Chapter 1

Linear Operators

1.1 Unbounded operators

Let E and F be two Banach spaces.

1.1.1 Remainder about bounded operators

We denote by $\mathcal{L}(\mathsf{E},\mathsf{F})$ the set of bounded (continuous) linear maps from E to F, and for $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$ we set

$$\|A\|_{\mathcal{L}(\mathsf{E},\mathsf{F})} = \sup_{\varphi \in \mathsf{E}\setminus\{0\}} \frac{\|A\varphi\|_{\mathsf{F}}}{\|\varphi\|_{\mathsf{E}}}.$$

We write $\mathcal{L}(\mathsf{E})$ for $\mathcal{L}(\mathsf{E},\mathsf{E})$.

If G is a third Banach space then for $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$ and $B \in \mathcal{L}(\mathsf{F},\mathsf{G})$ we have

$$\|BA\|_{\mathcal{L}(\mathsf{E},\mathsf{G})} \leqslant \|A\|_{\mathcal{L}(\mathsf{E},\mathsf{F})} \|B\|_{\mathcal{L}(\mathsf{F},\mathsf{G})}. \tag{1.1}$$

Example 1.1. If E has finite dimension then all the linear maps from E to F are continuous.

Example 1.2. We consider on $\ell^2(\mathbb{N})$ the operators S_r and S_ℓ defined by

$$S_r(u_0, u_1, \dots, u_n, \dots) = (0, u_0, \dots, u_{n-1}, \dots)$$

and

$$S_\ell(u_0, u_1, \dots, u_n, \dots) = (u_1, u_2, \dots, u_{n+1}, \dots).$$

Then S_r and S_ℓ are bounded operators on $\ell^2(\mathbb{N})$ with $\|S_r\|_{\mathcal{L}(\ell^2(\mathbb{N}))} = \|S_\ell\|_{\mathcal{L}(\ell^2(\mathbb{N}))} = 1$.

Example 1.3. Let $a = (a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$. For $u = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ we define $M_a u \in \ell^2(\mathbb{N})$ by

$$\forall n \in \mathbb{N}, \quad (M_a u)_n = a_n u_n.$$

We have $M_a \in \mathcal{L}(\ell^2(\mathbb{N}))$ with $\|M_a\|_{\mathcal{L}(\ell^2(\mathbb{N}))} = \sup_{n\in\mathbb{N}} |a_n|$.

Example 1.4. Let Ω be an open subset of $\mathbb{R}^d$, endowed with the Lebesgue measure Leb. We work on the Banach (Hilbert) space $L^2(\Omega) = L^2(\Omega;\mathbb{C})$. Let $w \in L^\infty(\Omega)$. We consider on $L^2(\Omega)$ the multiplication operator $M_w : u \mapsto uw$. Then $M_w \in \mathcal{L}(L^2(\Omega))$ and

$$\|M_w\|_{\mathcal{L}(L^2(\Omega))} = \|w\|_{L^\infty(\Omega)}.$$

Definition 1.5. *We say that $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$ is invertible if there exists $B \in \mathcal{L}(\mathsf{F},\mathsf{E})$ such that $BA = \mathrm{Id}_{\mathsf{E}}$ and $AB = \mathrm{Id}_{\mathsf{F}}$.*

The following result is a consequence of the open mapping theorem (see for instance [Bre11, Cor. 2.7]).

Proposition 1.6. *Let $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$. Assume that A is bijective. Then its inverse is necessarily continuous, so A is invertible.*

Example 1.7.
- S_r is not surjective and S_ℓ is not injective, so these two operators are not invertible.
- Given $a = (a_n) \in \ell^\infty(\mathbb{N})$, the operator M_a is invertible if and only if
$$0 \notin \overline{\{a_n, n \in \mathbb{N}\}}.$$
- Given $w \in L^\infty(\Omega)$, the operator M_w is invertible in $L^2(\Omega)$ if and only if there exists $\varepsilon > 0$ such that
$$\mathsf{Leb}\big(\{x \in \Omega : |w(x)| \leqslant \varepsilon\}\big) = 0. \tag{1.2}$$

  Assume that (1.2) holds. Then w^{-1} is well defined almost everywhere and $\|w^{-1}\|_{L^\infty(\Omega)} \leqslant \frac{1}{\varepsilon}$. Then $M_{w^{-1}} \in \mathcal{L}(L^2(\Omega))$ is an inverse for M_w. Conversely, assume that M_w is invertible. Assume by contradiction that (1.2) does not hold. Then for all $n \in \mathbb{N}^*$ we can consider $A_n \in \mathcal{B}(\mathcal{O})$ such that $\mathsf{Leb}(A_n) \in]0, +\infty[$ and
$$\forall x \in A_n, \quad |w(x)| \leqslant \frac{1}{n}.$$
  Then we set
$$u_n = \frac{\mathbb{1}_{A_n}}{\mathsf{Leb}(A_n)^{\frac{1}{2}}},$$
  so that $\|u_n\|_{L^2(\Omega)} = 1$ and
$$\|M_w u_n\|_{L^2(\Omega)}^2 = \frac{1}{\mathsf{Leb}(A_n)} \int_{A_n} |w(x)|^2 \, \mathrm{d}x \leqslant \frac{1}{n^2}.$$
  Then
$$\|u_n\|_{L^2(\Omega)} = \big\|M_w^{-1} M_w u_n\big\|_{L^2(\Omega)} \leqslant \frac{1}{n} \|M_w^{-1}\|_{\mathcal{L}(L^2(\Omega))} \xrightarrow[n \to +\infty]{} 0,$$
  which gives a contradiction.

Proposition 1.8. *Let $A \in \mathcal{L}(\mathsf{E})$ with $\|A\|_{\mathcal{L}(\mathsf{E})} < 1$. Then $\mathrm{Id}_\mathsf{E} - A$ is invertible and*
$$(\mathrm{Id}_\mathsf{E} - A)^{-1} = \sum_{n=0}^{+\infty} A^n.$$

Proof. We first observe that
$$\sum_{n=0}^{+\infty} \|A^n\|_{\mathcal{L}(\mathsf{E})} \leqslant \sum_{n=0}^{+\infty} \|A\|_{\mathcal{L}(\mathsf{E})}^n < \infty,$$
so the sum $\sum_{n=0}^{+\infty} A^n$ is convergent in $\mathcal{L}(\mathsf{E})$. Then for all $N \in \mathbb{N}$ we have
$$(\mathrm{Id}_\mathsf{E} - A) \sum_{n=0}^{N} A^n = \mathrm{Id}_\mathsf{E} - A^{N+1}.$$
Taking the limit $N \to +\infty$ gives
$$(\mathrm{Id}_\mathsf{E} - A) \sum_{n=0}^{+\infty} A^n = \mathrm{Id}_\mathsf{E}.$$
Ex. 1.1 We similarly see that $\sum_{n=0}^{+\infty} A^n (\mathrm{Id}_\mathsf{E} - A) = \mathrm{Id}_\mathsf{E}$, and the conclusion follows. ∎

1.1.2 Unbounded operators

Definition 1.9. *A linear operator (or unbounded operator) from* E *to* F *is a linear map* A *from a linear subspace* $\mathsf{Dom}(A)$ *of* E *(the domain of* A*) to* F*. An operator on* E *is an operator from* E *to itself.*

Definition 1.10. *We say that the operator* A *is densely defined if* $\mathsf{Dom}(A)$ *is dense in* E*.*

Example 1.11. A bounded operator $A \in \mathcal{L}(\mathsf{E}, \mathsf{F})$ is a particular case of an unbounded operator with $\mathsf{Dom}(A) = \mathsf{E}$.

Example 1.12. Let Ω be an open subset of $\mathbb{R}^d$, endowed with the Lebesgue measure Leb. Let w be a measurable function on Ω. We consider on $L^2(\Omega)$ the multiplication operator

$$M_w : u \mapsto wu,$$

defined on the domain

$$\mathsf{Dom}(M_w) = \left\{ u \in L^2(\Omega) \, : \, wu \in L^2(\Omega) \right\}. \tag{1.3}$$

Example 1.13. Let w be a measurable function on $\mathbb{R}^d$. We consider on $L^2(\mathbb{R}^d)$ the operator $P_w = \mathcal{F}^{-1} M_w \mathcal{F}$ (where $\mathcal{F}u = \hat{u}$ is the Fourier transform of u) defined on the domain

$$\mathsf{Dom}(A) = \left\{ u \in L^2(\mathbb{R}^d) \, : \, w\hat{u} \in L^2(\mathbb{R}^d) \right\}.$$

Then for $u \in \mathsf{Dom}(A)$ and $\xi \in \mathbb{R}^d$ we have

$$\widehat{Au}(\xi) = w(\xi)\hat{u}(\xi).$$

Remark 1.14. One has to be careful when dealing with unbounded operators. For instance, if A_1 and A_2 are two operators on E, then the sum $A_1 + A_2$ is only defined on the domain $\mathsf{Dom}(A_1) \cap \mathsf{Dom}(A_2)$ (which can be $\{0\}$) and the composition $A_2 \circ A_1$ is defined on $\{\varphi \in \mathsf{Dom}(A_1) \, : \, A_1\varphi \in \mathsf{Dom}(A_2)\}$.

Definition 1.15. *Let* A *and* B *be two linear operators from* E *to* F*. We say that* A *is an extension of* B *and we write* $B \subset A$ *if* $\mathsf{Dom}(B) \subset \mathsf{Dom}(A)$ *and* $A\varphi = B\varphi$ *for all* $\varphi \in \mathsf{Dom}(B)$*.*

Example 1.16. Let w be a continuous function on Ω and let M_w be the multiplication operator by w as in Example 1.12. We can define M_w^0 by $M_w^0 u = wu$ for $u \in \mathsf{Dom}(M_w^0) = C_0^\infty(\Omega)$. Then we have $M_w^0 \subset M_w$.

Example 1.17. Let Ω be an open subset of class C^2 in $\mathbb{R}^d$. We denote by H_0, $\tilde{H}$, H_D and H_N the operators on $L^2(\Omega)$ which all act as $u \mapsto -\Delta u$, but defined on different domains:

- $\mathsf{Dom}(H_0) = C_0^\infty(\Omega)$,
- $\mathsf{Dom}(\tilde{H}) = H^2(\Omega)$,
- $\mathsf{Dom}(H_D) = H^2(\Omega) \cap H_0^1(\Omega)$,
- $\mathsf{Dom}(H_N) = \left\{ u \in H^2(\Omega) \, : \, \partial_\nu u = 0 \text{ on } \partial\Omega \right\}$.

These four operators are densely defined. Moreover we have $H_0 \subset H_D \subset \tilde{H}$ and $H_0 \subset H_N \subset \tilde{H}$.

Definition 1.18. *Let* A *be an operator from* E *to* F*. The graph of* A *is*

$$\mathsf{Gr}(A) = \{(\varphi, A\varphi), \varphi \in \mathsf{Dom}(A)\} \subset \mathsf{E} \times \mathsf{F}.$$

Remark 1.19. If A and T are two linear operators from E to F then $T \subset A$ (in the sense of Definition 1.15 if and only if $\mathsf{Gr}(T) \subset \mathsf{Gr}(A)$ (in the usual sense of inclusion for subsets of $\mathsf{E} \times \mathsf{F}$).

