Large time behavior of finite difference schemes for the transport equation

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We consider the transport equation ($c > 0$)

$$\partial_t u + c\partial_x u = 0, \quad t \geq 0, \quad x \in \mathbb{R},$$
$$u_{t=0} = u_0.$$

We set $\Delta t, \Delta x > 0, \lambda := \frac{\Delta t}{\Delta x}$.

Example of explicit one-step finite difference approximation:

We consider $n \in \mathbb{N}$, $j \in \mathbb{Z}$.

- Upwind scheme:

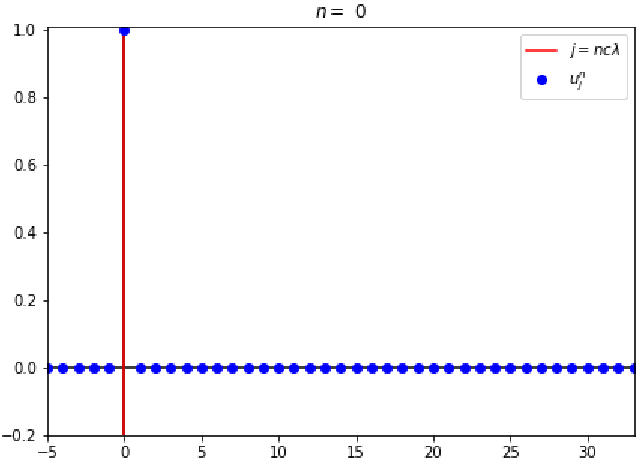
$$u_j^{n+1} = c\lambda u_{j-1}^n + (1 - c\lambda)u_j^n.$$

- Modified Lax Friedrichs scheme: ($0 < D < \frac{1}{2\lambda}$)

$$u_j^{n+1} = \left(\lambda D + \frac{c\lambda}{2}\right) u_{j-1}^n + (1 - 2D\lambda)u_j^n + \left(\lambda D - \frac{c\lambda}{2}\right) u_{j+1}^n.$$

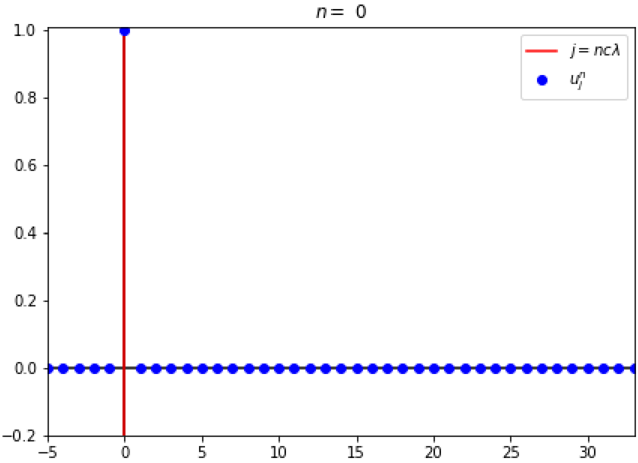
- O3 scheme

$$u_j^{n+1} = -\frac{c\lambda(1-(c\lambda)^2)}{6} u_{j-2}^n + \frac{c\lambda(1+c\lambda)(2-c\lambda)}{2} u_{j-1}^n + \frac{(1-(c\lambda)^2)(2-c\lambda)}{2} u_j^n - \frac{c\lambda(1-c\lambda)(2-c\lambda)}{6} u_{j+1}^n.$$



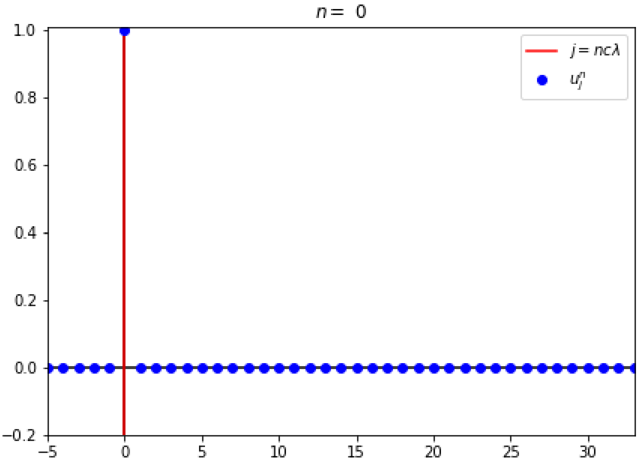
Modified Lax Friedrichs scheme: $c\lambda = \frac{1}{2}$ and $u^0 = \delta := (\delta_{j,0})_{j\in\mathbb{Z}}$

