Final Exam

Monday, November 28 (3h)

Five pages of notes are allowed. French or English can be used for the answers. Unless otherwise specified, all the answers have to be justified and the clarity of the writing will be taken into account.

Exercise 1. We consider on $\ell^2(\mathbb{N})$ the operator A defined on the domain

$$\mathsf{Dom}(A) = \left\{ u = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \, : \, \sum_{n=0}^{\infty} n^2 |u_n|^2 < +\infty \right\}$$

by

$$\forall u = (u_n)_{n\in\mathbb{N}} \in \mathsf{Dom}(A), \quad Au = (ne^{in} u_n)_{n\in\mathbb{N}}.$$

1. Prove that A is densely defined.
2. Prove that A is closed.
3. What is the adjoint of A ?

Correction : **1.** Let $u = (u_n) \in \ell^2(\mathbb{N})$ and $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{+\infty} |u_n|^2 \leqslant \varepsilon$. We define $v = (v_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ by

$$v_n = \begin{cases} u_n & \text{if } n \leqslant N, \\ 0 & \text{if } n > N. \end{cases}$$

Then $v \in \mathsf{Dom}(A)$ and $\|u - v\|_{\ell^2(\mathbb{N})}^2 \leqslant \varepsilon$, which proves that $\mathsf{Dom}(A)$ is dense in $\ell^2(\mathbb{N})$.
2. Assume that we have a family (u^k) of sequences in $\mathsf{Dom}(A)$ such that $u^k \to u$ and $Au^k \to v$ for some u and v in $\ell^2(\mathbb{N})$. Then for all $n \in \mathbb{N}$ we have

$$\left|u_n^k - u_n\right|^2 \leqslant \left\|u^k - u\right\|_{\ell^2(\mathbb{N})}^2 \xrightarrow[k\to\infty]{} 0 \quad \text{and} \quad \left|ne^{in}u_n^k - v_n\right|^2 \leqslant \left\|Au^k - v\right\|_{\ell^2(\mathbb{N})}^2 \xrightarrow[k\to\infty]{} 0.$$

This implies that $v_n = ne^{in}u_n$ for all $n \in \mathbb{N}$. In particular,

$$\sum_{n\in\mathbb{N}} n^2 |u_n|^2 = \sum_{n\in\mathbb{N}} |v_n|^2 < +\infty.$$

This proves that $u \in \mathsf{Dom}(A)$. We also have $Au = v$, which proves that A is closed.
3. Let $v = (v_n)_{n\in\mathbb{N}} \in \mathsf{Dom}(A^*)$. We set $w = (w_n)_{n\in\mathbb{N}} = A^*v$. Then for all $u = (u_n)_{n\in\mathbb{N}} \in \mathsf{Dom}(A)$ we have

$$\sum_{n=0}^{\infty} u_n \overline{w_n} = \left\langle u, A_0^* v \right\rangle_{\ell^2(\mathbb{N})} = \langle A_0 u, v\rangle = \sum_{n=0}^{\infty} ne^{in} u_n \overline{v_n} = \sum_{n=0}^{\infty} u_n \overline{ne^{-in} v_n}.$$

Applied with the sequence $e_k = (e_{k,n})_{n\in\mathbb{N}}$ defined by $e_{k,k} = 1$ and $e_{k,n} = 0$ if $n \neq k$, this proves that

$$\forall n \in \mathbb{N}, \quad w_n = ne^{-in} v_n.$$

In particular,

$$\sum_{n\in\mathbb{N}} n^2 |v_n|^2 = \sum_{n\in\mathbb{N}} |w_n|^2 < +\infty,$$

so $v \in \mathsf{Dom}(A)$. Conversely, assume that $v = (v_n)_{n\in\mathbb{N}}$ belongs to $\mathsf{Dom}(A)$. Then for all $u = (u_n)_{n\in\mathbb{N}} \in \mathsf{Dom}(A)$ we have by the Cauchy-Schwarz inequality

$$|\langle Au, v\rangle| \leqslant \sum_{n\in\mathbb{N}} n |u_n| |v_n| \leqslant \|u\|_{\ell^2(\mathbb{N})} \sqrt{\sum_{n\in\mathbb{N}} n^2 |v_n|^2}.$$

This proves that $v \in \mathsf{Dom}(A^*)$. Finally we have proved that $\mathsf{Dom}(A^*) = \mathsf{Dom}(A)$ and that for $v = (v_n)_{n\in\mathbb{N}}$ we have

$$A^* v = (ne^{-in} v_n)_{n\in\mathbb{N}}.$$

□

Exercise 2. Let E_1, E_2 and E_3 be three Banach spaces such that $\mathsf{E}_1 \subset \mathsf{E}_2 \subset \mathsf{E}_3$. We assume that the embedding $i : \mathsf{E}_1 \to \mathsf{E}_2$ is compact and that the embedding $j : \mathsf{E}_2 \to \mathsf{E}_3$ is continuous. Let $\varepsilon > 0$. Prove that there exists $C_\varepsilon > 0$ such that for all $\varphi \in \mathsf{E}_1$ we have

$$\|\varphi\|_{\mathsf{E}_2} \leqslant \varepsilon \|\varphi\|_{\mathsf{E}_1} + C_\varepsilon \|\varphi\|_{\mathsf{E}_3} .$$

