Green's function pointwise estimates for spectrally stable discrete shock profiles

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Conservation law, shocks and finite difference schemes

We consider a one-dimensional scalar conservation law

$$\begin{aligned} \partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \end{aligned} \tag{1}$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

The result that will be presented also holds for systems of conservations laws.

This type of PDE tends to have solutions with discontinuities.

Larger goal: We want to know if numerical schemes obtained by discretizing (1) can approach correctly those discontinuous solutions.

Steady Lax shock: We consider $(u^-, u^+) \in \mathbb{R}^2$ such that

$$f(u^-) = f(u^+) \quad \text{(Rankine-Hugoniot condition)}$$

which implies that the function u defined by

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) := \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{else,} \end{cases}$$

is a weak solution of (1).

We will also consider that

$$f'(u^+) < 0 < f'(u^-). \quad \text{(Lax shock)}$$

This translates the fact that the characteristic curves for the states u^- and u^+ enter the shock.

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$) and the shock associated to the states $u^- = 1$ and $u^+ = -1$.

We introduce a conservative one-step explicit finite difference scheme $\mathcal{N} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ such that for $u = (u_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$

$$\mathcal{N}(u)_j := u_j - \nu \left(F\left(\nu; u_{j-p+1}, \ldots, u_{j+q}\right) - F\left(\nu; u_{j-p}, \ldots, u_{j+q-1}\right)\right),$$

where $p, q \in \mathbb{N}\backslash\{0\}$, the numerical flux $F :]0, +\infty[\times \mathbb{R}^{p+q} \to \mathbb{R}^d$ is a smooth function and we fix $\nu = \frac{\Delta t}{\Delta x} > 0$ satisfying a well-chosen CFL condition.

Assumptions:

- $\forall u \in \mathbb{R}, \quad F(\nu; u, \ldots, u) = f(u)$ (consistency condition)
- ℓ^2-stability for some constant states
- The scheme introduces numerical diffusion (numerical viscosity) rather than numerical dispersion (at least for the states $u^{\pm}$).

Example : We consider the modified Lax Friedrichs scheme

$$\forall u \in \mathbb{R}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \quad \mathcal{N}(u)_j := \frac{u_{j+1} + u_j + u_{j-1}}{3} - \nu \frac{f(u_{j+1}) - f(u_{j-1})}{2}.$$

Stationary discrete shock profiles: Definition and existence

We are interested in solutions of

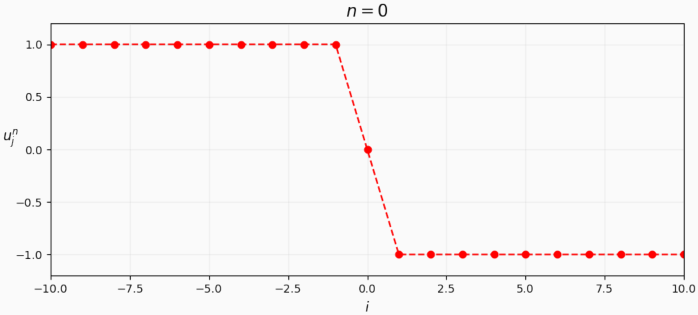
$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}(u^n), \qquad u^0 \in \mathbb{R}^{\mathbb{Z}}. \tag{2}$$

Stationary discrete shock profile (SDSP): We suppose that there exists a sequence $\overline{u} = (\overline{u}_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ that satisfies

$$\mathcal{N}(\overline{u}) = \overline{u} \quad \text{and} \quad \overline{u}_j \underset{j\to\pm\infty}{\rightarrow} u^{\pm}.$$

Example : We consider the initial condition (mean of the standing shock on each cell $\left[(j-\frac{1}{2})\Delta x,(j+\frac{1}{2})\Delta x\right[$)

$$\forall j \in \mathbb{Z}, \quad u_j^0 := \begin{cases} 1 & \text{if } j \leq -1, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j \geq 1. \end{cases}$$