Definition 1.20. *Let* A *be an operator on* E*. We define on* $\mathsf{Dom}(A)$ *the graph norm by*

$$\|\varphi\|_A^2 := \|(\varphi, A\varphi)\|_{\mathsf{E}\times\mathsf{F}}^2 = \|A\varphi\|_{\mathsf{F}}^2 + \|\varphi\|_{\mathsf{E}}^2.$$

Remark 1.21. If $A \in \mathcal{L}(\mathsf{E})$ then the graph norm is equivalent to the original norm on E.

Example 1.22. We consider on $L^2(\mathbb{R}^d)$ the Laplace operator $H = -\Delta$, with domain $\mathsf{Dom}(H) = H^2(\mathbb{R}^d)$. Then the graph norm of H is equivalent to the usual Sobolev norm:

$$\|-\Delta u\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{L^2(\mathbb{R}^d)}^2 \simeq \|u\|_{H^2(\mathbb{R}^d)}^2.$$

This is not the case on a general open subset Ω of $\mathbb{R}^d$.

1.1.3 Boundedly invertible operators

Definition 1.23. *Let A be a linear operator from E to F. We say that A is invertible (or that it is boundedly invertible, or that it has a bounded inverse) if there exists $B \in \mathcal{L}(\mathsf{F}, \mathsf{E})$ such that $\mathsf{Ran}(B) \subset \mathsf{Dom}(A)$, $BA = \mathrm{Id}_{\mathsf{Dom}(A)}$ and $AB = \mathrm{Id}_{\mathsf{F}}$. In this case we write $B = A^{-1}$.*

Remark 1.24. Let $A \in \mathcal{L}(\mathsf{E}, \mathsf{F})$. Then A is boundedly invertible (in the sense of Definition 1.23) if and only if it is invertible in the sense of Definition 1.5, and in this case the two definitions of A^{-1} coincide.

Remark 1.25. Notice that if A is invertible then it is a bijective map from $\mathsf{Dom}(A)$ to F. But if $\mathsf{Dom}(A) \neq \mathsf{E}$ then A^{-1} is only a right inverse of A.

Remark 1.26. If A is injective we can always define an (unbounded) inverse A^{-1}, even if A is not surjective. We define A^{-1} as an operator from F to E with domain $\mathsf{Dom}(A^{-1}) = \mathsf{Ran}(A)$ and we have $A^{-1}A = \mathrm{Id}_{\mathsf{Dom}(A)}$, $AA^{-1} = \mathrm{Id}_{\mathsf{Ran}(A)}$. Unless explicitely mentioned, we will not consider unbounded inverses in this course. Notice however that in many references an operator is said to be invertible as soon as it has an unbounded inverse.

Example 1.27. Let M_w be the multiplication operatoron $L^2(\Omega)$ defined in Example 1.12. Then, as in the bounded case, M_w is invertible if and only if

$$\exists \varepsilon > 0, \quad \mathsf{Leb}\big(\{x \in \Omega : |w(x)| \leqslant \varepsilon\}\big) = 0,$$

and in this case we have $M_w^{-1} = M_{w^{-1}}$. Moreover M_w is injective if and only if

$$\mathsf{Leb}\big(\{x \in \Omega : w(x) = 0\}\big) = 0,$$

In particular, it may happen that M_w is injective but does not have a bounded inverse. Notice also that the operator M_w^0 as defined in Example 1.16 is not invertible since its range is never equal to $L^2(\Omega)$.

Example 1.28. We consider on $L^2(\mathbb{R}^d)$ the operator $A = -\Delta + 1$ with domain $\mathsf{Dom}(A) = H^2(\mathbb{R}^d)$. Then A has a bounded inverse A^{-1}. For $f \in L^2(\mathbb{R}^d)$ the function $u = A^{-1}f$ satisfies, for almost all $\xi \in \mathbb{R}^d$,

Ex. 1.2

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 + 1}.$$

1.1.4 Closed operators

The notion of bounded operator is too restrictive for the applications. On the other hand, we will see that we cannot say much with spectral theory for general unbounded operators. It turns out that a good intermediate choice is to consider the class of closed operators. Roughly speaking, if $\varphi_n \to \varphi$ then we do not require that $A\varphi_n \to A\varphi$, but if $A\varphi_n$ has a limit then it must be $A\varphi$ (in particular, this implies that $A\varphi$ should be defined).

Proposition-Definition 1.29. *Let A be an operator from E to F. We say that A is closed if the following equivalent assertions are satisfied.*

(i) *If a sequence $(\varphi_n)_{n\in\mathbb{N}} \in \mathsf{Dom}(A)^{\mathbb{N}}$ is such that φ_n goes to some φ in E and $A\varphi_n$ goes to some ψ in F, then φ belongs to $\mathsf{Dom}(A)$ and $A\varphi = \psi$;*

(ii) *$\mathsf{Gr}(A)$ is closed in $\mathsf{E} \times \mathsf{F}$;*

(iii) *$\mathsf{Dom}(A)$, endowed with the norm $\|\cdot\|_A$, is complete (hence a Banach space).*

Ex. 1.3

Remark 1.30. Let A be a closed operator on E. Then A defines a bounded operator from the Banach space $\mathsf{Dom}(A)$ to E.

Example 1.31. A bounded operator is closed.

Example 1.32.
- We consider on $L^2(\mathbb{R})$ the operator A defined on the domain $\mathsf{Dom}(A) = C_0^\infty(\mathbb{R})$ by $(Au)(x) = x^2 u(x)$, $x \in \mathbb{R}$. We define $v : \mathbb{R} \to \mathbb{R}$ by $v(x) = e^{-x^2}$. Let $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ be equal to 1 on $[-1, 1]$. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$ we set $\chi_n(x) = \chi(x/n)$. Then $\chi_n v$ goes to v in $L^2(\mathbb{R})$, $\chi_n v \in \mathsf{Dom}(A)$ for all $n \in \mathbb{N}^*$ and $A(\chi_n v)$ has a limit in $L^2(\mathbb{R})$. However v does not belong to $\mathsf{Dom}(A)$. This proves that A is not closed.

- We now consider the operator $A : u \mapsto x^2 u$ on the domain
$$\mathsf{Dom}(A) = \left\{ u \in L^2(\mathbb{R}) \,:\, x^2 u \in L^2(\mathbb{R}) \right\}.$$
Assume that $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathsf{Dom}(A)$ which goes to some u in $L^2(\mathbb{R})$ and such that Au_n has a limit $v \in L^2(\mathbb{R})$. The function $x^2 u$ belongs to $L^2_{\text{loc}}(\mathbb{R})$ and for all $\phi \in C_0^\infty(\mathbb{R})$ we have
$$\int_{\mathbb{R}} x^2 u(x)\phi(x)\,\mathrm{d}x = \lim_{n\to+\infty} \int_{\mathbb{R}} x^2 u_n(x)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} v(x)\phi(x)\,\mathrm{d}x.$$
This proves that $x^2 u(x) = v(x)$ for almost all $x \in \mathbb{R}$. In particular, $u \in \mathsf{Dom}(A)$ and $Au = v$. This proves that A is closed.

Ex. 1.4

Example 1.33. The Laplace operator $-\Delta$ with domain $C_0^\infty(\mathbb{R}^d)$ is not closed in $L^2(\mathbb{R}^d)$. Let $u \in H^2(\mathbb{R}^d)\setminus C_0^\infty(\mathbb{R}^d)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^d)$ which goes to u in $H^2(\mathbb{R}^d)$. Then u_n goes to u in $L^2(\mathbb{R}^d)$, the sequence $(-\Delta u_n)_{n\in\mathbb{N}}$ has a limit in $L^2(\mathbb{R}^d)$ but $u \notin C_0^\infty(\mathbb{R}^d)$. This proves that the Laplace operator is not closed if the domain is $C_0^\infty(\mathbb{R}^d)$. However it is closed with domain $H^2(\mathbb{R}^d)$.

Example 1.34. This example generalizes Examples 1.32 and 1.33. Let Ω be an open subset of $\mathbb{R}^d$. Let $m \in \mathbb{N}$ and consider smooth functions b_α on Ω for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leqslant m$. Then we consider the differential operator
$$P = \sum_{|\alpha|\leqslant m} b_\alpha(x)\partial_x^\alpha. \tag{1.4}$$
We denote by P^* the formal adjoint of P. It is defined for $\phi \in C_0^\infty(\Omega)$ by
$$P^*\phi = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|}\partial_x^\alpha(\overline{b_\alpha}\phi) = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (\partial_x^{\alpha-\beta}\overline{b_\alpha})\partial^\beta\phi. \tag{1.5}$$
Given $u \in L^2(\Omega)$, we have $Pu \in L^2(\Omega)$ (in the sense of distributions) if and only if there exists $v \in L^2(\Omega)$ such that
$$\forall \phi \in C_0^\infty(\Omega), \qquad \int_\Omega u\overline{P^*\phi}\,\mathrm{d}x = \int_\Omega v\overline{\phi}\,\mathrm{d}x,$$
and in this case we write $Pu = v$.

We define an unbounded operator A on $L^2(\Omega)$ by setting $Au = Pu$ for any u in the domain
$$\mathsf{Dom}(A) = \left\{ u \in L^2(\Omega) \,:\, Pu \in L^2(\Omega) \right\},$$
where Pu is understood in the sense of distributions. This operator A is closed. Indeed, let (u_n) be a sequence in $\mathsf{Dom}(A)$ such that u_n goes to some u and Au_n goes to some v in $L^2(\Omega)$. For $\phi \in C_0^\infty(\Omega)$ we have
$$\begin{aligned}\int_\Omega u(x)\overline{(P^*\phi)(x)}\,\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega u_n(x)\overline{(P^*\phi)(x)}\,\mathrm{d}x = \lim_{n\to\infty}\int_\Omega (Pu_n)(x)\overline{\phi(x)}\,\mathrm{d}x \\ &= \lim_{n\to\infty}\int_\Omega (Au_n)(x)\overline{\phi(x)}\,\mathrm{d}x = \int_\Omega v(x)\overline{\phi(x)}\,\mathrm{d}x.\end{aligned}$$
This proves that in the sense of distributions we have $Pu = v \in L^2(\Omega)$. Therefore $u \in \mathsf{Dom}(A)$ and $Au = v$. This proves that A is closed.

The reason why we are interested in closed operators is the following result.

