Chapter 5

Semigroups and evolution equations

[Draft version, November 21, 2022]

In this chapter we discuss the properties of (strongly continuous) semigroups. This is motivated by the analysis of (linear but also non-linear) evolution (time-dependant) problems.

More precisely, given a Banach space E, an operator A on E and $\varphi_0 \in \mathsf{E}$, we consider the linear Cauchy problem

$$\begin{cases} \varphi'(t) = A\varphi(t), & \forall t \geqslant 0, \\ \varphi(0) = \varphi_0. \end{cases} \tag{5.1}$$

Definition 5.1. *Let I be an interval of $\mathbb{R}$ which contains 0. A strong solution of (5.1) on I is a function $\varphi \in C^1(I; \mathsf{E}) \cap C^0(I; \mathsf{Dom}(A))$ which satisfies (5.1) in the natural sense.*

We can also consider the inhomogeneous problem

$$\begin{cases} \varphi'(t) - A\varphi(t) = f(t), & \forall t \geqslant 0, \\ \varphi(0) = \varphi_0. \end{cases} \tag{5.2}$$

or the semilinear problem

$$\begin{cases} \varphi'(t) - A\varphi(t) = F(\varphi(t)), & \forall t \geqslant 0, \\ \varphi(0) = \varphi_0. \end{cases} \tag{5.3}$$

5.1 Exponential of a bounded operator

If A is a bounded operator on E, we can set for all $t \in \mathbb{R}$

$$e^{tA} = \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!} \tag{5.4}$$

The following results are consequencies of the properties of power series in a Banach space.

Proposition 5.2. (i) *For $t \in \mathbb{R}$ we have $e^{tA} \in \mathcal{L}(\mathsf{E})$ and $\left\| e^{tA} \right\|_{\mathcal{L}(\mathsf{E})} \leqslant e^{|t| \|A\|_{\mathcal{L}(\mathsf{E})}}$.*

(ii) *We have $e^{0A} = \mathrm{Id}_{\mathsf{E}}$.*

(iii) *For $s, t \in \mathbb{R}$ we have $e^{tA} e^{sA} = e^{(t+s)A} = e^{sA} e^{tA}$.*

(iv) *If $B \in \mathcal{L}(\mathsf{E})$ commutes with A, then it commutes with e^{tA} for all $t \geqslant 0$.*

(v) *The map*

$$\begin{cases} \mathbb{R} & \rightarrow & \mathcal{L}(\mathsf{E}) \\ t & \mapsto & e^{tA} \end{cases}$$

is of class C^∞ *and*

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A.$$

In particular, for $\varphi_0 \in \mathsf{E}$ *the function* $t \mapsto e^{tA}\varphi_0$ *is a strong solution of* (5.1) *on* $\mathbb{R}$.

Ex. 5.1

Remark 5.3. Let $f \in C^0(\mathbb{R}_+, \mathsf{E})$. Assume that $\varphi \in C^1(I, \mathsf{E})$ is a solution of (5.2). Then for all $t \in I$ we have the Duhamel formula

$$\varphi(t) = e^{tA}\varphi_0 + \int_0^t e^{(t-s)A} f(s)\,\mathrm{d}s.$$

The purpose of this chapter is to generalize these properties for an unbounded operator A on E (in this case the exponential cannot be defined by the power series (5.4)).

5.2 Strongly continuous semigroups

The notion of strongly continuous semigroup generalizes some properties of the family $(e^{tA})_{t\geqslant 0}$ and will be at the heart of the discussion.

Definition 5.4. *We say that the family* $(S_t)_{t\geqslant 0}$ *of operators in* $\mathcal{L}(\mathsf{E})$ *is a* C^0*-semigroup (or strongly continuous semigroup) if*

(i) $S_0 = \mathrm{Id}_{\mathsf{E}}$ *;*

(ii) $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$ *for all* $s, t \geqslant 0$ *;*

(iii) the map $t \mapsto S_t$ *is strongly continuous on* $\mathbb{R}_+$ *(for all* $\varphi \in \mathsf{E}$ *the map* $t \mapsto S_t\varphi \in \mathsf{E}$ *is continuous on* $\mathbb{R}_+$*).*

Remark 5.5. The second property implies that S_{t_1} commutes with S_{t_2} for all $t_1, t_2 \geqslant 0$.

Remark 5.6. Notice that we do not require the continuity of the map $t \mapsto S_t$ for the topology of $\mathcal{L}(\mathsf{E})$.

Proposition 5.7. *Let* $(S_t)_{t\geqslant 0}$ *be a* C^0*-semigroup. There exist* $M \geqslant 0$ *and* $\omega \in \mathbb{R}$ *such that for all* $t \in \mathbb{R}_+$ *we have*

$$\|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant Me^{\omega t}. \tag{5.5}$$

Moreover, if for some $t_0 \in \mathbb{R}_+$ *we have* $\|S_{t_0}\|_{\mathcal{L}(\mathsf{E})} < 1$ *then* (5.5) *holds for some* $\omega < 0$.

Proof. • Let $\varphi \in \mathsf{E}$. By continuity, there exists $C_\varphi > 0$ such that

$$\forall t \in [0,1], \quad \|S_t\varphi\|_{\mathsf{E}} \leqslant C_\varphi \|\varphi\|_{\mathsf{E}}.$$

By the uniform boundedness principle, there exists $C \geqslant 1$ such that

$$\forall t \in [0,1], \quad \|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant C.$$

Then, for all $N \in \mathbb{N}^*$ and $t \in [N-1, N]$ we get

$$\|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant C^N \leqslant C^{t+1} = Ce^{t\ln(C)}.$$

This gives the first statement with $M = C$ and $\omega = \ln(C)$.
• Now assume that $\alpha = \|S_{t_0}\|_{\mathcal{L}(\mathsf{E})} \in]0,1[$ for some $t_0 > 0$. Let $C = \sup_{t\in[0,t_0]} \|S_t\|_{\mathcal{L}(\mathsf{E})}$. Then for $N \in \mathbb{N}^*$ and $t \in [(N-1)t_0, Nt_0]$ we have

$$\|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant \|S_{t_0}\|_{\mathcal{L}(\mathsf{E})}^{N-1} \left\|S_{t-(N-1)t_0}\right\| \leqslant C\alpha^{N-1} \leqslant \frac{M}{\alpha}\alpha^{\frac{t}{t_0}} = \frac{C}{\alpha}e^{t\frac{\ln(\alpha)}{t_0}}.$$

Then (5.5) holds with $M = \frac{C}{\alpha}$ and $\omega = \frac{\ln(\alpha)}{t_0} < 0$. ∎

Remark 5.8. To prove the continuity of $\varphi \mapsto S_t\varphi$ it is enough to prove that $S_t\varphi \to \varphi$ in E as $t \to 0^+$. Indeed, let $\varphi \in \mathsf{E}$ and $t_0 > 0$. For the right-continuity we simply write, for $h > 0$,

$$S_{t_0+h}\varphi - S_{t_0}\varphi = S_{t_0}\big(S_h\varphi - \varphi\big) \xrightarrow[h\to 0^+]{} 0.$$

