Chapter 1

Spectrum of bounded operators

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In this chapter we introduce the basic notions of spectral theory for bounded operators. We do not too far in the general properties since many aspects will be common with the theory for unbounded operators, discussed in the next chapter.

1.1 Bounded operators - Examples

Let E and F be two Banach spaces. We denote by $\mathcal{L}(\mathsf{E},\mathsf{F})$ the set of bounded linear maps from E to F, and for $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$ we set

$$\|A\|_{\mathcal{L}(\mathsf{E},\mathsf{F})} = \sup_{\varphi \in \mathsf{E}\setminus\{0\}} \frac{\|A\varphi\|_{\mathsf{F}}}{\|\varphi\|_{\mathsf{E}}}.$$

We write $\mathcal{L}(\mathsf{E})$ for $\mathcal{L}(\mathsf{E},\mathsf{E})$.

Remark 1.1. We recall that a linear map from E to F is continuous if and only if it is bounded.

Remark 1.2. Let G be a third Banach space. For $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$ and $B \in \mathcal{L}(\mathsf{F},\mathsf{G})$ we have

$$\|BA\|_{\mathcal{L}(\mathsf{E},\mathsf{G})} \leqslant \|A\|_{\mathcal{L}(\mathsf{E},\mathsf{F})} \|B\|_{\mathcal{L}(\mathsf{F},\mathsf{G})}.$$

Example 1.3. If E has finite dimension then all the linear maps from E to F are continuous.

Example 1.4. We consider on $\ell^2(\mathbb{N})$ the operators S_r and S_ℓ defined by

$$S_r(u_0, u_1, \dots, u_n, \dots) = (0, u_0, \dots, u_{n-1}, \dots)$$

and

$$S_\ell(u_0, u_1, \dots, u_n, \dots) = (u_1, u_2, \dots, u_{n+1}, \dots).$$

Then S_r and S_ℓ are bounded operators on $\ell^2(\mathbb{N})$ with $\|S_r\|_{\mathcal{L}(\ell^2(\mathbb{N}))} = \|S_\ell\|_{\mathcal{L}(\ell^2(\mathbb{N}))} = 1$.

Example 1.5. Let $a = (a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$. For $u = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ we define $M_a u \in \ell^2(\mathbb{N})$ by

$$\forall n \in \mathbb{N}, \quad (M_a u)_n = a_n u_n.$$

We have $M_a \in \ell^2(\mathbb{N})$ with $\|M_a\|_{\mathcal{L}(\ell^2(\mathbb{N}))} = \sup_{n\in\mathbb{N}} |a_n|$.

Example 1.6. More generally, let (Ω, μ) be a measure space. Let $w \in L^\infty(\Omega)$. We consider on $L^2(\Omega)$ the multiplication operator $M_w : u \mapsto uw$. Then we have $M_w \in \mathcal{L}(L^2(\Omega))$ with

$$\|M_w\|_{\mathcal{L}(L^2(\Omega))} = \|w\|_{L^\infty(\Omega)}.$$

Definition 1.7. *We say that $A \in \mathcal{L}(\mathsf{E},\mathsf{F})$ is invertible if there exists $B \in \mathcal{L}(\mathsf{F},\mathsf{E})$ such that $BA = \mathrm{Id}_{\mathsf{E}}$ and $AB = \mathrm{Id}_{\mathsf{F}}$.*

Example 1.8.
- S_r is not surjective and S_ℓ is not injective, so these two operators are not invertible.

- Given $a = (a_n) \in \ell^\infty(\mathbb{N})$, the operator M_a is invertible if and only if

$$0 \notin \overline{\{a_n, n \in \mathbb{N}\}}.$$

- Given $w \in L^\infty(\Omega, \mu)$, the operator M_w is invertible in $L^2(\Omega, \mu)$ if and only if there exists $\varepsilon > 0$ such that

$$\mu\big(\{x \in \Omega : |w(x)| \leqslant \varepsilon\}\big) = 0. \tag{1.1}$$

  Assume that (1.1) holds. Then w^{-1} is well defined almost everywhere and $\|w^{-1}\|_{L^\infty(\Omega)} \leqslant \frac{1}{\varepsilon}$. Then $M_{w^{-1}} \in \mathcal{L}(L^2(\Omega))$ is an inverse for M_w. Conversely, assume that M_w is invertible. Assume by contradiction that (1.1) does not hold. Then for all $n \in \mathbb{N}^*$ we set

$$A_n = w^{-1}\left(D\left(0, \frac{1}{n}\right)\right) \quad \text{and} \quad u_n = \frac{\mathbb{1}_{A_n}}{\mu(A_n)^{\frac{1}{2}}}.$$

  Then $\|u_n\|_{L^2(\Omega)} = 1$ and

$$\|M_w u_n\|^2_{L^2(\Omega)} = \frac{1}{\mu(A_n)} \int_{A_n} |w(x)|^2 \, \mathrm{d}\mu(x) \leqslant \frac{1}{n^2}.$$

  Then

$$\|u_n\|_{L^2(\Omega)} = \left\|M_w^{-1} M_w u_n\right\|_{L^2(\Omega)} \leqslant \frac{1}{n}\|M_w^{-1}\|_{\mathcal{L}(L^2(\Omega))} \xrightarrow[n\to+\infty]{} 0,$$

  which gives a contradiction.

The following result is a consequence of the open mapping theorem (see for instance [Bre11, Cor. 2.7]).

Proposition 1.9. *Let $A \in \mathcal{L}(\mathsf{E}, \mathsf{F})$. Assume that A is bijective. Then its inverse is necessarily continuous.*

1.2 Spectrum of bounded operators - Resolvent

Let E be a Banach space.

1.2.1 Definition and basic properties

Definition 1.10. *Let $A \in \mathcal{L}(\mathsf{E})$.*

(i) *The resolvent set $\rho(A)$ of A is the set of $z \in \mathbb{C}$ such that $(A - z) = (A - z\,\mathrm{Id}_{\mathsf{E}})$ is invertible.*

(ii) *The spectrum $\sigma(A)$ of A is the complementery set of $\rho(A)$ in $\mathbb{C}$.*

Definition 1.11. *Let $A \in \mathcal{L}(\mathsf{E})$. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of A if $(A - \lambda)$ is not injective. In other words, there exists $u \in \mathsf{E}\setminus\{0\}$ such that $Au = \lambda u$. Such a vector u is called an eigenvector of A for the eigenvalue λ. The geometric multiplicity of λ is the dimension of $\ker(A - \lambda)$. We denote by $\sigma_{\mathsf{p}}(A)$ the set of eigenvalues of A.*

Ex. 1.1 *Remark* 1.12. We have $\sigma_{\mathsf{p}}(A) \subset \sigma(A)$, but the inclusion can be strict.