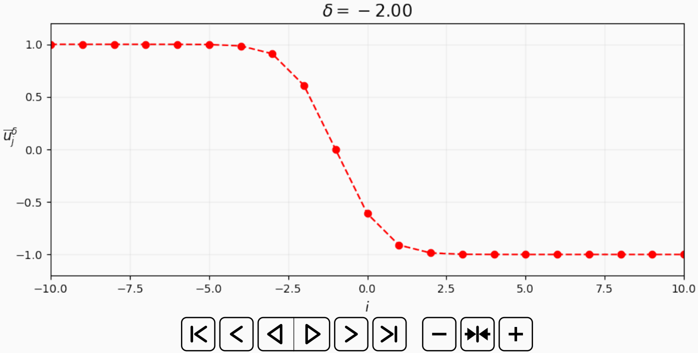


Existence of DSPs

For standing Lax shocks, in some cases, we have the proof of the existence of a continuous one-parameter family $(\overline{u}^{\delta})_{\delta \in I}$ of SDSPs.

- Jennings, *Discrete shocks* (1974)
  - scalar case
  - conservative monotone scheme
- Majda and Ralston, *Discrete Shock Profiles for Systems of Conservation Laws* (1979)
  - system case
  - weak Lax shocks
- Different cases: Smyrlis (1990), Liu-Yu (1999), Serre (2004) etc...

Example : We consider the same initial condition u^0 as before but add a mass δ at $j = 0$. We look at the limit of the solution of the numerical scheme.



We will use the terms "translation of the profile" and "derivative of the profile" even tough we are in a discrete setting.

Stability of discrete shock profiles

The end goal would be to prove a property of nonlinear orbital stability for some SDSPs:

For admissible perturbations h, prove that the solution u^n of the numerical scheme for the initial condition $u^0 = \overline{u} + h$ converges towards the set of translations of the SDSP $\left\{\overline{u}^\delta, \delta \in I\right\}$.

We are going to present a possible first step towards such a result.

- Jennings, *Discrete shocks* (1974)
  - scalar case
  - conservative monotone scheme
- Liu-Xin, *L^1-stability of stationary discrete shocks*, (1993)
  - system case
  - Lax-Friedrichs scheme
  - weak Lax shocks
  - zero mass perturbation (dropped in Ying (1997))
- Different cases: Liu-Yu (1999), etc...

$\Rightarrow$ Extension of the result of Pauline Lafitte, *Green's function pointwise estimates for the modified Lax-Friedrichs scheme*, (2003)

Linearization of the numerical scheme about the SDSP

We define the bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ obtained by linearizing $\mathcal{N}$ about $\overline{u}$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k},$$

with $a_{j,k} \to a_k^{\pm}$ as $j \to \pm\infty$. We are interested in solutions of the linearized numerical scheme

$$\forall n \in \mathbb{N}, \quad h^{n+1} = \mathcal{L}h^n, \qquad h^0 \in \ell^2(\mathbb{Z}).$$

We define the Green's function

$$\forall n \in \mathbb{N}, \forall j_0 \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, \cdot) = (\mathcal{G}(n, j_0, j))_{j \in \mathbb{Z}} := \mathcal{L}^n \delta_{j_0} \in \ell^2(\mathbb{Z}).$$

Spectral assumptions on $\mathcal{L}$

- 1 is a simple eigenvalue of the operator $\mathcal{L}$.

  "$\mathcal{N}(\overline{u}^\delta) = \overline{u}^\delta$ and thus $\mathcal{L}\frac{\partial \overline{u}^\delta}{\partial \delta} = \frac{\partial \overline{u}^\delta}{\partial \delta}$."

- The operator $\mathcal{L}$ has no other eigenvalue of modulus equal or larger than 1. (Spectral stability)

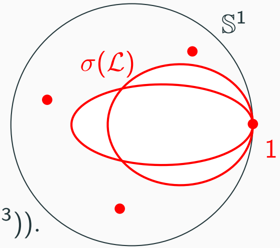
- We assume that

$$\forall \kappa \in \mathbb{S}^1 \backslash \{1\}, \quad \left| \sum_{k=-p}^{q} \kappa^k a_k^\pm \right| < 1$$

and that there exists a complex number $\beta_\pm$ with positive real part such that

$$\sum_{k=-p}^{q} a_k^\pm e^{i\xi k} \underset{\xi \to 0}{=} \exp(-i\alpha_\pm \xi - \beta_\pm \xi^2 + O(|\xi|^3)).$$

with $\alpha_\pm := f'(u^\pm)\nu$.