Correction : Assume by contradiction that the statement is not true. Then for all $n \in \mathbb{N}$ there exists $\varphi_n \in \mathsf{E}_1$ such that

$$\|\varphi_n\|_{\mathsf{E}_2} > \varepsilon \|\varphi_n\|_{\mathsf{E}_1} + n \|\varphi_n\|_{\mathsf{E}_3} .$$

In particular $\|\varphi_n\|_{\mathsf{E}_2} \neq 0$, so $\|\varphi_n\|_{\mathsf{E}_1} \neq 0$. After dividing by $\|\varphi_n\|_{\mathsf{E}_1}$ if necessary, we can assume without loss of generality that $\|\varphi_n\|_{\mathsf{E}_1} = 1$ for all $n \in \mathbb{N}$.

Then the sequence (φ_n) is bounded in E_1, so it has a convergent subsequence in E_2. After extracting a subsequence if necessary, we can assume that φ_n goes to some $\varphi \in \mathsf{E}_2$. And by continuity of the injection of E_2 in E_3, φ_n also goes to φ in E_3. Then

$$\|\varphi\|_{\mathsf{E}_3} = \lim_{n\to\infty} \|\varphi_n\|_{\mathsf{E}_3} \leqslant \lim_{n\to\infty} \frac{\|\varphi_n\|_{\mathsf{E}_2}}{n} = 0.$$

This proves that $\varphi = 0$. This gives a contradiction with

$$\forall n \in \mathbb{N}, \quad \|\varphi_n\|_{\mathsf{E}_2} > \varepsilon,$$

and concludes the proof by contradiction. □

Exercise 3. For $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we define $H_0 u \in \ell^2(\mathbb{Z})$ by

$$\forall n \in \mathbb{Z}, \quad (H_0 u)_n = 2u_n - u_{n+1} - u_{n-1}.$$

1. Prove that this defines a bounded operator H_0 on $\ell^2(\mathbb{Z})$.
2. We denote by L^2_{per} the space of 2π-periodic functions in $L^2_{\mathrm{loc}}(\mathbb{R})$ (this is equivalent to considering $L^2(S^1)$, where S^1 is the circle, or one dimensional torus). It is endowed with the norm defined by

$$\|v\|^2_{L^2_{\mathsf{per}}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |v(x)|^2 \, \mathrm{d}x.$$

For $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we define $\mathcal{F}u \in L^2_{\mathsf{per}}$ by

$$\forall x \in \mathbb{R}, \quad (\mathcal{F}u)(x) = \sum_{n\in\mathbb{Z}} u_n e^{-inx}.$$

We recall that $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2_{\mathsf{per}}$ is a unitary operator. Prove that $\mathcal{F} H_0 \mathcal{F}^{-1}$ is the operator M of multiplication by $2(1 - \cos(x))$ on L^2_{per}.
3. Give without proof the spectrum of M.
4. Prove that $\sigma(H_0) = \sigma(M)$.
5. Prove that H_0 has no eigenvalue.
6. Let $(\beta_n)_{n\in\mathbb{Z}}$ be a real-valued sequence such that $\beta_n > 0$ for all $n \in \mathbb{Z}$ and $\beta_n \to 0$ as $n \to \pm\infty$. We denote by B the operator on $\ell^2(\mathbb{Z})$ which maps $u = (u_n) \in \ell^2(\mathbb{Z})$ to $Bu = (\beta_n u_n)_{n\in\mathbb{N}}$. For $\alpha \in \mathbb{R}$ we set $H_\alpha = H_0 + \alpha B$. Prove that H_α is selfadjoint for all $\alpha \in \mathbb{R}$.
7. Let $\alpha \in \mathbb{R}$. What is the essential spectrum of H_α ?
8. Prove that there exists $\alpha \in \mathbb{R}$ such that H_α has at least one eigenvalue.
9. Let $N \in \mathbb{N}^*$. Prove that there exists $\alpha \in \mathbb{R}$ such that H_α has at least N eigenvalues (counted with multiplicities).