Decay rate : $\dfrac{1}{\sqrt{n}}$



O3 scheme: $c\lambda = \frac{1}{2}$ and $u^0 = \delta := (\delta_{j,0})_{j \in \mathbb{Z}}$

Decay rate : $\dfrac{1}{n^{\frac{1}{4}}}$



Lax Wendroff scheme: $c\lambda = \frac{1}{2}$ and $u^0 = \delta := (\delta_{j,0})_{j\in\mathbb{Z}}$

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$$u_j^{n+1} = \underbrace{\left(\lambda D + \frac{c\lambda}{2}\right)}_{= \mathbf{a}_1} u_{j-1}^n + \underbrace{(1 - 2D\lambda)}_{= \mathbf{a}_0} u_j^n + \underbrace{\left(\lambda D - \frac{c\lambda}{2}\right)}_{= \mathbf{a}_{-1}} u_{j+1}^n$$

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Thus,

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathbf{a} * u^n,$$

where $\mathbf{a} = (\cdots, 0, \mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1, 0, \cdots) \in \mathbb{C}^{\mathbb{Z}}$.

An explicit one-step finite difference approximation can be written as

$$\forall n \in \mathbb{N}, \quad u^n = \mathbf{a}^n * u^0,$$
$$u^0 \in \mathbb{C}^{\mathbb{Z}},$$

where $\mathbf{a} \in \mathbb{C}^{\mathbb{Z}}$ is finitely supported and $\mathbf{a}^n = \mathbf{a} * \cdots * \mathbf{a}$.

Goal: We want to study $\mathbf{a}^n$ for large values of n to better understand the large time behavior of the finite difference scheme.

A detour through Probability - The Local Limit Theorem

We consider a random walk

$$S_n := X_1 + \cdots + X_n$$

where X_n are i.i.d. random variables (same law as some random variable X with values in $\mathbb{Z}$) and we use the notation

$$\forall j \in \mathbb{Z}, \quad \mathbf{a}_j := P(X = j).$$

We then have

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad \mathbf{a}_j^n = P(S_n = j).$$

Central Limit Theorem :

$$\sqrt{n}\left(\frac{S_n}{n} - \mathbb{E}(X)\right) \overset{\mathcal{L}}{\to} \mathcal{N}(0, V(X)).$$

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Local Limit Theorem : Under suitable conditions on the sequence **a**

$$\mathbf{a}_j^n - \frac{1}{\sqrt{2\pi V(X) n}} \exp\left(-\frac{|j - n\mathbb{E}(X)|^2}{2nV(X)}\right) \underset{n\to+\infty}{=} o\left(\frac{1}{\sqrt{n}}\right) \quad \text{uniformly on } \mathbb{Z}.$$

A detour through Probability - The Local Limit Theorem

$$\mathbf{a}_j^n - \frac{1}{\sqrt{2\pi V(X) n}} \exp\left(-\frac{|j - n\mathbb{E}(X)|^2}{2nV(X)}\right) - \frac{1}{n} q\left(\frac{j - n\mathbb{E}(X)}{\sqrt{V(X)n}}\right) \underset{n\to+\infty}{=} o\left(\frac{1}{n}\right),$$

where

$$\forall x \in \mathbb{R}, \quad q(x) := C(X)(x^3 - 3x)e^{-\frac{x^2}{2}}$$

with $C(X)$ a constant depending on the random variable X.

Problem: For a finite difference scheme, the sequence $\mathbf{a}$ does not always have non negative or even real coefficients.
Thus, we are looking for a generalization of the Local Limit Theorem.
We also want to find precise bounds for $\mathbf{a}_j^n$.

