Chapter 5

Semigroups and evolution equations

In this chapter we discuss the properties of (strongly continuous) semigroups. This is motivated by the analysis of (linear but also non-linear) evolution (time-dependant) problems.

More precisely, given a Banach space E, an operator A on E and $\varphi_0 \in \mathsf{E}$, we consider the linear Cauchy problem

$$\begin{cases} \varphi'(t) = A\varphi(t), & \forall t \geqslant 0, \\ \varphi(0) = \varphi_0. \end{cases} \tag{5.1}$$

Definition 5.1. *Let I be an interval of $\mathbb{R}$ which contains 0. A (strong) solution of (5.1) on I is a function $\varphi \in C^1(I;\mathsf{E}) \cap C^0(I;\mathsf{Dom}(A))$ which satisfies (5.1) in the natural sense.*

5.1 Exponential of a bounded operator

If A is a bounded operator on E, we can set for all $t \in \mathbb{R}$

$$e^{tA} = \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!} \tag{5.2}$$

The following results are consequencies of the properties of power series in a Banach space.

Proposition 5.2. (i) *For $t \in \mathbb{R}$ we have $e^{tA} \in \mathcal{L}(\mathsf{E})$ and $\left\| e^{tA} \right\|_{\mathcal{L}(\mathsf{E})} \leqslant e^{|t| \|A\|_{\mathcal{L}(\mathsf{E})}}$.*

(ii) *We have $e^{0A} = \mathrm{Id}_{\mathsf{E}}$.*

(iii) *For $s, t \in \mathbb{R}$ we have $e^{tA} e^{sA} = e^{(t+s)A} = e^{sA} e^{tA}$.*

(iv) *If $B \in \mathcal{L}(\mathsf{E})$ commutes with A, then it commutes with e^{tA} for all $t \geqslant 0$.*

(v) *The map*

$$\left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathcal{L}(\mathsf{E}) \\ t & \mapsto & e^{tA} \end{array} \right.$$

is of class C^∞ and

$$\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A.$$

In particular, for $\varphi_0 \in \mathsf{E}$ the function $t \mapsto e^{tA}\varphi_0$ is a strong solution of (5.1) on $\mathbb{R}$.

The purpose of this chapter is to generalize these properties for an unbounded operator A on E (in this case the exponential cannot be defined by the power series (5.2)). Ex. 5.1-5.2

5.2 Strongly continuous semigroups

When A is bounded, the solution of the problem (5.1) is given by a family of operators (e^{tA}) with good properties given in Proposition 5.2. The notion of strongly continuous semigroup generalizes these properties and will be at the heart of the discussion.

Definition 5.3. *We say that the family $(S_t)_{t\geqslant 0}$ of operators in $\mathcal{L}(\mathsf{E})$ is a C^0-semigroup (or strongly continuous semigroup) if*

(i) $S_0 = \mathrm{Id}_{\mathsf{E}}$ *;*

(ii) $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$ *for all $s, t \geqslant 0$;*

(iii) the map $t \mapsto S_t$ is strongly continuous on $\mathbb{R}_+$ (for all $\varphi \in \mathsf{E}$ the map $t \mapsto S_t\varphi \in \mathsf{E}$ is continuous on $\mathbb{R}_+$).

Remark 5.4. The second property implies that S_{t_1} commutes with S_{t_2} for all $t_1, t_2 \geqslant 0$.

Remark 5.5. Notice that we do not require the continuity of the map $t \mapsto S_t$ for the topology of $\mathcal{L}(\mathsf{E})$.

Proposition 5.6. *Let $(S_t)_{t\geqslant 0}$ be a C^0-semigroup. There exist $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that for all $t \in \mathbb{R}_+$ we have*

$$\|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant M e^{\omega t}. \tag{5.3}$$

Moreover, if for some $t_0 \in \mathbb{R}_+$ we have $\|S_{t_0}\|_{\mathcal{L}(\mathsf{E})} < 1$ then (5.3) holds for some $\omega < 0$.

Proof. • Let $\varphi \in \mathsf{E}$. By continuity, there exists $C_\varphi > 0$ such that

$$\forall t \in [0,1], \quad \|S_t\varphi\|_{\mathsf{E}} \leqslant C_\varphi \|\varphi\|_{\mathsf{E}}.$$

By the uniform boundedness principle, there exists $C \geqslant 1$ such that

$$\forall t \in [0,1], \quad \|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant C.$$

Then, for all $N \in \mathbb{N}^*$ and $t \in [N-1, N]$ we get

$$\|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant C^N \leqslant C^{t+1} = Ce^{t\ln(C)}.$$

This gives the first statement with $M = C$ and $\omega = \ln(C)$.

• Now assume that $\alpha = \|S_{t_0}\|_{\mathcal{L}(\mathsf{E})} \in]0,1[$ for some $t_0 > 0$. Let $C = \sup_{t\in[0,t_0]} \|S_t\|_{\mathcal{L}(\mathsf{E})}$. Then for $N \in \mathbb{N}^*$ and $t \in [(N-1)t_0, Nt_0]$ we have

$$\|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant \|S_{t_0}\|_{\mathcal{L}(\mathsf{E})}^{N-1} \left\|S_{t-(N-1)t_0}\right\| \leqslant C\alpha^{N-1} \leqslant \frac{M}{\alpha}\alpha^{\frac{t}{t_0}} = \frac{C}{\alpha}e^{t\frac{\ln(\alpha)}{t_0}}.$$

Then (5.3) holds with $M = \frac{C}{\alpha}$ and $\omega = \frac{\ln(\alpha)}{t_0} < 0$. □

Remark 5.7. To prove the continuity of $\varphi \mapsto S_t\varphi$ it is enough to prove that $S_t\varphi \to \varphi$ in E as $t \to 0^+$. Indeed, let $\varphi \in \mathsf{E}$ and $t_0 > 0$. For the right-continuity we simply write, for $h > 0$,

$$S_{t_0+h}\varphi - S_{t_0}\varphi = S_{t_0}\big(S_h\varphi - \varphi\big) \xrightarrow[h\to 0^+]{} 0.$$

On the other hand, by Proposition 5.6 S_{t_0-h} is bounded uniformly in $h \in]0, t_0]$, so

$$S_{t_0-h}\varphi - S_{t_0}\varphi = S_{t_0-h}\big(\varphi - S_h\varphi\big) \xrightarrow[h\to 0^+]{} 0.$$

Remark 5.8. Let $(S_t)_{t\geqslant 0}$ be a strongly continuous semigroup. The map

$$\begin{cases} \mathbb{R}_+ \times \mathsf{E} & \to & \mathsf{E} \\ (t,\varphi) & \mapsto & S_t\varphi \end{cases}$$

is continous. Let $(t,\varphi) \in \mathbb{R}_+ \times \mathsf{E}$. For $(\tau,\psi) \in \mathbb{R}_+ \times \mathsf{E}$ we have

$$\|S_\tau\psi - S_t\varphi\|_{\mathsf{E}} \leqslant \|S_\tau\psi - S_\tau\varphi\|_{\mathsf{E}} + \|S_\tau\varphi - S_t\varphi\|_{\mathsf{E}}$$

The first term is smaller than $\|S_\tau\|_{\mathcal{L}(\mathsf{E})} \|\psi - \varphi\|_{\mathsf{E}}$, and $\|S_\tau\|_{\mathcal{L}(\mathsf{E})}$ is uniformly bounded for $\tau \in [t-1, t+1]$ by Proposition 5.6. The second term goes to 0 as $\tau \to t$ by strong continuity. This proves that

$$\|S_\tau\psi - S_t\varphi\|_{\mathsf{E}} \xrightarrow[(\tau,\psi)\to(t,\varphi)]{} 0.$$

Definition 5.9. *We say that the family $(S_t)_{t\in\mathbb{R}}$ of operators in $\mathcal{L}(\mathsf{E})$ is a C^0-group (or strongly continuous group) if*

(i) $S_0 = \mathrm{Id}_{\mathsf{E}}$,

(ii) $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$ *for all* $s,t \in \mathbb{R}$,

(iii) the map $t \mapsto S_t$ *is strongly continuous on* $\mathbb{R}$.

