbation (dropped in Ying (1997))
- Different cases: Liu-Yu (1999), etc...

$\Rightarrow$ Extension of the result of Lafitte-Godillon, *Green's function pointwise estimates for the modified Lax-Friedrichs scheme*, (2003)

Linearization of the numerical scheme about the SDSP

We define the bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about $\overline{u}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k},$$

with $a_{j,k} \to a_k^{\pm}$ as $j \to \pm\infty$. We are interested in solutions of the linearized numerical scheme

$$\forall n \in \mathbb{N}, \quad h^{n+1} = \mathcal{L}h^n, \qquad h^0 \in \ell^2(\mathbb{Z}).$$

We define the Green's function

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Spectral assumptions on $\mathcal{L}$

- 1 is a simple eigenvalue of the operator $\mathcal{L}$.

$$"\mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta \text{ and thus } \mathcal{L}\frac{\partial \overline{u}^\delta}{\partial \delta} = \frac{\partial \overline{u}^\delta}{\partial \delta}."$$

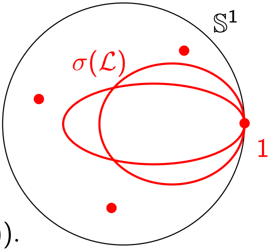
- The operator $\mathcal{L}$ has no other eigenvalue of modulus equal or larger than 1. (Spectral stability)
- We assume that

$$\forall \kappa \in \mathbb{S}^1 \backslash \{1\}, \quad \left| \sum_{k=-p}^{q} \kappa^k a_k^\pm \right| < 1$$

and that there exist an integer $\mu \in \mathbb{N} \backslash \{0\}$ and a complex number $\beta_\pm$ with positive real part such that

$$\sum_{k=-p}^{q} a_k^\pm e^{i\xi k} \underset{\xi \to 0}{=} \exp(-i\alpha_\pm \xi - \beta_\pm \xi^{2\mu} + O(|\xi|^{2\mu+1})).$$

with $\alpha_\pm := f'(u^\pm)\nu$.



Little detour

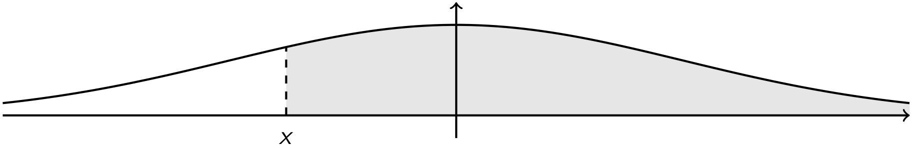
For $\beta \in \mathbb{C}$ with positive real part, we define the functions $H^{\beta}_{2\mu}, E^{\beta}_{2\mu} : \mathbb{R} \to \mathbb{C}$ via

$$\forall x \in \mathbb{R}, \quad H^{\beta}_{2\mu}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixu} e^{-\beta u^{2\mu}} du,$$

$$\forall x \in \mathbb{R}, \quad E^{\beta}_{2\mu}(x) := \int_x^{+\infty} H^{\beta}_{2\mu}(y) dy.$$

We have

$$\lim_{x \to +\infty} E^{\beta}_{2\mu}(x) = 0 \quad \text{and} \quad \lim_{x \to -\infty} E^{\beta}_{2\mu}(x) = 1.$$