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Fourier Series

For $\mathbf{a} \in \ell^1(\mathbb{Z})$, we introduce $F_{\mathbf{a}}$ the Fourier series associated to $\mathbf{a}$:

$$\forall \kappa \in \mathbb{S}^1, \quad F_{\mathbf{a}}(\kappa) := \sum_{k \in \mathbb{Z}} \mathbf{a}_k \kappa^k.$$

If $\mathbf{a}$ is finitely supported then $F_{\mathbf{a}}$ can be holomorphically extended on $\mathbb{C} \backslash \{0\}$.

The consistency of the scheme implies

$$F_{\mathbf{a}}(1) = 1 \quad \text{and} \quad F'_{\mathbf{a}}(1) = c\lambda.$$

The ℓ^2-stability is equivalent to having that

$$\max_{\mathbb{S}^1} |F_{\mathbf{a}}| \leq 1. \quad \text{(Von Neumann condition)}$$

Stronger hypothesis (Dissipation)

We will suppose that

$$\forall \kappa \in \mathbb{S}^1 \backslash \{1\}, \quad |F_{\mathbf{a}}(\kappa)| < 1.$$

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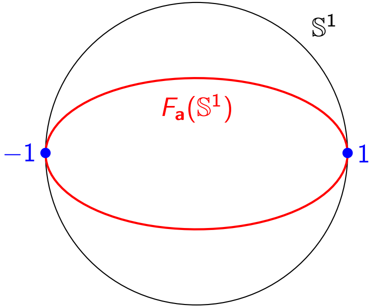
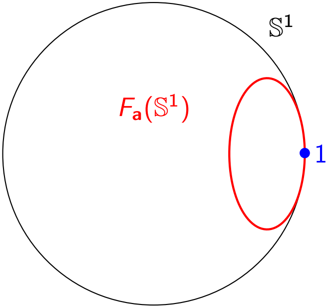


Lax-Friedrichs scheme, $c\lambda = \frac{1}{2}$

$$F_{\mathbf{a}}(e^{it}) = \cos(t) + ic\lambda\sin(t).$$



Lax-Wendroff scheme, $c\lambda = \frac{1}{2}$

$$F_{\mathbf{a}}(e^{it}) = 1 - 2(c\lambda)^2 \sin^2\left(\frac{t}{2}\right) + ic\lambda \sin(t).$$

Thomée's result

The goal of Thomée was to clasify the ℓ^∞ stable schemes. It is equivalent to searching for the sequences **a** such that $(\mathbf{a}^n)_n$ is bounded in $\ell^1(\mathbb{Z})$.

The logarithm of $F_\mathbf{a}$ at 1 as an asymptotic expansion of the form

$$F_\mathbf{a}(e^{it}) \underset{t\to 0}{=} \exp\left(ic\lambda t \quad +iq(t) \quad -\beta t^{2\mu} \quad + o(t^{2\mu})\right)$$

where $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, $\mu \in \mathbb{N}$ and q is a polynomial with real coefficients

$$q(t) = b_2 t^2 + \cdots + b_{2\mu-1} t^{2\mu-1}.$$

- $ic\lambda$: Consistency of the scheme
- $iq(t) - \beta t^{2\mu}$: Dissipation Hypothesis

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Thomée '65 (generalizeable for multiple tangency points)

The family $(\mathbf{a}^n)_{n\in\mathbb{N}}$ is bounded in $\ell^1(\mathbb{Z})$ (and the scheme is ℓ^∞ stable) if and only if $q = 0$. In this case, there exists algebraic bounds on $\mathbf{a}_j^n$.

We can observe that the condition $q = 0$ implies that the modified equation associated to our scheme is

$$\partial_t u + c\partial_x u = (-1)^{\mu+1}\frac{\beta}{\Delta t}\Delta x^{2\mu}\partial_x^{2\mu} u.$$

We have

$$F_{\mathbf{a}}(e^{it}) \underset{t\to 0}{=} \exp\left(ic\lambda t \quad + iq(t) \quad - \beta t^{2\mu} \quad + o(t^{2\mu})\right).$$

Scheme	$iq(t)$	2μ	ℓ^∞ stability
Upwind scheme	0	2	Yes
Modified Lax Friedrichs scheme	0	2	Yes
O3 scheme	0	4	Yes
Lax Wendroff scheme	$-i\frac{c\lambda(1-(c\lambda)^2)}{6}t^3$	4	No

Recent developments We assume that the logarithm of $F_{\mathbf{a}}$ has an asymptotic expansion of the form

$$F_{\mathbf{a}}(e^{i\xi}) \underset{\xi \to 0}{=} \exp\left(ic\lambda\xi - \beta\xi^{2\mu} + o(\xi^{2\mu})\right)$$

where $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$ and $\mu \in \mathbb{N}$.