Proposition 1.35. *Let A be an operator from E to F.*

(i) *If A is not closed then it does not have a bounded inverse.*

(ii) *If A is closed and defines a bijection from $\mathsf{Dom}(A)$ to F then it has a bounded inverse.*

Proof. • Assume that A has a bounded inverse $A^{-1} \in \mathcal{L}(\mathsf{F}, \mathsf{E})$. Then A^{-1} is closed, which implies that A is closed ($\mathsf{Gr}(A)$ is closed in $\mathsf{E} \times \mathsf{F}$ if and only if $\mathsf{Gr}(A^{-1})$ is closed in $\mathsf{F} \times \mathsf{E}$). We can also give a direct proof. Assume that (φ_n) is a sequence in E such that φ_n has a limit φ in E and $A\varphi_n$ has a limit ψ in F. Then $A\varphi_n \to \psi$ and $A^{-1}(A\varphi_n) \to \varphi$. Since A^{-1} is closed, this implies that $\psi \in \mathsf{Dom}(A^{-1}) = \mathsf{F}$ (nothing new here) and $\varphi = A^{-1}\psi$, so $\varphi \in \mathsf{Ran}(A^{-1}) = \mathsf{Dom}(A)$ and $A\varphi = \psi$. This proves that A is closed.
• Now assume that A is closed and bijective from $\mathsf{Dom}(A)$ to F. Then $\mathsf{Dom}(A)$ is a Banach space for the graph norm and $A \in \mathcal{L}(\mathsf{Dom}(A), \mathsf{F})$. By Proposition 1.6, its inverse is continuous from F to $\mathsf{Dom}(A)$, hence from F to E, so A is boundedly invertible. ☐

Proposition 1.36. *Let A be an operator from E to F. Assume that there exists $\alpha > 0$ such that*

$$\forall \varphi \in \mathsf{Dom}(A), \quad \|A\varphi\|_{\mathsf{F}} \geqslant \alpha \|\varphi\|_{\mathsf{E}} . \tag{1.6}$$

Then

(i) *A is injective ;*

(ii) *If A is closed, then A has closed range ;*

(iii) *If A is boundedly invertible then $\|A^{-1}\|_{\mathcal{L}(\mathsf{F},\mathsf{E})} \leqslant \frac{1}{\alpha}$.*

Proof. We prove the second statement. Let (ψ_n) be a sequence in $\mathsf{Ran}(A)$ which converges to some ψ in F. For $n \in \mathbb{N}$ we consider $\varphi_n \in \mathsf{Dom}(A)$ such that $A\varphi_n = \psi_n$. Since $(A\varphi_n)$ is a Cauchy sequence in F, (φ_n) is a Cauchy sequence in E by (2.1). Since E is complete, φ_n converges to some φ in E. Finally, since A is closed, $\varphi \in \mathsf{Dom}(A)$ and $\psi = A\varphi \in \mathsf{Ran}(A)$. This proves that $\mathsf{Ran}(A)$ is closed in F. ☐

1.1.5 Closable operators

We have seen in Examples 1.32 and 1.33 that an operator which is not closed can be extended to a closed operator on a bigger domain.

Definition 1.37. *We say that on operator A is closable if it has a closed extension.*

Of course, a closed operator is closable.

Proposition 1.38. *Let A be an operator from E to F. The following assertions are equivalent.*

(i) *A is closable;*

(ii) *For any sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathsf{Dom}(A)$ such that $\varphi_n \to 0$ in E and $A\varphi_n$ has a limit ψ in F, then $\psi = 0$;*

(iii) *$\overline{\mathsf{Gr}(A)}$ is the graph of a closed operator $\overline{A}$ from E to F.*

Definition 1.39. *If the assertions of Proposition 1.38 are satisfied, then the closure of A is the operator $\overline{A}$ such that $\mathsf{Gr}(\overline{A}) = \overline{\mathsf{Gr}(A)}$.*

Proof. • Assume (i) and let $\tilde{A}$ be a closed extension of A. Let (φ_n) be a sequence in $\mathsf{Dom}(A)$ such that $\varphi_n \to 0$ in E and $A\varphi_n \to \psi$ in F. Then (φ_n) is also a sequence in $\mathsf{Dom}(\tilde{A})$ going to 0 and $\tilde{A}\varphi_n \to \psi$. Since $\tilde{A}$ is closed we necessarily have $\psi = 0$, which proves (ii).
• Now assume (ii). We denote by $\overline{\mathsf{D}}$ the closure of $\mathsf{Dom}(A)$ for the graph norm. Let $\varphi \in \overline{\mathsf{D}}$ and let (φ_n) be a sequence in $\mathsf{Dom}(A)$ which goes to φ for the graph norm. Then $(A\varphi_n)$ is a Cauchy sequence in F, and we denote by $\overline{A}\varphi$ its limit. This definition does not depend on the choice of the sequence (φ_n), since if (ζ_n) is another sequence which goes to φ for the graph norm, we have $\varphi_n - \zeta_n \to 0$ and $A\varphi_n - A\zeta_n$ has a limit, so this limit is 0. This defines a linear map $\overline{A}$ from $\overline{\mathsf{D}}$ to F, and $\overline{A}$ is an extension of A with $\mathsf{Dom}(\overline{A}) = \overline{\mathsf{D}}$.

By definition we have $\mathsf{Gr}(\overline{A}) \subset \overline{\mathsf{Gr}(A)}$. Now let $(\varphi, \psi) \in \overline{\mathsf{Gr}(A)}$. There exists a sequence (φ_n, ψ_n) in $\mathsf{Gr}(A)$ such that $\varphi_n \to \varphi$ in E and $\psi_n = A\varphi_n \to \psi$ in F. By definition of $\overline{A}$ we have $\varphi \in \mathsf{Dom}(\overline{A})$ and $\psi = \overline{A}\varphi$, so $(\varphi, \psi) \in \mathsf{Gr}(\overline{A})$. This proves that $\mathsf{Gr}(\overline{A}) = \overline{\mathsf{Gr}(A)}$. Since $\overline{A}$ has a closed graph, this is a closed operator and (iii) is proved.
• Finally, assume (iii). Since $\mathsf{Gr}(A) \subset \mathsf{Gr}(\overline{A})$, $\overline{A}$ is an extension of A. Since $\overline{A}$ is closed, (i) is proved. ☐

We have already seen examples of operators which are not closed but closable.

Example 1.40. We consider on $L^2(\mathbb{R}^d)$ the operators H_0 and H which acts as $-\Delta$ on the domains

$$\mathsf{Dom}(H_0) = C_0^\infty(\mathbb{R}^d) \quad \mathsf{Dom}(H) = H^2(\mathbb{R}^d).$$

Then $H = \overline{H_0}$.

Here is an example of operator which is not closable. Ex. 1.5

Example 1.41. We consider the operator A from $L^2(\mathbb{R})$ to $\mathbb{C}$ defined on $\mathsf{Dom}(A) = C_0^\infty(\mathbb{R})$ by $Au = u(0)$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R})$ such that $u_n \to 0$ in $L^2(\mathbb{R})$ but $u_n(0) \to 1$ in $\mathbb{R}$, so A is not closable.

Proposition 1.42. *If A is a closable operator, then $\overline{A}$ is the smallest closed extension of A (if B is a closed extension of A we have $\overline{A} \subset B$ or, equivalently, $\mathsf{Gr}(\overline{A}) \subset \mathsf{Gr}(B)$).*

Proof. Let B be a closed extension. Then $\mathsf{Gr}(B)$ is closed and contains $\mathsf{Gr}(A)$, so it contains $\mathsf{Gr}(\overline{A}) = \overline{\mathsf{Gr}(A)}$. ☐

Definition 1.43. *Let A be a closed operator from E to F. Let $\mathcal{D}$ be a linear subspace of $\mathsf{Dom}(A)$. We say that $\mathcal{D}$ is a core of A if $\overline{A_{|D}} = A$. Equivalently, $\mathcal{D}$ is dense in $\mathsf{Dom}(A)$ for the graph norm, or for any $\varphi \in \mathsf{Dom}(A)$ there exists a sequence (φ_n) in $\mathcal{D}$ such that $\varphi_n \to \varphi$ in E and $A\varphi_n \to A\varphi$ in F.*

Example 1.44. We consider on $L^2(\mathbb{R}^d)$ the Laplacian $A = -\Delta$, $\mathsf{Dom}(A) = H^2(\mathbb{R}^d)$. Any subspace $\mathcal{D}$ of $H^2(\mathbb{R}^d)$ which contains $C_0^\infty(\mathbb{R}^d)$ is a core of A.

1.1.6 Reducing subspaces

Proposition 1.45. *Let A be an operator on E. Let Π be a projection of E such that*

$$\Pi A \subset A\Pi$$

(for all $\varphi \in \mathsf{Dom}(A)$ we have $\Pi\varphi \in \mathsf{Dom}(A)$ and $A\Pi\varphi = \Pi A\varphi$). Let $\mathsf{F} = \mathsf{Ran}(\Pi) = \ker(1-\Pi)$ and $\mathsf{G} = \ker(\Pi)$.

(i) *F and G are closed subspaces of E and $\mathsf{E} = \mathsf{F} \oplus \mathsf{G}$.*

(ii) *A maps $\mathsf{Dom}(A) \cap \mathsf{F}$ to F and $\mathsf{Dom}(A) \cap \mathsf{G}$ to G. We denote by A_F and A_G the restrictions of A to F and G, with $\mathsf{Dom}(A_\mathsf{F}) = \mathsf{Dom}(A) \cap \mathsf{F}$ and $\mathsf{Dom}(A_\mathsf{G}) = \mathsf{Dom}(A) \cap \mathsf{G}$ (then for $\varphi \in \mathsf{E}$ and $(\varphi_\mathsf{F}, \varphi_\mathsf{G}) \in \mathsf{F} \times \mathsf{G}$ (unique) such that $\varphi = \varphi_\mathsf{F} + \varphi_\mathsf{G}$ we have $A\varphi = A_\mathsf{F}\varphi_\mathsf{F} + A_\mathsf{G}\varphi_\mathsf{G}$). We can write $A = A_\mathsf{F} \oplus A_\mathsf{G}$.*

(iii) *If $\mathsf{Dom}(A)$ is dense in E then $\mathsf{Dom}(A_\mathsf{F})$ is dense in F and $\mathsf{Dom}(A_\mathsf{G})$ is dense in G.*

(iv) *If A is closed then A_F and A_G are closed.*

(v) *A is boundedly invertible if and only if both A_F and A_G are. In this case F and G are left invariant by A^{-1} and the restrictions of A^{-1} to F and G are given by $(A_\mathsf{F})^{-1}$ and $(A_\mathsf{G})^{-1}$. In other words, $A^{-1} = A_\mathsf{F}^{-1} \oplus A_\mathsf{G}^{-1}$.*

Proof. • G is closed since it is the kernel of the bounded operator Π, and F is closed since it is the kernel of $(1-\Pi)$. Let $\varphi \in \mathsf{F} \cap \mathsf{G}$. We have $\varphi = \Pi\varphi = 0$, so $\mathsf{F} \cap \mathsf{G} = \{0\}$. On the other hand, for $\varphi \in \mathsf{E}$ we have $\varphi = \Pi\varphi + (\varphi - \Pi\varphi)$ with $\Pi\varphi \in \mathsf{F}$ and $\varphi - \Pi\varphi \in \mathsf{G}$, so $\mathsf{F} + \mathsf{G} = \mathsf{E}$.
• For $\varphi \in \mathsf{Dom}(A) \cap \mathsf{F}$ we have $\Pi A\varphi = A\Pi\varphi = A\varphi$, so $A\varphi \in \ker(1-\Pi) = \mathsf{F}$. Similarly, for $\varphi \in \mathsf{Dom}(A) \cap \mathsf{G}$ we have $\Pi A\varphi = A\Pi\varphi = 0$, so $A\varphi \in \mathsf{G}$.
• Assume that $\mathsf{Dom}(A)$ is dense in E. Let $\varphi \in \mathsf{E}$. There exists a sequence (φ_n) in $\mathsf{Dom}(A)$ which converges to φ in E. Since $\mathsf{Dom}(A)$ is left invariant by Π, $\Pi\varphi_n$ and $(1-\Pi)\varphi_n$ belong to $\mathsf{Dom}(A)$ for all $n \in \mathbb{N}$. Then $\Pi\varphi_n \in \mathsf{Dom}(A) \cap \mathsf{F}$ and $(1-\Pi)\varphi_n \in \mathsf{Dom}(A) \cap \mathsf{G}$. Finally, $\Pi\varphi_n \to \Pi\varphi$ (this is φ if $\varphi \in \mathsf{F}$) and $(1-\Pi)\varphi_n \to (1-\Pi)\varphi$ (this is φ if $\varphi \in \mathsf{G}$).
• Assume that A is closed. Let (φ_n) be a sequence in $\mathsf{Dom}(A_\mathsf{F})$ such that $\varphi_n \to \varphi$ and $A_\mathsf{F}\varphi_n \to \psi$ in F. Then $\varphi_n \to \varphi$ and $A\varphi \to \psi$ in E. Since A is closed, this proves that $\varphi \in \mathsf{Dom}(A)$ and $A\varphi = \psi$. Since $\varphi \in \mathsf{F}$ we also have $\varphi \in \mathsf{Dom}(A_\mathsf{F})$ and $A_F\varphi = \psi$. This proves that A_F is closed.