Correction : **1.** Let $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. We define $u_\pm$ by $u_{\pm,n} = u_{n\pm1}$. In particular we have $u_\pm \in \ell^2(\mathbb{Z})$ and $\|u_\pm\|_{\ell^2(\mathbb{Z})} = \|u\|_{\ell^2(\mathbb{Z})}$. Since $H_0 u = 2u - u_- - u_+$ we have by the Cauchy-Schwarz inequality

$$\|H_0 u\|_{\ell^2(\mathbb{Z})} \leqslant 2 \|u\|_{\ell^2(\mathbb{Z})} + \|u_+\|_{\ell^2(\mathbb{Z})} + \|u_-\|_{\ell^2(\mathbb{Z})} \leqslant 4 \|u\|_{\ell^2(\mathbb{Z})} .$$

This proves that H_0 is bounded on $\ell^2(\mathbb{Z})$ and $\|H_0\|_{\mathcal{L}(\ell^2(\mathbb{Z}))} \leqslant 4$.
2. Let $u = (u_n) \in \ell^2(\mathbb{Z})$. For $x \in \mathbb{R}$ we have

$$\begin{aligned}
(\mathcal{F} H_0 u)(x) &= \sum_{n\in\mathbb{Z}} (2u_n - u_{n-1} - u_{n+1}) e^{-inx} \\
&= \sum_{n\in\mathbb{Z}} \left(2e^{-inx} - e^{-i(n+1)x} - e^{-i(n-1)x}\right) u_n \\
&= \sum_{n\in\mathbb{Z}} 2(1 - \cos(x)) e^{-inx} u_n \\
&= 2(1 - \cos(x))(\mathcal{F}u)(x).
\end{aligned}$$

This proves that $\mathcal{F}H_0 = M\mathcal{F}$.
3. We know that the spectrum of the operator of multiplication by a function is the closure of the image of this function. In this case, the spectrum of the multiplication M by $2(1-\cos(x))$ on $[-\pi,\pi]$ is $[0,4]$.
4. Let $z \in \mathbb{C}$. We have

$$H_0 - z\,\mathrm{Id}_{\ell^2(\mathbb{Z})} = \mathcal{F}^{-1}M\mathcal{F} - z\,\mathrm{Id}_{\ell^2(\mathbb{Z})} = \mathcal{F}^{-1}\big(M - z\,\mathrm{Id}_{L^2_{\mathsf{per}}}\big)\mathcal{F}.$$

Then $H_0 - z\,\mathrm{Id}_{\ell^2(\mathbb{Z})}$ is invertible if and only if $M - z\,\mathrm{Id}_{L^2_{\mathsf{per}}}$ is, so $\sigma(H_0) = \sigma(M) = [0,4]$.
5. Similarly, λ is an eigenvalue of H_0 if and only if it is an eigenvalue of M. However, for $\lambda \in [0,4]$ we have

$$\lambda(\{x \in [-\pi,\pi] \,:\, 2(1-\cos(x)) = \lambda\}) = 0.$$

Then M has no eigenvalue, so H_0 has no eigenvalue. We recall the proof of this fact. Assume that $v \in L^2_{\mathsf{per}}$ and $\lambda \in \mathbb{C}$ are such that $Mv = \lambda v$. Then for almost all $x \in \mathbb{R}$ we have

$$\big(2(1-\cos(x)) - \lambda\big)v(x) = 0.$$

This implies that $v = 0$ almost everywhere. Thus λ is not an eigenvalue of M.
6. The sequence $(\beta_n)_{n\in\mathbb{Z}}$ goes to 0 at infinity, so it is bounded. For $u = (u_n)_{n\in\mathbb{N}}$ we have

$$\sum_{n\in\mathbb{Z}} |\beta_n u_n|^2 \leqslant \sup_{n\in\mathbb{Z}} |\beta_n|^2 \sum_{n\in\mathbb{Z}} |u_n|^2 .$$

This proves that B is bounded. Then for any $\alpha \in \mathbb{R}$ the operator H_α is bounded.
Let $u = (u_n)$ and $v = (v_n)$ in $\ell^2(\mathbb{Z})$. We have

$$\begin{aligned}\langle H_\alpha u, v\rangle_{\ell^2(\mathbb{Z})} &= 2\sum_{n\in\mathbb{Z}} u_n\overline{v_n} - \sum_{n\in\mathbb{Z}} u_{n+1}\overline{v_n} - \sum_{n\in\mathbb{Z}} u_{n-1}\overline{v_n} + \sum_{n\in\mathbb{Z}} \alpha\beta_n u_n\overline{v_n}\\ &= 2\sum_{n\in\mathbb{Z}} u_n\overline{v_n} - \sum_{n\in\mathbb{Z}} u_n\overline{v_{n-1}} - \sum_{n\in\mathbb{Z}} u_n\overline{v_{n+1}} + \sum_{n\in\mathbb{Z}} u_n\overline{\alpha\beta_n v_n}\\ &= \langle u, H_\alpha v\rangle_{\ell^2(\mathbb{Z})} .\end{aligned}$$

