# Chapter 4

# Compact operators, compact resolvents

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## 4.1 Compact operators

### 4.1.1 Definition and properties

Let $\mathsf{E}$ and $\mathsf{F}$ be two Banach spaces.

**Definition 4.1.** *Let $A$ be a linear map from $\mathsf{E}$ to $\mathsf{F}$. We say that $A$ is compact if one of the following equivalent assertions is satisfied.*

(i) *For any bounded sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathsf{E}$, the sequence $(A\varphi_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathsf{F}$.*

(ii) *$\overline{A(B_{\mathsf{E}})}$ is compact in $\mathsf{F}$ (we have denoted by $B_{\mathsf{E}}$ the unit ball in $\mathsf{E}$).*

(iii) *$\overline{A(B)}$ is compact in $\mathsf{F}$ for any bounded subset $B$ of $\mathsf{E}$.*

*We denote by $\mathcal{K}(\mathsf{E},\mathsf{F})$ the set of compact operators from $\mathsf{E}$ to $\mathsf{F}$. We also write $\mathcal{K}(\mathsf{E})$ for $\mathcal{K}(\mathsf{E},\mathsf{E})$.*

For the proof of the equivalences we recall that a subset $\Omega$ of a metric space is compact if and only if any sequence in $\Omega$ has a convergent subsequence in $\Omega$.

*Example* 4.2. Finite rank operators are compact.

*Example* 4.3. The identity operator on $\mathsf{E}$ is compact if and only if $\mathsf{E}$ has finite dimension.

**Proposition 4.4.** *Let $\mathsf{E}$ and $\mathsf{F}$ be two Banach spaces.*

(i) *A compact operator is a bounded operator ($\mathcal{K}(\mathsf{E},\mathsf{F}) \subset \mathcal{L}(\mathsf{E},\mathsf{F})$)*

(ii) *$\mathcal{K}(\mathsf{E},\mathsf{F})$ is a closed subspace of $\mathcal{L}(\mathsf{E},\mathsf{F})$.*

(iii) *For $A \in \mathcal{K}(\mathsf{E},\mathsf{F})$, $B_1 \in \mathcal{B}(\mathsf{E}_1,\mathsf{E})$ and $B_2 \in \mathcal{B}(\mathsf{F},\mathsf{F}_2)$ we have $A \circ B_1 \in \mathcal{K}(\mathsf{E}_1,\mathsf{F})$ and $B_2 \circ A \in \mathcal{K}(\mathsf{E},\mathsf{F}_2)$.*

(iv) *For $A \in \mathcal{K}(\mathsf{E},\mathsf{F})$ we have $A^* \in \mathcal{K}(\mathsf{F}^*,\mathsf{E}^*)$.*

*Proof.* • Let $A \in \mathcal{K}(\mathsf{E},\mathsf{F})$ and assume by contradiction that $A$ is not bounded. Then there exists a sequence $(\varphi_n)$ in $\mathsf{E}$ such that $\|\varphi_n\|_{\mathsf{E}} = 1$ for all $n$ and $\|A\varphi_n\|_{\mathsf{F}} \to \infty$ as $n \to \infty$. Then $(A\varphi_n)$ cannot have a convergent subsequence in $\mathsf{F}$, which gives a contradiction.
• The fact that $\mathcal{K}(\mathsf{E},\mathsf{F})$ is a subspace of $\mathcal{L}(\mathsf{E},\mathsf{F})$ is clear. Let $(A_n)$ be a sequence in $\mathcal{K}(\mathsf{E},\mathsf{F})$ which converges to some $A$ in $\mathcal{L}(\mathsf{E},\mathsf{F})$. Let $(\varphi_n)$ be a bounded sequence in $\mathsf{E}$. Let $M > 0$ such that $\|\varphi_n\| \leqslant M$ for all $n \in \mathbb{N}$. There exists a subsequence $(\varphi_{n(1,k)})_{k\in\mathbb{N}}$ such that

$(A_1 \varphi_{n(1,k)})$ is convergent in $\mathsf{F}$. From this subsequence we can extract a subsequence $(\varphi_{n(2,k)})$ such that $(A_2 \varphi_{n(2,k)})$ is convergent (and $(A_1 \varphi_{n(2,k)})$ is also convergent). By induction on $m$, we construct a subsequence $(\varphi_{n(m,k)})$ of $(\varphi_{n(m-1,k)})$ such that $(A_m \varphi_{n(m,k)})_{k \in \mathbb{N}}$ is convergent. Then by the Cantor diagonal argument, if we set $n_k = n(k,k)$ for all $k \in \mathbb{N}$, then the sequence $(A_j \varphi_{n_k})_{k \in \mathbb{N}}$ is convergent for all $j \in \mathbb{N}$.

Let $\varepsilon > 0$. Let $j \in \mathbb{N}$ such that $\|A_j - A\|_{\mathcal{L}(\mathsf{E},\mathsf{F})} \leqslant \frac{\varepsilon}{3M}$. Let $N \in \mathbb{N}$ such that $\left\|A_j(\varphi_{n_{k_1}} - \varphi_{n_{k_2}})\right\|_{\mathsf{F}} \leqslant \frac{\varepsilon}{3}$ for all $k_1, k_2 \geqslant N$. Then for $k_1, k_2 \geqslant N$ we have

$$\left\|A\varphi_{n_{k_1}} - A\varphi_{n_{k_2}}\right\|_{\mathsf{F}} \leqslant \left\|(A - A_j)\varphi_{n_{k_1}}\right\|_{\mathsf{F}} + \left\|A_j(\varphi_{n_{k_1}} - \varphi_{n_{k_2}})\right\|_{\mathsf{F}} + \left\|(A_j - A)\varphi_{n_{k_2}}\right\|_{\mathsf{F}} \leqslant \varepsilon.$$

This proves that $(A\varphi_{n(k)})$ is a Cauchy sequence, and hence convergent in $\mathsf{F}$.