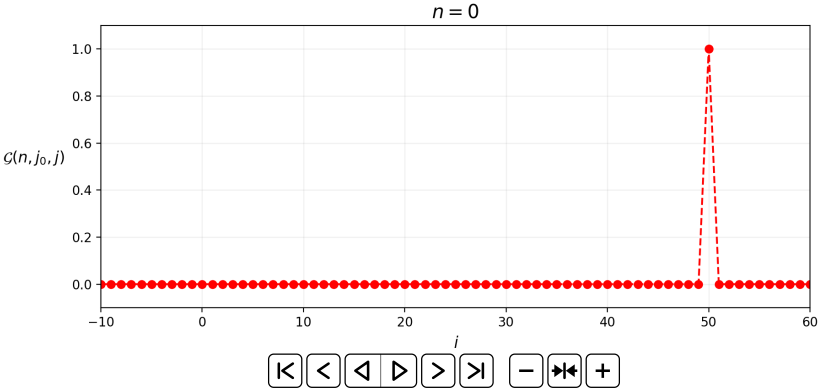
Theorem

Under some more precise assumptions, there exists a positive constant c and a sequence $V \in \ker(Id - \mathcal{L})$ such that for all $n \in \mathbb{N} \setminus \{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

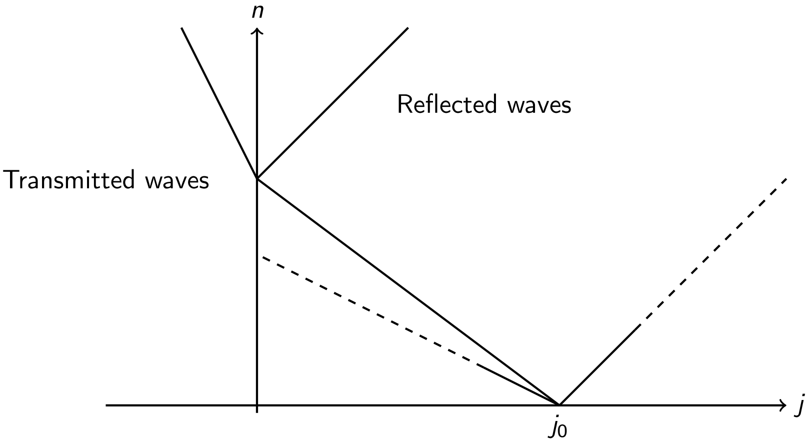
$$
\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=& E_{2\mu}^{\beta_+}\left(\frac{j_0 + n\alpha_+}{n^{\frac{1}{2\mu}}}\right) V(j) \quad \text{(Excited eigenvector)} \\
&+ \mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{|n\alpha_+ - (j - j_0)|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right)\right) \quad \text{(Gaussian wave)} \\
&+ \mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{1}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{|n\alpha_+ + j_0|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right) e^{-c|j|}\right) \quad \text{(Exponential residual)} \\
&+ O(e^{-cn - c|j - j_0|})
\end{aligned}
$$

There is a similar result for $j_0 \in -\mathbb{N}$.

We choose $j_0 = 50$.



Case of systems



• Using the inverse Laplace tranform with Γ a path that surrounds the spectrum $\sigma(\mathcal{L})$, we have

$$\forall n \in \mathbb{N}\backslash\{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi}\int_\Gamma z^n \left((zId - \mathcal{L})^{-1}\delta_{j_0}\right)_j dz. \qquad (2)$$

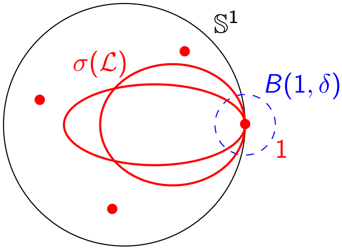
• We rewrite the eigenvalue problem

$$(zId - \mathcal{L})u = 0$$

as a discrete dynamical system

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z)W_j. \qquad (3)$$

We are interested in solutions of (3) that tend towards 0 as j tends to $+\infty$ or $-\infty$ (Jost solutions, geometric dichotomy) and use them to express $(zId - \mathcal{L})^{-1}\delta_{j_0}$.



• Using this idea and a good choice of path Γ, we prove sharp estimates on the temporal Green's function.

Perspective / Open questions

We have thus a precise description of the Green's function for the linearized scheme about spectrally stable stationnary discrete shock profiles.

- Existence of spectrally stable SDSPs?
- Can we now prove nonlinear orbital stability ? (at least in the scalar case?)
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Study of the stability for multi-dimensional conservation laws (Carbuncle phenomenon)
- ...

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