Generalized Gaussian Bounds

Under the previous hypotheses (consistency, dissipation, ,ℓ^2 and ℓ^∞ stability), there exist two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}\setminus\{0\}, \forall j \in \mathbb{Z}, \quad |\mathbf{a}_j^n| \leq \frac{C}{n^{\frac{1}{2\mu}}} \exp\left(-c\left(\frac{|j - nc\lambda|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

Due to Diaconis, Saloff-Coste '14, generalization in Coulombel, Faye '21

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We define

$$\forall x \in \mathbb{R}, \quad H_{2\mu}^{\beta}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu} e^{-\beta u^{2\mu}} du.$$

It is the fundamental solution of

$$\partial_t u + (-1)^{\mu}\beta\partial_x^{2\mu} u = 0.$$

For $\mu = 1$, we have

$$H_{2\mu}^{\beta}(x) = \frac{1}{\sqrt{4\pi\beta}}\exp\left(-\frac{|x|^2}{4\beta}\right).$$

It is the heat kernel / normal distribution.

We define

$$\forall x \in \mathbb{R}, \quad H^{\beta}_{2\mu}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu} e^{-\beta u^{2\mu}} du.$$

Generalized Local Limit Theorem

Under the previous hypotheses, we have

$$\mathbf{a}^n_j - \frac{1}{n^{\frac{1}{2\mu}}} H^{\beta}_{2\mu}\left(\frac{j - nc\lambda}{n^{\frac{1}{2\mu}}}\right) \underset{n\to+\infty}{=} o\left(\frac{1}{n^{\frac{1}{2\mu}}}\right)$$

where the error term is uniform on $\mathbb{Z}$

Due to Randles, Saloff-Coste '15 (also '17 for a multidimensional generalization)

Main Theorem

Under the previous hypotheses, there exist two constants $C, c > 0$ such that for all $n \in \mathbb{N} \backslash \{0\}$ and $j \in \mathbb{Z}$

$$\left| \mathbf{a}_j^n - \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}^{\beta} \left(\frac{j - nc\lambda}{n^{\frac{1}{2\mu}}} \right) \right| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c \left(\frac{|j - nc\lambda|}{n^{\frac{1}{2\mu}}} \right)^{\frac{2\mu}{2\mu-1}} \right).$$

Corollary

Under the previous hypotheses, there exists a constant $C > 0$ such that for all $u_0 \in \ell^2(\mathbb{Z})$

$$\forall n \in \mathbb{N} \backslash \{0\}, \quad \| \mathbf{a}^n * u_0 - H^n * u_0 \|_{\ell^2(\mathbb{Z})} \leq \frac{C}{n^{\frac{1}{2\mu}}} \| u_0 \|_{\ell^2(\mathbb{Z})},$$

where

$$H^n = \left(\frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}^{\beta} \left(\frac{j - nc\lambda}{n^{\frac{1}{2\mu}}} \right) \right)_{j \in \mathbb{Z}}.$$

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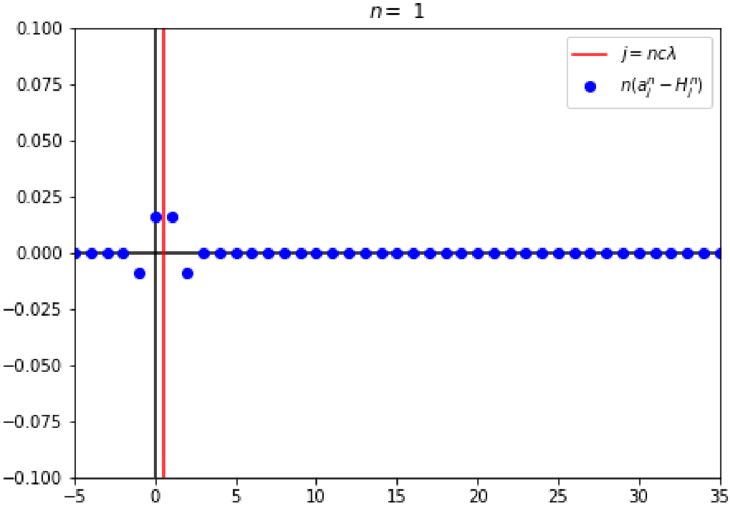

