Pointwise estimates of Green's function for discrete shock profiles

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1 – Conservation laws and shocks

We consider a one-dimensional scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, \qquad u : \mathbb{R}_+ \times \mathbb{R} \to \mathcal{U}, \tag{1}$$

where the space of states $\mathcal{U}$ is an open set of $\mathbb{R}$, the flux $f : \mathcal{U} \to \mathbb{R}$ is a smooth function.

Those PDEs tend to have discontinuous solutions, even for smooth initial data.

Shocks: For $(u^-, u^+; s) \in \mathcal{U}^2 \times \mathbb{R}$, we define the $(u^-, u^+; s)$-shock

$$\forall t \in \mathbb{R}_+, \forall x \in \mathbb{R}, \quad u(t,x) = \begin{cases} u^- & \text{if } x < st, \\ u^+ & \text{if } x \geq st. \end{cases}$$

It is a solution of (1) if and only if the Rankine-Hugoniot condition is satisfied

$$f(u^+) - f(u^-) = s(u^+ - u^-).$$

We also impose an entropy conditon (Oleinik's condition E)

$$\forall u \in]u^-, u^+[, \quad \frac{f(u) - f(u^+)}{u - u^+} < \frac{f(u^-) - f(u^+)}{u^- - u^+}.$$

Example : We can consider the Burgers equation ($f(u) = \frac{u^2}{2}$). The shock $(1, -1; 0)$ satisfies the Rankine-Hugoniot condition.

2 – Finite difference scheme and discrete shock profiles

We fix a mesh grid $\Delta x > 0$ and a time step $\Delta t > 0$. We introduce a conservative one-step explicit finite difference scheme $\mathcal{N} : \mathcal{U}^{\mathbb{Z}} \to \mathcal{U}^{\mathbb{Z}}$ such that for $u = (u_j)_{j \in \mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$

$$(\mathcal{N}u)_j := u_j - \frac{\Delta t}{\Delta x}\left(F\left(\frac{\Delta t}{\Delta x}; u_{j-p+1}, \dots, u_{j+q}\right) - F\left(\frac{\Delta t}{\Delta x}; u_{j-p}, \dots, u_{j+q-1}\right)\right), \tag{2}$$

where $p, q \in \mathbb{N}^*$ and the numerical flux $F : (\lambda; u_{-p}, \dots, u_{q-1}) \in \mathbb{R}_+^* \times \mathcal{U}^{p+q} \to \mathbb{R}^d$ is a smooth function. We will consider that it satisfies a standard consistency condition (for smooth/constant solutions) and ℓ^2-stability for some constant states. We are interested in solutions of

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathcal{N}u^n, \qquad u^0 \in \mathcal{U}^{\mathbb{Z}}. \tag{3}$$

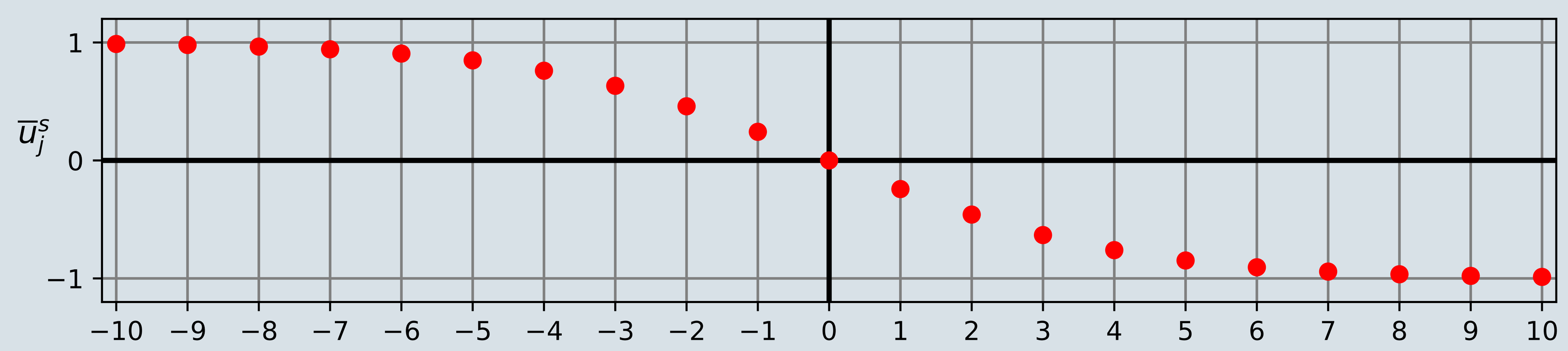
Is there an enhanced consistency condition on the numerical scheme for discontinuous/shock solutions? $\Rightarrow$ Discrete shock profiles (DSP) Traveling waves solutions of (3) that link two states $u^\pm$.