- The third statement is left as an exercice.
- Let $(\varphi_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\mathsf{F}^*$. Since $A$ is compact, $\overline{A(B_{\mathsf{E}})}$ is a compact metric space, and the functions $\varphi_n$, $n \in \mathbb{N}$, are equicontinuous thereon. Then, by the Ascoli-Arzelà Theorem, there exists a subsequence $(\varphi_{n_k})_{k \in \mathbb{N}}$ convergent in $C^0(\overline{A(B_{\mathsf{E}})})$. We denote by $\varphi \in C^0(\overline{A(B_{\mathsf{E}})})$ the limit. In particular we have

$$\sup_{\|x\|_{\mathsf{E}} \leqslant 1} |\varphi_{n_k}(A(x)) - \varphi(A(x))| \xrightarrow[k \to +\infty]{} 0.$$

We deduce that $(\varphi_{n_k} \circ A) = (A^* \circ \varphi_{n_k})$ is a Cauchy sequence in $\mathsf{E}^*$. Since $\mathsf{E}^*$ is a Banach space, it has a limit in $\mathsf{E}^*$. This proves that $A^* \in \mathcal{K}(\mathsf{F}^*, \mathsf{E}^*)$. ☐

*Example* 4.5. Let $a = (a_n)_{n \in \mathbb{N}}$ be a sequence which converges to 0. We consider on $\ell^2(\mathbb{N})$ the multiplication operator $M_a$ by $a$ (see Example 1.5). Then $M_a$ is compact on $\ell^2(\mathbb{N})$. Indeed, for $N \in \mathbb{N}$ we denote by $a_N$ the sequence defined by

$$\alpha_N = \begin{cases} a_n & \text{if } n \leqslant N, \\ 0 & \text{if } n > N. \end{cases}$$

Then the multiplication $M_{\alpha_N}$ by $\alpha_N$ is of finite rank, hence compact, for all $N \in \mathbb{N}$. Moreover

$$\|M_a - M_{\alpha_N}\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \leqslant \sup_{n > N} |a_n| \xrightarrow[N \to \infty]{} 0.$$

Since $\mathcal{K}(\ell^2(\mathbb{N}))$ is closed, this proves that $M_a$ is compact.

**Proposition 4.6.** *Let $A \in \mathcal{K}(\mathsf{E},\mathsf{F})$ and let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence in $\mathsf{E}$ which converges weakly to some $\varphi \in \mathsf{E}$ (i.e. for any $\ell \in \mathsf{E}^*$ we have $\ell(\varphi_n) \to \ell(\varphi)$). Then $A\varphi_n$ converges (in norm) to $A\varphi$.*

*Proof.* Assume by contradiction that $A\varphi_n$ does not converges to $A\varphi$. There exists $\varepsilon > 0$ and a subsequence $\varphi_{n_k}$ such that $\|A\varphi_{n_k} - A\varphi\|_{\mathsf{F}} \geqslant \varepsilon$ for all $k$. The sequence $(\varphi_k)$ has a weak limit so it is bounded (see Proposition 3.5.(iii) in [Brézis]). Since $A$ is compact, after extracting another subsequence if necessary, we can assume that $(A\varphi_{n_k})$ has a limit $w$ in $\mathsf{F}$. Since $A\varphi_{n_k}$ goes weakly to $A\varphi$ (if $\ell \in \mathsf{F}'$ then $\ell \circ A \in \mathsf{E}'$), this implies that $w = A\varphi$ and gives a contradiction. ☐

**Proposition 4.7.** *Let $\mathcal{H}$ be a separable Hilbert space. Then any compact operator $A$ is the limit in $\mathcal{L}(\mathcal{H})$ of a sequence of operators of finite ranks.*

*Proof.* Let $(\varphi_n)_{n \in \mathbb{N}}$ be a Hilbert basis of $\mathcal{H}$. For $n \in \mathbb{N}$ we set $\mathsf{F}_n = \mathsf{span}(\varphi_0, \dots, \varphi_n)$ and we denote by $\Pi_n$ the orthogonal projection on $\mathsf{F}_n$. Then we set $A_n = A\Pi_n$. Assume by contradiction that

$$\rho = \liminf \|A - A_n\| > 0.$$

Then for all $n \in \mathbb{N}$ large enough (in fact for all $n$ since the sequence $(\|A - A_n\|)$ is non-increasing) there exists $\psi_n \in \mathsf{F}_n^\perp$ such that $\|\psi_n\| = 1$ and $\|A\psi_n\| = \|(A - A_n)\psi_n\| \geqslant \frac{\rho}{2}$. For $\psi \in \mathcal{H}$ we have

$$|\langle \psi, \psi_n \rangle| \leqslant \|(1 - \Pi_n)\psi\| \leqslant \left( \sum_{k=n+1}^{\infty} |\langle \varphi_k, \psi \rangle|^2 \right)^{\frac{1}{2}} \xrightarrow[n \to \infty]{} 0.$$

This proves that the sequence $(\varphi_n)$ goes weakly to 0. This gives a contradiction with Proposition 4.6 since $(A\varphi_n)$ does not go to 0. ☐

### 4.1.2 Examples of compact operators and compact embeddings

We finish this paragraph with more examples of compact operators. Here we discuss the sets of regular functions.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ and $k \in \mathbb{N}$. We recall that $C^k(\overline{\Omega})$ is the set of restrictions to $\Omega$ of functions in $C^k(\mathbb{R}^d)$. It is endowed with the norm defined by

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|_{L^\infty(\Omega)} .$$

**Proposition 4.8.** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ and $k \in \mathbb{N}$. Then $C^{k+1}(\overline{\Omega})$ is compactly embedded in $C^k(\overline{\Omega})$.*

*Proof.* Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C^{k+1}(\overline{\Omega})$. Let $M$ be such that $\|u_n\|_{C^{k+1}(\overline{\Omega})} \leqslant M$.

Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. Since $\|\nabla\partial^\alpha u_n\|_{L^\infty(\Omega)}$ is uniformly bounded, the sequence $(\partial^\alpha u_n)$ is uniformly Lipschitz (in particular equicontinuous) on $\Omega$. By the Ascoli-Arzelà Theorem, it has a subsequence which converges uniformly to some $v_\alpha$ in $C^0(\overline{\Omega})$. Then there exists an increasing sequence $(n_k)$ such that $\partial^\alpha u_{n_k}$ goes to $v_\alpha$ when $n \to \infty$ for all $|\alpha| \leqslant k$.

Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$ and $j \in [\![1, d]\!]$. Let $x \in \Omega$. For $t \in \mathbb{R}$ small enough we have

$$\begin{aligned} v_\alpha(x + te_j) - v_\alpha(x) &= \lim_{k\to+\infty} \partial^\alpha u_{n_k}(x + te_j) - \partial^\alpha u_{n_k}(x) \\ &= \lim_{k\to+\infty} \int_0^t \partial^{\alpha+e_j} u_{n_k}(x + se_j)\,\mathrm{d}s. \end{aligned}$$

Since the map $s \mapsto \partial^{\alpha+e_j} u_{n_k}(x + se_j)$ converges uniformly to $s \mapsto v_{\alpha+e_j}(x + se_j)$ on $[0, t]$ we get

$$v_\alpha(x + te_j) - v_\alpha(x) = \int_0^t v_{\alpha+e_j}(x + se_j)\,\mathrm{d}s.$$

This proves that $\partial_j v_\alpha = v_{\alpha+e_j}$. Finally for all $|\alpha| \leqslant k$ we have $\partial^\alpha v_0 = v_\alpha$, so

$$\|u_{n_k} - v_0\|_{C^k(\overline{\Omega})} \xrightarrow[k\to+\infty]{} 0.$$

□

Ex. 4.2

*Example* 4.9. Let $K \in C^0([0, 1]^2)$. For $u \in C^0([0, 1])$ and $x \in [0, 1]$ we set

$$(Au)(x) = \int_0^1 K(x, y)u(u)\,\mathrm{d}y.$$

Let $M > 0$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C^0([0, 1])$ such that $\|u_n\|_\infty \leqslant M$ for all $n \in \mathbb{N}$. Let $x \in [0, 1]$ and $\varepsilon > 0$. Since $K$ is uniformly continuous there exists $\delta > 0$ such that for all $(x_1, y_1), (x_2, y_2) \in [0, 1]^2$ we have

$$|x_1 - x_2| + |y_1 - y_2| \leqslant \delta \quad \Longrightarrow \quad |K(x_1, y_1) - K(x_2, y_2)| \leqslant \frac{\varepsilon}{M}.$$

Then for $n \in \mathbb{N}$ and $x' \in [0, 1]$ such that $|x - x'| \leqslant \delta$ we have

$$\left|(Au_n)(x) - (Au_n)(x')\right| \leqslant \int_0^1 \left|K(x, y) - K(x', y)\right| |u_n(y)|\,\mathrm{d}y \leqslant \varepsilon.$$

This proves that the family $(Au_n)_{n\in\mathbb{N}}$ is equicontinuous on $[0, 1]$. By the Ascoli-Arzelà Theorem it has a convergent subsequence in $C^0([0, 1])$, which proves that $A$ is compact on $C^0([0, 1])$.

It is not the purpose of this course to study Sobolev spaces in details. Here we are going to use the following result.

**Theorem 4.10** (Rellich)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$.*

(i) *$H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ ;*

(ii) *if $\Omega$ is of class $C^1$ then $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Ex. 4.3

### 4.1.3 Fredholm alternative

Let $\mathsf{E}$ and $\mathsf{F}$ be two Banach spaces. Let $\mathcal{H}$ be a Hilbert space.

We recall that if $G$ is a subspace of $F$ then the codimension $\operatorname{codim}(G)$ of $G$ (in $\mathsf{F}$) is the dimension of the quotient $\mathsf{F}/G$. It is the dimension of any subspace $\tilde{G}$ of $\mathsf{F}$ such that $F = G \oplus \tilde{G}$.

**Definition 4.11.** *A bounded operator $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$ is said to be Fredholm if $\mathsf{dim}(\ker(A)) < +\infty$, $\mathsf{Ran}(A)$ is closed in $\mathsf{F}$ and $\operatorname{codim}(\mathsf{Ran}(A)) < +\infty$. In this case, we define the index of $A$ by*

$$\mathsf{ind}(A) = \mathsf{dim}(\ker(A)) - \operatorname{codim}(\mathsf{Ran}(A)) \quad \in \mathbb{Z}.$$

*We denote by $\mathsf{Fred}(\mathsf{E},\mathsf{F})$ the set of Fredholm operators from $\mathsf{E}$ to $\mathsf{F}$.*

*Remark* 4.12. In fact it is not necessary to assume that $\mathsf{Ran}(A)$ since it can be deduced from the other assumptions.

*Remark* 4.13. If $\mathsf{F}$ is a Hilbert space then $\operatorname{codim}(\mathsf{Ran}(A)) = \mathsf{Ran}(A)^\perp$.

*Example* 4.14. A bijective bounded operator is Fredholm of index 0.

*Example* 4.15. If $\mathsf{E}$ and $\mathsf{F}$ have finite dimensions then any $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$ is Fredholm with index $\mathsf{ind}(A) = \mathsf{dim}(\mathsf{E}) - \mathsf{dim}(\mathsf{F})$.

*Example* 4.16. We consider the shift operators of Example 1.4. Then $S_r$ is Fredholm of index -1 and $S_\ell$ is Fredholm of index 1.

**Proposition 4.17.** *Let $A \in \mathcal{L}(\mathcal{H})$. Assume that $\ker(A)$ and $\ker(A^*)$ have finite dimensions and that $\mathsf{Ran}(A)$ is closed. Then $A$ is a Fredholm operator.*

*Proof.* Since $\mathsf{Ran}(A)$ is closed we have by Proposition 1.33

$$\operatorname{codim}(\mathsf{Ran}(A)) = \mathsf{dim}(\mathsf{Ran}(A)^\perp) = \mathsf{dim}(\ker(A^*)) < +\infty.$$

This proves that $A$ is Fredholm. □

**Proposition 4.18.** *Let $A \in \mathcal{L}(\mathcal{H})$ be a compact operator. Then $\operatorname{Id} - A \in \mathsf{Fred}(\mathcal{H})$ and $\mathsf{ind}(\operatorname{Id} - A) = 0$. In particular, $(\operatorname{Id} - A)$ is invertible if and only if it is injective.*

*Proof.* • Since the restriction of $A$ to $\ker(\operatorname{Id} - A)$ is compact and is equal to $\operatorname{Id}$, $\ker(\operatorname{Id} - A)$ has finite dimension.

• Since $A^*$ is also a compact operator, $\ker((\operatorname{Id} - A)^*) = \ker(\operatorname{Id} - A^*)$ is also of finite dimension.

• We prove that $\mathsf{Ran}(\operatorname{Id} - A)$ is closed. Let $\psi_n$ be a sequence in $\mathsf{Ran}(\operatorname{Id} - A)$ which has a limit $\psi$ in $\mathcal{H}$. For $n \in \mathbb{N}$ there exists $\varphi_n \in \ker(\operatorname{Id} - A)^\perp$ such that $\varphi_n - A\varphi_n = \psi_n$.