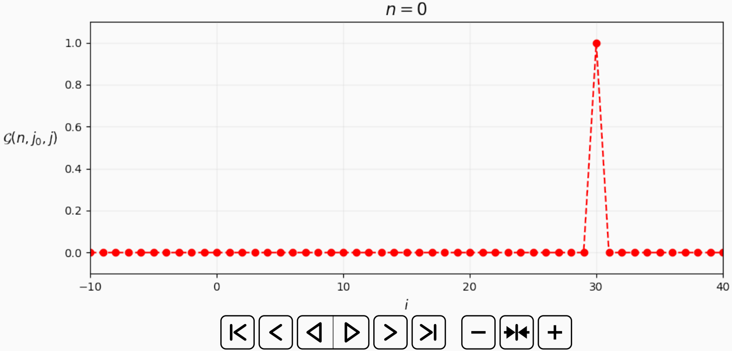
Theorem

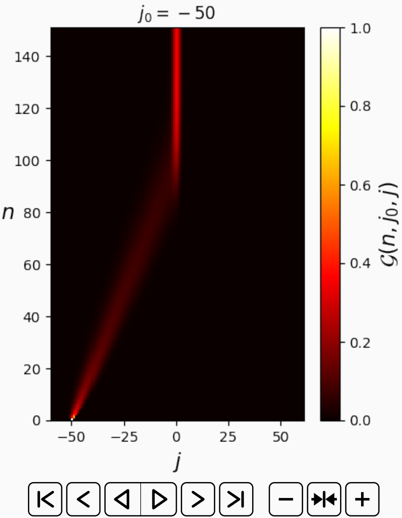
Under some more precise assumptions, there exist a positive constant c, an element V of $\ker(Id - \mathcal{L})$ and an (explicit) function $E : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N} \setminus \{0\}$, $j_0 \in \mathbb{N}$ and $j \in \mathbb{Z}$

$$
\begin{aligned}
&\mathcal{G}(n, j_0, j) \\
=&E\left(\frac{n\alpha_+ + j_0}{\sqrt{n}}\right) V(j) \quad \text{(Excited eigenvector)} \\
+&\mathbb{1}_{j\in\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|n\alpha_+ - (j - j_0)|^2}{n}\right)\right)\right) \quad \text{(Gaussian wave)} \\
+&\mathbb{1}_{j\in-\mathbb{N}} O\left(\frac{1}{\sqrt{n}} \exp\left(-c\left(\frac{|n\alpha_+ + j_0|^2}{n}\right)\right) e^{-c|j|}\right) \quad \text{(Exponential residual)} \\
+&O(e^{-cn-c|j-j_0|})
\end{aligned}
$$

where $E(x) \underset{x\to-\infty}{\rightarrow} 1$ and $E(x) \underset{x\to+\infty}{\rightarrow} 0$.

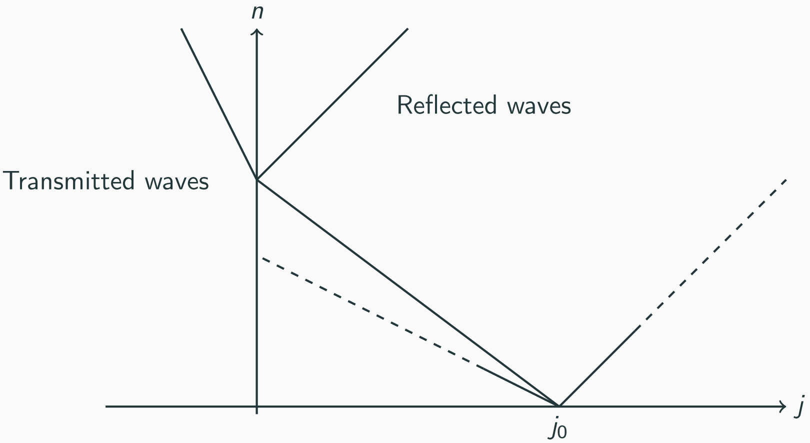
There is a similar result for $j_0 \in -\mathbb{N}$.