On the other hand, by Proposition 5.7 S_{t_0-h} is bounded uniformly in $h \in]0, t_0]$, so

$$S_{t_0-h}\varphi - S_{t_0}\varphi = S_{t_0-h}\big(\varphi - S_h\varphi\big) \xrightarrow[h\to 0^+]{} 0.$$

Remark 5.9. Let $(S_t)_{t\geqslant 0}$ be a strongly continuous semigroup. The map

$$\begin{cases} \mathbb{R}_+ \times \mathsf{E} & \to & \mathsf{E} \\ (t,\varphi) & \mapsto & S_t\varphi \end{cases}$$

is continous. Let $(t,\varphi) \in \mathbb{R}_+ \times \mathsf{E}$. For $(\tau,\psi) \in \mathbb{R}_+ \times \mathsf{E}$ we have

$$\|S_\tau\psi - S_t\varphi\|_{\mathsf{E}} \leqslant \|S_\tau\psi - S_\tau\varphi\|_{\mathsf{E}} + \|S_\tau\varphi - S_t\varphi\|_{\mathsf{E}}$$

The first term is smaller than $\|S_\tau\|_{\mathcal{L}(\mathsf{E})}\|\psi - \varphi\|_{\mathsf{E}}$, and $\|S_\tau\|_{\mathcal{L}(\mathsf{E})}$ is uniformly bounded for $\tau \in [t-1, t+1]$ by Proposition 5.7. The second term goes to 0 as $\tau \to t$ by strong continuity. This proves that

$$\|S_\tau\psi - S_t\varphi\|_{\mathsf{E}} \xrightarrow[(\tau,\psi)\to(t,\varphi)]{} 0.$$

Definition 5.10. *We say that the family $(S_t)_{t\in\mathbb{R}}$ of operators in $\mathcal{L}(\mathsf{E})$ is a C^0-group (or strongly continuous group) if*

(i) $S_0 = \mathrm{Id}_{\mathsf{E}}$,

(ii) $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$ *for all* $s, t \in \mathbb{R}$,

(iii) the map $t \mapsto S_t$ *is strongly continuous on* $\mathbb{R}$.

Remark 5.11. If $(S_t)_{t\in\mathbb{R}}$ is a strongly continuous group then $S_{-t} = S_t^{-1}$ for all $t \in \mathbb{R}$. Moreover, $(S_t)_{t\geqslant 0}$ and $(S_{-t})_{t\geqslant 0}$ are strongly continuous semigroups.

Definition 5.12.

- *A unitary group on $\mathcal{H}$ is a strongly continuous group $(U_t)_{t\in\mathbb{R}}$ such that U_t is unitary on $\mathcal{H}$ for all $t \in \mathbb{R}$.*
- *A contractions semigroup on E is a strongly continuous semigroup $(S_t)_{t\geqslant 0}$ such that $\|S_t\|_{\mathcal{L}(\mathsf{E})} \leqslant 1$ for all $t \geqslant 0$.*

Example 5.13 (Translation). For $t \in \mathbb{R}$ we consider on $L^2(\mathbb{R})$ the operator S_t such that for $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$(S_tu)(x) = u(x+t).$$

This defines a unitary group on $L^2(\mathbb{R})$.

Example 5.14 (Dilation). For $t \in \mathbb{R}$ we consider on $L^2(\mathbb{R})$ the operator S_t such that for $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$(S_tu)(x) = e^{2t}u(e^tx).$$

This defines a unitary group on $L^2(\mathbb{R})$.

Example 5.15. For $t \geqslant 0$, $u \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ we set

$$(S_tu)(x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}u(y)\,\mathrm{d}y.$$

Then $(S_t)_{t\geqslant 0}$ is a contractions semigroup on $L^2(\mathbb{R})$.

5.3 Dissipative operators

We set
$$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}.$$

Definition 5.16. *Let A be an operator on E. We say that A is dissipative if*

Ex. 5.2
$$\forall \varphi \in \mathsf{Dom}(A), \forall z \in \mathbb{C}_+, \quad \|(A-z)\varphi\|_{\mathsf{E}} \geqslant \operatorname{Re}(z)\|u\|_{\mathsf{E}}.$$

Remark 5.17. In particular, if A is dissipative then any $z \in \mathbb{C}_+$ is a regular point of A.

Example 5.18. A skew-symmetric operator on the Hilbert space $\mathcal{H}$ is dissipative (see Proposition 3.7).

Proposition 5.19. *Let A be an operator on $\mathcal{H}$. Then A is dissipative if and only if*
$$\forall \varphi \in \mathsf{Dom}(A), \quad \operatorname{Re}\langle A\varphi, \varphi\rangle \leqslant 0. \tag{5.6}$$

Proof. Let $\varphi \in \mathsf{Dom}(A)$. For $z = \tau + i\mu \in \mathbb{C}_+$ with $\tau > 0$ and $\mu \in \mathbb{R}$ we have
$$\begin{aligned} \|(A-z)\varphi\|_{\mathcal{H}}^2 &= \|(A-i\mu)\varphi\|_{\mathcal{H}}^2 - 2\operatorname{Re}\langle (A-i\mu)\varphi, \tau\varphi\rangle_{\mathcal{H}} + \tau^2\|u\|_{\mathcal{H}}^2 \\ &= \|(A-i\mu)\varphi\|_{\mathcal{H}}^2 - 2\tau\operatorname{Re}\langle A\varphi, \varphi\rangle_{\mathcal{H}} + \tau^2\|u\|_{\mathcal{H}}^2. \end{aligned} \tag{5.7}$$

If (5.6) holds, this gives
$$\|(A-z)\varphi\|_{\mathcal{H}}^2 \geqslant \tau^2\|u\|_{\mathcal{H}}^2,$$
so A is dissipative. Conversely, if A is dissipative then (5.7) gives
$$2\tau\operatorname{Re}\langle A\varphi, \varphi\rangle_{\mathcal{H}} - \|(A-i\mu)\varphi\|_{\mathcal{H}}^2 = \tau^2\|u\|_{\mathcal{H}}^2 - \|(A-z)\varphi\|_{\mathcal{H}}^2 \leqslant 0.$$
We divide by τ and let τ go to $+\infty$. This gives (5.6). ☐

Definition 5.20. *Let A be a dissipative operator on E. We say that A is maximal dissipative if it is dissipative and any $z \in \mathbb{C}_+$ belongs to its resolvent set.*

Example 5.21. If A is a skew-adjoint operator on the Hilbert space $\mathcal{H}$, then A and $-A$ are maximal dissipative. In particular, if A is selfadjoint then iA and $-iA$ are maximal dissipative.

Example 5.22. The Laplacian Δ with domain $\mathsf{Dom}(\Delta) = H^2(\mathbb{R}^d)$ is maximal dissipative on $L^2(\mathbb{R}^d)$. More generally, a selfadjoint and non-positive operator is maximal dissipative.

Remark 5.23.
- If A is maximal dissipative then for all $z \in \mathbb{C}_+$ we have
$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{1}{\operatorname{Re}(z)}. \tag{5.8}$$
- If A is an operator such that $\mathbb{C}_+ \subset \rho(A)$ and (5.8) holds, then A is maximal dissipative.

