Chapter 4

Distributions

The purpose of this chapter is to introduce distributions. Distributions generalize the notion of function of one or several real variables. In particular, we will extend the usual notion of derivability, which turns out to be too rigid for applications.

The definition of functions (we only consider here functions of one or several real variables) and of the regularity of these functions (continuity, etc.) have taken a long time to stabilize to the precise and general notions as they are now understood. For example, a function f from $\mathbb{R}^3$ to $\mathbb{R}$ is any correspondance that associates to any element $x \in \mathbb{R}^3$ a unique element $f(x)$ of $\mathbb{R}$. This is a very abstract and general notion.

And yet, this notion shows its limits and is not always adapted to the calculations that one may have to make. Typically, when f represents a physical quantity as a function of the position x. For example, if f denotes a mass density or an electric density, and if we are interested in a very localized mass or charge, we model it by a point mass or charge. This greatly simplifies the calculations, but the density f... is no longer a function. Indeed, in this case the density is what is improperly called “a Dirac function”, zero except at one point but of strictly positive integral. This cannot be realized by any function with the Lebesgue theory for integrals. Thus, to make a simpler calculation, we have to use an object that seems more complicated. So what should we do ? Giving up rigorous calculations, or giving up a model with which we can actually do the computation? Neither, obviously, and it is the aim of the distributions to propose a rigorous, efficient and sufficiently general framework to include in particular the functions in the usual sense and the Dirac function. In fact we have already solved this problem by introducing the measures, since the Dirac function has been replaced by the Dirac measure. But the distributions go further and will include in particular the measures.

Another aspect for which the usual theory of functions seems too restrictive is the following. Let us consider a simple partial differential equation, namely the transport problem

$$\forall (t,x) \in \mathbb{R}^2, \quad \frac{\partial u}{\partial t}(t,x) + \frac{\partial u}{\partial x}(t,x) = 0, \tag{4.1}$$

with a given initial condition:

$$\forall x \in \mathbb{R}, \quad u(0,x) = u_0(x). \tag{4.2}$$

The study of this type of problem will come later, but an important question before looking for a solution is to ask in which set we are working. In which space do we choose the initial data u_0 ? And in which space do we look for the solution u ? A natural choice is to look for u in $C^1(\mathbb{R}^2)$ and then to consider u_0 in $C^1(\mathbb{R})$. We can then verify that the unique solution of the problem is given by

$$\forall (t,x) \in \mathbb{R}^2, \quad u(t,x) = u_0(x-t). \tag{4.3}$$

And now, what happens if we consider an initial data u_0 which is not differentiable ? We can still define u by (4.3), physically it will do exactly the same thing (translation of the profile u_0 to the right when t grows), but on the other hand u is no longer differentiable and we can no longer put it into (4.1). What is the problem? Should we exclude such a solution, which seems physically reasonable but which is not a solution to the problem as it was posed, or should we rethink the way we pose the problem?

As for the Dirac function via the unit approximation sequences, one could approximate in a suitable sense an irregular function by a sequence of regular functions. But it is easier to make calculations with a Dirac than with a sequence of unit approximations, and the same will be true for functions that we will call "derivable in the sense of distributions". Thus it is quite relevant to introduce these new spaces of "functions".

The change of point of view on functions that leads to the definition of distributions is the following. Rather than characterizing a function of x (for example) by evaluating its value at each point x in $\mathbb{R}$, it is characterized by all its averages weighted by a function with compact support. In other words, instead of focusing on every $f(x)$ for $x \in \mathbb{R}$, we will focus on every $\int_{\mathbb{R}} f(s)\phi(s)\,\mathrm{d}s$ for $\phi \in C_0^\infty(\mathbb{R})$.

The characterization of a function by its value at each point had already been challenged in integration, where one began to consider that two functions that differ only at one point must be considered as equal.