Remark 5.10. If $(S_t)_{t\in\mathbb{R}}$ is a strongly continuous group then $S_{-t} = S_t^{-1}$ for all $t \in \mathbb{R}$. Moreover, $(S_t)_{t\geqslant 0}$ and $(S_{-t})_{t\geqslant 0}$ are strongly continuous semigroups.

Definition 5.11.

- *A unitary group on* $\mathcal{H}$ *is a strongly continuous group* $(U_t)_{t\in\mathbb{R}}$ *such that* U_t *is unitary on* $\mathcal{H}$ *for all* $t \in \mathbb{R}$.
- *A contractions semigroup on* E *is a strongly continuous semigroup* $(S_t)_{t\geqslant 0}$ *such that* $\|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant 1$ *for all* $t \geqslant 0$.

Example 5.12 (Translation). For $t \in \mathbb{R}$ we consider on $L^2(\mathbb{R})$ the operator S_t such that for $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$(S_t u)(x) = u(x+t).$$

This defines a unitary group on $L^2(\mathbb{R})$.

Example 5.13 (Dilation). For $t \in \mathbb{R}$ we consider on $L^2(\mathbb{R})$ the operator S_t such that for $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$(S_t u)(x) = e^{2t} u(e^t x).$$

This defines a unitary group on $L^2(\mathbb{R})$.

Example 5.14 (Heat semigroup). We set $S_0 = \mathrm{Id}_{L^2(\mathbb{R})}$. For $t > 00$, $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ we set

$$(S_t u)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} u(y)\,\mathrm{d}y.$$

Then we have $S_t u = G_t * u$ with

$$G(s) = \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}}.$$

We have $\|G_t\|_{L^1(\mathbb{R})} = 1$, $G_{t_1} * G_{t_2} = G_{t_1+t_2}$ and G_t is an approximation of δ when $t \to 0$. Thus from the properties of the convolution product we deduce that $(S_t)_{t\geqslant 0}$ is a contractions semigroup on $L^2(\mathbb{R})$.

5.3 Dissipative operators

We set

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}.$$

Definition 5.15. *Let A be an operator on E. We say that A is dissipative if*

$$\forall \varphi \in \mathsf{Dom}(A), \forall z \in \mathbb{C}_+, \quad \|(A-z)\varphi\|_{\mathsf{E}} \geqslant \mathrm{Re}(z)\,\|u\|_{\mathsf{E}}.$$

Remark 5.16. In particular, if A is dissipative then any $z \in \mathbb{C}_+$ is a regular point of A.

Example 5.17. A skew-symmetric operator on the Hilbert space $\mathcal{H}$ is dissipative (see Proposition 3.7).

Proposition 5.18. *Let A be an operator on $\mathcal{H}$. Then A is dissipative if and only if*

$$\forall \varphi \in \mathsf{Dom}(A), \quad \mathrm{Re}\,\langle A\varphi, \varphi\rangle \leqslant 0. \tag{5.4}$$

Proof. Let $\varphi \in \mathsf{Dom}(A)$. For $z = \tau + i\mu \in \mathbb{C}_+$ with $\tau > 0$ and $\mu \in \mathbb{R}$ we have

$$\begin{aligned} \|(A-z)\varphi\|_{\mathcal{H}}^2 &= \|(A-i\mu)\varphi\|_{\mathcal{H}}^2 - 2\,\mathrm{Re}\,\langle (A-i\mu)\varphi, \tau\varphi\rangle_{\mathcal{H}} + \tau^2\,\|u\|_{\mathcal{H}}^2 \\ &= \|(A-i\mu)\varphi\|_{\mathcal{H}}^2 - 2\tau\,\mathrm{Re}\,\langle A\varphi, \varphi\rangle_{\mathcal{H}} + \tau^2\,\|u\|_{\mathcal{H}}^2. \end{aligned} \tag{5.5}$$

If (5.4) holds, this gives

$$\|(A-z)\varphi\|_{\mathcal{H}}^2 \geqslant \tau^2 \|u\|_{\mathcal{H}}^2 ,$$

so A is dissipative. Conversely, if A is dissipative then (5.5) gives

$$2\tau \operatorname{Re}\langle A\varphi,\varphi\rangle_{\mathcal{H}} - \|(A-i\mu)\varphi\|_{\mathcal{H}}^2 = \tau^2\|u\|_{\mathcal{H}}^2 - \|(A-z)\varphi\|_{\mathcal{H}}^2 \leqslant 0.$$

We divide by τ and let τ go to $+\infty$. This gives (5.4). □

Definition 5.19. *Let A be a dissipative operator on E. We say that A is maximal dissipative if it is dissipative and any $z \in \mathbb{C}_+$ belongs to its resolvent set.*

Ex. 5.3

Example 5.20. If A is a skew-adjoint operator on the Hilbert space $\mathcal{H}$, then A and $-A$ are maximal dissipative. In particular, if A is selfadjoint then iA and $-iA$ are maximal dissipative.

Example 5.21. The Laplacian Δ with domain $\mathsf{Dom}(\Delta) = H^2(\mathbb{R}^d)$ is maximal dissipative on $L^2(\mathbb{R}^d)$. More generally, a selfadjoint and non-positive operator is maximal dissipative.

Remark 5.22.

- If A is maximal dissipative then for all $z \in \mathbb{C}_+$ we have

$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{1}{\operatorname{Re}(z)}. \tag{5.6}$$

- If A is an operator such that $\mathbb{C}_+ \subset \rho(A)$ and (5.6) holds, then A is maximal dissipative. However, we may have $\mathbb{C}_+ \subset \rho(A)$ even if A is not maximal dissipative.

Proposition 5.23. *Let A be a dissipative operator on E. Assume that A is closed and that $\mathsf{Ran}(A-z_0)$ is dense in E for some $z_0 \in \mathbb{C}_+$. Then A is maximal dissipative.*

In particular, if $\rho(A) \cap \mathbb{C}_+ \neq \varnothing$, then A is maximal dissipative.

Proof. Since A is closed and dissipative, $(A-z_0)$ is injective with closed range by Proposition 1.36. By assumption $(A-z_0)$ is then bijective, and $z_0 \in \rho(A)$.

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\rho(A) \cap \mathbb{C}_+$ which goes to some $z \in \mathbb{C}_+$. We have

$$\limsup_{n\in\mathbb{N}} \left\|(A-z)^{-1}\right\| \leqslant \frac{1}{\operatorname{Re}(z)} < +\infty.$$

This implies that $z \in \rho(A)$. Then $\rho(A)$ is closed in $\mathbb{C}_+$. Since it is also open and $\mathbb{C}_+$ is connected, we have $\mathbb{C}_+ \subset \rho(A)$. □

Proposition 5.24. *Let A be a densely defined and closed operator on the Hilbert space $\mathcal{H}$. Assume that A and A^* are dissipative. Then A is maximal dissipative.*

Proof. By Proposition 5.23, it is enough to show that $\mathsf{Ran}(A-1)$ is dense in $\mathcal{H}$. Since A^* is dissipative, (A^*-1) is injective and $\overline{\mathsf{Ran}(A-1)} = \ker(A^*-1)^\perp = \mathcal{H}$. □

Proposition 5.25. *Let A be a maximal dissipative operator on the Hilbert space $\mathcal{H}$. Then A is densely defined.*