Assume by contradiction that $(\varphi_n)$ is not bounded. After extracting a subsequence if necessary, we can assume that $\|\varphi_n\|_{\mathcal{H}} \to +\infty$. For $n \in \mathbb{N}$ large enough we set $\tilde{\varphi}_n = \varphi_n / \|\varphi_n\|$. Then $\tilde{\varphi}_n - A\tilde{\varphi}_n \to 0$. On the other hand the sequence $(\tilde{\varphi}_n)$ is bounded so, after extracting a new subsequence, we can assume that $A\tilde{\varphi}_n$ goes to some $\zeta$ in $\mathcal{H}$. Then $\tilde{\varphi}_n \to \zeta$ and

$$\zeta - A\zeta = \lim_{n\to\infty} \tilde{\varphi}_n - A\tilde{\varphi}_n = 0.$$

This proves that $\zeta \in \ker(\operatorname{Id} - A)$. Since $\tilde{\varphi}_n \in \ker(\operatorname{Id} - A)^\perp$ for all $n$, we have $\zeta = 0$. Thus $\tilde{\varphi}_n \to 0$, which gives a contradiction, so $(\varphi_n)$ is bounded.

After extracting a subsequence if necessary, we can assume that $A\varphi_n$ goes to some $\theta$ in $\mathcal{H}$. Then $\varphi_n \to \psi + \theta$ and

$$\psi = \lim_{n\to\infty} \big(\varphi_n - A\varphi_n\big) = (\psi + \theta) - A(\psi + \theta) \in \mathsf{Ran}(\operatorname{Id} - A).$$

This proves that $\mathsf{Ran}(\operatorname{Id} - A)$ is closed.

• Now assume that $(\operatorname{Id} - A)$ is injective, and assume by contradiction that $\mathcal{H}_1 = (\operatorname{Id} - A)(\mathcal{H})$ is not equal to $\mathcal{H}$. Since $\mathcal{H}_1$ is closed, it is a Hilbert space with the structure inherited from $\mathcal{H}$, and by restriction, $A$ defines a compact operator on $\mathcal{H}_1$. We set $\mathcal{H}_2 = (\operatorname{Id} - A)(\mathcal{H}_1)$. Then $\mathcal{H}_2$ is closed, and since $(\operatorname{Id} - A)$ is injective, we have $\mathcal{H}_2 \subsetneq \mathcal{H}_1$ (take $\varphi \in \mathcal{H} \backslash \mathcal{H}_1$, then $(\operatorname{Id} - A)u$ belongs to $\mathcal{H}_1 \backslash \mathcal{H}_2$). By induction we set $\mathcal{H}_k = (\operatorname{Id} - A)(\mathcal{H}_{k-1})$ for all $k \geqslant 2$. Then $\mathcal{H}_k$ is closed and $\mathcal{H}_{k+1} \subsetneq \mathcal{H}_k$ for all $k \in \mathbb{N}^*$. In particular, for all $k \in \mathbb{N}^*$ we can find $\varphi_k \in \mathcal{H}_k$ such that $\|\varphi_k\|_{\mathcal{H}} = 1$ and $\varphi_k \in \mathcal{H}_{k+1}^\perp$. Then for $k \in \mathbb{N}^*$ and $j > k$ we have

$$A\varphi_j - A\varphi_k = -(\varphi_j - A\varphi_j) + (\varphi_k - A\varphi_k) + \varphi_j - \varphi_k.$$

Since $-(\varphi_j - A\varphi_j) + (\varphi_k - A\varphi_k) + \varphi_j \in \mathcal{H}_{k+1}$ this yields

$$\|A\varphi_j - A\varphi_k\| \geqslant 1.$$

This gives a contradiction since $A$ is compact. Thus, if $(\mathrm{Id} - A)$ is injective, then it is also surjective.

- It remains to prove that $\mathrm{Ker}(\mathrm{Id} - A)$ and $\mathrm{Ker}(\mathrm{Id} - A^*)$ have the same dimension. Assume by contradiction that $\mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - A)) < \mathsf{dim}(\mathsf{Ran}(\mathrm{Id} - A)^\perp)$. There exists a bounded operator $T : \mathrm{Ker}(\mathrm{Id} - A) \to \mathsf{Ran}(\mathrm{Id} - A)^\perp$ injective but not surjective. We extend $T$ by 0 on $\mathrm{Ker}(\mathrm{Id} - A)^\perp$. This defines an operator $T$ on $\mathcal{H}$ which has a finite dimensional range included in $\mathsf{Ran}(\mathrm{Id} - A)^\perp$. In particular it is compact, and so is $\tilde{A} = A + T$. Let $\varphi \in \mathrm{Ker}(\mathrm{Id} - \tilde{A})$. We have $\varphi - A\varphi = T\varphi$. Since $\varphi - A\varphi \in \mathsf{Ran}(\mathrm{Id} - A)$ and $T\varphi \in \mathsf{Ran}(\mathrm{Id} - A)^\perp$, we have $\varphi - A\varphi = T\varphi = 0$. Therefore $\varphi = 0$ since $T$ is injective on $\mathrm{Ker}(\mathrm{Id} - A)$. Then $(\mathrm{Id} - \tilde{A})$ is injective, and hence surjective. However for $\psi \in \mathsf{Ran}(\mathrm{Id} - A)^\perp \backslash \mathsf{Ran}(T)$ the equation

$$\varphi - (A\varphi + T\varphi) = \psi$$

cannot have a solution. This gives a contradiction and proves that

$$\mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - A)) \geqslant \mathsf{dim}(\mathsf{Ran}(\mathrm{Id} - A)^\perp) = \mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - A^*)).$$

We get the opposite inequality by interchanging the roles of $A$ and $A^*$, and the proof is complete. ☐

## 4.2 Spectrum of compact operators

### 4.2.1 General properties

**Theorem 4.19.** *Let $\mathcal{H}$ be a separable Hilbert space of infinite dimension. Let $A$ be a compact operator on $\mathcal{H}$. Then $\sigma_{\mathsf{ess}}(A) = \{0\}$.*

*Remark* 4.20.
- 0 always belongs to the spectrum of $A$. With examples of the form given in Example 1.5 (see Example 4.5), we see that 0 is not necessarily an eigenvalue, it can be an eigenvalue of infinite multiplicity or an eigenvalue of finite multiplicity.
- A non-zero element of the spectrum is necessarily an isolated eigenvalue of finite algebraic multiplicity. The non-zero spectrum if finite or is given by a sequence going to 0.