Proposition 5.24. *Let A be a dissipative operator on E. Assume that A is closed and that $\mathsf{Ran}(A-z_0)$ is dense in $\mathcal{H}$ for some $z_0 \in \mathbb{C}_+$. Then A is maximal dissipative.*

Proof. Since A is closed and dissipative, $(A-z_0)$ is injective with closed range by Proposition 2.34. By assumption $(A-z_0)$ is then bijective, and $z_0 \in \rho(A)$.

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\rho(A) \cap \mathbb{C}_+$ which goes to some $z \in \mathbb{C}_+$. By (5.8), we have
$$\limsup_{n\in\mathbb{N}} \left\|(A-z)^{-1}\right\| \leqslant \frac{1}{\operatorname{Re}(z)} < +\infty.$$
This implies that $z \in \rho(A)$. Then $\rho(A)$ is closed in $\mathbb{C}_+$. Since it is also open and $\mathbb{C}_+$ is connected, we have $\mathbb{C}_+ \subset \rho(A)$. ☐

Proposition 5.25. *Let A be a densely defined and closed operator on the Hilbert space $\mathcal{H}$. Assume that A and A^* are dissipative. Then A is maximal dissipative.*

Proof. By Proposition 5.24, it is enough to show that $\mathsf{Ran}(A-1)$ is dense in $\mathcal{H}$. Since A^* is dissipative, (A^*-1) is injective and $\overline{\mathsf{Ran}(A-1)} = \ker(A^*-1)^\perp = \mathcal{H}$. □

Proposition 5.26. *Let A be a maximal dissipative operator on the Hilbert space $\mathcal{H}$. Then A is densely defined.*

Proof. Let $\varphi \in \mathsf{Dom}(A)^\perp$ and $\psi = (A-1)^{-1}\varphi \in \mathsf{Dom}(A)$. We have

$$0 = \langle \varphi, \psi \rangle_{\mathcal{H}} = \langle A\psi - \psi, \psi \rangle_{\mathcal{H}},$$

so

$$\langle A\psi, \psi \rangle_{\mathcal{H}} = \|\psi\|_{\mathcal{H}}^2 \geqslant 0.$$

This implies that $\psi = 0$ and hence $\varphi = 0$. □

Proposition 5.27. *Let A be a maximal dissipative operator. Let B be a dissipative operator. Assume that B is A-bounded with bound smaller than 1. Then $A+B$ is maximal dissipative.*

Proof. See the proof of Theorem 3.41. □

Example 5.28. Let $V \in L^\infty(\mathbb{R}^d, \mathbb{C})$ be such that $\mathrm{Im}(V(x)) \leqslant 0$. We consider the Schrödinger operator $H = H_0 + V(x)$, where H_0 is the free Laplacian. Then $-iH$ is a maximal dissipative operator. Indeed $-iH_0$ is skew-adjoint and $-iV$ is dissipative and bounded, so $-iH$ is maximal dissipative by Proposition 5.27.

Example 5.29. Let $m > 0$. We consider on $\mathscr{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ the norm defined by

$$\|(u,v)\|_{\mathscr{H}}^2 = \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + m\,\|u\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^2(\mathbb{R}^d)}^2.$$

Then we define on $\mathscr{H}$ the operator

$$\mathcal{W}_a = \begin{pmatrix} 0 & 1 \\ \Delta - m & -a \end{pmatrix},$$

with domain

$$\mathsf{Dom}(\mathcal{W}) = H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d).$$

We know by Exercise 3.4 that $\mathcal{W}_0$ is skew-adjoint on $\mathscr{H}$. Since the operator

$$\begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}$$

is bounded and dissipative on $\mathscr{H}$, we get by Proposition 5.27 that $\mathcal{W}_a$ is maximal dissipative on $\mathscr{H}$.

Ex. 5.3

Proposition 5.30. *Let A be an operator on $\mathcal{H}$. Then A is skew-adjoint if and only if A and $-A$ are maximal dissipative.*

Proof. • Assume that A is skew-adjoint. By Proposition 5.19, A and $-A$ are dissipative. Moreover 1 belongs to the resolvent set of A and $-A$, so they are both maximal dissipative by Proposition 5.24.

• Conversely, assume that A and $-A$ are maximal dissipative. By Proposition 5.19 we have $\mathrm{Re}\langle A\varphi, \varphi \rangle = 0$ for all $\varphi \in \mathsf{Dom}(A)$, so A is skew-symmetric by Remark 3.2. By definition, 1 belongs to the resolvent sets of A and $-A$, so A is skew-adjoint by Proposition 3.21. □

5.4 Generators of C^0-semigroups

Definition 5.31. *Let $(S_t)_{t\geqslant 0}$ be a C^0-semigroup on E. We denote by $\mathsf{Dom}(A)$ the set of $\varphi \in \mathsf{E}$ such that the limit*

$$\lim_{t\to 0^+} \frac{S_t\varphi - \varphi}{t}$$

exists in E. In this case, we denote by $A\varphi$ this limit. This defines an operator A on E with domain $\mathsf{Dom}(A)$. We say that A is the generator of $(S_t)_{t\geqslant 0}$.

Example 5.32. Let $A \in \mathcal{L}(\mathsf{E})$. For $t \geqslant 0$ we set $S_t = e^{tA}$, as defined by (5.4). Then the generator of (S_t) is... A.

In general, if A is the generator of the semigroup $(S_t)_{t\geqslant 0}$ then for all $t \geqslant 0$ we can write $S_t = e^{tA}$.

Proposition 5.33. *Let $(S_t)_{t\geqslant 0}$ be a C^0-semigroup on E. Let A be its generator.*

(i) *Let $\varphi \in \mathsf{Dom}(A)$. The map $t \mapsto S_t\varphi$ is differentiable on $\mathbb{R}_+$, we have $S_t\varphi \in \mathsf{Dom}(A)$ for all $t \in \mathbb{R}_+$ and*
$$\frac{d}{dt}(S_t\varphi) = S_t A\varphi = AS_t\varphi.$$

(ii) *Let $\varphi \in \mathsf{E}$. For $t \geqslant 0$ we have*
$$\int_0^t S_\tau\varphi \,\mathrm{d}\tau \quad \in \mathsf{Dom}(A)$$
and
$$S_t\varphi - \varphi = A\int_0^t S_\tau\varphi \,\mathrm{d}\tau.$$
If $\varphi \in \mathsf{Dom}(A)$ we also have
$$S_t\varphi - \varphi = A\int_0^t S_\tau\varphi \,\mathrm{d}\tau = \int_0^t S_\tau A\varphi \,\mathrm{d}\tau.$$

Proof. • Let $t \geqslant 0$. For $\tau > 0$ we have
$$\frac{S_\tau - \mathrm{Id}}{\tau} S_t\varphi = S_t \frac{S_\tau - \mathrm{Id}}{\tau}\varphi \xrightarrow[\tau\to 0^+]{} S_t A\varphi.$$
This proves that $S_t\varphi \in \mathsf{Dom}(A)$ and $AS_t\varphi = S_tA\varphi$. Now let $t > 0$. For $\tau > 0$ we have
$$\frac{S_{t+\tau}\varphi - S_t\varphi}{\tau} \xrightarrow[\tau\to 0]{} S_tA\varphi.$$
and, for $\tau \in]0, t]$,
$$\frac{S_{t-\tau}\varphi - S_t\varphi}{-\tau} = S_{t-\tau}\frac{S_\tau\varphi - \varphi}{\tau} \xrightarrow[\tau\to 0]{} S_tA\varphi.$$
This proves that the map $t \mapsto S$ is differentiable and
$$\frac{d}{dt}(S_t\varphi) = S_tA\varphi.$$