Example 1.13. We consider the multiplication operator M_w defined in Example 1.6. Let $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of M_w is and only if

$$\mu\left(\{x \in \Omega : w(x) = \lambda\}\right) > 0.$$

On the other hand, since $M_w - \lambda = M_{w-\lambda}$, we see that λ belongs to $\sigma(M_w)$ if and only if for all $\varepsilon > 0$ we have

$$\mu\left(\{x \in \Omega : |w(x) - \lambda| \leqslant \varepsilon\}\right) > 0.$$

Ex. 1.2

Proposition 1.14. *Let $A \in \mathcal{L}(\mathsf{E})$.*

(i) $\sigma(A) \subset D(0, \|A\|_{\mathcal{L}(\mathsf{E})})$.

(ii) *$\rho(A)$ is open. For $z_0 \in \rho(A)$ and $|z - z_0| < \left\|(A - z_0)^{-1}\right\|_{\mathcal{L}(\mathsf{E})}^{-1}$ we have $z \in \rho(A)$ and*

$$(A - z)^{-1} = \sum_{n \in \mathbb{N}} (z - z_0)^n (A - z_0)^{-(n+1)}. \tag{1.2}$$

(iii) *$\sigma(A)$ is compact.*

Proof. • Let $z \in \mathbb{C}$ such that $|z| > \|A\|$. Then we have

$$A - z = -z \left(\operatorname{Id} - \frac{A}{z} \right).$$

Since

$$\left\| \frac{A}{z} \right\| = \frac{\|A\|}{|z|} < 1,$$

the operator $\operatorname{Id} - \frac{A}{z}$ is invertible with inverse given by the Neumann series $\sum_{k \in \mathbb{N}} (\frac{A}{z})^k$. This proves that $A - z$ is invertible with inverse

$$(A - z)^{-1} = -\sum_{k \in \mathbb{N}} \frac{A^k}{z^{k+1}}.$$

• Let $z_0 \in \rho(A)$. For $z \in D\big(z_0, \left\|(A - z_0)^{-1}\right\|_{\mathcal{L}(\mathsf{E})}^{-1}\big)$ we have

$$A - z = (A - z_0) - (z - z_0) = \big(1 - (z - z_0)(A - z_0)^{-1}\big)(A - z_0).$$

Since $(z - z_0)(A - z_0)^{-1}$ has norm less that 1, the operator $1 - (z - z_0)(A - z_0)^{-1}$ is invertible with inverse

$$\big(1 - (z - z_0)(A - z_0)^{-1}\big)^{-1} = \sum_{n \in \mathbb{N}} (z - z_0)^n (A - z_0)^{-n}.$$

Then $A - z$ is invertible and $(A - z)^{-1}$ is given by (1.2). This proves in particular that $\rho(A)$ is open.

• Finally, $\sigma(A) = \mathbb{C} \backslash \rho(A)$ is closed by (ii) and bounded by (i), so it is compact. □

Proposition 1.15. *Let $A \in \mathcal{L}(\mathsf{E})$ be invertible. Then*

$$\sigma(A^{-1}) = \left\{ \lambda^{-1}, \lambda \in \sigma(A) \right\}.$$

Proof. We already know that 0 is in $\rho(A) \cap \rho(A^{-1})$. For $\lambda \in \mathbb{C} \backslash \{0\}$ we have

$$(A - \lambda) = \lambda A (\lambda^{-1} - A^{-1})$$

so $(A^{-1} - \lambda^{-1})$ is invertible if and only if $(A - \lambda)$ is invertible. □

Ex. 1.3

Proposition 1.16. *Let $A \in \mathcal{L}(\mathsf{E})$. Let $z \in \mathbb{C}$. Assume that there exists $c_0 > 0$ such that*

$$\forall \varphi \in \mathsf{E}, \quad \|(A - z)\varphi\|_{\mathsf{E}} \geqslant c_0 \|\varphi\|_{\mathsf{E}}. \tag{1.3}$$

We say that z is a regular point of A. Then

(i) *$(A - \lambda)$ is injective ;*

(ii) *$(A - \lambda)$ has closed range;*

(iii) *If $(A - \lambda)$ is invertible then $\left\|(A - \lambda)^{-1}\right\| \leqslant c_0^{-1}$.*

This means that if z is a regular point of A, then $z \in \rho(A)$ if and only if $\mathsf{Ran}(A - \lambda)$ is dense in E. Moreover, in this case we already have a bound for the inverse.

Proof. We prove the second statement. Let (ψ_n) be a sequence in $\mathsf{Ran}(A - z)$ which converges to some ψ in E. For $n \in \mathbb{N}$ we consider $\varphi_n \in \mathsf{E}$ such that $(A - z)\varphi_n = \psi_n$. Since $((A - z)\varphi_n)$ is a Cauchy sequence, so is (φ_n) by (1.3). Since E is complete, φ_n converges to some φ in E. Finally, since A is continuous, $\psi = (A - z)\varphi \in \mathsf{Ran}(A - z)$. This proves that $\mathsf{Ran}(A - z)$ is closed in E. □

1.2.2 Resolvent

Definition 1.17. *Let $A \in \mathcal{L}(\mathsf{E})$. The resolvent of A is the map*

$$\begin{cases} \rho(A) & \to & \mathcal{L}(\mathsf{E}), \\ z & \mapsto & (A-z)^{-1}. \end{cases}$$

Proposition 1.18 (Resolvent Identity)**.** *Let $A \in \mathcal{L}(\mathsf{E})$. For $z_1, z_2 \in \rho(A)$ we have*

$$\begin{aligned}(A-z_1)^{-1} - (A-z_2)^{-1} &= (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1} \\ &= (z_1 - z_2)(A-z_2)^{-1}(A-z_1)^{-1}.\end{aligned}$$

Proof. We have $(A-z_2)-(A-z_1) = z_1 - z_2$. The first equality follows after composition by $(A-z_1)^{-1}$ on the left and by $(A-z_2)^{-1}$ on the right. The second equality is similar. ☐

Remark 1.19. The resolvent identity proves in particular that $(A-z_1)^{-1}$ and $(A-z_2)^{-1}$ commute.