Proof. Let $\varphi \in \mathsf{Dom}(A)^\perp$ and $\psi = (A-1)^{-1}\varphi \in \mathsf{Dom}(A)$. We have

$$0 = \langle \varphi,\psi\rangle_{\mathcal{H}} = \langle A\psi - \psi,\psi\rangle_{\mathcal{H}} ,$$

so

$$\|\psi\|_{\mathcal{H}}^2 = \operatorname{Re}\|\psi\|_{\mathcal{H}}^2 = \operatorname{Re}\langle A\psi,\psi\rangle_{\mathcal{H}} \leqslant 0.$$

This implies that $\psi = 0$ and hence $\varphi = 0$. □

Proposition 5.26. *Let A be a maximal dissipative operator. Let B be a dissipative operator. Assume that B is A-bounded with bound smaller than 1. Then $A+B$ is maximal dissipative.*

Proof. The proof is similar to the proof of Theorem 3.44. □

Example 5.27. Let $V \in L^\infty(\mathbb{R}^d, \mathbb{C})$ be such that $\mathrm{Im}(V(x)) \leqslant 0$. We consider the Schrödinger operator $H = H_0 + V(x)$, where H_0 is the free Laplacian. Then $-iH$ is a maximal dissipative operator. Indeed $-iH_0$ is skew-adjoint and $-iV$ is dissipative and bounded, so $-iH$ is maximal dissipative by Proposition 5.26.

Example 5.28. Let $m > 0$. We consider on $\mathscr{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ the norm defined by

$$\|(u,v)\|_{\mathscr{H}}^2 = \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + m\,\|u\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^2(\mathbb{R}^d)}^2 .$$

Then we define on $\mathscr{H}$ the operator

$$\mathcal{W}_a = \begin{pmatrix} 0 & 1 \\ \Delta - m & -a \end{pmatrix},$$

with domain

$$\mathsf{Dom}(\mathcal{W}) = H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d).$$

We know by Exercise 3.5 that $\mathcal{W}_0$ is skew-adjoint on $\mathscr{H}$. Since the operator

$$\begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}$$

is bounded and dissipative on $\mathscr{H}$, we get by Proposition 5.26 that $\mathcal{W}_a$ is maximal dissipative on $\mathscr{H}$.

✐ Ex. 5.5

Proposition 5.29. *Let A be an operator on $\mathcal{H}$. Then A is skew-adjoint if and only if A and $-A$ are maximal dissipative.*

Proof. • Assume that A is skew-adjoint. By Proposition 5.18, A and $-A$ are dissipative. Moreover 1 belongs to the resolvent set of A and $-A$, so they are both maximal dissipative by Proposition 5.23.

• Conversely, assume that A and $-A$ are maximal dissipative. By Proposition 5.18 we have $\mathrm{Re}\langle A\varphi, \varphi\rangle = 0$ for all $\varphi \in \mathsf{Dom}(A)$, so A is skew-symmetric by Remark 3.2. By definition, 1 belongs to the resolvent sets of A and $-A$, so A is skew-adjoint by Proposition 3.22. ☐

5.4 Generators of C^0-semigroups

Definition 5.30. *Let $(S_t)_{t\geqslant 0}$ be a C^0-semigroup on E. We denote by $\mathsf{Dom}(A)$ the set of $\varphi \in \mathsf{E}$ such that the limit*

$$\lim_{t\to 0^+} \frac{S_t\varphi - \varphi}{t}$$

exists in E. In this case, we denote by $A\varphi$ this limit. This defines an operator A on E with domain $\mathsf{Dom}(A)$. We say that A is the generator of $(S_t)_{t\geqslant 0}$.

Example 5.31. Let $A \in \mathcal{L}(\mathsf{E})$. For $t \geqslant 0$ we set $S_t = e^{tA}$, as defined by (5.2). Then the generator of (S_t) is... A.

In general, if A is the generator of the semigroup $(S_t)_{t\geqslant 0}$ then for all $t \geqslant 0$ we can write $S_t = e^{tA}$.

Proposition 5.32. *Let $(S_t)_{t\geqslant 0}$ be a C^0-semigroup on E. Let A be its generator.*

(i) *Let $\varphi \in \mathsf{Dom}(A)$. The map $t \mapsto S_t\varphi$ is differentiable from $\mathbb{R}_+$ to E, we have $S_t\varphi \in \mathsf{Dom}(A)$ for all $t \in \mathbb{R}_+$ and*

$$\frac{d}{dt}\big(S_t\varphi\big) = S_t A\varphi = AS_t\varphi.$$

(ii) *Let $\varphi \in \mathsf{E}$. For $t \geqslant 0$ we have*

$$\int_0^t S_\tau\varphi\,\mathrm{d}\tau \quad \in \mathsf{Dom}(A)$$

and

$$S_t\varphi - \varphi = A\int_0^t S_\tau\varphi \,\mathrm{d}\tau.$$

If $\varphi \in \mathsf{Dom}(A)$ *we also have*

$$S_t\varphi - \varphi = A\int_0^t S_\tau\varphi \,\mathrm{d}\tau = \int_0^t S_\tau A\varphi \,\mathrm{d}\tau.$$

Proof. • Let $t \geqslant 0$. For $\tau > 0$ we have

$$\frac{S_\tau - \mathrm{Id}}{\tau} S_t\varphi = S_t \frac{S_\tau - \mathrm{Id}}{\tau}\varphi \xrightarrow[\tau\to 0^+]{} S_t A\varphi.$$

This proves that $S_t\varphi \in \mathsf{Dom}(A)$ and $AS_t\varphi = S_tA\varphi$. Now let $t > 0$. For $\tau > 0$ we have

$$\frac{S_{t+\tau}\varphi - S_t\varphi}{\tau} \xrightarrow[\tau\to 0]{} S_t A\varphi.$$

and, for $\tau \in]0,t]$,

$$\frac{S_{t-\tau}\varphi - S_t\varphi}{-\tau} = S_{t-\tau}\frac{S_\tau\varphi - \varphi}{\tau} \xrightarrow[\tau\to 0]{} S_t A\varphi.$$

This proves that the map $t \mapsto S_t\varphi$ is differentiable and

$$\frac{d}{dt}\big(S_t\varphi\big) = S_t A\varphi.$$

• For $h > 0$ we have

$$\begin{aligned}
\frac{1}{h}\left(S_h\int_0^t S_\tau\varphi\,\mathrm{d}\tau - \int_0^t S_\tau\varphi\,\mathrm{d}\tau\right) &= \frac{1}{h}\left(\int_0^t S_{\tau+h}\varphi\,\mathrm{d}\tau - \int_0^t S_\tau\varphi\,\mathrm{d}\tau\right)\\
&= \frac{1}{h}\left(\int_h^{t+h} S_\tau\varphi\,\mathrm{d}\tau - \int_0^t S_\tau\varphi\,\mathrm{d}\tau\right)\\
&= \frac{1}{h}\left(\int_t^{t+h} S_\tau\varphi\,\mathrm{d}\tau - \int_0^h S_\tau\varphi\,\mathrm{d}\tau\right)\\
&\xrightarrow[h\to 0]{} S_t\varphi - \varphi.
\end{aligned}$$

This proves the first part of the second statement. Now assume that $\varphi \in \mathsf{Dom}(A)$. Since

$$S_\tau \frac{S_h\varphi - \varphi}{h} \xrightarrow[h\to 0]{} S_\tau A\varphi$$

uniformly in $\tau \in [0,t]$ (by Proposition 5.6), we have

$$\frac{S_h - \mathrm{Id}}{h}\int_0^t S_\tau\varphi\,\mathrm{d}\tau = \int_0^t S_\tau\frac{S_h\varphi - \varphi}{h}\,\mathrm{d}\tau \xrightarrow[h\to 0]{} \int_0^t S_\tau A\varphi\,\mathrm{d}\tau,$$

and the proof is complete. □

Remark 5.33. If A is not closed we cannot just write $A\int_0^t S_\tau\varphi\,\mathrm{d}\tau = \int_0^t AS_\tau\varphi\,\mathrm{d}\tau$ to prove the last statement of the proposition. We are actually going to use this property to prove that A is closed.