- Assume that A is invertible. Let $\psi \in \mathsf{F}$. Let $(\varphi_\mathsf{F}, \varphi_\mathsf{G}) \in (\mathsf{Dom}(A) \cap \mathsf{F}) \times (\mathsf{Dom}(A) \cap \mathsf{G})$ such that $A^{-1}\psi = \varphi_\mathsf{F} + \varphi_\mathsf{G}$. Then $\psi = A\varphi_\mathsf{F} + A\varphi_\mathsf{G}$. We necessarily have $A\varphi_\mathsf{G} = 0$, so $\varphi_\mathsf{G} = 0$. Thus A^{-1} maps F into itself, and $(A^{-1})_{|\mathsf{F}}$ is a bounded inverse for A_F. Similarly, A_G is invertible. Conversely, if A_F and A_G are invertible then $A_\mathsf{F}^{-1} \oplus A_\mathsf{G}^{-1}$ defines a bounded inverse for A. □

Example 1.46. Let Ω be an open subset of $\mathbb{R}^d$ and let ω be a Borel subset of Ω. The multiplication by $\mathbb{1}_\Omega$ defines a projection Π of $L^2(\Omega)$ and for any measurable function w on Ω we have $\Pi M_w \subset M_w \Pi$ (where M_w is as defined in Example 1.12). Moreover we have

$$\mathsf{Ran}(\Pi) = \left\{ u \in L^2(\Omega) \,:\, u(x) = 0 \text{ for almost all } x \in \Omega \backslash \omega \right\}$$

and

$$\ker(\Pi) = \left\{ u \in L^2(\Omega) \,:\, u(x) = 0 \text{ for almost all } x \in \omega \right\}.$$

Notice that Π is an orthogonal projection in this case, which is not necessarily the case in Proposition 1.45.

Example 1.47. Let ω be an open subset of $\mathbb{R}^d$. The operator $\mathcal{F}^{-1}\mathbb{1}_\omega\mathcal{F}$ defines a projection Π of $L^2(\mathbb{R}^d)$. Then for a measurable function w and $P_w = \mathcal{F}^{-1} M_w \mathcal{F}$ (see Example 1.13) we have $\Pi P_w \subset P_w \Pi$.

1.2 Adjoint of an operator

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces.

1.2.1 Definition

Definition 1.48. *Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\psi \in \mathcal{H}_2$. We say that ψ belongs to $\mathsf{Dom}(A^*)$ if there exists $\psi^* \in \mathcal{H}_1$ such that*

$$\forall \varphi \in \mathsf{Dom}(A), \quad \langle A\varphi, \psi \rangle_{\mathcal{H}_2} = \langle \varphi, \psi^* \rangle_{\mathcal{H}_1}.$$

In this case ψ^* is unique and we set $A^*\psi = \psi^*$. This defines an operator A^* from $\mathcal{H}_2$ to $\mathcal{H}_1$
Ex. 1.6 *with domain $\mathsf{Dom}(A^*)$. We say that A^* is the adjoint of A.*

Notice that if A is not densely defined, then $A^*\psi$ is not uniquely defined. We will never consider this situation.

Remark 1.49. By definition, we have

$$\forall \varphi \in \mathsf{Dom}(A), \forall \psi \in \mathsf{Dom}(A^*), \quad \langle A\varphi, \psi \rangle_{\mathcal{H}_2} = \langle \varphi, A^*\psi \rangle_{\mathcal{H}_1}.$$

Remark 1.50. Let $\psi \in \mathcal{H}_2$. By the Riesz representation theorem, ψ belongs to $\mathsf{Dom}(A^*)$ if and only if there exists $C > 0$ such that

$$\forall \varphi \in \mathsf{Dom}(A), \quad \left| \langle A\varphi, \psi \rangle_{\mathcal{H}_2} \right| \leqslant C \left\| \varphi \right\|_{\mathcal{H}_1}.$$

Moreover, in this case we have $\|A^*\psi\|_{\mathcal{H}_1} \leqslant C$.

Ex. 1.10-1.11

1.2.2 Adjoint of a bounded operator

We begin with examples and properties for the adjoint of bounded operators.

Example 1.51. Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are of finite dimensions $n_1, n_2 \in \mathbb{N}^*$. Let β_1 and β_2 be orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, and let $M = (m_{j,k})_{\substack{1 \leqslant j \leqslant n_2 \\ 1 \leqslant k \leqslant n_1}}$ be the matrix of A in β_1 and β_2. Then the matrix of A^* in β_2 and β_1 is

$$M^* = \overline{M}^{\mathsf{T}} = (\overline{m_{k,j}})_{\substack{1 \leqslant j \leqslant n_2 \\ 1 \leqslant k \leqslant n_1}}.$$

Example 1.52. Let $w \in L^\infty(\Omega)$ and let M_w be the multiplication operator as in Example 1.4. Then the adjoint of M_w is $M_w^* = M_{\overline{w}}$.

Example 1.53. The shift operators S_r and S_ℓ (see Example 1.2) are adjoint of each other on $\ell^2(\mathbb{N})$.

Proposition 1.54. *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$.*

(i) $A^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ *;*

(ii) $(A^*)^* = A$ *;*

(iii) $\|A^*\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)} = \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$.

Ex. 1.7

Proof. • Let $\psi \in \mathcal{H}_2$. Then for all $\varphi \in \mathcal{H}_1$ we have

$$|\langle A\varphi, \psi\rangle| \leqslant \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \|\varphi\|_{\mathcal{H}_1} \|\psi\|_{\mathcal{H}_2} ,$$

so $\psi \in \mathsf{Dom}(A^*)$ and

$$\|A^*\psi\|_{\mathcal{H}_1} \leqslant \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \|\psi\|_{\mathcal{H}_2} .$$

This proves that $A^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ and

$$\|A^*\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)} \leqslant \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$$

• Let $\varphi \in \mathcal{H}_1$. For all $\psi \in \mathcal{H}_2$ we have

$$\langle A^*\psi, \varphi\rangle_{\mathcal{H}_1} = \overline{\langle \varphi, A^*\psi\rangle_{\mathcal{H}_1}} = \overline{\langle A\varphi, \psi\rangle_{\mathcal{H}_2}} = \langle \psi, A\varphi\rangle_{\mathcal{H}_2}$$

This proves that $A^{**}\varphi = A\varphi$.

• Then

$$\|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} = \|A^{**}\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leqslant \|A^*\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)} ,$$

and finally, $\|A^*\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)} = \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$. ☐

Proposition 1.55. *For $A_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $A_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ we have $(A_2A_1)^* = A_1^*A_2^*$.*

Proof. Let $\varphi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_3$. We have

$$\langle A_2A_1\varphi, \psi\rangle_{\mathcal{H}_3} = \langle A_1\varphi, A_2^*\psi\rangle_{\mathcal{H}_2} = \langle \varphi, A_1^*A_2^*\psi\rangle_{\mathcal{H}_1} ,$$

and the conclusion follows. ☐

Ex. 1.9

Proposition 1.56. *Let U be a closed and densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ which has a bounded inverse given by $U^{-1} = U^*$. Then $\mathsf{Dom}(U) = \mathcal{H}_1$ and $\|U\varphi\|_{\mathcal{H}_2} = \|\varphi\|_{\mathcal{H}_1}$ for all $\varphi \in \mathcal{H}_1$. We say that U is unitary.*

Proof. Let $\varphi \in \mathsf{Dom}(U)$. Since $U^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ we have

$$\|U\varphi\|_{\mathcal{H}_2}^2 = \langle U\varphi, U\varphi\rangle_{\mathcal{H}_2} = \langle U^*U\varphi, \varphi\rangle_{\mathcal{H}_1} = \|\varphi\|_{\mathcal{H}_1}^2 .$$

Since U is closed, this proves that $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ (see Exercise 1.3). ☐

1.2.3 Main properties of the adjoint

In this paragraph we give general properties for the adjoint of a linear operator.

Proposition 1.57. *Let A and T be two densely defined operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $T \subset A$. Then $A^* \subset T^*$.*

Proof. Let $\psi \in \mathsf{Dom}(A^*)$. For all $\varphi \in \mathsf{Dom}(T)$ we have

$$\langle T\varphi, \psi\rangle_{\mathcal{H}_2} = \langle A\varphi, \psi\rangle_{\mathcal{H}_2} = \langle \varphi, A^*\psi\rangle_{\mathcal{H}_1} .$$

This proves that $\psi \in \mathsf{Dom}(T^*)$ and $T^*\psi = A^*\psi$. ☐

Proposition 1.58. *Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then we have*

$$\ker(A^*) = \mathsf{Ran}(A)^\perp, \quad \ker(A^*)^\perp = \overline{\mathsf{Ran}(A)}.$$

Proof. Let $\varphi \in \ker(A^*)$. Then for all $\psi \in \mathsf{Dom}(A)$ we have

$$\langle A\psi, \varphi\rangle_{\mathcal{H}_2} = \langle \psi, A^*\varphi\rangle_{\mathcal{H}_1} = 0,$$

so $\varphi \in \mathsf{Ran}(A)^\perp$. Conversely, if $\varphi \in \mathsf{Ran}(A)^\perp$ then the same computation shows that $\varphi \in \ker(A^*)$. This gives the first inequality. Then[1] we have

$$\ker(A^*)^\perp = (\mathsf{Ran}(A)^\perp)^\perp = \overline{\mathsf{Ran}(A)},$$

and the proof is complete. □

Proposition 1.59. *Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then A^* is closed.*

Proof. Let (ψ_n) be a sequence in $\mathsf{Dom}(A^*)$ such that ψ_n goes to some ψ in $\mathcal{H}_2$ and $A^*\psi_n$ goes to some ζ in $\mathcal{H}_1$. For $\varphi \in \mathsf{Dom}(A)$ we have

$$\langle A\varphi, \psi\rangle_{\mathcal{H}_2} = \lim_{n\to+\infty} \langle A\varphi, \psi_n\rangle_{\mathcal{H}_2} = \lim_{n\to\infty} \langle \varphi, A^*\psi_n\rangle_{\mathcal{H}_1} = \langle \varphi, \zeta\rangle_{\mathcal{H}_1}.$$

This proves that $\psi \in \mathsf{Dom}(A^*)$ and $A^*\psi = \zeta$. Thus A^* is closed. □

Proposition 1.60. *Let A be a densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then A is closable if and only if $\mathsf{Dom}(A^*)$ is dense in $\mathcal{H}_2$. Moreover, in this case we have $(\overline{A})^* = A^*$ and $\overline{A} = (A^*)^*$. In particular, A is closed if and only if $A = (A^*)^*$.*

We can write A^{**} instead of $(A^*)^*$.