Proposition 1.20. *Let $A \in \mathcal{L}(\mathsf{E})$. The resolvent $R_A : z \mapsto (A-z)^{-1}$ is analytic on $\rho(A)$ and $R_A' = R_A^2$.*

Proof. This follows from (1.2). ☐

Proposition 1.21. *Let $A \in \mathcal{L}(\mathsf{E})$. Then $\sigma(A) \neq \varnothing$.*

Proof. Assume by contradiction that $\rho(A) = \mathbb{C}$. For $z \in \mathbb{C}$ such that $|z| \geqslant 2\,\|A\|_{\mathcal{L}(\mathsf{E})}$ we have

$$\left\|(A-z)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} = \frac{1}{|z|}\left\|\left(\frac{A}{z}-1\right)^{-1}\right\|_{\mathcal{L}(\mathsf{E})} \leqslant \frac{1}{|z|}\sum_{k=0}^{\infty}\left(\frac{\|A\|_{\mathcal{L}(\mathsf{E})}}{|z|}\right)^k \leqslant \frac{2}{|z|}. \tag{1.4}$$

Let $\varphi \in \mathsf{E}$ and $\ell \in \mathsf{E}'$. The map $z \mapsto \ell((A-z)^{-1}\varphi)$ is holomorphic on $\mathbb{C}$ and bounded. Thus it is constant by the Liouville Theorem. By the previous estimate, its value must be 0. In particular, $\ell(A^{-1}\varphi) = 0$ for all φ and all $\ell \in \mathsf{E}'$. By the Hahn-Banach Theorem, we have $A^{-1}\varphi$ for all $\varphi \in \mathsf{E}$. This gives a contradiction and proves that $\rho(A) \neq \mathbb{C}$. ☐

Remark 1.22. In the real case we know from the finite dimensional case that the spectrum of a bounded operator can be empty.

1.2.3 Spectral radius

Definition 1.23. *Let $A \in \mathcal{L}(\mathsf{E})$. The spectral radius of A is*

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|\,.$$

By Proposition 1.14 we already know that $r(A) \leqslant \|A\|_{\mathcal{L}(\mathsf{E})}$. The equality is not true in general. Consider for instance the matrix

$$A_\alpha = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

for $\alpha \in \mathbb{C}$. We have $\sigma(A) = \{1\}$ and $\|A\|_{\mathcal{L}(\mathbb{C}^2)} \to +\infty$ as $|\alpha| \to +\infty$. In general we have at least the following result.

Proposition 1.24 (Gelfand's Formula)**.** *Let $A \in \mathcal{L}(\mathsf{E})$. We have*

$$r(A) = \inf_{n \in \mathbb{N}^*} \|A^n\|_{\mathcal{L}(\mathsf{E})}^{\frac{1}{n}} = \lim_{n \to \mathbb{N}^*} \|A^n\|_{\mathcal{L}(\mathsf{E})}^{\frac{1}{n}}\,.$$

Example 1.25. Check that A_α satisfies the Gelfand Formula.

Proof. • Assume that there exists $N \in \mathbb{N}$ such that $A^N = 0$. Then $A^n = 0$ for all $n \geqslant N$. Let $z \in \mathbb{C} \setminus \{0\}$. Then $(z^{-1}A - 1)$ is invertible with inverse

$$\left(\frac{A}{z} - 1\right)^{-1} = -\sum_{n=0}^{N-1} \left(\frac{A}{z}\right)^n.$$

This proves that $A - z = z(z^{-1}A - 1)$ is invertible. Thus $\sigma(A) \subset \{0\}$. Since $\sigma(A) \neq \emptyset$, we have $\sigma(A) = \{0\}$ and the proposition is proved in this case. Now we assume that $A^n \neq 0$ for all $n \in \mathbb{N}$.

• For $n \in \mathbb{N}$ we set $u_n = \ln(\|A^n\|)$. For $m, p \in \mathbb{N}^*$ we have by Remark 1.2

$$u_{m+p} \leqslant u_m + u_p.$$

Let $p \in \mathbb{N}^*$. Let $n \in \mathbb{N}^*$ and $(q, r) \in \mathbb{N} \times [\![0, p-1]\!]$ such that $n = qp + r$. Then we have

$$\frac{u_n}{n} \leqslant \frac{qu_p + u_r}{qp + r} \leqslant \frac{u_p}{p} + \frac{u_r}{n},$$

so

$$\limsup_{n\to\infty} \frac{u_n}{n} \leqslant \frac{u_p}{p}.$$

Then for all $p \in \mathbb{N}^*$ we have

$$\limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leqslant \|A^p\|^{\frac{1}{p}}$$

Thus

$$\limsup_{n\in\infty} \|A^n\|^{\frac{1}{n}} \leqslant \inf_{p\in\mathbb{N}^*} \|A^p\|^{\frac{1}{p}}.$$

This implies that

$$\|A^n\|^{\frac{1}{n}} \xrightarrow[n\to\infty]{} \inf_{p\in\mathbb{N}^*} \|A^p\|^{\frac{1}{p}},$$

which gives the second inequality of the proposition.

• We set $\tilde{r}(A) = \lim \|A^n\|^{\frac{1}{n}}$. For $z \in \mathbb{C}$ we have $\ker(A - z) \subset \ker(A^n - z^n)$ and

$$A^n - z^n = (A - z) \sum_{k=0}^{n-1} z^k A^{n-1-k},$$

so $\mathsf{Ran}(A^n - z^n) \subset \mathsf{Ran}(A - z)$. Thus, if $A^n - z^n$ is bijective, then so is $A - z$. Now let $\lambda \in \sigma(A)$. We have $\lambda^n \in \sigma(A^n)$. By Proposition 1.14 we have $|\lambda|^n = |\lambda^n| \leqslant \|A^n\|$, so $|\lambda| \leqslant \|A^n\|^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, and hence $|\lambda| \leqslant \tilde{r}(A)$. This proves that $r(A) \leqslant \tilde{r}(A)$.

• Let $z \in \mathbb{C}$ with $|z| > \tilde{r}(A)$. Then the power series

$$-\sum_{n\in\mathbb{N}} \frac{A^n}{z^{n+1}}$$

is convergent in $\mathcal{L}(\mathsf{E})$ and defines a bounded inverse for $(A - z)$. This proves that $\tilde{r}(A) \leqslant r(A)$ and concludes the proof. □

1.3 Adjoint of a bounded operator

Let $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces.

1.3.1 Definition and basic properties

Definition 1.26. *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Let $\psi \in \mathcal{H}_2$. We denote by $A^*\psi$ the unique vector in $\mathcal{H}_1$ such that*

$$\forall \varphi \in \mathcal{H}_1, \quad \langle A\varphi, \psi\rangle_{\mathcal{H}_2} = \langle \varphi, A^*\psi\rangle_{\mathcal{H}_1}. \tag{1.5}$$

The definition is justified by the Riesz representation theorem. Indeed, since $\varphi \mapsto \langle A\varphi, \psi\rangle_{\mathcal{H}_2}$ is a continuous semilinear map on $\mathcal{H}_1$, there exists a unique $A^*\psi$ such that (1.5) holds.