Proposition 5.34. *The generator of a* C^0*-semigroup is a closed and densely defined operator that determines the semigroup uniquely.*

Proof. • Let $\varphi \in \mathsf{E}$. By Proposition 5.32, we have for all $h > 0$

$$\frac{1}{h}\int_0^h S_\tau\varphi\,\mathrm{d}\tau \in \mathsf{Dom}(A).$$

Since this goes to φ as $h \to 0$, this proves that $\mathsf{Dom}(A)$ is dense in E.

• Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{Dom}(A)$ such that φ_n goes to some φ and $A\varphi_n$ goes to some ψ in E. For $n \in \mathbb{N}$ and $h > 0$ we have by Proposition 5.32

$$S_h\varphi_n - \varphi_n = \int_0^h S_\tau A\varphi_n \, \mathrm{d}\tau.$$

Taking the limit $n \to +\infty$ and dividing by h, and then taking the limit $h \to 0$, we get

$$\frac{S_h\varphi - \varphi}{h} = \frac{1}{h}\int_0^h S_\tau \psi \, \mathrm{d}\tau \xrightarrow[h\to 0]{} \psi.$$

This proves that $\varphi \in \mathsf{Dom}(A)$ with $A\varphi = \psi$. Thus A is closed.

• Assume that $(\tilde{S}_t)_{t\geqslant 0}$ is a C^0-semigroup whose generator is A. Let $\varphi \in \mathsf{Dom}(A)$ and $t > 0$. For $\theta \in [0, t]$ we set

$$\psi(\theta) = \tilde{S}_{t-\theta} S_\theta \varphi \in \mathsf{E}.$$

For $\theta \in [0, t]$ and $h \in \mathbb{R}^*$ such that $\theta + h \in [0, t]$ we have

$$\begin{aligned}\frac{\psi(\theta+h) - \psi(\theta)}{h} &= \tilde{S}_{t-\theta-h}\left(\frac{S_{\theta+h}\varphi - S_\theta\varphi}{h} - AS_\theta\varphi\right) \\ &\quad + \tilde{S}_{t-\theta-h} A S_\theta \varphi \\ &\quad + \frac{\tilde{S}_{t-\theta-h} - \tilde{S}_{t-\theta}}{h} S_\theta \varphi.\end{aligned}$$

Since $\tilde{S}_{t-\theta-h}$ is bounded uniformly in $h \in [-1, 1]\setminus\{0\}$ by Proposition 5.6, this gives by Proposition 5.32

$$\frac{\psi(\theta+h) - \psi(\theta)}{h} \xrightarrow[h\to 0]{} \tilde{S}_{t-\theta} A S_\theta \varphi - A\tilde{S}_{t-\theta} S_\theta \varphi = 0.$$

Then $S_t\varphi = \psi(t) = \psi(0) = \tilde{S}_t\varphi$. Since $\mathsf{Dom}(A)$ is dense in E, this proves that $\tilde{S}_t = S_t$ for all $t \geqslant 0$. ☐

Proposition 5.35. *Let A be the generator of a C^0-semigroup $(e^{tA})_{t\geqslant 0}$. If D is a subspace of $\mathsf{Dom}(A)$ dense in E and invariant by S_t for all $t \geqslant 0$, then it is a core of A.*

Proof. We have to prove that D is dense in $\mathsf{Dom}(A)$ (for the graph norm). Let $\varphi \in \mathsf{Dom}(A)$ and $\varepsilon > 0$. Let (φ_n) be a sequence in D which goes to φ in E. By Proposition 5.32 there exists $t > 0$ such that

$$\left\| \frac{1}{t}\int_0^t e^{sA}\varphi \, \mathrm{d}s - \varphi \right\|_{\mathsf{Dom}(A)} = \left\| \frac{1}{t}\int_0^t e^{sA}\varphi \, \mathrm{d}s - \varphi \right\|_{\mathsf{E}} + \left\| \frac{1}{t}\int_0^t e^{sA}A\varphi \, \mathrm{d}s - A\varphi \right\|_{\mathsf{E}} \leqslant \frac{\varepsilon}{3}.$$

Again by Proposition 5.32 we have

$$A\left(\frac{1}{t}\int_0^t e^{sA}(\varphi_n - \varphi)\, \mathrm{d}s\right) = \frac{S_t - \mathrm{Id}}{t}(\varphi_n - \varphi) \xrightarrow[n\to\infty]{} 0,$$

so there exists $n \in \mathbb{N}$ such that

$$\left\| \frac{1}{t}\int_0^t e^{sA}\varphi_n \, \mathrm{d}s - \frac{1}{t}\int_0^t e^{sA}\varphi \, \mathrm{d}s \right\|_{\mathsf{Dom}(A)} \leqslant \frac{\varepsilon}{3}.$$

We see the integral $\frac{1}{t}\int_0^t e^{sA}\varphi_n \, \mathrm{d}s$ as a Riemann integral. In particular, there exists $n \in \mathbb{N}^*$ such that

$$\left\| \frac{1}{t}\int_0^t e^{sA}\varphi_n \, \mathrm{d}s - \frac{1}{N}\sum_{k=1}^N e^{\frac{tkA}{N}}\varphi_n \right\|_{\mathsf{Dom}(A)} \leqslant \frac{\varepsilon}{3}.$$

Since D is invariant by $e^{\frac{tkA}{N}}$ for all k, we have $\frac{1}{N}\sum_{k=1}^N e^{\frac{tkA}{N}}\varphi_n \in D$ and the conclusion follows. ☐

Example 5.36. Let A be the generator of the translation semigroup (Example 5.12). Let $u \in C_0^\infty(\mathbb{R})$. Then we have

$$\left\| \frac{u(\cdot + h) - u(\cdot)}{h} - u'(\cdot) \right\|_{L^2(\mathbb{R})} \xrightarrow[h \to 0]{} 0,$$

so $u \in \mathsf{Dom}(A)$ and $Au = u'$. Since $C_0^\infty(\mathbb{R})$ is left invariant by translations and is dense in $L^2(\mathbb{R})$, it is a core of A by Proposition 5.35. This implies that A is the derivative operator, set on $\mathsf{Dom}(A) = H^1(\mathbb{R})$.

Theorem 5.37. *Let A be the generator of a C^0-semigroup $(S_t)_{t \geqslant 0}$. Let $M \geqslant 1$ and $\omega \in \mathbb{R}$ be given by Proposition 5.6. Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \omega$. Then $z \in \rho(A)$ and for $\varphi \in \mathsf{E}$ we have*

$$(A - z)^{-1} \varphi = - \int_0^{+\infty} e^{-tz} S_t \varphi \, \mathrm{d}t = - \int_0^{+\infty} e^{t(A-z)} \varphi \, \mathrm{d}t.$$

Moreover,

$$\left\| (A - z)^{-1} \right\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{M}{\mathrm{Re}(z) - \omega}.$$

In particular, if $(S_t)_{t \geqslant 0}$ is a contractions semigroup, then A is maximal dissipative.

The integrals have to be understood in the sense of Riemann integrals for continuous functions

$$\int_0^{+\infty} e^{t(A-z)} \varphi \, \mathrm{d}t = \lim_{T \to +\infty} \int_0^T e^{t(A-z)} \varphi \, \mathrm{d}t.$$

It is well defined since for all $t \geqslant 0$ we have $\left\| e^{t(A-z)} \right\|_{\mathcal{L}(\mathsf{E})} \leqslant M e^{t(\omega - \mathrm{Re}(z))}$.