This proves that H_α is symmetric. Then it is selfadjoint.
7. Let $N \in \mathbb{N}$. We denote by B_N the operator which maps $u = (u_n)_{n\in\mathbb{Z}}$ to the sequence $B_N u$ such that

$$(B_N u)_{n\in\mathbb{Z}} = \begin{cases}\beta_n u_n & \text{if } |n| \leqslant N,\\ 0 & \text{if } |n| > N.\end{cases}$$

Then B_N is of finite rank, so it is a compact operator on $\ell^2(\mathbb{Z})$. On the other hand we have

$$\|B - B_N\|_{\mathcal{L}(\ell^2(\mathbb{Z}))} = \sup_{|n|>N} |\beta_n| \xrightarrow[N\to\infty]{} 0,$$

so B is also a compact operator on $\ell^2(\mathbb{Z})$.
In particular, it is H_0-compact. By the Weyl Theorem, we have $\sigma_{\mathsf{ess}}(H_\alpha) = \sigma_{\mathsf{ess}}(H_0)[0,4]$.
8. For $\alpha \in \mathbb{R}$ we set $\eta_\alpha = \min\sigma(H_\alpha)$. For $k \in \mathbb{Z}$ we define the sequence $e^k = (e^k_n)_{n\in\mathbb{Z}}$ such that $e^k_k = 1$ and $e^k_n = 0$ if $n \neq k$. We have $\|e^k\|_{\ell^2(\mathbb{Z})} = 1$ for all $k \in \mathbb{Z}$. We have

$$\left\langle H_\alpha e^0, e^0\right\rangle = \left\langle H_0 e^0, e^0\right\rangle + \alpha\beta_0 \xrightarrow[\alpha\to-\infty]{} -\infty.$$

In particular, there exists $\alpha \in \mathbb{R}$ such that $\left\langle H_\alpha e^k, e^k\right\rangle < 0$. By the Min-max Theorem we have $\eta_\alpha < 0$. Then $\eta_\alpha \in \sigma(H_\alpha)$ but $\eta_\alpha \notin \sigma_{\mathsf{ess}}(H_\alpha)$. This implies that η_α is an eigenvalue of H_α.
9. Let $N \in \mathbb{N}$. We set $F_N = \mathsf{span}(e^0,\dots,e^{N-1})$. For $u = \sum_{n=0}^{N-1} u_n e^n$ and $\alpha < 0$ we have

$$\begin{aligned}\langle H_\alpha u, u\rangle_{\ell^2(\mathbb{Z})} &= \sum_{0\leqslant j,k\leqslant N-1} u_j\overline{u_k}\left\langle H_\alpha e^j, e^k\right\rangle\\ &= \sum_{0\leqslant j,k\leqslant N-1} u_j\overline{u_k}\left\langle H_0 e^j, e^k\right\rangle + \alpha\sum_{k=0}^{N-1}\beta_k|u_k|^2\\ &\leqslant \big(C + (-\alpha)\inf_{0\leqslant k\leqslant N-1}\beta_k\big)\|u\|^2_{\ell^2(\mathbb{Z})} ,\end{aligned}$$

where $C = \sup_{0\leqslant j,k\leqslant N-1}\left|\left\langle H_0 e^j, e^k\right\rangle\right|$. This proves that

$$\sup_{\substack{u\in F_N\\ \|u\|=1}} \langle H_\alpha u, u\rangle \xrightarrow[\alpha\to-\infty]{} -\infty.$$

In particular, there exists $\alpha \in \mathbb{R}$ such that the left-hand side is negative, and in particular smaller than $\inf\sigma_{\mathsf{ess}}(H_\alpha)$. In this case, the Min-max Theorem ensures that H_α has at least N eigenvalues (counted with multiplicities) under the essential spectrum.

□

Exercise 4. We consider on $\mathcal{H} = L^2(0,1)$ the operator A defined by

$$\mathsf{Dom}(A) = \left\{ u \in H^2(0,1) \,:\, u(0) = 0 \text{ and } u'(1) = 0 \right\}$$

and $Au = -u''$ for all $u \in \mathsf{Dom}(A)$. We recall that if $u \in L^2(0,1)$ is such that $u'' \in L^2(0,1)$ then $u' \in L^2(0,1)$, and moreover the graph norm on $\mathsf{Dom}(A)$ is equivalent to the norm $\|\cdot\|_{H^2(0,1)}$.

1. Prove that A is selfadjoint.
2. Prove that $A \geqslant 0$.
3. Prove that $(-A)$ generates a contractions semigroup on $L^2(0,1)$.
4. Prove that $\ker(A) = \{0\}$ (we recall that if $u \in H^2(0,1)$ satisfies $-u'' = 0$ in the sense of distributions, then it is of class C^2).
5. Prove that $\min \sigma(A) > 0$.
6. Prove that there exists $\gamma > 0$ such that for all $t \geqslant 0$ we have $\left\|e^{-tA}\right\|_{\mathcal{L}(L^2(0,1))} \leqslant e^{-t\gamma}$.