• For $h > 0$ we have
$$\begin{aligned}\frac{1}{h}\left(S_h\int_0^t S_\tau\varphi\,\mathrm{d}\tau - \int_0^t S_\tau\varphi\,\mathrm{d}\tau\right) &= \frac{1}{h}\left(\int_0^t S_{\tau+h}\varphi\,\mathrm{d}\tau - \int_0^t S_\tau\varphi\,\mathrm{d}\tau\right)\\ &= \frac{1}{h}\left(\int_h^{t+h} S_\tau\varphi\,\mathrm{d}\tau - \int_0^t S_\tau\varphi\,\mathrm{d}\tau\right)\\ &= \frac{1}{h}\left(\int_t^{t+h} S_\tau\varphi\,\mathrm{d}\tau - \int_0^h S_\tau\varphi\,\mathrm{d}\tau\right)\\ &\xrightarrow[h\to 0]{} S_t\varphi - \varphi.\end{aligned}$$
This proves the first part of the second statement. Now assume that $\varphi \in \mathsf{Dom}(A)$. Since
$$S_\tau\frac{S_h\varphi - \varphi}{h} \xrightarrow[h\to 0]{} S_\tau A\varphi$$
uniformly in $\tau \in [0, t]$ (by Proposition 5.7), we have
$$\frac{S_h - \mathrm{Id}}{h}\int_0^t S_\tau\varphi\,\mathrm{d}\tau = \int_0^t S_\tau\frac{S_h\varphi - \varphi}{h}\,\mathrm{d}\tau \xrightarrow[h\to 0]{} \int_0^t S_\tau A\varphi\,\mathrm{d}\tau,$$
and the proof is complete. ☐

Remark 5.34. If A is not closed we cannot just write $A\int_0^t S_\tau\varphi\,\mathrm{d}\tau = \int_0^t AS_\tau\varphi\,\mathrm{d}\tau$ to prove the last statement of the proposition. We are actually going to use this property to prove that A is closed.

Proposition 5.35. *The generator of a C^0-semigroup is a closed and densely defined operator that determines the semigroup uniquely.*

Proof. • Let $\varphi \in \mathsf{E}$. By Proposition 5.33, we have for all $h > 0$

$$\frac{1}{h}\int_0^h S_\tau\varphi\,\mathrm{d}\tau \in \mathsf{Dom}(A).$$

Since this goes to φ as $h \to 0$, this proves that $\mathsf{Dom}(A)$ is dense in E.
• Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{Dom}(A)$ such that φ_n goes to some φ and $A\varphi_n$ goes to some ψ in E. For $n \in \mathbb{N}$ and $h > 0$ we have by Proposition 5.33

$$S_h\varphi_n - \varphi_n = \int_0^h S_\tau A\varphi_n\,\mathrm{d}\tau.$$

Taking the limit $n \to +\infty$ and dividing by h, we get

$$\frac{S_h\varphi - \varphi}{h} = \frac{1}{h}\int_0^h S_\tau\psi\,\mathrm{d}\tau \xrightarrow[h\to 0]{} \psi.$$

This proves that $\varphi \in \mathsf{Dom}(A)$ with $A\varphi = \psi$. Thus A is closed.
• Assume that $(\tilde{S}_t)_{t\geqslant 0}$ is a C^0-semigroup whose generator is A. Let $\varphi \in \mathsf{Dom}(A)$ and $t > 0$. For $\theta \in [0,t]$ we set

$$\psi(\theta) = \tilde{S}_{t-\theta}S_\theta\varphi \in \mathsf{E}.$$

For $\theta \in [0,t]$ and $h \in \mathbb{R}^*$ such that $\theta + h \in [0,t]$ we have

$$\begin{aligned}\frac{\psi(\theta+h)-\psi(\theta)}{h} &= \tilde{S}_{t-\theta-h}\left(\frac{S_{\theta+h}\varphi - S_\theta\varphi}{h} - AS_\theta\varphi\right)\\ &\quad + \tilde{S}_{t-\theta-h}AS_\theta\varphi\\ &\quad + \frac{\tilde{S}_{t-\theta-h}-\tilde{S}_{t-\theta}}{h}S_\theta\varphi\end{aligned}$$

Since $\tilde{S}_{t-\theta-h}$ is bounded uniformly in $h \in [-1,1]\setminus\{0\}$ by Proposition 5.7, this gives by Proposition 5.33

$$\frac{\psi(\theta+h)-\psi(\theta)}{h} \xrightarrow[h\to 0]{} \tilde{S}_{t-\theta}AS_\theta\varphi - A\tilde{S}_{t-\theta}S_\theta\varphi = 0.$$

Then $S_t\varphi = \psi(t) = \psi(0) = \tilde{S}_t\varphi$. Since $\mathsf{Dom}(A)$ is dense in E, this proves that $\tilde{S}_t = S_t$ for all $t \geqslant 0$. □

Proposition 5.36. *Let A be the generator of a C^0-semigroup $(S_t)_{t\geqslant 0}$. If D is a subspace of $\mathsf{Dom}(A)$ dense in E and invariant by S_t for all $t \geqslant 0$, then it is a core of A.*

Proof. We have to prove that D is dense in $\mathsf{Dom}(A)$ (for the graph norm). Let $\varphi \in \mathsf{Dom}(A)$ and $\varepsilon > 0$. Let (φ_n) be a sequence in D which goes to φ in E. By Proposition 5.33 there exists $t > 0$ such that

$$\left\|\frac{1}{t}\int_0^t e^{sA}\varphi\,\mathrm{d}s - \varphi\right\|_{\mathsf{Dom}(A)} = \left\|\frac{1}{t}\int_0^t e^{sA}\varphi\,\mathrm{d}s - \varphi\right\|_{\mathsf{E}} + \left\|\frac{1}{t}\int_0^t e^{sA}A\varphi\,\mathrm{d}s - A\varphi\right\|_{\mathsf{E}} \leqslant \frac{\varepsilon}{3}.$$

Again by Proposition 5.33 we have

$$A\left(\frac{1}{t}\int_0^t e^{sA}(\varphi_n - \varphi)\,\mathrm{d}s\right) = \frac{S_t - \mathrm{Id}}{t}(\varphi_n - \varphi) \xrightarrow[n\to\infty]{} 0,$$

so there exists $n \in \mathbb{N}$ such that

$$\left\|\frac{1}{t}\int_0^t e^{sA}\varphi_n\,\mathrm{d}s - \frac{1}{t}\int_0^t e^{sA}\varphi\,\mathrm{d}s\right\|_{\mathsf{Dom}(A)} \leqslant \frac{\varepsilon}{3}.$$

We see the integral $\frac{1}{t}\int_0^t e^{sA}\varphi_n \,\mathrm{d}s$ as a Riemann integral. In particular, there exists $n \in \mathbb{N}^*$ such that

$$\left\| \frac{1}{t}\int_0^t e^{sA}\varphi_n \,\mathrm{d}s - \frac{1}{N}\sum_{k=1}^{N} e^{\frac{tkA}{N}}\varphi_n \right\|_{\mathsf{Dom}(A)} \leqslant \frac{\varepsilon}{3}.$$