*Proof.* • Assume that 0 belongs to the resolvent set of $A$. Then Id is the composition of the compact operator $A$ with the bounded operator $A^{-1}$, so Id is a compact operator, which gives a contradiction since $\mathsf{dim}(\mathcal{H}) = +\infty$.

- Let $\lambda \in \mathbb{C} \backslash \{0\}$. Then we have $A - \lambda = \lambda(\lambda^{-1}A - \mathrm{Id})$. Since $\lambda^{-1}A$ is compact, Proposition 4.18 shows that $(A - \lambda)$ is invertible if and only if it is injective, so $\lambda \in \sigma(A)$ if and only if it is an eigenvalue. Moreover, in this case we have $\mathsf{dim}(\mathrm{Ker}(A - \lambda)) = \mathsf{dim}(\mathrm{Ker}(\lambda^{-1}A - \mathrm{Id})) < +\infty$.
- Since $A$ is a bounded operator, the set of eigenvalues of $A$ is bounded in $\mathbb{C}$. Assume that $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of distinct non-zero eigenvalues of $A$ converging to some $\lambda \in \mathbb{C}$. We prove that $\lambda = 0$. For $n \in \mathbb{N}$ we consider $w_n \in \ker(A - \lambda_n) \backslash \{0\}$. Then for $n \in \mathbb{N}$ we set $\mathcal{H}_n = \mathsf{span}(w_0, \dots, w_{n-1})$ and we consider $u_n \in \mathcal{H}_n$ such that $\|u_n\| = 1$ and $u_n \in \mathcal{H}_{n-1}^\perp$ if $n \geqslant 1$. Then for $j \in \mathbb{N}$ and $k > j$ we have

$$\left\| \frac{Au_k}{\lambda_k} - \frac{Au_j}{\lambda_j} \right\|_{\mathcal{H}} = \left\| \frac{Au_k - \lambda_k u_k}{\lambda_k} - \frac{Au_j - \lambda_j u_j}{\lambda_j} + u_k - u_j \right\|_{\mathcal{H}} \geqslant 1,$$

since $Au_k - \lambda_k u_k, Au_j - \lambda_j u_j, u_j \in \mathcal{H}_{k-1}$. If $\lambda \neq 0$ we obtain a contradiction with the compactness of $A$.

- Assume that $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $A$. Let $r > 0$ such that $D(\lambda, 2r) \setminus \{\lambda\} \subset \rho(A)$. Let

$$M = 1 + \sup_{|z-\lambda|=r} \left\| (A-z)^{-1} \right\|.$$

By Proposition 4.7 there exists a finite rank operator $T$ such that $\|A - T\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{1}{2M^2}$. Then for $z \in \mathcal{C}(\lambda, r)$ we have

$$T - z = (A - z)\big(1 - (A-z)^{-1}(A-T)\big),$$

so $z \in \rho(F)$ and

$$\begin{aligned} \left\| (A-z)^{-1} - (T-z)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} &\leqslant \sum_{j=1}^{\infty} \left\| \big((A-z)^{-(j+1)}(A-T)\big)^j \right\| \leqslant M \sum_{j=1}^{\infty} (2M)^{-j} \\ &\leqslant \frac{2M}{4M-2} < 1. \end{aligned}$$

We set

$$\Pi_A = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} (A-\zeta)^{-1} \, \mathrm{d}\zeta \quad \text{and} \quad \Pi_T = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} (T-\zeta)^{-1} \, \mathrm{d}\zeta.$$

Then we have

$$\|\Pi_A(\lambda) - \Pi_F(\lambda)\| < 1.$$

This implies that

$$\ker(\Pi_F) \cap \mathsf{Ran}(\Pi_A) = \ker(\Pi_R) \cap \ker(\mathrm{Id} - \Pi_A) = \{0\},$$

so the retriction of $\Pi_F$ to $\mathsf{Ran}(\Pi_A)$ defines an injective map from $\mathsf{Ran}(\Pi_A)$ to $\mathsf{Ran}(\Pi_F)$.

On the other hand, by Proposition 2.66 we have $\mathsf{Ran}(\Pi_F) \cap \ker(F) = \{0\}$, so $F$ defines by restriction an injective map on $\mathsf{Ran}(\Pi_F)$ and hence $\Pi_F$ has finite rank.

Finally, $\Pi_A$ has finite rank and $\lambda$ has finite algebraic multiplicity, so $\lambda \in \sigma_{\mathsf{disc}}(A)$. □

### 4.2.2 Spectral theorem for compact normal operators

**Theorem 4.21.** *Assume that* $\dim(\mathcal{H}) = \infty$. *Let $A$ be a compact and normal operator on $\mathcal{H}$. Let $(\lambda_k)_{1 \leqslant k \leqslant N, k \in \mathbb{N}^*}$ with $N \in \mathbb{N} \cup \{\infty\}$ be the sequence of non-zero eigenvalues of $A$. We set $\lambda_0 = 0$. Then we have*

$$\mathcal{H} = \overline{\bigoplus_{k=0}^{N} \ker(A - \lambda_k)}$$

*and*

$$A = \sum_{k=1}^{N} \lambda_k \Pi_k,$$

*where $\Pi_k$ is the orthogonal projection on* $\ker(A - \lambda_k)$. *If moreover $\mathcal{H}$ is separable, then there exists a Hilbert basis of eigenvectors of $A$.*

Notice that the sum for $A$ is convergent in $\mathcal{L}(\mathcal{H})$ if $N = \infty$. Indeed, we set $A_n = \sum_{k=1}^{n} \lambda_k \Pi_k$ then

$$\|A - A_n\| = r(A - A_n) = \sup_{k>n} |\lambda_k| \xrightarrow[n\to\infty]{} 0.$$

In particular the sum does not depend on the order of summation.