Proof. • We define

$$\Theta : \begin{cases} \mathcal{H}_1 \times \mathcal{H}_2 & \to & \mathcal{H}_2 \times \mathcal{H}_1, \\ (x_1, x_2) & \mapsto & (-x_2, x_1). \end{cases}$$

Then $\Theta^* = \Theta^{-1} = -\Theta : (y_2, y_1) \mapsto (y_1, -y_2)$.

• Let $(\psi, \tilde{\psi}) \in \mathcal{H}_2 \times \mathcal{H}_1$. We have

$$\begin{aligned}
(\psi, \tilde{\psi}) \in \mathsf{Gr}(A^*) &\iff \forall \varphi \in \mathsf{Dom}(A), \quad -\langle A\varphi, \psi\rangle_{\mathcal{H}_2} + \langle \varphi, \tilde{\psi}\rangle_{\mathcal{H}_1} = 0 \\
&\iff \forall \varphi \in \mathsf{Dom}(A), \quad \langle \Theta(\varphi, A\varphi), (\psi, \tilde{\psi})\rangle_{\mathcal{H}_2\times\mathcal{H}_1} = 0 \\
&\iff (\psi, \tilde{\psi}) \in (\Theta\mathsf{Gr}(A))^\perp,
\end{aligned}$$

so

$$\mathsf{Gr}(A^*) = \big(\Theta\mathsf{Gr}(A)\big)^\perp = \Theta\big(\mathsf{Gr}(A)^\perp\big). \tag{1.7}$$

Then

$$\mathsf{Gr}(A^*)^\perp = \overline{\Theta\mathsf{Gr}(A)} = \Theta\overline{\mathsf{Gr}(A)}.$$

After composition by Θ^* we get

$$\overline{\mathsf{Gr}(A)} = \Theta^*\big(\mathsf{Gr}(A^*)^\perp\big) = \Theta\big(\mathsf{Gr}(A^*)^\perp\big). \tag{1.8}$$

• Assume that $\mathsf{Dom}(A^*)$ is dense in $\mathcal{H}_2$. Then we can define $A^{**} = (A^*)^*$. By Proposition 1.59, this defines a closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\varphi \in \mathsf{Dom}(A)$. For all $\psi \in \mathsf{Dom}(A^*)$ we have

$$\langle A^*\psi, \varphi\rangle = \langle \psi, A\varphi\rangle,$$

[1]

Proposition. *Let $\mathcal{H}$ be a Hilbert space and let F be a subspace of $\mathcal{H}$. Then we have*

$$(F^\perp)^\perp = \overline{F}.$$

(see Proposition 1.9 in [Bre11] for the version in normed vector spaces)

Proof. • We have $F \subset F^{\perp\perp}$ and $F^{\perp\perp}$ is closed, so $\overline{F} \subset F^{\perp\perp}$

• We have $F^\perp = \overline{F}^\perp$ and $\mathcal{H} = \overline{F} \oplus \overline{F}^\perp$. Let $\varphi \in F^{\perp\perp}$. There exist $\overline{\varphi} \in \overline{F}$ and $\overline{\varphi}^\perp \in \overline{F}^\perp = F^\perp$ such that $\varphi = \overline{\varphi} + \overline{\varphi}^\perp$. Then $0 = \langle \varphi, \overline{\varphi}^\perp\rangle = \|\overline{\varphi}^\perp\|^2$, so $\varphi = \overline{\varphi} \in \overline{F}$. □

so $\varphi \in \mathsf{Dom}(A^{**})$ and $A^{**}\varphi = A\varphi$. This proves that A^{**} is an extension of A, and in particular A is closable.

- Now assume that A is closable and let $\psi \in \mathsf{Dom}(A^*)^\perp$. Then, by (1.8),

$$(0,\psi) = \Theta^*(-\psi,0) \in \Theta^*(\mathsf{Gr}(A^*)^\perp) = \overline{\mathsf{Gr}(A)} = \mathsf{Gr}(\overline{A}).$$

so $\psi = 0$. Thus $\mathsf{Dom}(A^*)$ is dense in $\mathcal{H}_2$. Moreover, by (1.7) applied with $\overline{A}$ we have

$$\mathsf{Gr}((\overline{A})^*) = \Theta\big(\mathsf{Gr}(\overline{A})^\perp\big) = \Theta\big(\overline{\mathsf{Gr}(A)}^\perp\big) = \Theta\big(\mathsf{Gr}(A)^\perp\big) = \mathsf{Gr}(A^*).$$

This proves that $(\overline{A})^* = A^*$. Since A^* is densely defined, we can consider its adjoint A^{**}. By (1.7) applied first to A^* (with Θ replaced by $-\Theta^*$) and then to A, we have

$$\mathsf{Gr}(A^{**}) = \Theta^*\big(\mathsf{Gr}(A^*)^\perp\big) = \Theta^*\Big(\big(\Theta\overline{\mathsf{Gr}(A)}^\perp\big)^\perp\Big) = \big(\overline{\mathsf{Gr}(A)}^\perp\big)^\perp = \overline{\mathsf{Gr}(A)} = \mathsf{Gr}(\overline{A}).$$

This proves that $A^{**} = \overline{A}$. □

Proposition 1.61. *Let A be a closed and densely defined operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then $A^* : \mathsf{Dom}(A^*) \to \mathcal{H}_1$ is boundedly invertible if and only if $A : \mathsf{Dom}(A) \to \mathcal{H}_2$ is, and in this case we have $(A^*)^{-1} = (A^{-1})^*$.*

Proof. Assume that A has a bounded inverse. Then the adjoint $(A^{-1})^*$ of A^{-1} is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\varphi \in \mathsf{Dom}(A^*)$. For all $\psi \in \mathcal{H}_2$ we have

$$\big\langle (A^{-1})^* A^*\varphi, \psi\big\rangle_{\mathcal{H}_2} = \big\langle A^*\varphi, A^{-1}\psi\big\rangle_{\mathcal{H}_1} = \big\langle \varphi, AA^{-1}\psi\big\rangle_{\mathcal{H}_2} = \langle \varphi,\psi\rangle_{\mathcal{H}_2}.$$

This proves that $(A^{-1})^*A^{-1}\varphi = \varphi$, and we deduce that $(A^{-1})^*A^{-1} = \mathrm{Id}_{\mathsf{Dom}(A)}$. Now let $\psi \in \mathcal{H}_1$. For all $\varphi \in \mathsf{Dom}(A)$ we have

$$\big\langle A\varphi, (A^{-1})^*\psi\big\rangle = \big\langle A^{-1}A\varphi, \psi\big\rangle = \langle \varphi,\psi\rangle,$$

so $(A^{-1})^*\psi \in \mathsf{Dom}(A^*)$ and $A^*(A^{-1})^*\psi = \psi$. This proves that $A^*(A^{-1})^* = \mathrm{Id}_{\mathcal{H}_1}$. Finally we have proved that A^* is boundedly invertible and $(A^*)^{-1} = (A^{-1})^*$.

Now assume that A^* is boundedly invertible. Then $A = A^{**}$ is boundedly invertible, and the proof is complete. □

Remark 1.62. If A is bounded and boundedly invertible, then by Proposition 1.55 we can simply write

$$A^*(A^{-1})^* = (A^{-1}A)^* = \mathrm{Id}^* = \mathrm{Id},$$

and similarly $(A^{-1})^*A^* = \mathrm{Id}$, so A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

1.2.4 Examples: adjoints of some differential operators

General differential operators with smooth and bounded coefficients

Let Ω be an open subset of $\mathbb{R}^d$. We define on $\mathcal{H} = L^2(\Omega)$ the operator A_0 which acts as the differential operator P (see (1.4)) on the domain $\mathsf{Dom}(A_0) = C_0^\infty(\Omega)$. Then $v \in L^2(\Omega)$ belongs to $\mathsf{Dom}(A_0^*)$ if and only if there exists $w \in L^2(\Omega)$ such that

$$\forall \phi \in C_0^\infty(\Omega), \qquad \int_\Omega P\phi(x)\overline{v(x)}\,\mathrm{d}x = \int_\Omega \phi(x)\overline{w(x)}\,\mathrm{d}x.$$

By definition, this means that $P^*v = w$ (see (1.5)) in the sense of distributions. Then A_0^* acts as P^* on the domain

$$\mathsf{Dom}(A_0^*) = \big\{v \in L^2(\Omega) \,:\, P^*v \in L^2(\Omega)\big\}.$$

Then A_0 is closed by Proposition 1.59 or by Example 1.34. The domain of A_0^* contains $C_0^\infty(\Omega)$, so it is dense. By Proposition 1.60 this implies that A_0 is closable. This is consistent with the fact that we already know by Example 1.34 that A_0 has a closed extension. Notice that A_0 may have several closed extensions (see for instance the discussion of Section 3.1.5).

The Laplace operator

As a particular case, we consider the Laplace operator. We define the operators H_0 and H which act as $-\Delta$ on the domains

$$\mathsf{Dom}(H_0) = C_0^\infty(\Omega) \quad \text{and} \quad \mathsf{Dom}(H) = \left\{ u \in L^2(\Omega) \,:\, \Delta u \in L^2(\Omega) \right\}.$$

When $\Omega = \mathbb{R}^d$, the domain of H is just $H^2(\mathbb{R}^d)$. We recall that this is not true for a general Ω (it can happen that $u \in L^2(\Omega)$ and $\Delta u \in L^2(\Omega)$ but u is not in $H^2(\Omega)$).

Since the formal adjoint of the Laplacian is the Laplacian itself we have in general $H_0^* = H$. Since $H_0 \subset H$ we have $H^* \subset H_0^* = H$ by Proposition 1.57.

When $\Omega = \mathbb{R}^d$ we actually have $H^* = H_0^* = H$. Several proofs are possible.

We can directly prove that $H \subset H^*$. Let $\psi \in \mathsf{Dom}(H) = H^2(\mathbb{R}^d)$. For $\varphi \in \mathsf{Dom}(H)$ we have by the Green formula

$$\langle H\varphi, \psi \rangle = \langle -\Delta\varphi, \psi \rangle = \langle \varphi, -\Delta\psi \rangle = \langle \varphi, H\psi \rangle,$$

so $\psi \in \mathsf{Dom}(H^*)$ and $H^*\psi = H\psi$. Alternatively, we can use the fact that $H = \overline{H_0}$ (see Example 1.33) and Proposition 1.60 which gives $H^* = \overline{H_0}^* = H_0^*$.

In general, since functions in $\mathsf{Dom}(H)$ or $\mathsf{Dom}(H_0^*)$ are not necessarily in $H^2(\Omega)$, we cannot apply the usual Green formula.