Since D is invariant by $e^{\frac{tkA}{N}}$ for all k, we have $\frac{1}{N}\sum_{k=1}^{N} e^{\frac{tkA}{N}}\varphi_n \in D$ and the conclusion follows. □

Example 5.37. Let A be the generator of the translation semigroup (Example 5.13). Let $u \in C_0^\infty(\mathbb{R})$. Then we have

$$\left\| \frac{u(\cdot + h) - u(\cdot)}{h} - u'(\cdot) \right\|_{L^2(\mathbb{R})} \xrightarrow[h\to 0]{} 0,$$

so $u \in \mathsf{Dom}(A)$ and $Au = u'$. Since $C_0^\infty(\mathbb{R})$ is left invariant by translations and is dense in $L^2(\mathbb{R})$, it is a core of A by Proposition 5.36. This implies that A is the derivative operator, set on $\mathsf{Dom}(A) = H^1(\mathbb{R})$.

Theorem 5.38. *Let A be the generator of a C^0-semigroup $(S_t)_{t\geqslant 0}$. Let $M \geqslant 1$ and $\omega \in \mathbb{R}$ be given by Proposition 5.7. Then for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \omega$ we have $z \in \rho(A)$, and for $\varphi \in \mathsf{E}$,*

$$(A - z)^{-1}\varphi = -\int_0^{+\infty} e^{-tz} S_t\varphi \,\mathrm{d}t = -\int_0^{+\infty} e^{t(A-z)}\varphi \,\mathrm{d}t.$$

Moreover,

$$\left\|(A - z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{M}{\mathrm{Re}(z) - \omega}.$$

In particular, if $(S_t)_{t\geqslant 0}$ is a contractions semigroup, then A is maximal dissipative.

The integrals have to be understood in the sense of Riemann integrals for continuous functions

$$\int_0^{+\infty} e^{t(A-z)}\varphi \,\mathrm{d}t = \lim_{T\to+\infty}\int_0^{T} e^{t(A-z)}\varphi \,\mathrm{d}t.$$

Proof. • For $\varphi \in \mathsf{E}$ we set

$$I(\varphi) = \int_0^{+\infty} e^{t(A-z)}\varphi \,\mathrm{d}t.$$

We have

$$\begin{aligned}
\frac{e^{hA} - \mathrm{Id}}{h} I(\varphi) &= \frac{1}{h}\left(\int_0^{+\infty} e^{-tz}e^{(t+h)A}\varphi \,\mathrm{d}t - \int_0^{+\infty} e^{-tz}e^{tA}\varphi \,\mathrm{d}t\right)\\
&= \frac{1}{h}\left(e^{hz}\int_h^{+\infty} e^{t(A-z)}\varphi \,\mathrm{d}t - \int_0^{+\infty} e^{t(A-z)}\varphi \,\mathrm{d}t\right)\\
&= -\frac{e^{hz}}{h}\int_0^{h} e^{t(A-z)}\varphi \,\mathrm{d}t + \frac{e^{hz}-1}{h}\int_0^{+\infty} e^{t(A-z)}\varphi \,\mathrm{d}t\\
&\xrightarrow[h\to 0]{} -\varphi + zI(\varphi).
\end{aligned}$$

This proves that $I(\varphi) \in \mathsf{Dom}(A)$ and

$$(A - z)I(\varphi) = -\varphi.$$

Now let $\psi \in \mathsf{Dom}(A)$. We have

$$\int_0^{T} e^{t(A-z)}\psi \,\mathrm{d}t \xrightarrow[T\to+\infty]{} I(\psi),$$

and

$$(A - z)\int_0^{T} e^{t(A-z)}\varphi \,\mathrm{d}t = \int_0^{T} e^{t(A-z)}(A - z)\varphi \,\mathrm{d}t \xrightarrow[T\to+\infty]{} I\big((A - z)\psi\big).$$

Since $(A-z)$ is closed this proves that $I\big((A-z)\psi\big) = (A-z)I(\psi) = -\psi$. Thus $(A-z)$ is invertible and its inverse is given by $(A-z)^{-1}\varphi = -I(\varphi)$. Then

$$\begin{aligned}\left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} &\leqslant \int_0^{+\infty} e^{-t\operatorname{Re}(z)} \left\|e^{tA}\right\|_{\mathcal{L}(\mathsf{E})} \,\mathrm{d}t\\ &\leqslant M\int_0^{+\infty} e^{-t(\operatorname{Re}(z)-\omega)}\,\mathrm{d}t\\ &\leqslant \frac{M}{\operatorname{Re}(z)-\omega}.\end{aligned}$$

Finally, the fact that the generator of contractions semigroup ($M=1$ and $\omega=0$) is maximal dissipative follows from Remark 5.23. ☐

Definition 5.39. *Let $(S_t)_{t\geqslant 0}$ a strongly continuous group. Then we denote by* $\mathsf{Dom}(A)$ *the set of $\varphi\in\mathsf{E}$ such that the map $t\mapsto S_t\varphi$ is differentiable at $t=0$, and for $\varphi\in\mathsf{Dom}(A)$ we denote by $A\varphi$ the derivative at 0.*

Theorem 5.40. *The generator of a unitary group on the Hilbert space $\mathcal{H}$ is skew-adjoint.*

Proof. Let $(U_t)_{t\in\mathbb{R}}$ be a unitary group and let A be its generator. A is in particular the generator of the contractions semigroup $(U_t)_{t\geqslant 0}$, so it is maximal dissipative. On the other hand, the generator of the contractions semigroup $(U_{-t})_{t\geqslant 0}$ is $-A$, which is also maximal dissipative. Then A is skew-adjoint by Proposition 5.30. ☐

5.5 Hille-Yosida Theorem

Lemma 5.41. *Let A be a densely defined operator. Assume that there exist $\omega\in\mathbb{R}$ and $M>0$ such that $[\omega,+\infty[\subset\rho(A)$ and $\left\|(A-\lambda)^{-1}\right\|_{\mathcal{L}(\mathsf{E})}\leqslant\frac{M}{\lambda}$ for all $\lambda\geqslant\omega$.*

(i) *For $\varphi\in\mathsf{E}$ we have $-\lambda(A-\lambda)^{-1}\varphi\to\varphi$ as $\lambda\to+\infty$.*

(ii) *For $\varphi\in\mathsf{Dom}(A)$ we have $-\lambda A(A-\lambda)^{-1}\varphi = -\lambda(A-\lambda)^{-1}A\varphi\to A\varphi$ as $\lambda\to+\infty$.*

Proof. For $\varphi\in\mathsf{Dom}(A)$ we have

$$\left\|-\lambda(A-\lambda)^{-1}\varphi-\varphi\right\|_{\mathsf{E}} = \left\|(A-\lambda)^{-1}A\varphi\right\| \leqslant \frac{M\left\|A\varphi\right\|_{\mathsf{E}}}{\lambda}\xrightarrow[\lambda\to+\infty]{} 0.$$

Since $\lambda(A-\lambda)^{-1}$ is bounded uniformly in $\lambda\geqslant\omega$, we deduce the first statement for all $\varphi\in\mathsf{E}$. Then for $\varphi\in\mathsf{Dom}(A)$ we apply the first statement to $A\varphi$ to get the second. ☐

Theorem 5.42 (Hille-Yosida)**.** *Let A be a densely defined operator. Assume that $]0,+\infty[\subset\rho(A)$ and*

$$\forall\lambda>0,\quad \left\|(A-\lambda)^{-1}\right\|_{\mathcal{L}(\mathsf{E})}\leqslant\frac{1}{\lambda}.$$

Then A generates a contractions semigroup. In particular, a densely defined and maximal dissipative operator generates a contractions semigroup.