*Proof.* We set $F = \overline{\bigoplus_{k=1}^{N} \ker(A - \lambda_k)}$. By Proposition 1.39, we have $F = \overline{\bigoplus_{k=1}^{N} \ker(A^* - \overline{\lambda_k})}$. We have $A^*(F) \subset F$, so $A(F^\perp) \subset F^\perp$. The restriction $A_0$ of $A$ to $F^\perp$ is a compact normal operator without non-zero eigenvalues, so $A_0 = 0$. Thus $F^\perp \subset \ker(A)$. Since $\ker(A) \subset \mathsf{F}^\perp$ by Proposition 1.39, we have $F^\perp = \ker(A)$ and the conclusion follows. □

## 4.3 Operators with compact resolvents

**Definition 4.22.** *Let $A$ be an operator on $\mathsf{E}$. We say that $A$ has compact resolvent if $\rho(A) \neq \varnothing$ and for some (hence any) $z \in \rho(A)$ the resolvent $(A - z)^{-1}$ is a compact operator on $\mathsf{E}$.*

We have to check that the compactness of $(A - z)^{-1}$ does not depend on $z \in \rho(A)$.

*Proof.* Assume that there exists $z_0 \in \rho(A)$ such that $(A - z_0)^{-1}$ is compact. Let $z \in \rho(A)$. By the resolvent identity we have

$$(A - z)^{-1} = (A - z_0)^{-1} - (z - z_0)(A - z_0)^{-1}(A - z)^{-1}.$$

Both terms of the right-hand side are compact, so $(A - z)^{-1}$ is compact. □

*Example* 4.23. Let $\Omega$ be an open bounded subset of $\mathbb{R}^d$ of class $C^2$. Then the Dirichlet Laplacian on $\Omega$ ($A = -\Delta$, $\mathsf{Dom}(A) = H^2(\Omega) \cap H^1_0(\Omega)$) has compact resolvent. Indeed, it is a selfadjoint operator so its resolvent set is not empty. Then for $z \in \rho(A)$ the resolvent $(A - z)^{-1}$ defines a bounded operator from $L^2(\Omega)$ to $H^2(\Omega)$. Since $H^2(\Omega)$ is compactly embedded in $L^2(\Omega)$, then $(A - z)^{-1}$ is a compact operator on $L^2(\Omega)$.

*Example* 4.24. We can prove that the domain of the harmonic oscillator on $\mathbb{R}$ (see (2.7)-(2.8)) is given by

$$\mathsf{Dom}(H) = \left\{ u \in H^2(\mathbb{R}) \,:\, x^2 u \in L^2 \right\}. \tag{4.1}$$

Note that it is not clear that this is equal to (2.8). From this we can deduce that $\mathsf{Dom}(H)$ is compactly embedded in $L^2(\mathbb{R})$ (see Exercise 4.4) and hence that $H$ has a compact resolvent.

Ex. 4.4

If $A$ has compact resolvent, we can deduce good spectral properties from the good spectral properties of its resolvent.

**Proposition 4.25.** *Let $A$ be a closed operator with non-empty resolvent set. Let $z_0 \in \rho(A)$. Let $R = (A - z_0)^{-1} \in \mathcal{L}(\mathsf{E})$. Let $z \in \mathbb{C} \setminus \{0\}$. Then $z$ belongs to $\sigma(R)$ ($\sigma_{\mathsf{disc}}(R)$, $\sigma_{\mathsf{ess}}(R)$, respectively) if and only if $z_0 + \frac{1}{z}$ belongs to $\sigma(A)$ ($\sigma_{\mathsf{disc}}(A)$, $\sigma_{\mathsf{ess}}(A)$, respectively).*

*Proof.* • It is clear that the map $z \mapsto z - z_0$ is a bijection between $\sigma(A)$ and $\sigma(A - z_0)$ which preserves the discrete and essential parts of the spectrum. Thus we can assume without loss of generality that $z_0 = 0$.
• We have

$$A^{-1} - z^{-1} = -z^{-1}(A - z)A^{-1}.$$

Then $z^{-1} \in \sigma(A^{-1})$ if and only if $(A - z) : \mathsf{Dom}(A) \to \mathsf{E}$ is invertible, hence if and only if $z \in \sigma(A)$. Moreover, if $z \in \rho(A)$ then

$$(A^{-1} - z^{-1})^{-1} = -zA(A - z)^{-1} = -z - z^2(A - z)^{-1}.$$

• It remains to prove that $\lambda \in \sigma_{\mathsf{disc}}(A)$ if and only if $\lambda^{-1} \in \sigma_{\mathsf{disc}}(A^{-1})$. The map $z \mapsto z^{-1}$ maps isolated points of $\sigma(A)$ to isolated points of $\sigma(A^{-1})$. Let $\lambda$ be an isolated point in $\sigma(A)$. Let $r \in ]0, |\lambda|[$ be such that $D(\lambda, 2r) \cap \sigma(A) = \{\lambda\}$. We have

$$\Pi_\lambda = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} (A - \zeta)^{-1} \, \mathrm{d}\zeta = \frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)} \frac{1}{\zeta^2} (A^{-1} - \zeta^{-1})^{-1} \, \mathrm{d}\zeta = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda,r)^{-1}} (A^{-1} - z) \, \mathrm{d}z,$$

where $\mathcal{C}(\lambda, r)^{-1} = \left\{ \zeta^{-1}, \zeta \in \mathcal{C}(\lambda, r) \right\}$. For $r > 0$ small, $\mathcal{C}(\lambda, r)$ is close to $\mathcal{C}(\lambda^{-1}, r / \left|\lambda^2\right|)$ and is also oriented in the direct sense. Thus the Riesz projections of $\lambda$ for the operator $A$ and of $\lambda^{-1}$ for $A^{-1}$ coincide. In particular, $\lambda \in \sigma_{\mathsf{disc}}(A)$ if and only if $\lambda^{-1} \in \sigma_{\mathsf{disc}}(A^{-1})$. □

**Theorem 4.26.** *Let $A$ be an operator on $\mathcal{H}$ with compact resolvent. Then $\sigma_{\mathsf{ess}}(A) = \varnothing$.*

*Proof.* Let $z_0 \in \rho(A)$. Since $(A - z_0)^{-1}$ is compact, we have $\sigma_{\mathsf{ess}}((A - z_0)^{-1}) \cap \mathbb{C}^* = \varnothing$ by Then by Proposition 4.25, we have $\sigma_{\mathsf{ess}}(A - z_0) \cap \mathbb{C}^* = \varnothing$, which implies that $\sigma_{\mathsf{ess}}(A - z_0)$ and then $\sigma_{\mathsf{ess}}(A)$ are empty. □

*Remark* 4.27. If $\mathsf{dim}(\mathsf{E}) = +\infty$, emptyness of $\sigma_{\mathsf{ess}}(A)$ implies that the spectrum of $A$ consists of a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues of finite multiplicities and such that

$$|\lambda_n| \xrightarrow[n\to\infty]{} +\infty.$$

We rewrite the theorem is the important particular case of a selfadjoint operator.