In dimension 1, it is still true that $\mathsf{Dom}(H_0^*) = \mathsf{Dom}(H) = H^2(\Omega)$. And we can see that in general we do not necessarily have $H^* = H_0^*$. We consider the case $\Omega =]0,1[$. Let $v \in \mathsf{Dom}(H^*)$ and $w = H^*v$. For all $u \in \mathsf{Dom}(H) = H^2(0,1)$ we have

$$-\int_0^1 u''(x)\overline{v(x)}\,\mathrm{d}x = \langle Hu, v \rangle_{L^2(0,1)} = \langle u, w \rangle_{L^2(0,1)} = \int_0^1 u(x)\overline{w(x)}\,\mathrm{d}x.$$

On the other hand, we also have $v \in \mathsf{Dom}(H_0^*) = H^2(0,1)$, so by the Green formula

$$\begin{aligned}
-\int_0^1 u(x)''\overline{v(x)}\,\mathrm{d}x &= -u'(1)\overline{v(1)} + u'(0)\overline{v(1)} + \int_0^1 u'(x)\overline{v'(x)}\,\mathrm{d}x \\
&= -u'(1)\overline{v(1)} + u'(0)\overline{v(1)} + u(1)\overline{v'(1)} - u(0)\overline{v'(0)} - \int_0^1 u(x)\overline{v''(x)}\,\mathrm{d}x.
\end{aligned}$$

This implies that $w = -v''$ and $v(0) = v(1) = v'(0) = v'(1) = 0$. Thus, a function in $H^2(0,1)$ which does not vanish at 0 or 1 belongs to $\mathsf{Dom}(H_0^*)$ but not to $\mathsf{Dom}(H^*)$. Then $\mathsf{Dom}(H_0^*)$ is not included in $\mathsf{Dom}(H^*)$.

Creation and annihilation operators

We consider on $\mathcal{H} = L^2(\mathbb{R})$ the creation and annihilation operators defined on the domain $C_0^\infty(\mathbb{R})$ by

$$\forall u \in C_0^\infty(\mathbb{R}), \quad \mathsf{a}_0 u = \frac{u' + xu}{\sqrt{2}} \quad \text{and} \quad \mathsf{c}_0 u = \frac{-u' + xu}{\sqrt{2}}.$$

Then we set

$$\mathsf{a} = \overline{\mathsf{a}_0} \quad \text{and} \quad \mathsf{c} = \overline{\mathsf{c}_0}.$$

We have

$$\mathsf{Dom}(\mathsf{a}) = \left\{ u \in L^2(\mathbb{R}) \,:\, u' + xu \in L^2(\mathbb{R}) \right\}, \quad \mathsf{Dom}(\mathsf{c}) = \left\{ u \in L^2(\mathbb{R}) \,:\, -u' + xu \in L^2(\mathbb{R}) \right\}.$$

Finally we have

$$\mathsf{a}^* = \mathsf{c} \quad \text{and} \quad \mathsf{c}^* = \mathsf{a}.$$

1.3 Operators and quadratic forms

1.3.1 Lax-Milgram Theorem

Let $\mathcal{V}$ be a Hilbert space. We denote by $\mathcal{V}'$ the space of semilinear forms on $\mathcal{V}$.

Definition 1.63. (i) *A sesquilinear form* q *on* $\mathcal{V}$ *is a map* $\mathsf{q} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ *such that*

- *for all* $\psi \in \mathcal{V}$ *the map* $\varphi \mapsto \mathsf{q}(\varphi, \psi)$ *is linear ;*
- *for all* $\varphi \in \mathcal{V}$ *the map* $\psi \mapsto \mathsf{q}(\varphi, \psi)$ *is semilinear.*

(ii) *The quadratic form associated to* q *is the map* $\varphi \mapsto \mathsf{q}(\varphi, \varphi)$*. It is usually also denoted by* q*.*

(iii) *We say that* q *is continuous if there exists* $C \geqslant 0$ *such that, for all* $\varphi, \psi \in \mathcal{V}$,

$$|\mathsf{q}(\varphi, \psi)| \leqslant C \|\varphi\|_{\mathcal{V}} \|\psi\|_{\mathcal{V}} . \tag{1.9}$$

(iv) *We say that* q *is coercive if there exists* $\alpha > 0$ *such that for all* $\varphi \in \mathcal{V}$ *we have*

$$|\mathsf{q}(\varphi, \varphi)| \geqslant \alpha \|\varphi\|_{\mathcal{V}}^2 . \tag{1.10}$$

(v) *The adjoint* q^* *of the form* q *is the sesquilinear form defined by*

$$\forall \varphi, \psi \in \mathcal{V}, \quad \mathsf{q}^*(\varphi, \psi) = \overline{\mathsf{q}(\psi, \varphi)}.$$

Remark 1.64. Coercivity is often defined by

$$\mathsf{q}(\varphi, \varphi) \geqslant \alpha \|\varphi\|_{\mathcal{V}}^2 .$$

We use a weaker property here.

Proposition 1.65 (Representation Theorem - Bounded case)**.** *Let* q *be a continuous sesquilinear form on* $\mathcal{V}$*. There exists a unique operator* $T \in \mathcal{L}(\mathcal{V})$ *such that*

$$\forall \varphi, \psi \in \mathcal{V}, \quad \mathsf{q}(\varphi, \psi) = \langle T\varphi, \psi \rangle_{\mathcal{V}} ,$$

and we have

$$\|T\|_{\mathcal{L}(\mathcal{V})} = \sup_{\varphi, \psi \in \mathcal{V} \setminus \{0\}} \frac{|\mathsf{q}(\varphi, \psi)|}{\|\varphi\|_{\mathcal{V}} \|\psi\|_{\mathcal{V}}}.$$

Moreover,

(i) the operator associated with the adjoint form q^* *is* T^* *;*

(ii) if q *is* α*-coercive then* T *is invertible and* $\|T^{-1}\|_{\mathcal{L}(\mathcal{V})} \leqslant \alpha$*.*

Proof. • Let $\varphi \in \mathcal{V}$. The map $\psi \mapsto \mathsf{q}(\varphi, \psi)$ is a continuous semilinear form on $\mathcal{V}$, so by the Riesz representation theorem there exists a unique element of $\mathcal{V}$, which we denote by $T\varphi$, such that

$$\forall \psi \in \mathcal{V}, \quad \mathsf{q}(\varphi, \psi) = \langle T\varphi, \psi \rangle_{\mathcal{V}} .$$

This defines a unique map $T : \mathcal{V} \to \mathcal{V}$.
• Let $\varphi_1, \varphi_2 \in \mathcal{V}$ and $\lambda \in \mathbb{R}$. For all $\psi \in \mathcal{V}$ we have

$$\begin{aligned}\langle T(\varphi_1 + \lambda\varphi_2), \psi \rangle_{\mathcal{V}} &= \mathsf{q}(\varphi_1 + \lambda\varphi_2, \psi) = \mathsf{q}(\varphi_1, \psi) + \lambda \mathsf{q}(\varphi_2, \psi) = \langle T\varphi_1, \psi \rangle_{\mathcal{V}} + \lambda \langle T\varphi_2, \psi \rangle_{\mathcal{V}} \\ &= \langle T\varphi_1 + \lambda T\varphi_2, \psi \rangle .\end{aligned}$$

This proves that $T(\varphi_1 + \lambda\varphi_2) = T\varphi_1 + \lambda T\varphi_2$, so T is linear.
• For $\varphi \in \mathcal{V}$ we have

$$\|T\varphi\|_{\mathcal{V}}^2 = \langle T\varphi, T\varphi \rangle_{\mathcal{V}} = \mathsf{q}(\varphi, T\varphi) \leqslant C \|\varphi\|_{\mathcal{V}} \|T\varphi\|_{\mathcal{V}} ,$$

where $C = \sup_{\varphi, \psi \in \mathcal{V} \setminus \{0\}} \frac{|\mathsf{q}(\varphi, \psi)|}{\|\varphi\|_{\mathcal{V}} \|\psi\|_{\mathcal{V}}}$, so $\|T\varphi\|_{\mathcal{V}} \leqslant C \|\varphi\|_{\mathcal{V}}$. This proves that $T \in \mathcal{L}(\mathcal{V})$ and $\|T\|_{\mathcal{L}(\mathcal{V})} \leqslant C$. Conversely, for $\varphi, \psi \in \mathcal{V} \setminus \{0\}$ we have

$$|\mathsf{q}(\varphi, \psi)| = |\langle T\varphi, \psi \rangle| \leqslant \|T\| \|\varphi\| \|\psi\| .$$

- Finally, let $\tilde{T} \in \mathcal{L}(\mathcal{V})$ be the operator associated to the adjoint form q^*. Let $\psi \in \mathcal{V}$. For all $\varphi \in \mathcal{V}$ we have

$$\langle T\varphi, \psi \rangle = \mathsf{q}(\varphi, \psi) = \overline{\mathsf{q}^*(\psi, \varphi)} = \overline{\langle \tilde{T}\psi, \varphi \rangle} = \langle \varphi, \tilde{T}\psi \rangle .$$

This proves that $T^*\psi = \tilde{T}\varphi$.

- Now we assume that q is coercive. For $\varphi \in \mathcal{V}$ we have

$$\alpha \|\varphi\|_{\mathcal{V}}^2 \leqslant |\mathsf{q}(\varphi, \varphi)| = |\langle T\varphi, \varphi \rangle_{\mathcal{V}}| \leqslant \|T\varphi\|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}} ,$$

so

$$\|T\varphi\|_{\mathcal{V}} \geqslant \alpha \|\varphi\|_{\mathcal{V}} . \tag{1.11}$$

By Proposition 1.36, T is injective with closed range. Now let $\psi \in \mathsf{Ran}(T)^\perp$. We have

$$0 = |\langle T\psi, \psi \rangle_{\mathcal{V}}| = |\mathsf{q}(\psi, \psi)| \geqslant \alpha \|\psi\|_{\mathcal{V}}^2 ,$$

so $\psi = 0$. Since $\mathsf{Ran}(T)$ is closed, this implies that $\mathsf{Ran}(T) = \mathcal{V}$. Thus T is bijective and by (1.11) we have $\|T^{-1}\|_{\mathcal{L}(\mathcal{V})} \leqslant \alpha^{-1}$.

□

Example 1.66. The map $\varphi \mapsto \|\varphi\|_{\mathcal{V}}^2$ is a (coercive) quadratic form on $\mathcal{V}$, and in this case the operator T in Proposition 1.65 is $T = \mathrm{Id}_{\mathcal{V}}$.

Ex. 1.12

Theorem 1.67 (Lax-Milgram)**.** *Let $\mathcal{V}$ be a Hilbert space. Let q be a continuous and coercive sesquilinear form on $\mathcal{V}$. Let ℓ be a bounded semilinear form on $\mathcal{V}$. Then there exists a unique $\varphi_\ell \in \mathcal{V}$ such that*

$$\forall \psi \in \mathcal{V}, \quad \mathsf{q}(\varphi_\ell, \psi) = \langle T\varphi_\ell, \psi \rangle = \ell(\psi).$$

Proof. Let T be given by Proposition 1.65. By the Riesz theorem there exists $\zeta \in \mathcal{V}$ such that $\langle \zeta, \psi \rangle = \ell(\psi)$ for all $\psi \in \mathcal{V}$. Then we set $\varphi_\ell = T^{-1}\zeta$. □

Example 1.68. Let $f \in L^2(\mathbb{R}^d)$. Then there exists a unique $u \in H^1(\mathbb{R}^d)$ such that

$$\forall v \in H^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \big(u'(x)\overline{v}'(x) + u(x)\overline{v}(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\overline{v}(x)\,\mathrm{d}x.$$

To see this we apply the Lax-Milgram Theorem to the quadratic form $(u, v) \mapsto \langle u, v \rangle_{H^1(\mathbb{R}^d)}$ (continuous and coercive on $H^1(\mathbb{R}^d)$) and the linear form $v \mapsto \langle f, v \rangle_{L^2(\mathbb{R}^d)}$ (continuous on $H^1(\mathbb{R}^d)$).