Correction :
1. For $u, v \in H^2(0,1)$ we have by the Green formula

$$-\int_0^1 u''(x)\overline{v(x)}\,\mathrm{d}x = -u'(1)\overline{v(1)} + u'(0)\overline{v(0)} + u(1)\overline{v'(1)} - u(0)\overline{v'(0)} - \int_0^1 u(x)\overline{v''(x)}\,\mathrm{d}x.$$

In particular, for $u, v \in \mathsf{Dom}(A)$ we have $\langle Au, v\rangle = \langle u, Av\rangle$, so A is symmetric.

Let $v \in \mathsf{Dom}(A^*)$. For all $\phi \in C_0^\infty(0,1)$ we have

$$-\int_0^1 \phi''(x)\overline{v(x)}\,\mathrm{d}x = \int_0^1 \phi(x)\overline{(A^*v)(x)}\,\mathrm{d}x.$$

This proves that in the sense of distributions we have $A^*v = -v'' \in L^2(0,1)$. We deduce in particular that $v \in H^2(0,1)$.

Then for all $u \in \mathsf{Dom}(A)$ we have by the computation above

$$0 = \langle Au, v\rangle - \left\langle u, A^*v\right\rangle = \left\langle -u'', v\right\rangle - \left\langle u, -v''\right\rangle = u'(0)\overline{v(0)} + u(1)\overline{v'(1)}.$$

Considering $u \in \mathsf{Dom}(A)$ such that $u'(0) = 0$ and $u(1) = 1$ we deduce that $v'(1) = 0$. Similarly, $v(0) = 0$, and finally $v \in \mathsf{Dom}(A)$. This proves that $A^* \subset A$, and hence A is selfadjoint.

2. For all $u \in \mathsf{Dom}(A)$ we have by the Green Formula

$$\langle Au, u\rangle_{\mathcal{H}} = -\int_0^1 u''(x)\overline{u(x)}\,\mathrm{d}x = \int_0^1 \left|u'(x)\right|^2\,\mathrm{d}x \geqslant 0,$$

so $A \geqslant 0$.

3. Since $(-A)$ is selfadjoint and non-positive, it is in particular maximal dissipative, and it generates a contractions semigroup.

4. Assume that $u \in H^2(0,1)$ is such that $u'' = 0$. There exist $\alpha, \beta \in \mathbb{C}$ such that $u(x) = \alpha x + \beta$ for almost all $x \in]0,1[$. Since $u \in \mathsf{Dom}(A)$, the boundary conditions imply that $\alpha = \beta = 0$, so $u = 0$ almost everywhere on $]0,1[$.

5. Since $\mathsf{Dom}(A)$ is continuously embedded in $H^2(0,1)$, it is compactly embedded in $\mathcal{H}$, and hence the operator A has compact resolvent (notice that the resolvent set is not empty since A is selfadjoint). Its spectrum consists of a sequence of isolated eigenvalues of finite multiplicities (and going to $+\infty$ since A is non-negative), so the essential spectrum of A is empty.

We denote by $(\lambda_k)_{k\in\mathbb{N}^*}$ the non-decreasing sequence of eigenvalues of A (counted with multiplicities, even if this is not important here). Since A is non-negative and $\ker(A) = \{0\}$, we have $\min\sigma(A) = \lambda_1 > 0$.

6. We set $B = A - \lambda_1$. Then B is selfadjoint and $\min\sigma(B) = 0$. This implies that $B \geqslant 0$. Then $(-B)$ also generates a contractions semigroup. Moreover, for all $t \geqslant 0$ we have

$$e^{-tA} = e^{-t(B+\lambda_1)} = e^{-t\lambda_1}e^{-tB}.$$

Indeed, $(e^{-\lambda_1}e^{-tB})$ defines a continuous semigroup on $\mathcal{H}$. For $\varphi \in \mathsf{Dom}(B)$ we have

$$\frac{d}{dt}e^{-t\lambda_1}e^{-tB} = -(B+\lambda_1)e^{-t\lambda_1}e^{-tB}\varphi,$$

so

$$e^{-t\lambda_1}e^{-tB}\varphi = e^{-t(B+\lambda_1)}\varphi.$$

By density of $\mathsf{Dom}(B)$ and continuity of the semigroups, this equality holds for all $\varphi \in \mathcal{H}$ and $t \geqslant 0$. We deduce

$$\left\|e^{-tA}\right\|_{\mathcal{L}(\mathcal{H})} = e^{-t\lambda_1}\left\|e^{-tB}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant e^{-t\lambda_1}.$$

$\square$

Exercise 5. Let $\mathcal{H}$ be a Hilbert space. Let $(S_t)_{t\geqslant 0}$ be a strongly continuous semigroup on $\mathcal{H}$ and let A be its generator. Prove that the generator of the semigroup $(S_t^*)_{t\geqslant 0}$ is A^* (the proof that $(S_t^*)_{t\geqslant 0}$ is a strongly continuous semigroup is not required).