**Theorem 4.28.** *Let $A$ be a selfadjoint operator with compact resolvent on $\mathcal{H}$. Then the spectrum of $A$ consists of a sequence $(\lambda_k)_{k\in\mathbb{N}^*}$ of eigenvalues with finite multiplicities and such that $|\lambda_k| \to +\infty$, and there is a Hilbert basis of $\mathcal{H}$ made with eigenvectors of $A$. If moreover $A$ is semibounded from below, then $\lambda_k \to +\infty$.*

## 4.4 Relatively compact operators - Weyl's Theorem

**Definition 4.29.** *Let $A$ be a closed operator on $\mathsf{E}$ with non-empty resolvent set. Let $B$ be an operator on $\mathsf{E}$. We say that $B$ is $A$-compact (or relatively compact with respect to $A$) if $\mathsf{Dom}(A) \subset \mathsf{Dom}(B)$ and one of the following equivalent assertions is satisfied.*

(i) *There exists $z_0 \in \rho(A)$ such that $B(A-z_0)^{-1}$ is compact.*

(ii) *For all $z \in \rho(A)$, the operator $B(A-z)^{-1}$ is compact.*

(iii) *For any sequence $(\varphi_n)$ bounded in $\mathsf{Dom}(A)$ (*i.e.* $(\varphi_n)$ and $(A\varphi_n)$ are bounded in $\mathsf{E}$) then $(B\varphi_n)$ has a convergent subsequence.*

*Proof.* • We prove that (iii) implies (ii). Let $z \in \rho(A)$. Let $(\psi_n)$ be a bounded sequence in $\mathsf{E}$. Then $((A-z)^{-1}\psi_n)$ is bounded in $\mathsf{Dom}(A)$, and hence $(B(A-z_0)^{-1}\psi_n)$ has a convergent subsequence in $\mathsf{E}$. This proves that $B(A-z_0)^{-1}$ is compact.
• Conversely, assume that $B(A-z_0)^{-1}$ is compact for some $z_0 \in \rho(A)$ and consider $(\psi_n)$ bounded in $\mathsf{Dom}(A)$. Then $(A-z_0)\psi_n$ is bounded in $\mathsf{E}$. Then $(B\psi_n) = (B(A-z_0)^{-1}(A-z_0)\psi_n)$ has a convergent subsequence in $\mathsf{E}$. This proves (iii). □

**Proposition 4.30.** *Assume that $B$ is closed and $A$-compact. Then it is relatively bounded with $A$-bound 0.*

*Proof.* Assume by contradiction that there exists $\varepsilon > 0$ and a sequence $(\varphi_n)$ in $\mathsf{Dom}(A) \subset \mathsf{Dom}(B)$ such that

$$\forall n \in \mathbb{N}, \quad \|B\varphi_n\| > \varepsilon \|A\varphi_n\| + n\|\varphi_n\|.$$

After extracting a subsequence if necessary, we can assume that $\|A\varphi_n\| > \|\varphi_n\|$ for all $n$, or that $\|A\varphi_n\| \leqslant \|\varphi_n\|$ for all $n$. In the first case we set $\psi_n = \varphi_n / \|A\varphi_n\|$, so that

$$\|B\psi_n\| > \varepsilon + n\|\psi_n\|, \quad \|\psi_n\| \leqslant 1.$$

After extracting a subsequence, $B\psi_n$ has a limit. In particular $(\|B\varphi_n\|)$ is bounded, so $\psi_n \to 0$. Since $B$ is closed, we have $B\psi_n \to 0$, which gives a contradiction. In the second case we similarly get a contradiction by setting $\psi_n = \varphi_n / \|\varphi_n\|$. □

**Lemma 4.31.** *Let $A_0$ and $A_1$ be two operators such that $\rho(A_0) \cap \rho(A_1) \neq \varnothing$. Let $B = A_1 - A_0$. Then $B$ is $A_0$-compact if and only if it is $A_1$-compact.*

*Proof.* Let $z_0 \in \rho(A_0) \cap \rho(A_1)$. Assume that $B$ is $A_0$-compact. We have

$$(A_1 - z_0)^{-1} = (A_0 - z_0)^{-1} - (A_1 - z_0)^{-1} B (A_0 - z_0)^{-1}$$

so

$$(A_1 - z_0)^{-1}\big(1 + B(A_0 - z_0)^{-1}\big) = (A_0 - z_0)^{-1}.$$

Let $\varphi \in \mathsf{E}$ such that $\varphi + B(A_0 - z_0)^{-1}\varphi = 0$. Then $\psi = (A_0 - z_0)^{-1}\varphi$ satisfies

$$(A_1 - z_0)\psi = (A_0 - z_0)\psi + B\psi = 0.$$

This implies that $\psi = 0$ and then $\varphi = 0$, so $1 + B(A_0 - z_0)^{-1}$ is injective. Since $B(A_0 - z_0)^{-1}$ is compact, we deduce by the Fredholm alternative that $1 + B(A_0 - z_0)^{-1}$ is invertible. Then

$$B(A_1 - z_0)^{-1} = B(A_0 - z_0)^{-1}\big(1 + B(A_0 - z_0)^{-1}\big)^{-1}$$

is the composition of a compact and a bounded opertor, so it is compact. This proves that $B$ is $A_1$-compact. We prove the converse by changing the roles of $A_0$ and $A_1$. □

**Theorem 4.32** (Weyl's Theorem for selfadjoint operators)**.** *Let $A_0$ and $A_1$ be two selfadjoint operators. Let $B = A_1 - A_0$ and assume that $B$ is $A_0$-compact. Then*

$$\sigma_{\mathsf{ess}}(A_1) = \sigma_{\mathsf{ess}}(A_0).$$

*Proof.* Let $\lambda \in \sigma_{\mathsf{ess}}(A_0)$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathsf{Dom}(A_0)$ such that $\|\varphi_n\| = 1$ for all $n \in \mathbb{N}$, $\varphi_n$ goes weakly to 0 and $\|(A_0 - \lambda)\varphi_n\| \to 0$ as $n \to \infty$ (see Proposition 3.50). Then

$$(A_0 - i)\varphi_n = (A_0 - \lambda)\varphi_n + (\lambda - i)\varphi_n \rightharpoonup 0.$$