$n = 0$
$\mathcal{G}(n, j_0, j)$
1.0
0.8
0.6
0.4
0.2
0.0
−10
0
10
20
30
40
i



$j_0 = -50$
140
120
100
80
60
40
20
0
n
−50
−25
0
25
50
j
1.0
0.8
0.6
0.4
0.2
0.0
$\mathcal{G}(n, j_0, j)$

Case of systems



• Using the inverse Laplace tranform with Γ a path that surrounds the spectrum $\sigma(\mathcal{L})$, we have

$$\forall n \in \mathbb{N} \setminus \{0\}, \forall j_0, j \in \mathbb{Z}, \quad \mathcal{G}(n, j_0, j) = \frac{1}{2i\pi} \int_\Gamma z^n \left((zId - \mathcal{L})^{-1} \delta_{j_0}\right)_j dz. \tag{3}$$

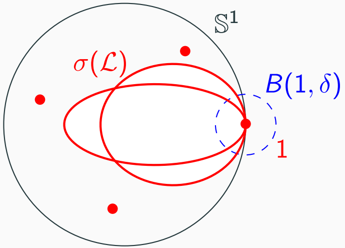
• We rewrite the eigenvalue problem

$$(zId - \mathcal{L})u = 0$$

as a discrete dynamical system

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z) W_j. \tag{4}$$

We are interested in solutions of (4) that tend towards 0 as j tends to $+\infty$ or $-\infty$ (Jost solutions, geometric dichotomy) and use them to express $(zId - \mathcal{L})^{-1}\delta_{j_0}$.



• Using this idea and a good choice of path Γ, we prove sharp estimates on the temporal Green's function.

About the theorem:

- Bounds uniform in j_0
- Very few limitation on the size of the stencil
- The result can be proved for systems
- The result can be proved for higher odd ordered schemes (not only for first order schemes)

Perspective:

- Can we now prove nonlinear orbital stability ? (at least in the scalar case?)
- Existence of spectrally stable SDSPs?
- What can we say for moving shocks (with rational speed)?
- What can we say for dispersive schemes? (Lax-Wendroff for instance)
- Study of the stability for multi-dimensional conservation laws (Carbuncle phenomenon)

P. Godillon.
Green's function pointwise estimates for the modified Lax-Friedrichs scheme.
M2AN, Math. Model. Numer. Anal., 37(1):1–39, 2003.

G. Jennings.
Discrete shocks.
Comm. Pure Appl. Math., 27:25–37, 1974.

J.-G. Liu and Z. P. Xin.
L^1-stability of stationary discrete shocks.
Math. Comp., 60(201):233–244, 1993.

T.-P. Liu and S.-H. Yu.
Continuum shock profiles for discrete conservation laws. I. Construction.
Comm. Pure Appl. Math., 52(1):85–127, 1999.

T.-P. Liu and S.-H. Yu.
Continuum shock profiles for discrete conservation laws. II. Stability.
Comm. Pure Appl. Math., 52(9):1047–1073, 1999.

A. Majda and J. Ralston.
Discrete shock profiles for systems of conservation laws.
Comm. Pure Appl. Math., 32(4):445–482, 1979.

D. Serre.
L^1-stability of nonlinear waves in scalar conservation laws.
In *Handbook of differential equations: Evolutionary equations. Vol. I*, pages 473–553. Amsterdam: Elsevier/North-Holland, 2004.

Y. S. Smyrlis.
Existence and stability of stationary profiles of the LW scheme.
Commun. Pure Appl. Math., 43(4):509–545, 1990.

L. Ying.
Asymptotic stability of discrete shock waves for the Lax-Friedrichs scheme to hyperbolic systems of conservation laws.
Japan J. Indust. Appl. Math., 14(3):437–468, 1997.