Upwind scheme: $c\lambda = \frac{1}{2}$ and $u^0 = \delta := (\delta_{j,0})_{j\in\mathbb{Z}}$

Idea of the proof

- Fourier Analysis : We have for $n \in \mathbb{N}$, $j \in \mathbb{Z}$,

$$\mathbf{a}_j^n = \frac{1}{2\pi} \int_0^{2\pi} e^{ijt} F_{\mathbf{a}}(e^{it})^n dt.$$

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- Other approach : The goal is to use another representation of the coefficients $\mathbf{a}_j^n$ using functional calculus.

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- Other approach : The goal is to use another representation of the coefficients $\mathbf{a}_j^n$ using functional calculus.

We introduce the Laurent operator

$$L_{\mathbf{a}} : u \in \ell^2(\mathbb{Z}) \mapsto a * u \in \ell^2(\mathbb{Z}).$$

Young's convolution inequality implies that this operator is well-defined and bounded. Fourier analysis implies that

$$\sigma(L_{\mathbf{a}}) = F_{\mathbf{a}}(\mathbb{S}^1).$$

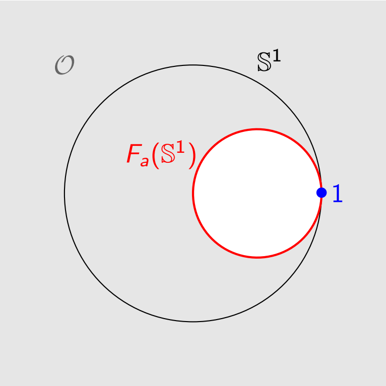
If you consider Γ a path that surrounds $F_{\mathbf{a}}(\mathbb{S}^1)$, then for $n \in \mathbb{N}^*$, $j \in \mathbb{Z}$,

$$\mathbf{a}_j^n = \frac{1}{2i\pi} \int_\Gamma z^n \left((zId - L_{\mathbf{a}})^{-1} \delta \right)_j dz. \quad \text{(Inverse Laplace transform)}$$

We try our best to choose Γ that is inside the unit disk so that the factor z^n decreases exponentially. However, this is not possible near 1.

We have

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad \mathbf{a}_j^n = \frac{1}{2i\pi} \int_\Gamma z^n \left((zId - L_\mathbf{a})^{-1} \delta \right)_j dz.$$



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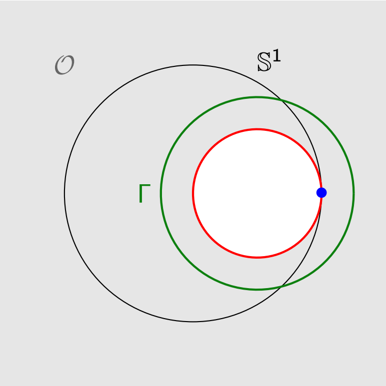

This will not be a good enough choice of Γ.

We have

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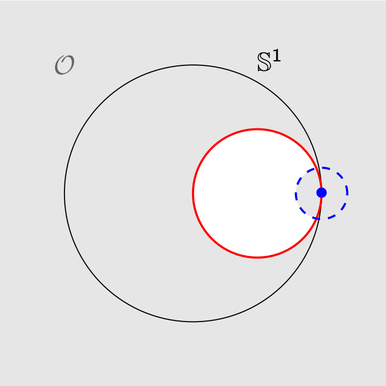

We need a holomorphic extension of $\left((zId - L_{\mathbf{a}})^{-1} \delta \right)_j$ in a neighborhood of 1 and a precise choice of Γ depending on n and j.

Perspective

- There are still some results that have not yet been generalized for implicit schemes.
- Could we expect a better asymptotic expansion of $\mathbf{a}_j^n$?
- Apply the same idea of proof to study the stability near discrete shock profiles for systems of conservation law. (Linearizing near particular non constant solutions, work in progress)

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