We have

$$(A_1 - \lambda)\varphi_n = (A_0 - \lambda)\varphi_n + B(A_0 - i)^{-1}(A_0 - i)\varphi_n.$$

Since $(A_0 - i)\varphi_n$ goes weakly to 0 and $B(A_0 - i)^{-1}$ is compact, the second term in the right-hand side goes strongly to 0 by Proposition 4.6. Then $(A_1 - \lambda)\varphi_n$ goes to 0 and $\lambda \in \sigma_{\mathsf{ess}}(A_1)$ by Proposition 3.50. This proves that $\sigma_{\mathsf{ess}}(A_0) \subset \sigma_{\mathsf{ess}}(A_1)$. Since $B$ is also $A_1$-compact by Proposition 4.31, we can prove the reverse inclusion by changing the roles of $A_0$ and $A_1$. □

*Example* 4.33. Let $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ such that $V(x) \to 0$ as $|x| \to 0$. We set $H_0 = -\Delta$ and $H_1 = -\Delta + V$, with $\mathsf{Dom}(H_0) = \mathsf{Dom}(H_1) = H^2(\mathbb{R}^d)$. Then we have

$$\sigma_{\mathsf{ess}}(H_1) = \sigma_{\mathsf{ess}}(H_0) = [0, +\infty[.$$

For this we prove that the multiplication by $V$ is $H_0$-compact.

With our definition of the essential spectrum, Theorem 4.32 is not true in general.

*Example* 4.34. We consider on $\ell^2(\mathbb{Z})$ the operators $A$ and $B$ defined by

$$A(\dots, u_{-2}, u_{-1}, u_0, u_1, u_2, \dots) = (\dots, u_{-1}, u_0, u_1, u_2, u_3, \dots)$$

and

$$B(\dots, u_{-2}, u_{-1}, u_0, u_1, u_2, \dots) = (\dots, 0, -u_0, 0, 0, 0, \dots),$$

so that

$$(A + B)(\dots, u_{-2}, u_{-1}, u_0, u_1, u_2, \dots) = (\dots, u_{-1}, 0, u_1, u_2, u_3, \dots)$$

The spectrum of $A$ is the unit circle $\mathcal{C}(0,1)$ (see Exercise 1.5) and $B$ is compact (it is of rank 1). On the other hand, as for the shift on the left in $\ell^2(\mathbb{N})$ (see Example 1.36), we can check that $\sigma(A + B) = \overline{D}(0,1)$.

However, we can prove the following result.

**Theorem 4.35.** *Let $A$ be a closed operator. Let $B$ be a $A$-compact operator. Let $\mathcal{U}$ be a connected component of $\mathbb{C}\backslash\sigma_{\mathsf{ess}}(A)$. Then we have*

$$\mathcal{U} \subset \mathbb{C}\backslash\sigma_{\mathsf{ess}}(A + B) \quad \text{or} \quad \mathcal{U} \subset \sigma_{\mathsf{ess}}(A + B).$$

*In particular, if $\mathcal{U} \cap \rho(A + B) \neq \varnothing$ then $\mathcal{U} \cap \sigma_{\mathsf{ess}}(A + B) = \varnothing$.*

## 4.5 Exercises

***Exercise* 4.1.** Let $(\alpha_n)$ be a sequence in $\mathbb{R}_+^*$ such that $\alpha_n \to +\infty$ as $n \to +\infty$. We set

$$\mathcal{V} = \left\{ (u_n)_{n\in\mathbb{N}} \, : \, \sum_{n\in\mathbb{N}} \alpha_n \left| u_n \right|^2 < +\infty \right\} \subset \ell^2(\mathbb{N}).$$

$\mathcal{V}$ is a Hilbert space for the inner product defined by

$$\langle u, v \rangle_{\mathcal{V}} = \sum_{n\in\mathbb{N}} \alpha_n u_n \overline{v_n}, \quad u = (u_n), v = (v_n).$$

Prove that $\mathcal{V}$ is compactly embedded in $\ell^2(\mathbb{N})$.

***Exercise* 4.2.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$. Let $k \in \mathbb{N}$ and $\theta \in ]0,1[$. We recall that $C^{k,\theta}(\Omega)$ is the set of functions $u \in C^k(\overline{\Omega})$ whose derivatives of order $k$ are Hölder-continuous of exponent $\theta$. It is endowed with the norm defined by

$$\|u\|_{C^{k,\theta}(\Omega)} = \sum_{\alpha \leqslant k} \|\partial^\alpha u\|_{L^\infty(\Omega)} + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\theta}.$$

Prove that $C^{k,\theta}(\Omega)$ is compactly embedded in $C_b^k(\Omega)$.

***Exercise* 4.3.** Let $V \in L^\infty(\mathbb{R})$. We assume that $V(x) \xrightarrow[|x|\to+\infty]{} 0$. Prove that the map

$$\left\{ \begin{array}{ccc} H^1(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ u & \mapsto & Vu \end{array} \right.$$

is compact.

***Exercise* 4.4.** **1.** Give an exemple of sequence $(u_n)$ bounded in $H^2(\mathbb{R})$ which has no limit in $L^2(\mathbb{R})$.
**2.** We consider a sequence $(u_n)$ in $H^2(\mathbb{R})$ such that $x^2 u_n$ belongs to $L^2(\mathbb{R})$ for all $n \in \mathbb{N}$. We assume that there exists $M \geqslant 0$ such that

$$\forall n \in \mathbb{N}, \quad \|u_n\|_{H^2(\mathbb{R})} + \left\|x^2 u_n\right\|_{L^2(\mathbb{R})} \leqslant M.$$

**3.** Prove that we can construct for all $m \in \mathbb{N}^*$ an extraction $(n_k(m))$ and $v_m \in L^2([-m,m])$ such that

- $\left\|u_{n_k(m)} - v_m\right\|_{L^2([-m,m])} \to 0$,
- $v_m$ and $v_\nu$ coincide on $[-m,m]$ whenever $\nu \geqslant m$.

**4.** Prove that there exists a subsequence $(u_{n_j})$ and $v \in L^2_{\text{loc}}(\mathbb{R})$ such that $\left\|u_{n_j} - v\right\|_{L^2([-R,R])} \to 0$ for all $R > 0$.
**5.** Prove that $u_{n_j}$ goes to $v$ in $L^2(\mathbb{R})$.