Proof. • We consider $R \in \mathcal{L}(\mathsf{E})$ defined by

$$\forall \varphi \in \mathsf{E}, \quad R\varphi = \int_0^{+\infty} e^{t(A-z)} \varphi \, \mathrm{d}t.$$

In particular,

$$\|R\|_{\mathcal{L}(\mathsf{E})} \leqslant \int_0^{+\infty} e^{-t\,\mathrm{Re}(z)} \left\| e^{tA} \right\|_{\mathcal{L}(\mathsf{E})} \, \mathrm{d}t \leqslant M \int_0^{+\infty} e^{t(\omega - \mathrm{Re}(z))} \, \mathrm{d}t = \frac{M}{\mathrm{Re}(z) - \omega}.$$

• We have

$$\begin{aligned}
\frac{e^{hA} - \mathrm{Id}}{h} R\varphi &= \frac{1}{h} \left(\int_0^{+\infty} e^{-tz} e^{(t+h)A} \varphi \, \mathrm{d}t - \int_0^{+\infty} e^{-tz} e^{tA} \varphi \, \mathrm{d}t \right) \\
&= \frac{1}{h} \left(e^{hz} \int_h^{+\infty} e^{t(A-z)} \varphi \, \mathrm{d}t - \int_0^{+\infty} e^{t(A-z)} \varphi \, \mathrm{d}t \right) \\
&= -\frac{e^{hz}}{h} \int_0^h e^{t(A-z)} \varphi \, \mathrm{d}t + \frac{e^{hz} - 1}{h} \int_0^{+\infty} e^{t(A-z)} \varphi \, \mathrm{d}t \\
&\xrightarrow[h \to 0]{} -\varphi + zR\varphi.
\end{aligned}$$

This proves that $\mathsf{Ran}(R) \subset \mathsf{Dom}(A)$ and

$$(A - z)R = -\mathrm{Id}\,.$$

• Now let $\psi \in \mathsf{Dom}(A)$. We have

$$\int_0^T e^{t(A-z)} \psi \, \mathrm{d}t \xrightarrow[T \to +\infty]{} R\psi,$$

and

$$(A - z) \int_0^T e^{t(A-z)} \varphi \, \mathrm{d}t = \int_0^T e^{t(A-z)} (A - z) \varphi \, \mathrm{d}t \xrightarrow[T \to +\infty]{} R(A - z)\psi.$$

Since $(A - z)$ is closed this proves that $R(A - z)\psi = (A - z)R\psi = -\psi$. Thus $(A - z)$ is invertible and its inverse is given by $(A - z)^{-1} = -R$.

• Finally, the fact that the generator of a contractions semigroup ($M = 1$ and $\omega = 0$) is maximal dissipative follows from Remark 5.22. ☐

Definition 5.38. *Let $(S_t)_{t\geqslant 0}$ a strongly continuous group. Then we denote by* $\mathsf{Dom}(A)$ *the set of $\varphi \in \mathsf{E}$ such that the map $t \mapsto S_t\varphi$ is differentiable at $t = 0$, and for $\varphi \in \mathsf{Dom}(A)$ we denote by $A\varphi$ the derivative at 0.*

Theorem 5.39. *The generator of a unitary group on the Hilbert space $\mathcal{H}$ is skew-adjoint.*

Proof. Let $(U_t)_{t\in\mathbb{R}}$ be a unitary group and let A be its generator. A is in particular the generator of the contractions semigroup $(U_t)_{t\geqslant 0}$, so it is maximal dissipative. On the other hand, the generator of the contractions semigroup $(U_{-t})_{t\geqslant 0}$ is $-A$, which is also maximal dissipative. Then A is skew-adjoint by Proposition 5.29. □

5.5 Hille-Yosida Theorem

Our question in this section is the following. Given an operator A on E, is there a strongly continuous semigroup on E whose generator is A ?

Lemma 5.40. *Let A be a densely defined operator. Assume that there exist $\omega \in \mathbb{R}$ and $M > 0$ such that $[\omega, +\infty[\subset \rho(A)$ and $\left\|(A-\lambda)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{M}{\lambda}$ for all $\lambda \geqslant \omega$.*

(i) *For $\varphi \in \mathsf{E}$ we have $-\lambda(A-\lambda)^{-1}\varphi \to \varphi$ as $\lambda \to +\infty$.*

(ii) *For $\varphi \in \mathsf{Dom}(A)$ we have $-\lambda A(A-\lambda)^{-1}\varphi = -\lambda(A-\lambda)^{-1}A\varphi \to A\varphi$ as $\lambda \to +\infty$.*

Proof. For $\varphi \in \mathsf{Dom}(A)$ we have

$$\left\|-\lambda(A-\lambda)^{-1}\varphi - \varphi\right\|_{\mathsf{E}} = \left\|(A-\lambda)^{-1}A\varphi\right\| \leqslant \frac{M\left\|A\varphi\right\|_{\mathsf{E}}}{\lambda} \xrightarrow[\lambda\to+\infty]{} 0.$$

Since $\lambda(A-\lambda)^{-1}$ is bounded uniformly in $\lambda \geqslant \omega$, we deduce the first statement for all $\varphi \in \mathsf{E}$. Then for $\varphi \in \mathsf{Dom}(A)$ we apply the first statement to $A\varphi$ to get the second. □

Theorem 5.41 (Hille-Yosida)**.** *Let A be a densely defined operator. Assume that* $]0, +\infty[\subset \rho(A)$ *and*

$$\forall \lambda > 0, \quad \left\|(A-\lambda)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{1}{\lambda}.$$

Then A generates a contractions semigroup. In particular, a densely defined and maximal dissipative operator generates a contractions semigroup.

Proof. For $n \in \mathbb{N}^*$ we consider the bounded operator

$$A_n = -nA(A-n)^{-1} = -n - n^2(A-n)^{-1}.$$

- For $t \geqslant 0$ we have

$$\left\|e^{tA_n}\right\|_{\mathcal{L}(\mathsf{E})} = e^{-nt}e^{tn^2\|(A-n)^{-1}\|_{\mathcal{L}(\mathsf{E})}} \leqslant e^{-nt}e^{nt} = 1.$$

Let $\varphi \in \mathsf{Dom}(A)$ and $t \geqslant 0$. A_n commutes with A_m and hence with e^{sA_m} for all $s \geqslant 0$, so

$$e^{tA_n}\varphi - e^{tA_m}\varphi = \int_0^t \frac{d}{ds}\left(e^{(t-s)A_m}e^{sA_n}\varphi\right) \mathrm{d}s = \int_0^t e^{(t-s)A_m}e^{sA_n}\left(A_n\varphi - A_m\varphi\right) \mathrm{d}s.$$

This gives

$$\left\|e^{tA_n}\varphi - e^{tA_m}\varphi\right\|_{\mathsf{E}} \leqslant t\left\|A_n\varphi - A_m\varphi\right\|_{\mathsf{E}}.$$

Since $(A_n\varphi)$ is a Cauchy sequence (by Lemma 5.40), the sequence $(e^{tA_n}\varphi)$ converges uniformly on $t \in [0, t_0]$ for any $t_0 > 0$. Since $\left\|e^{tA_n}\right\| \leqslant 1$, the same conclusion holds for any $\varphi \in \mathsf{E}$. We denote by $S_t\varphi$ the limit of $e^{tA_n}\varphi$.

- Let $\varphi \in \mathsf{E}$. Since the sequence of continuous maps $(e^{tA_n}\varphi)$ converges locally uniformly, the map $t \mapsto S_t\varphi$ is continuous on $\mathbb{R}_+$. Let $t, t_1, t_2 \geqslant 0$. For $n \in \mathbb{N}$ we have

$$\left\|e^{tA_n}\varphi\right\|_{\mathsf{E}} \leqslant \left\|\varphi\right\|_{\mathsf{E}} \quad \text{and} \quad e^{t_1A_n}e^{t_2A_n}\varphi = e^{(t_1+t_2)A_n}\varphi.$$

Taking the limit $n \to +\infty$ gives

$$\|S_t\varphi\|_{\mathsf{E}} \leqslant \|\varphi\|_{\mathsf{E}} \quad \text{and} \quad S_{t_1}S_{t_2}\varphi = S_{t_1+t_2}\varphi.$$

This proves that (S_t) is a C^0-semigroup on E.