Example 1.69. Let $f \in L^2(0,1)$. Then there exists a unique $u \in H^1(0,1)$ such that

$$\forall v \in H^1(0,1), \quad \int_0^1 \big(u'(x)\overline{v}'(x) + u(x)\overline{v}(x)\big)\,\mathrm{d}x = \int_0^1 f(x)\overline{v}(x)\,\mathrm{d}x.$$

and there exists a unique $u_0 \in H_0^1(0,1)$ such that

$$\forall v \in H_0^1(0,1), \quad \int_0^1 \big(u_0'(x)\overline{v}'(x) + u_0(x)\overline{v}(x)\big)\,\mathrm{d}x = \int_0^1 f(x)\overline{v}(x)\,\mathrm{d}x.$$

By the Poincaré inequality, the quadratic form $u \mapsto \|u'\|_{L^2(0,1)}^2$ is also coercive, so there also exists a unique $\tilde{u}_0 \in H_0^1(0,1)$ such that

$$\forall v \in H_0^1(0,1), \quad \int_0^1 \tilde{u}_0'(x)\overline{v}'(x)\,\mathrm{d}x = \int_0^1 f(x)\overline{v}(x)\,\mathrm{d}x.$$

Remark 1.70. It can be useful to see the quadratic forms in terms of operators in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$. More precisely, we can define a natural bijection between continuous sequilinear forms on $\mathcal{V}$ and operators in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$. Given a continuous sesquilinear form q on $\mathcal{V}$ we define $Q \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ by

$$\forall \varphi \in \mathcal{V}, \forall \psi \in \mathcal{V}, \quad (Q\varphi)(\psi) = \mathsf{q}(\varphi, \psi). \tag{1.12}$$

Conversely, given $Q \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$, we similarly define a corresponding continuous sesquilinear form q by (1.12).

Proposition 1.65 gives a link between quadratic forms and bounded operators on $\mathcal{V}$. We can directly define the natural bijection between $\mathcal{L}(\mathcal{V}, \mathcal{V}')$ and $\mathcal{L}(\mathcal{V})$ given by the Riesz representation theorem. Let

$$\mathcal{I} : \begin{cases} \mathcal{V} & \to & \mathcal{V}' \\ \varphi & \mapsto & \psi \mapsto \langle \varphi, \psi \rangle_{\mathcal{V}} \end{cases}$$

be the usual bijective isometry given by the Riesz theorem. Then the map

$$\begin{cases} \mathcal{L}(\mathcal{V}) & \to & \mathcal{L}(\mathcal{V}, \mathcal{V}') \\ T & \mapsto & \mathcal{I} \circ T \end{cases}$$

is also a bijective isometry. Moreover $T \in \mathcal{L}(\mathcal{V})$ is invertible if and only if $(\mathcal{I} \circ T) \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ is.

Notice that we can use $\mathcal{I}$ to identify $\mathcal{V}'$ with $\mathcal{V}$. It is on purpose that we do not use this possibility here.

1.3.2 A representation theorem

Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{V}$ be another Hilbert space, continuously embedded in $\mathcal{H}$. There exists $C_{\mathcal{V},\mathcal{H}}$ such that

$$\forall \varphi \in \mathcal{V}, \quad \|\varphi\|_{\mathcal{H}} \leqslant C_{\mathcal{V},\mathcal{H}} \|\varphi\|_{\mathcal{V}}.$$

We identify $\mathcal{H}$ with its dual $\mathcal{H}'$. Then we have

$$\mathcal{V} \subset \mathcal{H} \simeq \mathcal{H}' \subset \mathcal{V}'.$$

Notice that since we have already identified $\mathcal{H}$ with $\mathcal{H}'$ we cannot identify $\mathcal{V}$ with $\mathcal{V}'$.

Given a continuous and coercive form q on $\mathcal{V}$, we have associated in Proposition 1.65 an operator on $\mathcal{V}$. However, here our main space is $\mathcal{H}$ and our purpose is to define an operator corresponding to q on $\mathcal{H}$. For the typical examples 1.68-1.69, we have a quadratic form on H^1 and we want to define a corresponding operator on L^2.

Theorem 1.71 (Representation theorem)**.** *Let $\mathcal{H}$ and $\mathcal{V}$ be two Hilbert spaces such that $\mathcal{V}$ is densely and continuously embedded in $\mathcal{H}$. Let q be a continuous and coercive sesquilinear form on $\mathcal{V}$. We set*

$$\mathsf{Dom}(A) = \{\varphi \in \mathcal{V} \,:\, \exists C_\varphi > 0, \forall \psi \in \mathcal{V}, |\mathsf{q}(\varphi, \psi)| \leqslant C_\varphi \|\psi\|_{\mathcal{H}}\},$$

and for $\varphi \in \mathsf{Dom}(A)$ we define $A\varphi \in \mathcal{H}$ by

$$\forall \psi \in \mathcal{V}, \quad \mathsf{q}(\varphi, \psi) = \langle A\varphi, \psi \rangle_{\mathcal{H}}.$$

This defines on $\mathcal{H}$ an operator A with domain $\mathsf{Dom}(A)$ such that

(i) *$\mathsf{Dom}(A)$ is dense in $\mathcal{V}$ and in $\mathcal{H}$;*

(ii) *A is closed ;*

(iii) *A is invertible.*

Moreover, the operator on $\mathcal{H}$ associated to the form q^* is A^*.

Proof. • Let $\varphi \in \mathsf{Dom}(A)$. The map $\psi \mapsto \mathsf{q}(\varphi, \psi)$ extends to a bounded semilinear form on $\mathcal{H}$. Then, by the Riesz theorem, there exists a vector $A\varphi \in \mathcal{H}$ such that $\mathsf{q}(\varphi, \psi) = \langle A\varphi, \psi \rangle_{\mathcal{H}}$ for all $\psi \in \mathcal{V}$. This defines on $\mathcal{H}$ an operator A with domain $\mathsf{Dom}(A)$ (the linearity of A is left as an exercise).

• Let $\zeta \in \mathcal{H}$. The map $\psi \in \mathcal{V} \mapsto \langle \zeta, \psi \rangle_{\mathcal{H}}$ is a continuous semilinear map on $\mathcal{V}$ so, by the Lax-Milgram theorem, there exists $\varphi \in \mathcal{V}$ such that

$$\forall \psi \in \mathcal{V}, \quad \langle \zeta, \psi \rangle_{\mathcal{H}} = \mathsf{q}(\varphi, \psi).$$

Then we have $\varphi \in \mathsf{Dom}(A)$ and $A\varphi = \zeta$. This proves that A is surjective.

- For $\varphi \in \mathsf{Dom}(A)$ we have

$$\|A\varphi\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}} \geqslant |\langle A\varphi, \varphi\rangle_{\mathcal{H}}| = |\mathsf{q}(\varphi,\varphi)| \geqslant \alpha \|\varphi\|_{\mathcal{V}}^2 \geqslant \alpha C_{\mathcal{V},\mathcal{H}}^{-2} \|\varphi\|_{\mathcal{H}}^2 .$$

Thus,

$$\|A\varphi\|_{\mathcal{H}} \geqslant \alpha C_{\mathcal{V},\mathcal{H}}^{-2} \|\varphi\|_{\mathcal{H}} . \tag{1.13}$$

This proves in particular that A is injective. Since A is surjective, it is bijective. This inequality also implies that the inverse is bounded and $\|A^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \alpha^{-1} C_{\mathcal{V},\mathcal{H}}^2$. This implies that A is closed by Proposition 1.35.

- Let $\psi \in \mathcal{V}$ be in the orthogonal of $\mathsf{Dom}(A)$ in $\mathcal{V}$. Let $T \in \mathcal{L}(\mathcal{V})$ be given by Proposition 1.65. Since T^* is bijective (by Proposition 1.61), there exists $\zeta \in \mathcal{V}$ such that $T^*\zeta = \psi$. Then for all $\varphi \in \mathsf{Dom}(A)$ we have

$$0 = \langle \varphi, \psi\rangle_{\mathcal{V}} = \langle \varphi, T^*\zeta\rangle_{\mathcal{V}} = \langle T\varphi, \zeta\rangle_{\mathcal{V}} = \mathsf{q}(\varphi,\psi) = \langle A\varphi, \zeta\rangle_{\mathcal{H}} .$$

Since A is surjective, this implies that $\zeta = 0$, and hence $\psi = 0$. Then $\mathsf{Dom}(A)$ is dense in $\mathcal{V}$ for the topology of $\mathcal{V}$, and hence for the topology of $\mathcal{H}$. Since $\mathcal{V}$ is dense in $\mathcal{H}$, $\mathsf{Dom}(A)$ is also dense in $\mathcal{H}$.

- We denote by $\tilde{A}$ the operator associated to q^*. Since q^* is continuous and coercive, $\tilde{A}$ is also a densely defined, closed and invertible operator on $\mathcal{H}$. Let $\psi \in \mathsf{Dom}(\tilde{A})$. For all $\varphi \in \mathsf{Dom}(A)$ we have

$$\langle A\varphi, \psi\rangle = \mathsf{q}(\varphi,\psi) = \overline{\mathsf{q}^*(\psi,\varphi)} = \overline{\langle \tilde{A}\psi, \varphi\rangle} = \langle \varphi, \tilde{A}\psi\rangle .$$

This proves that $\tilde{A} \subset A^*$. Conversely, if $\psi \in \mathsf{Dom}(A^*)$ then for all $\varphi \in \mathsf{Dom}(A)$ we have

$$|\mathsf{q}^*(\psi,\varphi)| = |\mathsf{q}(\varphi,\psi)| = |\langle A\varphi,\psi\rangle| = |\langle \varphi, A^*\psi\rangle| \leqslant \|A^*\psi\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}} .$$

Since $\mathsf{Dom}(A)$ is dense in $\mathcal{V}$ and $\mathcal{H}$, we deduce that for all $\varphi \in \mathcal{V}$ we have

$$|\mathsf{q}^*(\psi,\varphi)| \leqslant \|A^*\psi\|_{\mathcal{H}} \|\varphi\|_{\mathcal{H}} ,$$

so $\psi \in \mathsf{Dom}(\tilde{A})$. This proves that $\mathsf{Dom}(A^*) \subset \mathsf{Dom}(\tilde{A})$, so $\tilde{A} = A^*$. □

Remark 1.72. Let q be a continuous quadratic form on $\mathcal{V}$. Assume that there exists $\beta \in \mathbb{C}$ such that the form $\mathsf{q}_\beta : \varphi \mapsto \mathsf{q}(\varphi) + \beta \|\varphi\|_{\mathcal{H}}$ is coercive on $\mathcal{V}$. Let A_β be the operator on $\mathcal{H}$ given by Theorem 1.71 and $A = A_\beta - \beta$ with $\mathsf{Dom}(A) = \mathsf{Dom}(A_\beta)$. Then A is closed and densely defined, and $(A+\beta)$ is invertible. Notice that this definition of A does not depend on the choice of β.