Example 1.27. Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are of finite dimensions $n_1, n_2 \in \mathbb{N}^*$. Let β_1 and β_2 be orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, and let $M = (m_{j,k})_{\substack{1\leqslant j\leqslant n_2\\ 1\leqslant k\leqslant n_1}}$ be the matrix of A in β_1 and β_2. Then the matrix of A^* in β_2 and β_1 is

$$M^* = \overline{M}^{\mathsf{T}} = (\overline{m_{k,j}})_{\substack{1\leqslant j\leqslant n_2\\ 1\leqslant k\leqslant n_1}}.$$

Example 1.28. Let $f \in L^\infty(\Omega, \mu)$ and let M_f be the multiplication operator as in Example 1.6. Then the adjoint of M_f is $M_f^* = M_{\overline{f}}$.

Example 1.29. The shift operators S_r and S_ℓ (see Example 1.4) are adjoint of each other on $\ell^2(\mathbb{N})$.

Proposition 1.30. *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$.*

(i) $(A^*)^* = A$.

(ii) $A^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ *and* $\|A^*\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)} = \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$.

Proof. • Let $\varphi \in \mathcal{H}_1$. For all $\psi \in \mathcal{H}_2$ we have

$$\langle A^*\psi, \varphi\rangle_{\mathcal{H}_1} = \overline{\langle \varphi, A^*\psi\rangle_{\mathcal{H}_1}} = \overline{\langle A\varphi, \psi\rangle_{\mathcal{H}_2}} = \langle \psi, A\varphi\rangle_{\mathcal{H}_2}$$

This proves that $A^{**}\varphi = A\varphi$.

• We leave the linearity of A^* as an exercise. For $\psi \in \mathcal{H}_2$, we have

$$\|A^*\psi\|_{\mathcal{H}_1}^2 = \langle AA^*\psi, \psi\rangle_{\mathcal{H}_1} \leqslant \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \|A^*\psi\|_{\mathcal{H}_1} \|\psi\|_{\mathcal{H}_2},$$

so $\|A^*\psi\|_{\mathcal{H}_1} \leqslant \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \|\psi\|_{\mathcal{H}_2}$. This proves that $A^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ and $\|A^*\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)} \leqslant \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$. Then

$$\|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} = \|A^{**}\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leqslant \|A^*\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)},$$

and finally, $\|A^*\|_{\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)} = \|A\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$. ☐

Proposition 1.31. *For $A_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $A_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ we have $(A_2A_1)^* = A_1^*A_2^*$.*

Proof. Let $\varphi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_3$. We have

$$\langle A_2A_1\varphi, \psi\rangle_{\mathcal{H}_3} = \langle A_1\varphi, A_2^*\psi\rangle_{\mathcal{H}_2} = \langle \varphi, A_1^*A_2^*\psi\rangle_{\mathcal{H}_1},$$

and the conclusion follows. ☐

Proposition 1.32. *Let $A \in \mathcal{L}(\mathcal{H})$. If F is a subspace of $\mathcal{H}$ such that $A(\mathsf{F}) \subset \mathsf{F}$, then $A^*(\mathsf{F}^\perp) \subset \mathsf{F}^\perp$.*

Proof. Let $\psi \in F^\perp$. Then for all $\varphi \in \mathsf{F}$ we have $\langle \varphi, A^*\psi\rangle = \langle A\varphi, \psi\rangle = 0$, so $A^*\psi \in F^\perp$. ☐

1.3.2 Spectrum of the adjoint

Proposition 1.33. *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$\ker(A^*) = \mathsf{Ran}(A)^\perp \quad \textit{and} \quad \ker(A^*)^\perp = \overline{\mathsf{Ran}(A)}.$$

Proof. Let $\varphi \in \ker(A^*)$. Then for all $\psi \in \mathcal{H}_1$ we have

$$\langle A\psi, \varphi\rangle_{\mathcal{H}_2} = \langle \psi, A^*\varphi\rangle_{\mathcal{H}_1} = 0,$$

so $\varphi \in \mathsf{Ran}(A)^\perp$. Conversely, if $\varphi \in \mathsf{Ran}(A)^\perp$ then the same computation shows that $\varphi \in \ker(A^*)$. This gives the first inequality. Then, by Proposition A.5 we have

$$\ker(A^*)^\perp = (\mathsf{Ran}(A)^\perp)^\perp = \overline{\mathsf{Ran}(A)},$$

and the proof is complete. ☐

Proposition 1.34. *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Then A^* is invertible if and only if A is, and in this case we have $(A^*)^{-1} = (A^{-1})^*$.*

Proof. • Assume that A is invertible. By Proposition 1.31 we have

$$A^*(A^{-1})^* = (A^{-1}A)^* = \mathrm{Id}^* = \mathrm{Id}\,.$$

and similarly $(A^{-1})^*A^* = \mathrm{Id}$, so A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.
• Similarly, if A^* is bijective then A^{**} is bijective. But $A^{**} = A$ by Proposition 1.30, and the proof is complete. □

Proposition 1.35. *Let $A \in \mathcal{L}(\mathcal{H})$. Then*

$$\sigma(A^*) = \{\overline{z}, z \in \sigma(A)\}\,.$$

Proof. Let $\lambda \in \mathbb{C}$. By Proposition 2.49 the operator $(A - \lambda)$ is bijective if and only if $(A - \lambda)^* = (A^* - \overline{\lambda})$ is bijective. □

Example 1.36. We consider on $\ell^2(\mathbb{N})$ the shift operators of Example 1.4. We have

$$\sigma_{\mathsf{p}}(S_r) = \varnothing \quad \text{and} \quad \sigma_{\mathsf{p}}(S_\ell) = D(0,1).$$

By Proposition 1.14, $\sigma(S_\ell)$ is closed and contained in $\overline{D}(0,1)$, so $\sigma(S_\ell) = \overline{D}(0,1)$. Finally, since $S_r^* = S_\ell$, we also have $\sigma(S_r) = \overline{D}(0,1)$ by Proposition 1.30.

1.3.3 Normal bounded operators

Definition 1.37. *We say that $A \in \mathcal{L}(\mathcal{H})$ is normal if $AA^* = A^*A$.*

Remark 1.38. If A is normal and invertible, then A^{-1} is normal.

Proposition 1.39. *Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator.*

(i) *For $\varphi \in \mathcal{H}$ we have $\|A\varphi\| = \|A^*\varphi\|$. In particular,* $\ker(A^*) = \ker(A)$.