- We denote by B (with domain $\mathsf{Dom}(B)$) the generator of the semigroup (S_t). Let $\varphi \in \mathsf{Dom}(A)$ and $t_0 > 0$. On $[0, t_0]$ the map $t \mapsto e^{tA_n}\varphi$ and its derivative $t \mapsto e^{tA_n}A_n\varphi$ converge uniformly to $t \mapsto S_t\varphi$ and $S_tA\varphi$. This implies that $S_t\varphi$ is differentiable at time 0 with derivative $A\varphi$. Thus $\varphi \in \mathsf{Dom}(B)$ and $B\varphi = A\varphi$. Now let $\varphi \in \mathsf{Dom}(B)$. Since $(A-1)$ is surjective, there exists $\psi \in \mathsf{Dom}(A)$ such that $(B-1)\varphi = (A-1)\psi = (B-1)\psi$. Since $(B-1)$ is injective, we have $\varphi = \psi \in \mathsf{Dom}(A)$ so $\mathsf{Dom}(B) \subset \mathsf{Dom}(A)$. This proves that $A = B$ is the generator of (S_t). □

Theorem 5.42. *A skew-adjoint operator A on $\mathcal{H}$ generates a unitary group.*

Proof. Since A are $-A$ are maximal dissipative, they generate two contractions semigroups $(S_t^+)_{t\geqslant 0}$ and $(S_t^-)_{t\geqslant 0}$.

Let $\varphi \in \mathsf{Dom}(A) = \mathsf{Dom}(-A)$. Let $t \in \mathbb{R}$. For $\tau \in \mathbb{R}\setminus\{t\}$ we have

$$\frac{S_\tau^- S_\tau^+\varphi - S_t^- S_t^+\varphi}{t-\tau} = S_\tau^- \frac{S_\tau^+\varphi - S_t^+\varphi}{t-\tau} + \frac{(S_\tau^- - S_t^-)S_t^+\varphi}{t-\tau}.$$

Since $\|S_\tau^-\| \leqslant 1$ and $S_t^+\varphi \in \mathsf{Dom}(A)$ we get

$$\frac{S_\tau^- S_\tau^+\varphi - S_t^- S_t^+\varphi}{t-\tau} \xrightarrow[\tau\to t]{} S_t^- A S_t^+\varphi - S_t^- A S_t^+ = 0.$$

This proves that for all $t \in \mathbb{R}$ we have

$$S_t^- S_t^+\varphi = \varphi.$$

Similarly, $S_t^+ S_t^-\varphi = \varphi$ for all $\varphi \in \mathsf{Dom}(A)$. By continuity of S_t^+ and S_t^- and by density of $\mathsf{Dom}(A)$, these equalities hold for all $\varphi \in \mathcal{H}$, so $S_t^- = (S_t^+)^{-1}$ for all $t \geqslant 0$. Fot $t \in \mathbb{R}$ we set

$$U_t = \begin{cases} S_t^+ & \text{if } t \geqslant 0, \\ S_{-t}^- & \text{if } t \leqslant 0. \end{cases}$$

This defines a strongly continuous group $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$. Finally for $t \in \mathbb{R}$ and $\varphi \in \mathcal{H}$ we have

$$\|\varphi\| = \|U_{-t}U_t\varphi\| \leqslant \|U_t\varphi\| \leqslant \varphi,$$

so U_t is an isometry. Since it is surjective, it is unitary and the proof is complete. □

5.6 Inversion formula and application to exponential decay

Let A be a maximal dissipative operator on $\mathcal{H}$. Theorem 5.37 gives an expression of the resolvent of A in terms of its propagator. We would like to write conversely the propagator in terms of the resolvent.

Let $\varphi \in \mathcal{H}$ and $\mu > 0$. By Theorem 5.37 we can write for all $\tau \in \mathbb{R}$

$$\big(A - (\mu + i\tau)\big)^{-1}\varphi = -\int_0^{+\infty} e^{-it\tau} e^{t(A-\mu)}\varphi \, dt.$$

This means that the map $\tau \mapsto \big(A - (\mu + i\tau)\big)^{-1}\varphi$ is the Fourier transform of the map $t \mapsto -\mathbb{1}_{\mathbb{R}_+}(t)e^{t(A-\mu)}\varphi$. We would like to inverse this relation. However, in general, these functions are not in $L^2(\mathbb{R};\mathcal{H})$ and the map $\tau \mapsto \big(A - (\mu + i\tau)\big)^{-1}\varphi$ is not integrable. The idea is to apply the inverse Fourier formula at least for "regular" vectors.

Lemma 5.43. *Let A be an operator on E with non-empty resolvent set and $z_0 \in \rho(A)$. Let $k \in \mathbb{N}^*$, $z \in \rho(A)$ and $\varphi \in \mathcal{H}$. If $\varphi \in \mathsf{Dom}(A^k)$ then we have*

$$(A-z)^{-k}\varphi = \frac{1}{(z-z_0)^k}\sum_{j=0}^{k} C_k^j (-1)^{k-j}(A-z)^{-j}(A-z_0)^j\varphi.$$

Proof. For $\varphi \in \mathsf{Dom}(A)$ we have

$$(A-z)\varphi - (A-z_0)\varphi = (z_0 - z)\varphi.$$

After composition by $(z_0-z)^{-1}(A-z)^{-1}$ on the left, we get on $\mathsf{Dom}(A)$

$$(A-z)^{-1} = \frac{1}{z-z_0}\big((A-z)^{-1}(A-z_0) - \mathrm{Id}\big).$$

This gives the case $k=1$. The general case follows by induction. □

Proposition 5.44. *Let A be the generator of a semigroup on E. Let $\mu \in \mathbb{R}$. Assume that $\mu + i\mathbb{R} \subset \rho(A)$ and*

$$\sup_{\mathrm{Re}(z)=\mu} \left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

For $k \in \mathbb{N}^*$, $\varphi \in \mathsf{Dom}(A^k)$ and $t > 0$ we have

$$e^{tA}\varphi = \frac{(-1)^{k+1}k!}{2i\pi\, t^k}\int_{\Gamma_\mu} e^{tz}(A-z)^{-(k+1)}\varphi\,\mathrm{d}z,$$

where $\Gamma_\mu : \tau \in \mathbb{R} \mapsto \mu + i\tau$.

Proof. Differentiating k times the equality of Theorem 5.37 we get

$$k!(A-z)^{-(k+1)}\varphi = (-1)^{k+1}\int_0^{+\infty} t^k e^{t(A-z)}\varphi\,\mathrm{d}t.$$

Then the map $\tau \mapsto k!(A-(\mu+i\tau))^{-(k+1)}\varphi$ is the Fourier transform of $t \mapsto (-1)^{k+1}\mathbb{1}_{\mathbb{R}_+}(t)t^k e^{t(A-\mu)}\varphi$. Since these functions are integrable we can apply the Inverse Fourier Formula, which gives

$$\forall t \in \mathbb{R}, \quad (-1)^{k+1}\mathbb{1}_{\mathbb{R}_+}(t)t^k e^{t(A-\mu)}\varphi = \frac{k!}{2\pi}\int_{\mathbb{R}} e^{it\tau}(A-(\mu+i\tau))^{-(k+1)}\varphi\,\mathrm{d}\tau,$$

or

$$\forall t \geqslant 0, \quad e^{tA}\varphi = \frac{(-1)^{k+1}k!}{2i\pi\, t^k}\int_{\Gamma_\mu} e^{tz}(A-z)^{-(k+1)}\varphi\,\mathrm{d}z.$$