Stationary discrete shock profile: We consider $u^-, u^+ \in \mathcal{U}$ such that the shock $(u^-, u^+; 0)$ satisfies the Rankine Hugoniot condition. We suppose that there exist a sequence $\overline{u}^s = (\overline{u}^s_j)_{j \in \mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$ that satisfies

$$\mathcal{N}(\overline{u}^s) = \overline{u}^s \quad \text{and} \quad \overline{u}^s_j \underset{j \to \pm\infty}{\to} u^\pm.$$

There are still a lot of questions about the existence and stability of DSPs. (see [2] for answers in the case of monotone schemes)

Example : We can consider the modified Lax-Friedrichs scheme for Burgers equation.



3 – Linear stability, Green's function

We define a bounded operator $\mathcal{L} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by linearizing $\mathcal{N}$ about $\overline{u}^s$:

$$\forall h \in \ell^2(\mathbb{Z}), \forall j \in \mathbb{Z}, \quad (\mathcal{L}h)_j := \sum_{k=-p}^{q} a_{j,k} h_{j+k}, \tag{4}$$

with $a_{j,k} \to a_k^\pm$ as $j \to \pm\infty$. We are interested in solutions of the linearized numerical scheme

$$\forall n \in \mathbb{N}, \quad h^{n+1} = \mathcal{L}h^n, \qquad h^0 \in \ell^2(\mathbb{Z}). \tag{5}$$

Green's function: We define the (temporal) Green's function

$$\forall l \in \mathbb{Z}, \qquad \mathcal{G}(0, l, \cdot) := \delta_l, \quad \forall n \in \mathbb{N}, \ \mathcal{G}(n+1, l, \cdot) := \mathcal{L}\mathcal{G}(n, l, \cdot). \tag{6}$$

Goal: Find sharp estimates on Green's function in order to prove orbital stability of the discrete shock profile.

A few hypotheses:

- We suppose that $f'(u^+) < 0 < f'(u^-)$. (Lax shock/ Entropy condition)
- We suppose that $\sigma(\mathcal{L}) \subset \{z \in \mathbb{C}, |z| < 1\} \cup \{1\}$. (Spectral stability)
- The scheme introduces numerical diffusion rather than numerical dispersion at the states $u^\pm$.

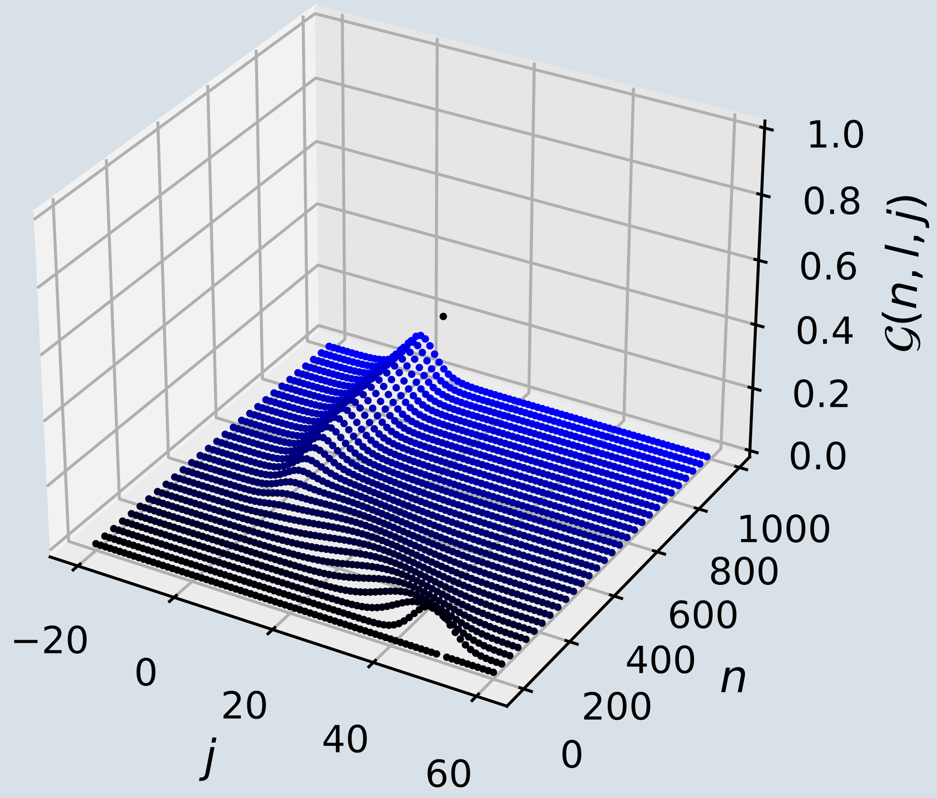
4 – Expected result (Work in progress)