(ii) *If λ and μ are two distinct eigenvalues of A, then* $\ker(A - \lambda)$ *and* $\ker(A - \mu)$ *are orthogonal.*

Proof. • Let $\varphi \in \mathcal{H}$. We have

$$\|A\varphi\|^2 = \langle A^*A\varphi, \varphi\rangle = \langle AA^*\varphi, \varphi\rangle = \|A^*\varphi\|^2\,,$$

which gives the first statement.
• Let $\varphi \in \ker(A - \lambda)$ and $\psi \in \ker(A - \mu)$. By the first statement we also have $\psi \in \ker((A - \mu)^*) = \ker(A^* - \overline{\mu})$. Then we have

$$(\lambda - \mu)\langle\varphi, \psi\rangle = \langle\lambda\varphi, \psi\rangle - \langle\varphi, \overline{\mu}\psi\rangle = \langle A\varphi, \psi\rangle - \langle\varphi, A^*\psi\rangle = 0.$$

Since $\lambda \neq \mu$, this proves that $\langle\varphi, \psi\rangle = 0$, so $\ker(A - \lambda)$ and $\ker(A - \mu)$ are orthogonal. □

Definition 1.40. *Let $A \in \mathcal{L}(\mathcal{H})$.*

(i) *We say that A is symmetric if*

$$\forall \varphi, \psi \in \mathcal{H}, \quad \langle A\varphi, \psi\rangle_{\mathcal{H}} = \langle\varphi, A\psi\rangle_{\mathcal{H}}\,.$$

(ii) *We say that A is selfadjoint if $A^* = A$.*

Ex. 1.4

Proposition 1.41. *Let $A \in \mathcal{L}(\mathcal{H})$. Then A is symmetric if and only if it is selfadjoint.*

Definition 1.42. *$A \in \mathcal{L}(\mathcal{H})$ is said to be skew-adjoint (or skew-symmetric) if $A^* = -A$. Notice that A is selfadjoint if and only if iA is skew-adjoint.*

Ex. 1.5 to 1.7

Definition 1.43. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. An operator $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be unitary if it is invertible and $A^{-1} = A^*$.*

Remark 1.44. Selfadjoint, skew-adjoint and unitary operators on $\mathcal{H}$ are normal.

Example 1.45. The multiplication operator M_w (see Example 1.6) is selfadjoint if and only if w is almost everywhere real valued.

In Section 1.2.3 we have said that the spectral radius of a bounded operator can be smaller that its norm. This is not the case for a normal operator.

Proposition 1.46. *Let $A \in \mathcal{L}(\mathsf{E})$ be normal. We have $r(A) = \|A\|_{\mathcal{L}(\mathcal{H})}$.*

Proof. • Assume that A is selfadjoint. We always have $\|A^2\| \leqslant \|A\|^2$. For $\varphi \in \mathcal{H}$ we have

$$\|A\varphi\|^2 = \langle A^*A\varphi, \varphi\rangle = \langle A^2\varphi, \varphi\rangle \leqslant \|A^2\| \|\varphi\|^2 .$$

This proves that $\|A\|^2 \leqslant \|A^2\|$, and hence $\|A\|^2 = \|A^2\|$. Since A^{2^k} is selfadjoint for all $k \in \mathbb{N}$, we deduce by induction that $\|A^{2^k}\| = \|A\|^{2^k}$ for all $k \in \mathbb{N}$. Then, by the Gelfand Formula we have

$$r(A) = \lim_{k\to\infty} \|A^{2^k}\|^{\frac{1}{2^k}} = \|A\| .$$

• Now we only assume that A is normal. We have $\|A^*A\| = \|A\|^2$ (exercise). On the other hand, since A^*A is selfadjoint we have $r(A^*A) = \|A^*A\|$, so $r(A^*A) = \|A\|^2$. On the other hand, since A is normal,

$$r(A^*A) = \lim_{n\to\infty} \|(A^*A)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(A^n)^*A^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|A^n\|^{\frac{2}{n}} = r(A)^2.$$

This proves that $r(A) = \|A\|$. □

Remark 1.47. If $A \in \mathcal{L}(\mathcal{H})$ is a normal operator such that $\sigma(A) = \{0\}$ then $A = 0$. This is not the case in general, since every nilpotent operator has spectrum $\{0\}$.

Theorem 1.48. *Let $A \in \mathcal{L}(\mathcal{H})$ a normal operator. For $z \in \rho(A)$ we have*

$$\|(A - z)^{-1}\|_{\mathcal{L}(\mathcal{H})} = \frac{1}{\mathsf{dist}(z, \sigma(A))}$$

Proof. For $\zeta \in \mathbb{C}\setminus\{z\}$ we have by Proposition 1.15

$$\sigma((A - z)^{-1}) = \left\{(\zeta - z)^{-1}, \zeta \in \sigma(A)\right\} .$$

Since $(A - z)^{-1}$ is normal, we deduce by Proposition 1.46

$$\|(A - z)^{-1}\| = r((A - z)^{-1}) = \sup_{\lambda\in\sigma(A)} |\lambda - z|^{-1} = \frac{1}{\inf_{\lambda\in\sigma(A)} |\lambda - z|} = \frac{1}{\mathsf{dist}(z, \sigma(A))}.$$ □

1.4 Polar decomposition

[Not discussed in class]

Definition 1.49. *Let $A \in \mathcal{L}(\mathcal{H})$ be a symmetric operator. We say that A is non-negative if $\langle A\varphi, \varphi\rangle_{\mathcal{H}} \geqslant 0$ for all $\varphi \in \mathcal{H}$.*

Proposition 1.50. *Let $A \in \mathcal{L}(\mathcal{H})$ be non-negative. Let $\varphi \in \mathcal{H}$. If $\langle A\varphi, \varphi\rangle = 0$ then $A\varphi = 0$.*

Proof. By the Cauchy-Schwarz inequality we have for all $\psi \in \mathcal{H}$

$$|\langle A\varphi, \psi\rangle| \leqslant \langle A\varphi, \varphi\rangle^{\frac{1}{2}} \langle A\psi, \psi\rangle^{\frac{1}{2}} = 0.$$

Then $\langle A\varphi, \psi\rangle = 0$ for all $\psi \in \mathcal{H}$, so $A\varphi = 0$. □

Proposition 1.51 (Square root of a bounded non-negative operator)**.** *Let $A \in \mathcal{L}(\mathcal{H})$ be non-negative. There exists a unique non-negative bounded operator S such that $S^2 = A$. Moreover, S commutes with A, and any operator which commutes with A also commutes with S. We can write $S = \sqrt{A}$.*

Proof. • Assume that the existence is proved when $\|A\| \leqslant 1$. Then in general we can multiply A by $\varepsilon = \|A\|^{-1}$, so that $\|\varepsilon A\| \leqslant 1$. Then we set $S = \varepsilon^{-\frac{1}{2}} S_\varepsilon$, where S_ε is the square root of εA. Then $S^2 = \varepsilon^{-1}\varepsilon A = A$ and, since S_ε commutes with εA, S commutes with A.
• Now assume that $\|A\| \leqslant 1$. We set $B = \mathrm{Id} - A$. For $\varphi \in \mathcal{H}$ we have

$$\langle B\varphi, \varphi\rangle = \|\varphi\|^2 - \langle A\varphi, \varphi\rangle \leqslant \|\varphi\|^2 .$$