□

Proposition 5.45. *Let A be the generator of a C^0-semigroup on $\mathcal{H}$. Let M and ω be given by Proposition 5.6. Let $\mu > \omega$. Then there exists $C > 0$ such that for $\varphi \in \mathcal{H}$ we have*

$$\int_{\tau\in\mathbb{R}} \left\|\big(A-(\mu+i\tau)\big)^{-1}\varphi\right\|_{\mathcal{H}}^2 \mathrm{d}\tau \leqslant C\left\|\varphi\right\|_{\mathcal{H}}^2.$$

Proof. Let $\varphi \in \mathcal{H}$. For $\tau \in \mathbb{R}$ we have by Theorem 5.37

$$(A-(\mu+i\tau))^{-1}\varphi = -\int_0^{+\infty} e^{t(A-(\mu+i\tau))}\varphi\,\mathrm{d}t = -\int_{\mathbb{R}} e^{-it\tau}\mathbb{1}_{\mathbb{R}_+}(t)e^{-t\mu}e^{tA}\varphi\,\mathrm{d}t. \tag{5.7}$$

The function $t \mapsto -\mathbb{1}_{\mathbb{R}_+}(t)e^{-t\mu}e^{tA}\varphi$ is in $L^2(\mathbb{R};\mathcal{H})$ and, by (5.7), its Fourier transform is $\tau \mapsto (A-(\mu+i\tau))^{-1}\varphi$. Then by the Plancherel inequality (which holds for a function with values in a Hilbert space) we have

$$\int_{\mathbb{R}} \left\|(A-(\mu+i\tau))^{-1}\varphi\right\|_{\mathcal{H}}^2 \,\mathrm{d}\tau = 2\pi\int_0^{+\infty} e^{-2t\mu}\left\|e^{tA}\varphi\right\|_{\mathcal{H}}^2\,\mathrm{d}t \leqslant C\left\|\varphi\right\|_{\mathcal{H}}^2,$$

with $C = \frac{\pi M^2}{\mu-\omega}$. □

Theorem 5.46 (Gearhart-Prüss). *Let A be the generator of a C^0-semigroup on the Hilbert space $\mathcal{H}$. Assume that $\mathbb{C}_+ \subset \rho(A)$ and that*

$$\beta = \sup_{z \in \mathbb{C}_+} \left\| (A - z)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

Let $\gamma < \frac{1}{\beta}$. Then there exists $C_\gamma > 0$ such that for $t \geqslant 0$ we have

$$\left\| e^{tA} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_\gamma e^{-\gamma t}.$$

Proof. • Let $\tilde{\gamma} \in]\gamma, \beta^{-1}[$. Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geqslant -\gamma$. There exists $z_0 \in \mathbb{C}_+$ such that $z \in D(z_0, \tilde{\gamma})$. Since $\mathsf{dist}(z_0, \sigma(A)) \geqslant \left\| (A - z_0)^{-1} \right\|^{-1} > |z - z_0|$ we have $z \in \rho(A)$. Then by the resolvent identity we have

$$(A - z)^{-1} \big(1 - (z - z_0)(A - z_0)^{-1} \big) = (A - z_0)^{-1}.$$

Since

$$\left\| (z - z_0)(A - z_0)^{-1} \right\| \leqslant \tilde{\gamma}\beta < 1,$$

this gives

$$\left\| (A - z)^{-1} \right\| \leqslant \left\| (A - z_0)^{-1} \right\| \left\| \big(1 - (z - z_0)(A - z_0)^{-1} \big)^{-1} \right\| \leqslant C_1 := \frac{\beta}{1 - \tilde{\gamma}\beta}. \tag{5.8}$$

- For $\tau \in \mathbb{R}$ we have by the resolvent identity

$$\big(A - (-\gamma + i\tau) \big)^{-1} = \big(1 - (\gamma + \mu)\big(A - (-\gamma + i\tau) \big)^{-1} \big) \big(A - (\mu + i\tau) \big)^{-1},$$

so with (5.8)

$$\left\| \big(A - (-\gamma + i\tau) \big)^{-1} \right\|^2 \leqslant \big(1 + (\gamma + \mu) C_1 \big)^2 \left\| \big(A - (\mu + i\tau) \big)^{-1} \right\|^2.$$

We denote by C_2 the constant given by Proposition 5.45. Then we have

$$\int_{\mathbb{R}} \left\| \big(A - (-\gamma + i\tau) \big)^{-1} \varphi \right\|_{\mathcal{H}}^2 \mathrm{d}\tau \leqslant C_3 \left\| \varphi \right\|_{\mathcal{H}}^2, \quad C_3 = C_2 \big(1 + (\gamma + \mu) C_1 \big)^2. \tag{5.9}$$

- Since A^* also satisfies the assumptions of the theorem, we also have for all $\psi \in \mathcal{H}$

$$\int_{\mathbb{R}} \left\| \big(A^* - (-\gamma + i\tau) \big)^{-1} \psi \right\|_{\mathcal{H}}^2 \mathrm{d}\tau \leqslant C_3 \left\| \psi \right\|_{\mathcal{H}}^2. \tag{5.10}$$

- Let $\varphi \in \mathsf{Dom}(A^2)$ and $\psi \in \mathcal{H}$. By Proposition 5.44 we have

$$\left\langle t e^{tA} \varphi, \psi \right\rangle = \frac{1}{2i\pi} \int_{\Gamma_\mu} e^{tz} \left\langle (A - z)^{-2} \varphi, \psi \right\rangle \mathrm{d}z.$$

Since the map $z \mapsto e^{tz} \left\langle (A - z)^{-2} \varphi, \psi \right\rangle$ is holomorphic on $\{\mathrm{Re}(z) > -\tilde{\gamma}\}$ and decays like $\mathrm{Im}(z)^{-2}$ as $|\mathrm{Im}(z)| \to +\infty$ (see Lemma 5.43), we can change the contour of integration from Γ_μ to $\Gamma_{-\gamma}$. This gives

$$\begin{aligned} \left\langle t e^{tA} \varphi, \psi \right\rangle &= \frac{1}{2i\pi} \int_{\Gamma_{-\gamma}} e^{tz} \left\langle (A - z)^{-2} \varphi, \psi \right\rangle \mathrm{d}z \\ &= \frac{1}{2i\pi} \int_{\Gamma_{-\gamma}} e^{tz} \left\langle (A - z)^{-1} \varphi, (A^* - \overline{z})^{-1} \psi \right\rangle \mathrm{d}z. \end{aligned}$$

Then, by the Cauchy-Schwarz inequality and (5.9)-(5.10) we get, for all $\varphi \in \mathsf{Dom}(A^2)$ and $\psi \in \mathcal{H}$,

$$\begin{aligned} \left| \left\langle t e^{tA} \varphi, \psi \right\rangle \right| &\leqslant \frac{e^{-\gamma t}}{2\pi} \left(\int_{\mathbb{R}} \left\| \big(A - (-\gamma + i\tau) \big)^{-1} \varphi \right\|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left\| \big(A^* - (-\gamma - i\tau) \big)^{-1} \psi \right\|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \\ &\leqslant \frac{C_3 e^{-\gamma t}}{2\pi} \left\| \varphi \right\| \left\| \psi \right\|. \end{aligned}$$

Since $\mathsf{Dom}(A^2)$ is dense in $\mathcal{H}$ (see Exercise 5.10), we have the same estimate for all $\varphi \in \mathcal{H}$, and

$$t \left\| e^{tA} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C_3 e^{-\gamma t}}{2\pi}.$$

This gives the estimate for $t \geqslant 1$. Since e^{tA} is bounded uniformly in $t \in [0, 1]$, we get the result by choosing a larger constant if necessary. □

5.7 Exercises

***Exercise* 5.1.** Compute e^{tA_j}, $t \in \mathbb{R}$, for the following matrices:

$$A_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

***Exercise* 5.2.** Prove Proposition 5.2.

***Exercise* 5.3.** **1.** Let A be a maximal dissipative operator on E. Assume that B is a dissipative extension of A. Prove that $A = B$.
2. Let A be a closed and dissipative operator on $\mathcal{H}$. Assume that A has no other dissipative extension than itself. Prove that A is maximal dissipative.