Remark 1.73. Let q be a continuous coercive quadratic form on $\mathcal{V}$ and $Q \in \mathcal{L}(\mathcal{V},\mathcal{V}')$ defined by (1.12) (invertible by Theorem 1.67). Let A the operator on $\mathcal{H}$ be given by Theorem 1.71. Then for all $\varphi \in \mathcal{H} \subset \mathcal{V}'$ we have $Q^{-1}\varphi = A^{-1}\varphi$.

1.3.3 Examples: Laplacian, Dirichlet and Neumann boundary conditions

Example 1.74. We consider on $H^1(\mathbb{R})$ the quadratic form

$$\mathsf{q} : u \mapsto \|u\|_{H^1(\mathbb{R})}^2 .$$

We apply Theorem 1.71 with $\mathcal{V} = H^1(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$. We have

$$\mathsf{Dom}(A) = \left\{ u \in H^1(\mathbb{R}) \, : \, u'' \in L^2(\mathbb{R}) \right\} = H^2(\mathbb{R}).$$

Indeed, if $u \in H^2(\mathbb{R})$ then for all $v \in H^1(\mathbb{R})$ we have

$$|q(u,v)| = \left| -\int_{\mathbb{R}} u''\overline{v}\,\mathrm{d}x + \int_{\mathbb{R}} u\overline{v}\,\mathrm{d}x \right| \leqslant (\|u''\| + \|u\|)\,\|v\| ,$$

so $u \in \mathsf{Dom}(A)$. Conversely, assume that $u \in \mathsf{Dom}(A)$. Then for all $v \in H^1(\mathbb{R})$ we have

$$\left| \int_{\mathbb{R}} u' \overline{v}' \,\mathrm{d}x \right| \leqslant |q(u,v)| + \|u\| \, \|v\| \leqslant (C_u + \|u\|) \, \|v\| \, .$$

This proves that $u'' \in L^2$, and hence $u \in H^2(\mathbb{R})$. Finally, for $u \in \mathsf{Dom}(A)$ we have

$$\forall v \in H^1(\mathbb{R}), \quad \langle Au, v \rangle = q(u,v) = \left\langle -u'' + u, v \right\rangle,$$

so

$$Au = -u'' + u.$$

Example 1.75. We consider on $H^1(0,1)$ the quadratic form

$$\mathsf{q}_N : u \mapsto \|u\|^2_{H^1(0,1)} \, .$$

We apply Theorem 1.71 with $\mathcal{V} = H^1(0,1)$ and $\mathcal{H} = L^2(0,1)$. We denote by A_N the corresponding operator. Let $u \in \mathsf{Dom}(A_N)$. For all $\phi \in C_0^\infty(]0,1[) \subset H^1(0,1)$ we have as above

$$\left| \int_0^1 u' \overline{\phi}' \,\mathrm{d}x \right| \leqslant (C_u + \|u\|) \, \|\phi\| \, .$$

This implies that $u'' \in L^2(0,1)$. Then for all $\phi \in C_0^\infty(]0,1[)$ we have

$$\langle A_N u, \phi \rangle = q_N(u,\phi) = \int_0^1 u' \overline{\phi}' \,\mathrm{d}x + \int_0^1 u \overline{\phi} \,\mathrm{d}x = \left\langle -u'' + u, \phi \right\rangle.$$

This proves that $A_N u = -u'' + u$. Then for all $v \in H^1(0,1)$ we have

$$\langle A_N u, v \rangle = q_N(u,v) = \int_0^1 u' \overline{v}' \,\mathrm{d}x + \int_0^1 u \overline{v} \,\mathrm{d}x = u'(1)\overline{v}(1) - u'(0)\overline{v}(0) + \left\langle -u'' + u, v \right\rangle$$

This proves that for all $v \in H^1(0,1)$

$$u'(1)\overline{v}(1) - u'(0)\overline{v}(0) = 0.$$

This implies that

$$u'(0) = u'(1) = 0. \tag{1.14}$$

Conversely, assume that $u \in H^2(0,1)$ satisfies (1.14). Then we can compute as above that

$$\forall v \in H^1(0,1), \quad q(u,v) = \left\langle -u'' + u, v \right\rangle.$$

Then $u \in \mathsf{Dom}(A_N)$. Finally we have

$$\mathsf{Dom}(A_N) = \left\{ u \in H^2(0,1) \, : \, u'(0) = u'(1) = 0 \right\}$$

and, for all $u \in \mathsf{Dom}(A_N)$,

$$A_N u = -u'' + u.$$

Example 1.76. We consider on $H_0^1(0,1)$ the quadratic form

$$\mathsf{q}_D : u \mapsto \|u\|^2_{H^1(0,1)} \, .$$

We apply Theorem 1.71 with $\mathcal{V} = H_0^1(0,1)$ and $\mathcal{H} = L^2(0,1)$. We denote by A_D the corresponding operator. Let $u \in \mathsf{Dom}(A_D)$. As above we see that $u \in H^2(0,1)$ and $A_D u = -u'' + u$. On the other hand, if $u \in H^2(0,1) \cap H_0^1(0,1)$ we have $q(u,v) = \langle -u'' + u, v \rangle$ for all $v \in H_0^1(0,1)$ (there are no boundary terms since u and v vanish at the boundary). Finally we have

$$\mathsf{Dom}(A_D) = H^2(0,1) \cap H_0^1(0,1),$$

and for all $u \in \mathsf{Dom}(A_D)$

$$A_D u = -u'' + u.$$

Example 1.77. By Remark 1.72 we can define the operators associated to the form

$$u \mapsto \int_0^1 |u(x)|^2 \,\mathrm{d}x$$

defined on $H^1(\mathbb{R})$ and $H^1(0,1)$ (note that this form is already coercive on $H_0^1(0,1)$). Ex. 1.13

1.4 Exercises

***Exercise* 1.1.** Let $A, B \in \mathcal{L}(\mathsf{E})$. Assume that A is invertible and

$$\|B - A\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{1}{2\,\|A^{-1}\|_{\mathcal{L}(\mathsf{E})}}.$$

Prove that B is invertible and

$$\left\|B^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant 2\left\|A^{-1}\right\|_{\mathcal{L}(\mathsf{E})}.$$

***Exercise* 1.2.** We consider on $L^2(\mathbb{R}^d)$ the Laplace operator $H_0 = -\Delta$ defined on the domain $\mathsf{Dom}(H_0) = H^2(\mathbb{R}^d)$.
1. Prove that H_0 is injective.
2. Prove that the range of H_0 is dense in $L^2(\mathbb{R}^d)$.
3. Prove that H_0 does not have a bounded inverse.

***Exercise* 1.3.** Let A be a densely defined operator from E to F. Assume that there exists $C > 0$ such that $\|A\varphi\|_{\mathsf{F}} \leqslant C\,\|\varphi\|_{\mathsf{E}}$ for all $\varphi \in \mathsf{Dom}(A)$.
1. Prove that A extends uniquely to a bounded operator $\tilde{A} \in \mathcal{L}(\mathsf{E},\mathsf{F})$ and that $\|\tilde{A}\|_{\mathcal{L}(\mathsf{E},\mathsf{F})} \leqslant C$.
2. Assume that A is closed. Prove that we already have $\mathsf{Dom}(A) = \mathsf{E}$ and $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$.

***Exercise* 1.4.** Prove that the multiplication operator M_w of Example 1.12 is closed. What about the operator M_w^0 of Example 1.16 ?

***Exercise* 1.5.** Let $\mathcal{H}$ be a Hilbert space. Assume that the family $(\beta_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a complex sequence. We consider on $\mathcal{H}$ the unique operator A_0 such that

$$\mathsf{Dom}(A_0) = \left\{ \sum_{n=0}^{N} \varphi_n \beta_n,\, N \in \mathbb{N} \text{ and } \varphi_0, \dots, \varphi_N \in \mathbb{C} \right\}$$

and

$$\forall n \in \mathbb{N}, \quad A\beta_n = \lambda_n \beta_n.$$

Prove that A_0 is closable and give its closure.

***Exercise* 1.6.** Let A be an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ with domain $\mathsf{Dom}(A)$. Prove that A^* is a linear operator.

***Exercise* 1.7.** Prove that the map

$$\begin{cases} \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2) & \to & \mathcal{L}(\mathcal{H}_2,\mathcal{H}_1) \\ A & \mapsto & A^* \end{cases}$$

is semi-linear.

***Exercise* 1.8.** Let $A \in \mathcal{L}(\mathcal{H})$. Prove that $\|A^*A\|_{\mathcal{L}(\mathcal{H})} = \|A\|^2_{\mathcal{L}(\mathcal{H})}$.

***Exercise* 1.9.** Let $A \in \mathcal{L}(\mathcal{H})$. Let F be a subspace of $\mathcal{H}$ such that $A(\mathsf{F}) \subset \mathsf{F}$. Prove that $A^*(\mathsf{F}^\perp) \subset \mathsf{F}^\perp$.

***Exercise* 1.10.** Let Ω be an open subset of $\mathbb{R}^d$. Let $w : \Omega \to \mathbb{C}$ be a continuous function. We consider on $L^2(\Omega)$ the multiplication operator M_w as in Example 1.12.
1. Prove that M_w is densely defined.
2. What is the adjoint of M_w ?

***Exercise* 1.11.** We consider the operator T from $L^2(\mathbb{R})$ to $\mathbb{C}$ defined by $\mathsf{Dom}(T) = C_0^\infty(\mathbb{R})$ and $T\phi = \phi(0)$ for all $\phi \in \mathsf{Dom}(T)$. Compute the adjoint T^* of T.

***Exercise* 1.12.** For $u, v \in H^1(\mathbb{R})$ we set $\delta(u,v) = u(0)\overline{v(0)}$.
1. Prove that this defines a sesquilinear form δ on $H^1(\mathbb{R})$.
2. Is δ coercive ?
3. By Proposition 1.65 there exists $T \in \mathcal{L}(H^1(\mathbb{R}))$ such that

$$\forall u, v \in H^1(\mathbb{R}), \qquad \int_{\mathbb{R}} \left((Tu)'(x)\overline{v'(x)} + (Tu)(x)\overline{v(x)} \right) \mathrm{d}x = \langle Tu, v\rangle_{H^1(\mathbb{R})} = \delta(u,v).$$

Give an explicit expression of Tu for all $u \in H^1(\mathbb{R})$. Is T injective ? Surjective ?

***Exercise* 1.13.** Let $\alpha \in \mathbb{C}$. For $u \in H^1(0,1)$ we set

$$\mathsf{q}_\alpha(u) = \int_0^1 \left|u'(x)\right|^2 \, \mathrm{d}x + \alpha \left|u(0)\right|^2 .$$

1. Prove that the quadratic form q_α is continuous on $H^1(0,1)$.
2. Prove that there exists $\beta \geqslant 0$ such that the form $\mathsf{q}_\alpha + \beta : u \mapsto \mathsf{q}_\alpha(u) + \beta \left\|u\right\|^2_{L^2(0,1)}$ is coercive.
3. We denote by A_α the operator on $L^2(0,1)$ associated with the form q_α by the representation theorem (see Remark 1.72). Describe A_α (domain and action on an element of this domain).