We also have

$$\langle B\varphi, \varphi\rangle = \|\varphi\|^2 - \langle A\varphi, \varphi\rangle \geqslant \|\varphi\|^2 - \|A\| \|u\|^2 \geqslant 0.$$

Then by the Cauchy-Schwarz inequality we have for $\varphi, \psi \in \mathcal{H}$,

$$|\langle B\varphi, \psi\rangle| \leqslant \langle B\varphi, \varphi\rangle^{\frac{1}{2}} \langle B\psi, \psi\rangle^{\frac{1}{2}} \leqslant \|\varphi\| \|\psi\| .$$

This proves that $\|B\| \leqslant 1$. Now we use the power series for the function $z \mapsto \sqrt{1-z}$, absolutely convergent [1] on $\overline{D}(0,1)$:

$$\forall z \in \overline{D}(0,1), \quad \sqrt{1-z} = 1 - \sum_{n=1}^{\infty} a_n z^n, \quad a_n = \frac{(2n)!}{(2n-1)(n!)^2 4^n}.$$

Then we set

$$S = 1 - \sum_{n=1}^{\infty} a_n B^n.$$

Then by Cauchy product for a power series we have $S^2 = \mathrm{Id} - B = A$. Moreover S commute with B and hence with A. Similarly, any operator which commutes with A commutes with B and hence with S.
• Now we prove uniqueness. Assume that S' is another solution. In particular S and S' commute. If we set

$$T = (S - S')S(S - S') \quad \text{and} \quad T' = (S - S')S'(S - S')$$

We observe that

$$T + T' = (S - S')(S + S')(S - S-) = (S - S')(S^2 - S'^2) = 0.$$

Since T and T' are non-negative, they are both 0 by Proposition 1.50. Then

$$(S - S')^4 = (S - S')(T - T') = 0.$$

This implies that $(S - S')^2 = 0$ and finally $S - S' = 0$. □

Definition 1.52. *For $A \in \mathcal{L}(\mathcal{H})$ we set $|A| = \sqrt{A^*A}$.*

This definition makes sense since A^*A is always a non-negative operator.

Definition 1.53. *We say that $U \in \mathcal{L}(\mathcal{H})$ is a partial isometry if for all $\varphi \in \ker(U)^\perp$ we have $\|U\varphi\| = \|\varphi\|$.*

[1] For $x \in [0,1[$ we have

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} a_n x^n.$$

Since all the coefficients are positive we have

$$\sum_{n=1}^{\infty} a_n = \lim_{x\to 1} \sum_{n=1}^{\infty} a_n x^n = 1 - \sqrt{1-1} = 1 < +\infty.$$

This proves that $\sum_{n=1}^{\infty} a_n < +\infty$.

Proposition 1.54. *Let $A \in \mathcal{L}(\mathcal{H})$. There exists a unique partial isometry U such that $\ker(U) = \ker(A)$ and*

$$A = U\,|A|\,.$$

Proof. • Assume that U_1 and U_2 are solutions. We have $U_1 |A| = U_2 |A|$ so $U_1 = U_2$ on $\mathsf{Ran}(|A|)$, and then on $\overline{\mathsf{Ran}(|A|)}$ by continuity. On the other hand, on $\overline{\mathsf{Ran}(|A|)}^{\perp} = \ker(|A|) = \ker(A)$ (see Proposition 1.33) we have $U_1 = U_2 = 0$ so, finally, $U_1 = U_2$.
• For $\varphi \in \mathcal{H}$ we have $\| |A| \varphi \| = \|A\varphi\|$. Then if $\varphi_1, \varphi_2 \in \mathcal{H}$ are such that $|A|\,\varphi_1 = |A|\,\varphi_2$, we also have $A\varphi_1 = A\varphi_2$. Thus we can define U on $\mathsf{Ran}(|A|)$ by

$$U\,|A|\,\varphi = A\varphi.$$

This is a linear isometry from $\mathsf{Ran}(|A|)$ to $\mathsf{Ran}(A)$. It can be extended to a linear isometry from $\overline{\mathsf{Ran}(|A|)}$ to $\overline{\mathsf{Ran}(A)}$. Then we extend U by 0 on $\overline{\mathsf{Ran}(|A|)}^{\perp} = \ker(A)$. In particular, $\ker(A) \subset \ker(U)$. On the other hand, since U is an isometry on $\ker(A)^{\perp}$, we can check that $\ker(U) = \ker(A)$. Then U is an isometry on $\ker(U)^{\perp}$, so this is a partial isometry. ☐

1.5 Operators and quadratic forms - Lax-Milgram Theorem

Let $\mathcal{V}$ be a Hilbert space. Let $\mathcal{V}'$ be the space of continuous semilinear forms on $\mathcal{V}$. We recall that

$$\mathcal{I} : \begin{cases} \mathcal{V} & \to & \mathcal{V}' \\ \varphi & \mapsto & \psi \mapsto \langle \varphi, \psi \rangle_{\mathcal{V}} \end{cases}$$

is a bijective isometry by the Riesz theorem. We can identify $\mathcal{V}$ and $\mathcal{V}'$ via this map, but we do not use this possibility here.

Then we can check that the map

$$\begin{cases} \mathcal{L}(\mathcal{V}) & \to & \mathcal{L}(\mathcal{V}, \mathcal{V}') \\ T & \mapsto & \mathcal{I} \circ T \end{cases}$$

is also a bijective isometry. Moreover $T \in \mathcal{L}(\mathcal{V})$ is invertible if and only if $(\mathcal{I} \circ T) \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ is.

Definition 1.55. *Let $\mathcal{V}$ be a Hilbert space.*

(i) *A sesquilinear form q on $\mathcal{V}$ is a map $\mathsf{q} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ such that*

- *for all $\psi \in \mathcal{V}$ the map $\varphi \mapsto \mathsf{q}(\varphi, \psi)$ is linear ;*
- *for all $\varphi \in \mathcal{V}$ the map $\psi \mapsto \mathsf{q}(\varphi, \psi)$ is semilinear.*

(ii) *The quadratic form associated to q is the map $\varphi \mapsto \mathsf{q}(\varphi, \varphi)$. It is usually also denoted by q.*

(iii) *We say that q is continuous if there exists $C \geqslant 0$ such that, for all $\varphi, \psi \in \mathcal{V}$,*

$$|\mathsf{q}(\varphi, \psi)| \leqslant C \,\|\varphi\|_{\mathcal{V}} \,\|\psi\|_{\mathcal{V}}\,. \tag{1.6}$$

(iv) *We say that q is coercive if there exists $\alpha > 0$ such that for all $\varphi \in \mathcal{V}$ we have*

$$|\mathsf{q}(\varphi, \varphi)| \geqslant \alpha \,\|\varphi\|_{\mathcal{V}}^2\,. \tag{1.7}$$

(v) *The adjoint q^* of the form q is the sesquilinear form defined by*

$$\forall \varphi, \psi \in \mathcal{V}, \quad \mathsf{q}^*(\varphi, \psi) = \overline{\mathsf{q}(\psi, \varphi)}.$$

Remark 1.56. Coercivity is often defined by

$$\mathsf{q}(\varphi, \varphi) \geqslant \alpha \,\|\varphi\|_{\mathcal{V}}^2\,.$$

We use a weaker property here.