Example: Burgers equation, modified Lax-Friedrichs scheme

We represent the temporal Green's function $\mathcal{G}(n, l, j)$ for $l = 50$.

Observations :

- We see a gaussian wave that travels at a speed $f'(u^+)\frac{\Delta t}{\Delta x}$, "entering" the shock that is located at 0. The spreading is caused by the numerical diffusion.
- When it reaches the shock, a residue forms that is independent from n. This happens because 1 is an eigenvalue of $\mathcal{L}$ (Lax shock).



The case of systems of conservation laws would be more complex, with multiple waves arising from the intial point and "refraction" and "reflection" effects when/if they reach the shock. (see [1])

5 – Spatial dynamics (based on [3] and [1])

Spatial Green's function: For $z \notin \sigma(\mathcal{L})$ and $l \in \mathbb{Z}$, we define

$$G(z, l, \cdot) := (zId - \mathcal{L})^{-1}\delta_l \in \ell^2(\mathbb{Z}). \tag{7}$$

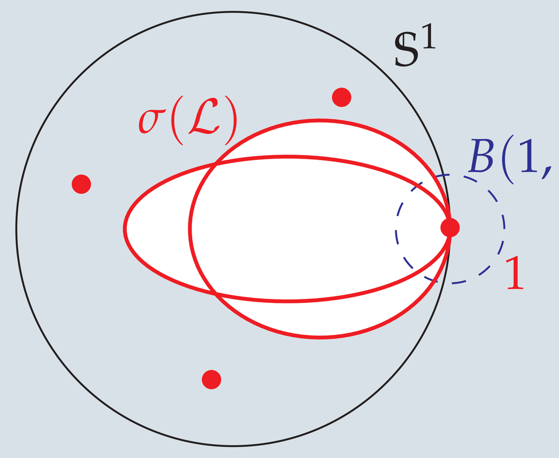
It is the Laplace transform of the temporal Green's function. Using the inverse Laplace tranform with Γ a path that surrounds the spectrum $\sigma(\mathcal{L})$, we have

$$\forall n \in \mathbb{N}^*, \forall l, j \in \mathbb{Z}, \quad \mathcal{G}(n, l, j) = \frac{1}{2i\pi}\int_\Gamma z^n G(z, l, j) dz. \tag{8}$$

- For any z_0 outside of $\sigma(\mathcal{L})$, there is a neighborhood U and two positive constants C, c such that for all $z \in U$

$$\forall j, l \in \mathbb{Z}, \quad |G(z, l, j)| \leq C\exp(-c|j - l|).$$

- We can meromorphically extend the spatial Green's function $G(\cdot, l, j)$ near 1 and decompose using particular solutions of the dynamical system (7).



Using these results and a good choice of path Γ, we hope to prove sharp estimates on the temporal Green's function. (Work in progress)

Idea of the proof: We rewrite the eigenvalue problem

$$(zId - \mathcal{L})u = 0$$

as a discrete dynamical system

$$\forall j \in \mathbb{Z}, \quad W_{j+1} = M_j(z)W_j. \tag{9}$$

We are interested in solutions of (9) that tend towards 0 as j tends to $+\infty$ or $-\infty$ (Jost solutions, geometric dichotomy) and use them to express the spatial Green's function.

6 – References

1. P. Godillon, Green's function pointwise estimates for the modified Lax-Friedrichs scheme, *M2AN, Math. Model. Numer. Anal., 37(1):1-39*, (2003).
2. G. Jennings, Discrete shocks, *Comm. Pure Appl. Math., 27:25-37*, (1974).
3. K. Zumbrun and P. Howard, Pointwise semigroup methods and stability of viscous shock waves, *Indiana Univ. Math. J., 47(3):741–871*, (1998).