Proof. For $n\in\mathbb{N}^*$ we consider the bounded operator

$$A_n = -nA(A-n)^{-1} = -n-n^2(A-n)^{-1}.$$

- For $t\geqslant 0$ we have

$$\left\|e^{tA_n}\right\|_{\mathcal{L}(\mathsf{E})} = e^{-nt}e^{tn^2\|(A-n)^{-1}\|_{\mathcal{L}(\mathsf{E})}}\leqslant e^{-nt}e^{nt}=1.$$

Let $\varphi\in\mathsf{Dom}(A)$ and $t\geqslant 0$. A_n commutes with A_m and hence with e^{sA_m} for all $s\geqslant 0$, so

$$e^{tA_n}\varphi - e^{tAm}\varphi = \int_0^t\frac{d}{ds}\big(e^{(t-s)A_m}e^{sA_n}\varphi\big)\,\mathrm{d}s = \int_0^t e^{(t-s)A_m}e^{sA_n}\big(A_n\varphi-A_m\varphi\big)\,\mathrm{d}s.$$

This gives

$$\left\| e^{tA_n}\varphi - e^{tA_m}\varphi \right\|_{\mathsf{E}} \leqslant t \left\| A_n\varphi - A_m\varphi \right\|_{\mathsf{E}}.$$

Since $(A_n\varphi)$ is a Cauchy sequence (by Lemma 5.41), the sequence $(e^{tA_n}\varphi)$ converges uniformly on $t \in [0, t_0]$ for any $t_0 > 0$. Since $\left\| e^{tA_n} \right\| \leqslant 1$, the same conclusion holds for any $\varphi \in \mathsf{E}$. We denote by $S_t\varphi$ the limit of $e^{tA_n}\varphi$.

- Let $\varphi \in \mathsf{E}$. Since the sequence of continuous maps $(e^{tA_n}\varphi)$ converges locally uniformly, the map $t \mapsto S_t\varphi$ is continuous on $\mathbb{R}_+$. Let $t, t_1, t_2 \geqslant 0$. For $n \in \mathbb{N}$ we have

$$\left\| e^{tA_n}\varphi \right\|_{\mathsf{E}} \leqslant \left\| \varphi \right\|_{\mathsf{E}} \quad \text{and} \quad e^{t_1A_n}e^{t_2A_n}\varphi = e^{(t_1+t_2)A_n}\varphi.$$

Taking the limit $n \to +\infty$ gives

$$\left\| S_t\varphi \right\|_{\mathsf{E}} \leqslant \left\| \varphi \right\|_{\mathsf{E}} \quad \text{and} \quad S_{t_1}S_{t_2}\varphi = S_{t_1+t_2}\varphi.$$

This proves that (S_t) is a C^0-semigroup on E.

- We denote by B (with domain $\mathsf{Dom}(B)$) the generator of the semigroup (S_t). Let $\varphi \in \mathsf{Dom}(A)$ and $t_0 > 0$. On $[0, t_0]$ the map $t \mapsto e^{tA_n}\varphi$ and its derivative $t \mapsto e^{tA_n}A_n\varphi$ converge uniformly to $t \mapsto S_t\varphi$ and $S_tA\varphi$. This implies that $S_t\varphi$ is differentiable at time 0 with derivative $A\varphi$. Thus $\varphi \in \mathsf{Dom}(B)$ and $B\varphi = A\varphi$. Now let $\varphi \in \mathsf{Dom}(B)$. Since $(A-1)$ is surjective, there exists $\psi \in \mathsf{Dom}(A)$ such that $(B-1)\varphi = (A-1)\psi = (B-1)\psi$. Since $(B-1)$ is injective, we have $\varphi = \psi \in \mathsf{Dom}(A)$ so $\mathsf{Dom}(B) \subset \mathsf{Dom}(A)$. This proves that $A = B$ is the generator of (S_t). □

Theorem 5.43. *A skew-adjoint operator A on $\mathcal{H}$ generates a unitary group.*

Proof. Since A are $-A$ are maximal dissipative, they generate two contractions semigroups $(S_t^+)_{t\geqslant 0}$ and $(S_t^-)_{t\geqslant 0}$.

Let $\varphi \in \mathsf{Dom}(A) = \mathsf{Dom}(-A)$. Let $t \in \mathbb{R}$. For $\tau \in \mathbb{R}\setminus\{t\}$ we have

$$\frac{S_\tau^- S_\tau^+\varphi - S_t^- S_t^+\varphi}{t-\tau} = S_\tau^- \frac{S_\tau^+\varphi - S_t^+\varphi}{t-\tau} + \frac{(S_\tau^- - S_t^-)S_t^+\varphi}{t-\tau}.$$

Since $\left\| S_\tau^- \right\| \leqslant 1$ and $S_t^+\varphi \in \mathsf{Dom}(A)$ we get

$$\frac{S_\tau^- S_\tau^+\varphi - S_t^- S_t^+\varphi}{t-\tau} \xrightarrow[\tau\to t]{} S_t^- AS_t^+\varphi - S_t^- AS_t^+ = 0.$$

This proves that for all $t \in \mathbb{R}$ we have

$$S_t^- S_t^+\varphi = \varphi.$$

Similarly, $S_t^+ S_t^-\varphi = \varphi$ for all $\varphi \in \mathsf{Dom}(A)$. By continuity of S_t^+ and S_t^- and by density of $\mathsf{Dom}(A)$, these equalities hold for all $\varphi \in \mathcal{H}$, so $S_t^- = (S_t^+)^{-1}$ for all $t \geqslant 0$. Fot $t \in \mathbb{R}$ we set

$$U_t = \begin{cases} S_t^+ & \text{if } t \geqslant 0, \\ S_{-t}^- & \text{if } t \leqslant 0. \end{cases}$$

This defines a strongly continuous group $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$. Finally for $t \in \mathbb{R}$ and $\varphi \in \mathcal{H}$ we have

$$\left\| \varphi \right\| = \left\| U_{-t}U_t\varphi \right\| \leqslant \left\| U_t\varphi \right\| \leqslant \varphi,$$

so U_t is an isometry. Since it is surjective, it is unitary and the proof is complete. □

5.6 Exponentially stable semigroups

In this section we give an example of a more advanced result, with a partial proof. It shows how we can use the properties of the resolvent of the generator to give properties on the time dependant problem.