Example 1.57. The map $\varphi \mapsto \|\varphi\|_{\mathcal{V}}^2$ is a (coercive) quadratic form on $\mathcal{V}$.

We can also define a bijection between continuous sequilinear forms on $\mathcal{V}$ and operators in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$. Given a continuous sesquilinar form q on $\mathcal{V}$ we define $Q \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ by

$$\forall \varphi \in \mathcal{V}, \forall \psi \in \mathcal{V}, \quad (Q\varphi)(\psi) = \mathsf{q}(\varphi, \psi). \tag{1.8}$$

Conversely, given $Q \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$, we similarly define a corresponding continuous sesquilinear form q.

Proposition 1.58. *Let q be a continuous sesquilinear form on $\mathcal{V}$. There exists a unique operator $T \in \mathcal{L}(\mathcal{V})$ such that*

$$\forall \varphi, \psi \in \mathcal{V}, \quad \mathsf{q}(\varphi, \psi) = \langle T\varphi, \psi \rangle_{\mathcal{V}}.$$

Moreover,

$$\sup_{\varphi, \psi \in \mathcal{V}\setminus\{0\}} \frac{|\mathsf{q}(\varphi, \psi)|}{\|\varphi\|_{\mathcal{V}} \|\psi\|_{\mathcal{V}}} = \|T\|_{\mathcal{L}(\mathcal{V})}.$$

The operator associated with the adjoint form q^* is T^*.

Proof. • Let $\varphi \in \mathcal{V}$. The map $\psi \mapsto \mathsf{q}(\varphi, \psi)$ is a continuous semilinear form on $\mathcal{V}$, so by the Riesz representation theorem there exists an element of $\mathcal{V}$, which we denote by $T\varphi$, such that

$$\forall \psi \in \mathcal{V}, \quad \mathsf{q}(\varphi, \psi) = \langle T\varphi, \psi \rangle_{\mathcal{V}}.$$

This defines a map $T : \mathcal{V} \to \mathcal{V}$.
• Let $\varphi_1, \varphi_2 \in \mathcal{V}$ and $\lambda \in \mathbb{R}$. For all $\psi \in \mathcal{V}$ we have

$$\begin{aligned}\langle T(\varphi_1 + \lambda\varphi_2), \psi \rangle_{\mathcal{V}} &= \mathsf{q}(\varphi_1 + \lambda\varphi_2, \psi) = \mathsf{q}(\varphi_1, \psi) + \lambda \mathsf{q}(\varphi_2, \psi) = \langle T\varphi_1, \psi \rangle_{\mathcal{V}} + \lambda \langle T\varphi_2, \psi \rangle_{\mathcal{V}} \\ &= \langle T\varphi_1 + \lambda T\varphi_2, \psi \rangle.\end{aligned}$$

This proves that $T(\varphi_1 + \lambda\varphi_2) = T\varphi_1 + \lambda T\varphi_2$, and hence that the map $\varphi \mapsto T\varphi$ is linear.
• For $\varphi \in \mathcal{V}$ we have

$$\|T\varphi\|_{\mathcal{V}}^2 = \langle T\varphi, T\varphi \rangle_{\mathcal{V}} = \mathsf{q}(\varphi, T\varphi) \leqslant C \|\varphi\|_{\mathcal{V}} \|T\varphi\|_{\mathcal{V}},$$

where $C = \sup_{\varphi, \psi \in \mathcal{V}\setminus\{0\}} \frac{|\mathsf{q}(\varphi, \psi)|}{\|\varphi\|_{\mathcal{V}} \|\psi\|_{\mathcal{V}}}$, so $\|T\varphi\|_{\mathcal{V}} \leqslant C \|\varphi\|_{\mathcal{V}}$. This proves that $T \in \mathcal{L}(\mathcal{V})$ and $\|T\|_{\mathcal{L}(\mathcal{V})} \leqslant C$. Conversely, for $\varphi, \psi \in \mathcal{V}\setminus\{0\}$ we have

$$|\mathsf{q}(\varphi, \psi)| = |\langle T\varphi, \psi \rangle| \leqslant \|T\| \|\varphi\| \|\psi\|.$$

• Finally, let $\tilde{T}$ be the operator associated to the adjoint form q^*. Let $\psi \in \mathcal{V}$. For all $\varphi \in \mathcal{V}$ we have

$$\langle T\varphi, \psi \rangle = \mathsf{q}(\varphi, \psi) = \overline{\mathsf{q}^*(\psi, \varphi)} = \overline{\langle \tilde{T}\psi, \varphi \rangle} = \langle \varphi, \tilde{T}\psi \rangle.$$

This proves that $\tilde{T} = T^*$. □

Theorem 1.59 (Lax-Milgram)**.** *Let $\mathcal{V}$ be a Hilbert space. Let q be a continuous and coercive sesquilinear form on $\mathcal{V}$. Let $T \in \mathcal{L}(\mathcal{V})$ be the corresponding operator. Then T is bijective and $\|T^{-1}\|_{\mathcal{L}(\mathcal{V})} \leqslant \alpha^{-1}$, where α is given by (1.7). In particular, if ℓ is a bounded semilinear form on $\mathcal{V}$ there exists a unique $\varphi_\ell \in \mathcal{V}$ such that*

$$\forall \psi \in \mathcal{V}, \quad \mathsf{q}(\varphi_\ell, \psi) = \langle T\varphi_\ell, \psi \rangle = \ell(\psi).$$

Proof. • For $\varphi \in \mathcal{V}$ we have

$$\alpha \|\varphi\|_{\mathcal{V}}^2 \leqslant |\mathsf{q}(\varphi, \varphi)| = |\langle T\varphi, \varphi \rangle_{\mathcal{V}}| \leqslant \|T\varphi\|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}},$$

so

$$\|T\varphi\|_{\mathcal{V}} \geqslant \alpha \|\varphi\|_{\mathcal{V}}. \tag{1.9}$$

This proves in particular that T is injective with closed range (see Proposition 1.16). Now let $\psi \in \mathsf{Ran}(T)^\perp$. In particular we have

$$0 = |\langle T\psi, \psi \rangle_{\mathcal{V}}| = |\mathsf{q}(\psi, \psi)| \geqslant \alpha \|\psi\|_{\mathcal{V}}^2,$$

so $\psi = 0$. Since $\mathsf{Ran}(T)$ is closed, this implies that $\mathsf{Ran}(T) = \mathcal{V}$. Thus T is bijective and by (1.9) we have $\|T^{-1}\|_{\mathcal{L}(\mathcal{V})} \leqslant \alpha^{-1}$.