Correction : We denote by B the generator of the semigroup $(S_t^*)_{t\geqslant 0}$.
• Let $\varphi \in \mathsf{Dom}(B)$. Let $\psi \in \mathsf{Dom}(A)$. We have

$$\left\langle \varphi, \frac{S_t\psi - \psi}{t} \right\rangle \xrightarrow[t\to 0]{} \langle \varphi, A\psi\rangle$$

and

$$\left\langle \varphi, \frac{S_t\psi - \psi}{t} \right\rangle = \left\langle \frac{S_t^*\varphi - \varphi}{t}, \psi \right\rangle \xrightarrow[t\to 0]{} \langle B\varphi, \psi\rangle .$$

This proves that $\langle \varphi, A\psi\rangle = \langle B\varphi, \psi\rangle$ for all $\psi \in \mathsf{Dom}(A)$, so $\varphi \in \mathsf{Dom}(A^*)$ and $A^*\varphi = B\varphi$.
• Let $\varphi \in \mathsf{Dom}(A^*)$. Let $\psi \in \mathcal{H}$. By Proposition 5.33 we have

$$\begin{aligned}\left\langle S_t^*\varphi - \varphi, \psi \right\rangle = \langle \varphi, S_t\psi - \psi\rangle = \left\langle \varphi, A\int_0^t S_\tau\psi \,\mathrm{d}\tau \right\rangle = \left\langle A^*\varphi, \int_0^t S_\tau\psi\,\mathrm{d}\tau \right\rangle \\ = \int_0^t \left\langle A^*\varphi, S_\tau\psi \right\rangle \mathrm{d}\tau = \int_0^t \left\langle S_\tau^* A^*\varphi, \psi \right\rangle \mathrm{d}\tau = \left\langle \int_0^t S_\tau^* A^*\varphi \,\mathrm{d}\tau, \psi \right\rangle .\end{aligned}$$

This gives

$$\frac{S_t^*\varphi - \varphi}{t} = \frac{1}{t}\int_0^t S_\tau A^*\varphi \,\mathrm{d}\tau \xrightarrow[t\to 0]{} A^*\varphi.$$

Thus $\varphi \in \mathsf{Dom}(B)$ (and $B\varphi = A^*\varphi$). □

Exercise 6. Let $\mathcal{H} = L^2(\mathbb{R})$.
1. We set $\mathsf{Dom}(T) = \{u \in C_0^\infty(\mathbb{R}) : u(0) = 0\}$, and for $u \in \mathsf{Dom}(T)$ we set $Tu = -u'' + u$. Prove that this defines a symmetric and non-negative operator T on $\mathcal{H}$.
2. Prove that T is not selfadjoint.
3. We set $\mathcal{V}_N = H^1(\mathbb{R})$. For $v \in \mathcal{V}_N$ we set $q_N(v) = \|v\|^2_{H^1(\mathbb{R})} = \|v'\|^2_{L^2(\mathbb{R})} + \|v\|^2_{L^2(\mathbb{R})}$. What is the operator A_N (domain and action) associated with the quadratic form q_N by the representation theorem on $\mathcal{H}$? Prove that A_N is a selfadjoint extension of T.
4. We set $\mathcal{V}_D = \{v \in H^1(\mathbb{R}) : v(0) = 0\}$. For $v \in \mathcal{V}_D$ we set $q_D(v) = \|v\|^2_{H^1(\mathbb{R})}$. What is the operator A_D (domain and action) associated with the quadratic form q_D by the representation theorem on $\mathcal{H}$? Prove that A_D is a selfadjoint extension of T.
5. Give all the selfadjoint extensions of T on $\mathcal{H}$.

Correction : **1.** Let $u, v \in \mathsf{Dom}(T)$. By integrations by parts we have

$$\begin{aligned}\langle Tu, v\rangle_{\mathcal{H}} &= -\int_{\mathbb{R}} u''(x)\overline{v(x)}\,\mathrm{d}x + \int_{\mathbb{R}} u(x)\overline{v(x)}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} u'(x)\overline{v'(x)}\,\mathrm{d}x + \int_{\mathbb{R}} u(x)\overline{v(x)}\,\mathrm{d}x \\ &= -\int_{\mathbb{R}} u(x)\overline{v''(x)}\,\mathrm{d}x\,\mathrm{d}x + \int_{\mathbb{R}} u(x)\overline{v(x)}\,\mathrm{d}x \\ &= \langle u, Tv\rangle_{\mathcal{H}} .\end{aligned}$$