Proposition 5.44. *Let (S_t) be a strongly continuous semigroup on $\mathcal{H}$ and let A be its generator. Then (S_t^*) is a strongly continuous semigroup whose generator is A^*.*

Theorem 5.45 (Gearhart-Prüss). *Let $(S_t)_{t\geqslant 0}$ be a C^0-semigroup on the Hilbert space $\mathcal{H}$. Let A be its generator. Assume that $\mathbb{C}_+ \subset \rho(A)$ and that*

$$\beta = \sup_{z\in\mathbb{C}_+} \left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

Let $\gamma < \frac{1}{\beta}$. Then there exists $C_\gamma > 0$ such that for $t \geqslant 0$ we have

$$\|S_t\|_{\mathcal{L}(\mathcal{H})} \leqslant C_\gamma e^{-\gamma t}.$$

Proof. • Let $\tilde{\gamma} \in]\gamma, \beta^{-1}[$. Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geqslant -\gamma$. There exists $z_0 \in \mathbb{C}_+$ such that $z \in D(z_0, \tilde{\gamma})$. Since $\mathsf{dist}(z_0, \sigma(A)) \geqslant \left\|(A-z_0)^{-1}\right\|^{-1} > |z-z_0|$ we have $z \in \rho(A)$. Then by the resolvent identity we have

$$(A-z)^{-1}\big(1-(z-z_0)(A-z_0)^{-1}\big) = (A-z_0)^{-1}.$$

Since

$$\left\|(z-z_0)(A-z_0)^{-1}\right\| \leqslant \tilde{\gamma}\beta < 1,$$

this gives

$$\left\|(A-z)^{-1}\right\| \leqslant \left\|(A-z_0)^{-1}\right\| \left\|\big(1-(z-z_0)(A-z_0)^{-1}\big)^{-1}\right\| \leqslant C_1 := \frac{\beta}{1-\tilde{\gamma}\beta}. \tag{5.9}$$

• Let M and ω be given by Proposition 5.7. Let $\mu > \omega$. Let $\varphi \in \mathcal{H}$. For $\tau \in \mathbb{R}$ we have by Theorem 5.38

$$(A-(\mu+i\tau))^{-1}\varphi = -\int_0^{+\infty} e^{t(A-(\mu+i\tau))}\varphi \,\mathrm{d}t = -\int_{\mathbb{R}} e^{-it\tau}\mathbb{1}_{\mathbb{R}_+}(t)e^{-t\mu}e^{tA}\varphi\,\mathrm{d}t. \tag{5.10}$$

The function $t \mapsto -\mathbb{1}_{\mathbb{R}_+}(t)e^{-t\mu}e^{tA}\varphi$ is in $L^2(\mathbb{R};\mathcal{H})$ and, by (5.10), its Fourier transform is $\tau \mapsto (A-(\mu+i\tau))^{-1}\varphi$. Then by the Plancherel inequality (which holds for a function with values in a Hilbert space) we have

$$\int_{\mathbb{R}} \left\|(A-(\mu+i\tau))^{-1}\varphi\right\|_{\mathcal{H}}^2 \mathrm{d}\tau = 2\pi\int_0^{+\infty} e^{-2t\mu}\left\|e^{tA}\varphi\right\|_{\mathcal{H}}^2 \mathrm{d}t \leqslant C_2\|\varphi\|_{\mathcal{H}}^2, \tag{5.11}$$

with $C_2 = \frac{\pi M^2}{\mu-\omega}$. For $\tau \in \mathbb{R}$ we have by the resolvent identity

$$\big(A-(-\gamma+i\tau)\big)^{-1} = \big(1-(\gamma+\mu)\big(A-(-\gamma+i\tau)\big)^{-1}\big)\big(A-(\mu+i\tau)\big)^{-1},$$

so with (5.9)

$$\left\|\big(A-(-\gamma+i\tau)\big)^{-1}\right\|^2 \leqslant \big(1+(\gamma+\mu)C_1\big)^2\left\|\big(A-(\mu+i\tau)\big)^{-1}\right\|^2.$$

Then, by (5.11)

$$\int_{\mathbb{R}}\left\|\big(A-(-\gamma+i\tau)\big)^{-1}\varphi\right\|_{\mathcal{H}}^2 \mathrm{d}\tau \leqslant C_3\|\varphi\|_{\mathcal{H}}^2, \quad C_3 = C_2\big(1+(\gamma+\mu)C_1\big)^2. \tag{5.12}$$

• Since A^* also satisfies the assumptions of the theorem, we also have for all $\psi \in \mathcal{H}$

$$\int_{\mathbb{R}}\left\|\big(A^*-(-\gamma+i\tau)\big)^{-1}\psi\right\|_{\mathcal{H}}^2 \mathrm{d}\tau \leqslant C_3\|\psi\|_{\mathcal{H}}^2. \tag{5.13}$$

• Let $\varphi \in \mathsf{Dom}(A^2)$. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geqslant -\gamma$ we have

$$(A-z)^{-1}\varphi = \frac{1}{z+2\gamma}\big((A-z)^{-1}(A+2\gamma)\varphi - \varphi\big),$$

and then

$$(A-z)^{-2}\varphi = \frac{1}{(z+2\gamma)^2}\big((A-z)^{-2}(A+2\gamma)^2\varphi - 2(A-z)^{-1}\varphi + \varphi\big).$$

In particular, the map $\tau \mapsto (A-(\lambda+i\tau))^{-2}\varphi$ is integrable on $\mathbb{R}$ for any $\lambda \geqslant -\gamma$.

- Let $\varphi \in \mathsf{Dom}(A^2)$ and $\psi \in \mathcal{H}$. By differentiation of (5.10) with respect to τ we get

$$\left\langle \big(A-(\mu+i\tau)\big)^{-2}\varphi,\psi\right\rangle = \int_0^{+\infty} e^{-it\tau}\left\langle te^{t(A-\mu))}\varphi,\psi\right\rangle \mathrm{d}t.$$

The inversion formula gives after multiplication by $e^{t\mu}$

$$\left\langle te^{tA}\varphi,\psi\right\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} e^{t(\mu+i\tau)}\left\langle (A-(\mu+i\tau))^{-2}\varphi,\psi\right\rangle \mathrm{d}\tau.$$

Since the map $\zeta \mapsto e^{t\zeta}\left\langle (A-\zeta)^{-2}\varphi,\psi\right\rangle$ is holomorphic on $\{\mathrm{Re}(\zeta) > -\tilde{\gamma}\}$ and decays like $\mathrm{Im}(\zeta)^{-2}$ as $|\mathrm{Im}(\zeta)| \to +\infty$, we can change the contour of integration from $\{\mathrm{Re}(\zeta)=\mu\}$ to $\{\mathrm{Re}(\zeta)=-\gamma\}$. This gives

$$\begin{aligned}\left\langle te^{tA}\varphi,\psi\right\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{t(-\gamma+i\tau)}\left\langle \big(A-(-\gamma+i\tau)\big)^{-2}\varphi,\psi\right\rangle \mathrm{d}\tau\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{t(-\gamma+i\tau)}\left\langle \big(A-(-\gamma+i\tau)\big)^{-1}\varphi,\big(A^*-(-\gamma-i\tau)\big)^{-1}\psi\right\rangle \mathrm{d}\tau.\end{aligned}$$