• By the Riesz theorem there exists $\zeta \in \mathcal{V}$ such that $\langle \zeta, \psi \rangle = \ell(\psi)$ for all $\psi \in \mathcal{V}$. Then we set $\varphi_\ell = T^{-1}\zeta$ to get the last statement. □

Example 1.60. We consider on $H^1(\mathbb{R})$ the quadratic form

$$\mathsf{q} : u \mapsto \|u\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \left(\left|u'(x)\right|^2 + |u(x)|^2 \right) \mathrm{d}x.$$

Let $f \in L^2(\mathbb{R})$. There exists $u \in H^1(\mathbb{R})$ such that

$$\forall v \in H^1(\mathbb{R}), \qquad \int_{\mathbb{R}} \left(u'(x)\overline{v}'(x) + u(x)\overline{v}(x) \right) \mathrm{d}x = \int_{\mathbb{R}} f(x)\overline{v}(x)\,\mathrm{d}x.$$

Example 1.61. We consider on $H^1(0,1)$ the quadratic form

$$\mathsf{q}_N : u \mapsto \|u\|^2_{H^1(0,1)} = \int_0^1 \left(\left|u'(x)\right|^2 + |u(x)|^2 \right) \mathrm{d}x..$$

We have the same result as above.

Example 1.62. We consider on $H^1_0(0,1)$ the quadratic form

$$\tilde{\mathsf{q}}_D : u \mapsto \|u\|^2_{H^1(0,1)}\,.$$

This is also a coercive form.

Example 1.63. We consider on $H^1_0(0,1)$ the quadratic form

$$\mathsf{q}_D : u \mapsto \|u'\|^2_{L^2(0,1)}.$$

By the Poincaré inequality, q_D is a coercive form on $H^1_0(0,1)$.

1.6 Exercises

***Exercise* 1.1.** We consider on $\ell^2(\mathbb{N}^*)$ the operator A defined by

$$A(u_1, u_2, u_3, \dots, u_k, \dots) = \left(0, \frac{u_1}{2}, \frac{u_2}{4}, \frac{u_3}{8}, \dots, \frac{u_k}{2^k}, \dots\right).$$

1. Prove that $A \in \mathcal{L}(\ell^2(\mathbb{N}^*))$ and compute $\|A\|_{\mathcal{L}(\ell^2(\mathbb{N}^*))}$.
2. Compute $\sigma(A)$.
3. Compute $\sigma_{\mathsf{p}}(A)$.

***Exercise* 1.2.** We define on $\mathbb{R}$ the function w defined by

$$w(x) = \begin{cases} \frac{1}{x+1} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

Then we consider on $L^2(\mathbb{R})$ the operator M_w of multiplication by w.
1. What is $\sigma(M_w)$?
2. What is $\sigma_{\mathsf{p}}(M_w)$? For each eigenvalue λ of M_w, give a corresponding eigenvector.

***Exercise* 1.3.** Let $A \in \mathcal{L}(\mathsf{E})$. Let $P \in \mathbb{C}[X]$. Prove that

$$\sigma(P(A)) = \{P(\lambda), \lambda \in \sigma(A)\}\,.$$

Let $\lambda \in \sigma(A)$. There exists $Q \in \mathbb{C}[X]$ such that $P(X) - P(\lambda) = Q(X)(X - \lambda) = (X - \lambda)Q(X)$.

***Exercise* 1.4.** Let $\Pi \in \mathcal{L}(\mathcal{H})$ be a projection of $\mathcal{H}$ ($\Pi^2 = \Pi$). Prove that Π is an orthogonal projection if and only if is selfadjoint.

***Exercise* 1.5.** For $u = (u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we set

$$S(\dots, u_{-2}, u_{-1}, u_0, u_1, u_2, \dots) = (\dots, u_{-1}, u_0, u_1, u_2, u_3, \dots).$$

1. Prove that this defines a unitary operator A on $\ell^2(\mathbb{Z})$.
2. Prove that $\sigma(S) \subset \mathbb{U} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.
3. Let $\lambda \in \mathbb{U}$. For $k \in \mathbb{N}$ we consider

$$u^{(k)} = (\dots, 0, 0, 1, \lambda, \lambda^2, \dots, \lambda^k, 0, 0, \dots).$$

Compute $\|u^{(k)}\|_{\ell^2(\mathbb{Z})}$ and $\|(S-\lambda)u^{(k)}\|_{\ell^2(\mathbb{Z})}$. Prove that $\lambda \in \sigma(S)$.

***Exercise* 1.6.** Let $A \in \mathcal{L}(\mathcal{H})$. Let $U \in \mathcal{L}(\mathcal{H})$ be unitary. Prove that

$$\sigma(U^*AU) = \sigma(A) \quad \text{and} \quad \sigma_{\mathsf{p}}(U^*AU) = \sigma(A).$$

***Exercise* 1.7.** We consider on $\ell^2(\mathbb{Z})$ the operator H_0 which maps the sequence $u = (u_n)_{n\in\mathbb{Z}}$ to the sequence $H_0 u$ defined by

$$\forall n \in \mathbb{Z}, \quad (H_0 u)_n = u_{n+1} + u_{n-1} - 2u_n.$$

1. Prove that $H_0 \in \mathcal{L}(\ell^2(\mathbb{Z}))$.
2. We denote by $L^2(\mathbb{S}^1)$ the set of L^2-functions on the torus $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Functions on $\mathbb{S}^1$ can also be seen as 2π-periodic functions on $\mathbb{R}$. For $v \in L^2(\mathbb{S}^1)$ we have

$$\|v\|^2_{L^2(S^1)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |v(s)|^2 \, \mathrm{d}s.$$

Given a sequence $u = (u_n)_{n\in\mathbb{Z}}$ we define $\Theta u \in L^2(\mathbb{S}^1)$ by

$$(\Theta u)(s) = \sum_{n\in\mathbb{Z}} u_n e^{ins}.$$

Prove that Θ is a unitary operator from $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{S}^1)$.
3. Prove that $\Theta H_0 \Theta^{-1}$ is a multiplication operator on $\mathbb{S}^1$.
4. Compute the spectrum of $\Theta H_0 \Theta^{-1}$ and deduce the spectrum of H_0 (use Exercise 1.6).