***Exercise* 5.4.** Let A be a densely defined and dissipative operator on $\mathcal{H}$. We define the operator T on $\mathsf{Dom}(T) = \mathsf{Ran}(A-1)$ by $T = (A+1)(A-1)^{-1}$ (since $(A-1)$ is injective, we can define $(A-1)^{-1}$ as an unbounded operator defined on $\mathsf{Ran}(A-1)^{-1}$, see Remark 1.26). T is called the Cayley transform of A.
1. Prove that $\|T\varphi\| \leqslant \|\varphi\|$ for all $\varphi \in \mathsf{Dom}(T)$. Deduce that we can extend T to a bounded operator $\tilde{T}$ on $\mathcal{H}$.
2. Prove that 1 is not an eigenvalue of T.
3. Prove that $A = (T+1)(T-1)^{-1}$ (where $(T-1)^{-1}$ is defined on $\mathsf{Ran}(T-1) = \mathsf{Dom}(A)$).
4. Let $\varphi \in \mathsf{Dom}(\tilde{T})$ such that $\tilde{T}\varphi = \varphi$.
  a. Prove that $\tilde{T}^*\varphi - \varphi = 0$.
  b. Prove that for all $\psi \in \mathsf{Dom}(A)$ we have $\langle \varphi, (A-1)\psi \rangle = \langle \varphi, (A+1)\psi \rangle$.
  c. Prove that 1 is not an eigenvalue of $\tilde{T}$.
5. Prove that $B = (\tilde{T}+1)(\tilde{T}-1)^{-1}$ (defined on $\mathsf{Dom}(B) = \mathsf{Ran}(\tilde{T}-1)$) is a maximal dissipative extension of A.

***Exercise* 5.5.** Let $\alpha \in \mathbb{C}$. We consider on $L^2(0,1)$ the Schrödinger operator with Robin condition, defined by

$$A_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathsf{Dom}(A_\alpha) = \left\{ u \in H^2(0,1) \, : \, u'(0) = \alpha u(0), u'(1) = -\alpha u(1) \right\}.$$

Prove that if $\mathrm{Im}(\alpha) \geqslant 0$ then iA_α is maximal dissipative.

***Exercise* 5.6.** Let A be a maximal dissipative operator on E. Let B be a bounded operator. Prove that $A + B$ (defined on $\mathsf{Dom}(A+B) = \mathsf{Dom}(A)$) generates a C^0-semigroup on E and that, for all $t \geqslant 0$,

$$\left\| e^{t(A+B)} \right\|_{\mathcal{L}(\mathsf{E})} \leqslant e^{t\|B\|_{\mathcal{L}(\mathsf{E})}}.$$

***Exercise* 5.7** (Generator of dilations)**.** For $t \in \mathbb{R}$ and $u \in L^2(\mathbb{R})$ we define the function $S_t u$ by

$$(S_t u)(x) = e^{\frac{t}{2}} u(e^t x).$$

1. Prove that this defines a unitary group $(S_t)_{t \in \mathbb{R}}$ on $L^2(\mathbb{R})$. We denote by A the generator of S_t.
2. Let $u \in C_0^\infty(\mathbb{R})$. Prove that $u \in \mathsf{Dom}(A)$ and that $Au = \frac{u}{2} + xu'$ (where we denote by xv the function $x \mapsto xv(x)$).
3. Prove that $C_0^\infty(\mathbb{R})$ is a core of A.
4. We set

$$\mathcal{D} = \left\{ u \in L^2(\mathbb{R}) \, : \, xu' \in L^2(\mathbb{R}) \right\}.$$

It is endowed with the norm defined by $\|u\|_{\mathcal{D}} = \|u\|_{L^2(\mathbb{R})} + \|xu'\|_{L^2(\mathbb{R})}$. Prove that $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{D}$.
5. Prove that $\mathsf{Dom}(A) = \mathcal{D}$.

***Exercise* 5.8.** Let A be the generator of a C^0-semigroup. Let $\varphi \in \mathsf{Dom}(A)$ and $\lambda \in \mathbb{C}$ such that $A\varphi = \lambda\varphi$. Prove that for all $t \geqslant 0$ we have $e^{tA}\varphi = e^{t\lambda}\varphi$.

***Exercise* 5.9** (Dilation by a general vector field)**.** Let X be a Lipschitzian vector field on $\mathbb{R}^d$. For $x_0 \in \mathbb{R}^d$ on note $t \mapsto \varphi(t; x_0)$ the solution on $\mathbb{R}$ of the problem

$$\begin{cases} y'_{x_0}(t) = X(y_{x_0}(t)), & \forall t \in \mathbb{R}, \\ y'_{x_0}(0) = x_0. \end{cases}$$

Then for $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ we set $\varphi^t(x_0) = y_{x_0}(t)$. Then we have $\varphi^0 = \mathrm{Id}_{\mathbb{R}^d}$ and $\varphi^{t+s} = \varphi^t \circ \varphi^s$ for all $s, t \in \mathbb{R}$. For $t \in \mathbb{R}$ and $u \in L^2(\mathbb{R}^d)$ we set

$$S_t u(x) = \det(d_x \varphi^t)^{\frac{1}{2}} u(\varphi^t x).$$

1. Prove that $(S_t)_{t \in \mathbb{R}}$ is a unitary group on $L^2(\mathbb{R}^d)$.
2. What is the generator of $(S_t)_{t \in \mathbb{R}}$?

***Exercise* 5.10.** Let A be the generator of a strongly continuous semigroup. We set

$$\mathsf{Dom}(A^\infty) = \bigcap_{n \in \mathbb{N}^*} \mathsf{Dom}(A^n)$$

(where, by induction, $\mathsf{Dom}(A^n) = \left\{\varphi \in \mathsf{Dom}(A^{n-1}) \,:\, A^{n-1}\varphi \in \mathsf{Dom}(A)\right\}$).
1. Prove that $\mathsf{Dom}(A^\infty)$ is a subspace of $\mathsf{Dom}(A)$, invariant by e^{tA} for all $t \geqslant 0$.
2. We denote by $\mathcal{C}$ the set of smooth functions on $\mathbb{R}$ compactly supported in $]0, +\infty[$. Let $\phi \in \mathcal{C}$ and $\psi \in \mathsf{E}$. We set

$$\psi_\phi = \int_0^{+\infty} \phi(s) e^{sA} \psi \, \mathrm{d}s.$$

Prove that $\psi_\phi \in \mathsf{Dom}(A)$ with

$$A\psi_\phi = -\int_0^{+\infty} \phi'(s) e^{sA} \psi \, \mathrm{d}s.$$

3. Prove that $\psi_\phi \in \mathsf{Dom}(A^\infty)$.
4. We set $D = \mathsf{span}\,\{\psi_\phi, \psi \in \mathsf{E}, \phi \in \mathcal{C}\}$. Assume by contradiction that D is not dense in E and consider $\ell \in \mathsf{E}'$ such that $\langle \ell, \psi \rangle_{\mathsf{E}',\mathsf{E}} = 0$ for all $\psi \in D$ (as given by the Hahn-Banach theorem).
  a. Prove that $\left\langle \ell, e^{sA}\psi \right\rangle_{\mathsf{E}',\mathsf{E}} = 0$ for all $s \geqslant 0$ and all $\psi \in \mathsf{E}$.
  b. Deduce that D is dense in E.
5. Prove that $\mathsf{Dom}(A^\infty)$ is a core for A.
6. Prove that $\mathsf{Dom}(A^n)$ is a core for A for all $n \in \mathbb{N}^*$.