This proves that T is symmetric. Moreover the computation gives

$$\langle Tu, u\rangle_{\mathcal{H}} = \int_{\mathbb{R}} \left|u'(x)\right|^2 \,\mathrm{d}x \geqslant 0,$$

which proves that T is non-negative.
2. Now let $v \in C_0^\infty(\mathbb{R})$ with $v(0) \neq 0$. The same computation as above shows that for all $u \in \mathsf{Dom}(T)$ we have

$$\langle Tu, v\rangle_{\mathcal{H}} = -\int_{\mathbb{R}} u(x)\overline{v''(x)}\,\mathrm{d}x\,\mathrm{d}x + \int_{\mathbb{R}} u(x)\overline{v(x)}\,\mathrm{d}x = \left\langle u, -v'' + v\right\rangle_{\mathcal{H}} .$$

This proves that $v \in \mathsf{Dom}(T^*)$. Since $v \notin \mathsf{Dom}(T)$, T cannot be selfadjoint.
3. Let $u \in \mathsf{Dom}(A_N) \subset \mathcal{V}_N$. For all $\phi \in C_0^\infty(\mathbb{R}) \subset \mathcal{V}_N$ we have

$$-\int_{\mathbb{R}} u'(x)\overline{\phi'(x)}\,\mathrm{d}x = -q_N(u,\phi) + \int_{\mathbb{R}} u(x)\overline{\phi(x)}\,\mathrm{d}x = \langle -A_N u + u, \phi\rangle .$$

This proves that u' has a derivative in $L^2(\mathbb{R})$, and $u'' = -A_N u + u$. In particular, $u \in H^2(\mathbb{R})$. Conversely, assume that $u \in H^2(\mathbb{R})$. Then for all $v \in H^1(\mathbb{R})$ we have by the Green formula

$$q_N(u,v) = \int_{\mathbb{R}} \left(u'(x)\overline{v'(x)} + u(x)\overline{v(x)}\right)\mathrm{d}x = \left\langle -u'' + u, v\right\rangle_{L^2(\mathbb{R})} .$$

This proves that $u \in \mathsf{Dom}(A_N)$ (and $A_N u = -u'' + u$). Finally we have $\mathsf{Dom}(A_N) = H^2(\mathbb{R})$ and for $u \in \mathsf{Dom}(A_N)$ we have $A_N = -u'' + u$.

4. Let $u \in \mathsf{Dom}(A_D)$. We denote by $u_\pm$ the restriction of u on $\mathbb{R}^*_\pm$. For $\phi \in C_0^\infty(\mathbb{R}^*_+)$ we have

$$-\int_0^{+\infty} u'_+(x)\overline{\phi'(x)}\,\mathrm{d}x = -q_D(u,\phi) + \int_0^{+\infty} u(x)\overline{\phi(x)}\,\mathrm{d}x = \langle -A_D u + u, \phi\rangle.$$

This proves that $u_+ \in H^2(\mathbb{R}^*_+)$ and $u''_+ = -A_D u + u$ on $\mathbb{R}^*_+$. Similarly, $u_- \in H^2(\mathbb{R}^*_-)$ and $u''_- = -A_D u + u$ on $\mathbb{R}^*_-$. Conversely let $u \in \mathcal{V}_D$ such that the restrictions u_+ and u_- of u to $\mathbb{R}^*_+$ and $\mathbb{R}^*_-$ belong to $H^2(\mathbb{R}^*_+)$ and $H^2(\mathbb{R}^*_-)$. Then for all $v \in \mathcal{V}_D$ we have

$$\begin{aligned} q_D(u,v) &= \int_{-\infty}^0 \big(u'_+(x)\overline{v'(x)} + u(x)\overline{v(x)}\big)\,\mathrm{d}x + \int_0^{+\infty} \big(u'_-(x)\overline{v'(x)} + u(x)\overline{v(x)}\big)\,\mathrm{d}x \\ &= \langle -u''_- + u_-, v\rangle_{L^2(\mathbb{R}^*_-)} + \langle -u''_+ + u_+, v\rangle_{L^2(\mathbb{R}^*_+)}. \end{aligned}$$

This proves that $u \in \mathsf{Dom}(A_D)$. Finally, we have

$$\mathsf{Dom}(A_D) = \left\{ u \in \mathcal{V}_D \,:\, u_{|\mathbb{R}^*_\pm} \in H^2(\mathbb{R}^*_\pm) \right\},$$

and for $u \in \mathsf{Dom}(A_D)$ we have, with $u_\pm = u_{|\mathbb{R}^*_\pm}$,

$$A_D u = \begin{cases} -u''_+ + u_+ & \text{on } \mathbb{R}^*_+, \\ -u''_- + u_- & \text{on } \mathbb{R}^*_-. \end{cases}$$