Then, by the Cauchy-Schwarz inequality and (5.12)-(5.13) we get, for all $\varphi \in \mathsf{Dom}(A^2)$ and $\psi \in \mathcal{H}$,

$$\begin{aligned}\left|\left\langle te^{tA}\varphi,\psi\right\rangle\right| &\leqslant \frac{e^{-\gamma t}}{2\pi}\left(\int_{\mathbb{R}}\left\|\big(A-(-\gamma+i\tau)\big)^{-1}\varphi\right\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\left\|\big(A^*-(-\gamma-i\tau)\big)^{-1}\psi\right\|^2 \mathrm{d}\tau\right)^{\frac{1}{2}}\\ &\leqslant \frac{C_3e^{-\gamma t}}{2\pi}\|\varphi\|\,\|\psi\|.\end{aligned}$$

Since $\mathsf{Dom}(A^2)$ is dense in $\mathcal{H}$ (see Exercise 5.8), we have the same estimate for all $\varphi \in \mathcal{H}$, and

$$t\left\|e^{tA}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{C_3e^{-\gamma t}}{2\pi}$$

This gives the estimate for $t \geqslant 1$. Since S_t is bounded uniformly in $[0,1]$, we get the result by choosing a larger constant if necessary. □

5.7 Exercises

***Exercise* 5.1.** Compute e^{tA_j}, $t \in \mathbb{R}$, for the following matrices:

$$A_1 = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix},\quad A_2 = \begin{pmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & \lambda\end{pmatrix},\quad A_3 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

***Exercise* 5.2.** Let A be a maximal dissipative operator on E. Assume that B is a dissipative extension of A. Prove that $A=B$.

***Exercise* 5.3.** Let $\alpha \in \mathbb{C}$. We consider on $L^2(0,1)$ the Schrödinger operator with Robin condition, defined by

$$A_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2},\quad \mathsf{Dom}(A_\alpha) = \left\{u \in H^2(0,1) \,:\, u'(0)=\alpha u(0), u'(1) = -\alpha u(1)\right\}.$$

Prove that if $\mathrm{Im}(\alpha) \geqslant 0$ then iA_α is maximal dissipative.

***Exercise* 5.4.** Let A be a maximal dissipative operator on E. Let B be a bounded operator. Prove that $A+B$ (defined on $\mathsf{Dom}(A+B) = \mathsf{Dom}(A)$) generates a C^0-semigroup on E and that, for all $t \geqslant 0$,

$$\left\|e^{t(A+B)}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant e^{t\|B\|_{\mathcal{L}(\mathsf{E})}}.$$

***Exercise* 5.5** (Generator of dilations)**.** For $t \in \mathbb{R}$ and $u \in L^2(\mathbb{R})$ we define the function $S_t u$ by

$$(S_t u)(x) = e^{\frac{t}{2}} u(e^t x).$$

1. Prove that this defines a unitary group $(S_t)_{t\in\mathbb{R}}$ on $L^2(\mathbb{R})$. We denote by A the generator of S_t.
2. Let $u \in C_0^\infty(\mathbb{R})$. Prove that $u \in \mathsf{Dom}(A)$ and that $Au = \frac{u}{2} + xu'$ (where we denote by xv the function $x \mapsto xv(x)$).
3. Prove that $C_0^\infty(\mathbb{R})$ is a core of A.
4. We set

$$\mathcal{D} = \left\{u \in L^2(\mathbb{R}) \,:\, xu' \in L^2(\mathbb{R})\right\}.$$

It is endowed with the norm defined by $\|u\|_{\mathcal{D}} = \|u\|_{L^2(\mathbb{R})} + \|xu'\|_{L^2(\mathbb{R})}$. Prove that $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{D}$.
5. Prove that $\mathsf{Dom}(A) = \mathcal{D}$.

***Exercise* 5.6.** Let A be the generator of a C^0-semigroup. Let $\varphi \in \mathsf{Dom}(A)$ and $\lambda \in \mathbb{C}$ such that $A\varphi = \lambda\varphi$. Prove that for all $t \geqslant 0$ we have $e^{tA}\varphi = e^{t\lambda}\varphi$.

***Exercise* 5.7** (Dilation by a general vector field)**.** Let X be a Lipschitzian vector field on $\mathbb{R}^d$. For $x_0 \in \mathbb{R}^d$ on note $t \mapsto \varphi(t; x_0)$ the solution on $\mathbb{R}$ of the problem

$$\begin{cases} y'_{x_0}(t) = X(y_{x_0}(t)), \quad \forall t \in \mathbb{R}, \\ y'_{x_0}(0) = x_0. \end{cases}$$

Then for $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ we set $\varphi^t(x_0) = y_{x_0}(t)$. Then we have $\varphi^0 = \mathrm{Id}_{\mathbb{R}^d}$ and $\varphi^{t+s} = \varphi^t \circ \varphi^s$ for all $s, t \in \mathbb{R}$. For $t \in \mathbb{R}$ and $u \in L^2(\mathbb{R}^d)$ we set

$$S_t u(x) = \det(d_x\varphi^t)^{\frac{1}{2}} u(\varphi^t x).$$

1. Prove that $(S_t)_{t\in\mathbb{R}}$ is a unitary group on $L^2(\mathbb{R}^d)$.
2. What is the generator of $(S_t)_{t\in\mathbb{R}}$?

***Exercise* 5.8.** Let A be the generator of a strongly continuous semigroup. We set

$$\mathsf{Dom}(A^\infty) = \bigcup_{n\in\mathbb{N}^*} \mathsf{Dom}(A^n)$$

(where, by induction, $\mathsf{Dom}(A^n) = \left\{\varphi \in \mathsf{Dom}(A^{n-1}) \,:\, A^{n-1}\varphi \in \mathsf{Dom}(A)\right\}$).
1. Prove that $\mathsf{Dom}(A^\infty)$ is a subspace of $\mathsf{Dom}(A)$, invariant by e^{tA} for all $t \geqslant 0$.
2. We denote by $\mathcal{C}$ the set of smooth functions on $\mathbb{R}$ compactly supported in $]0, +\infty[$. Let $\phi \in \mathcal{C}$ and $\psi \in \mathsf{E}$. We set

$$\psi_\phi = \int_0^{+\infty} \phi(s) e^{sA}\psi \,\mathrm{d}s.$$

Prove that $\psi_\phi \in \mathsf{Dom}(A)$ with

$$A\psi_\phi = -\int_0^{+\infty} \phi'(s) e^{sA} \,\mathrm{d}s.$$

3. Prove that $\psi_\phi \in \mathsf{Dom}(A^\infty)$.
4. We set $D = \mathsf{span}\{\psi_\phi, \psi \in \mathsf{E}, \phi \in \mathcal{C}\}$. Assume by contradiction that D is not dense in E and consider $\ell \in \mathsf{E}'$ such that $\langle \ell, \psi\rangle_{\mathsf{E}',\mathsf{E}} = 0$ for all $\psi \in D$ (as given by the Hahn-Banach theorem).
   a. Prove that $\left\langle \ell, e^{sA}\psi\right\rangle_{\mathsf{E}',\mathsf{E}} = 0$ for all $s \geqslant 0$ and all $\psi \in \mathsf{E}$.
   b. Deduce that D is dense in E.
5. Prove that $\mathsf{Dom}(A^\infty)$ is a core for A.
6. Prove that $\mathsf{Dom}(A^n)$ is a core for A for all $n \in \mathbb{N}^*$.