5. Let A be a selfadjoint extension of T. We have $T \subset A = A^* \subset T^*$. Let $v \in \mathsf{Dom}(A) \subset \mathsf{Dom}(T^*)$. As above we see that $v_+ = v_{|\mathbb{R}^*_+} \in H^2(\mathbb{R}^*_+)$ and $v_- = v_{|\mathbb{R}^*_-} \in H^2(\mathbb{R}^*_-)$. Moreover,

$$Av = \begin{cases} -v''_+ + v_+ & \text{on } \mathbb{R}^*_+, \\ -v''_- + v_- & \text{on } \mathbb{R}^*_-. \end{cases}$$

For all $u \in \mathsf{Dom}(T)$ we have

$$\begin{aligned} 0 &= \langle Tu, v\rangle - \langle u, Av\rangle \\ &= \int_{-\infty}^0 \big(-u''(x)\overline{v(x)} + u(x)\overline{v''(x)}\big)\,\mathrm{d}x + \int_0^{+\infty} \big(-u''(x)\overline{v(x)} + u(x)\overline{v''(x)}\big)\,\mathrm{d}x \\ &= -u'(0)\overline{v(0^-)} + u'(0)\overline{v(0^+)}. \end{aligned}$$

Choosing $u \in \mathsf{Dom}(T)$ such that $u'(0) \neq 0$, we deduce that $v(0^+) = v(0^-)$, and hence $v \in H^1(\mathbb{R})$.

We already know that A_D is a selfadjoint extension of T. We assume that $A \neq A_D$. Then there exists $u \in \mathsf{Dom}(A)$ such that $u(0) \neq 0$. Then for all $v \in \mathsf{Dom}(A)$ we have

$$\begin{aligned} 0 &= \langle Au, v\rangle - \langle u, Av\rangle \\ &= \int_{-\infty}^0 \big(-u''(x)\overline{v(x)} + u(x)\overline{v''(x)}\big)\,\mathrm{d}x + \int_0^{+\infty} \big(-u''(x)\overline{v(x)} + u(x)\overline{v''(x)}\big)\,\mathrm{d}x \\ &= -u'(0^-)\overline{v(0)} + u(0)\overline{v'(0^-)} + u'(0^+)\overline{v(0)} - u(0)\overline{v'(0^+)}. \end{aligned}$$

We set

$$\alpha = \frac{u'(0^+) - u'(0^-)}{u(0)}.$$

Then for all $v \in \mathsf{Dom}(A)$ we have

$$v'(0^+) - v'(0^-) = \alpha v(0).$$

Applied with $v = u$, this proves in particular that $\alpha \in \mathbb{R}$. Finally, there exists $\alpha \in \mathbb{R}$ such that $\mathsf{Dom}(A)$ is included in

$$\mathcal{D}_\alpha = \left\{ u \in H^1(\mathbb{R}) \,:\, u_{|\mathbb{R}^*_\pm} \in H^2(\mathbb{R}^*_\pm) \quad \text{and } u'(0^+) - u'(0^-) = \alpha u(0) \right\}.$$

For $\alpha \in \mathbb{R}$ we denote by A_α the operator defined by $\mathsf{Dom}(A_\alpha) = \mathcal{D}_\alpha$ and, for $u \in \mathsf{Dom}(A_\alpha)$ and $u_\pm = u_{|\mathbb{R}^*_\pm}$,

$$A_\alpha u = \begin{cases} -u''_+ + u_+ & \text{on } \mathbb{R}^*_+, \\ -u''_- + u_- & \text{on } \mathbb{R}^*_-. \end{cases}$$

If A is a selfadjoint extension of T, then there exists $\alpha \in \mathbb{R}$ such that $A \subset A_\alpha$, and hence $A = A_\alpha$. Conversely, we prove that for any $\alpha \in \mathbb{R}$ the operator A_α is a selfadjoint extension of T.

Let $\alpha \in \mathbb{R}$. We have $T \subset A_\alpha$. We check by direct computation that A_α is symmetric. Then we consider $v \in \mathsf{Dom}(A^*_\alpha)$. We have $v_\pm \in H^2(\mathbb{R}^*_\pm)$. Then for all $u \in \mathsf{Dom}(A_\alpha)$ we have

$$u'(0^+)\overline{v(0^+)} - u'(0^-)\overline{v(0^-)} - u(0)\overline{(v'(0^+) - v'(0^-))}.$$

With $u(0) = 0$ and $u'(0^+) = u'(0^-) \neq 0$ we see that $v(0^+) = v(0^-)$, so $v \in H^1(\mathbb{R})$. Then

$$u(0)\overline{\big(\alpha v(0) - (v'(0^+) - v'(0^-))\big)} = 0.$$

This implies that $(v'(0^+) - v'(0^-)) = \alpha v(0)$ and proves that $v \in \mathsf{Dom}(A_\alpha)$. Thus A_α is selfadjoint.

Finally, the selfadjoint extensions of T are the operators A_α for $\alpha \in \mathbb{R}$.

□