This new approach is not a simple mathematical artifice. On the contrary, it is quite natural if we look closely. Or rather a little less closely. Consider for example a function θ which describes the temperature of an infinite wire. What sense does it make to talk about the temperature at a specific point? The temperature measures the degree of agitation of particles. What sense would it make to measure the temperature with a precision greater than the typical distance traveled by each particle during the measure ? And even so, no device could measure it with infinite precision. What a thermometer measures, in the best case, is an average of the temperature over a small area around each x point. Considering that the function θ has a meaning, what is measured is not the value $\theta(x)$, but a quantity of the form

$$\int \theta(y)\phi(y)\,\mathrm{d}y,$$

where ϕ is a function which describes the weight with which the average is obtained. We call ϕ a test function.

In addition, this point of view perfectly fits what we intend to do with the Dirac function δ on $\mathbb{R}$. The aim of δ is to have a function such that

$$\int_{\mathbb{R}} \delta(x)\phi(x)\,\mathrm{d}x = \phi(0), \tag{4.4}$$

for any test function ϕ. Rather than trying to give a doubtful explanation to the left-hand side, we give up seeing δ as an usual function by assigning it values at each point of $\mathbb{R}$ and we *define* directly the *distribution* δ as the map $\phi \mapsto \phi(0)$. And it is in fact much simpler !

In the same spirit, we will define a weaker notion of solution for problems such that (4.1). We will say that u is a weak solution of (4.1) if

$$\forall \phi \in C^1(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} u(t,x) \left(\frac{\partial \phi}{\partial t}(t,x) + \frac{\partial \phi}{\partial x}(t,x) \right) \mathrm{d}t \, \mathrm{d}x = 0,$$

with the initial condition (4.2). Thus u can be a solution without being differentiable in the usual sense. We will say that u is a strong solution of (4.1) on $\mathbb{R}^2$ if it is a solution in the previous sense, that is, if u is of class C^1 on $\mathbb{R}^2$ and verifies (4.1). Using an integration by part, we see that a strong solution is a weak solution, and thus the notion of weak solution is a generalization of the usual notion. Defining derivation in the weak sense via integration by parts will be the key of the upcoming notion of derivation.

The aim of this chapter is to give a mathematical framework to all these ideas, by defining in particular the derivation in the sense of distributions.

4.1 Definitions

In this section, we introduce the notion of distribution. In these notes, we have chosen to gather all the examples in Section 4.2. The drawback of this choice is that the definitions will be given here without example. Thus, do not hesitate to read the Section 4.2 in parallel with this one. In particular, It is in section 4.2.1 that we will see that functions can be identified as examples of distributions, and that in that sense the notion of distribution "includes" in a suitable sense that of function.

4.1.1 Space of tests functions

Let Ω be an open set of $\mathbb{R}^d$. We start by collecting some properties that we will need later for the space of the test functions $C_0^\infty(\Omega)$.

Recall that $C_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for any $p \in [1, +\infty[$ (see Proposition 1.18). We have also proved a result of partition of unity with cut-off functions of class C^∞ (see Proposition 1.23).

The following properties of $C_0^\infty(\Omega)$ are elementary and the proofs are left as exercises for the reader.

Proposition 4.1. (i) *$C_0^\infty(\Omega)$ is a subspace of the space of functions from $\mathbb{R}^d$ to $\mathbb{C}$.*

(ii) *If $f \in C^\infty(\Omega)$ and $\phi \in C_0^\infty(\Omega)$ then $f\phi \in C_0^\infty(\Omega)$.*

(iii) *If $\phi \in C_0^\infty(\Omega)$ and $\alpha \in \mathbb{N}^d$ then $\partial^\alpha \phi \in C_0^\infty(\Omega)$.*

(iv) *Let $\phi \in C_0^\infty(\Omega)$. For $x \in \mathbb{R}^d$ we set*

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Then we have $\tilde{\phi} \in C_0^\infty(\mathbb{R}^d)$.

We now recall the Leibniz formula for C^∞ functions. For $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ in $\mathbb{N}^d$ we say that $\beta \leqslant \alpha$ if $\beta_j \leqslant \alpha_j$ for any $j \in [\![1, d]\!]$. Then we set

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}, \quad \text{where} \quad \alpha! = \alpha_1! \dots \alpha_d!.$$

Proposition 4.2 (Leibniz formula). *Let $u, v \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}^d$. Then we have*

$$\partial^\alpha(uv) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} u \, \partial^\beta v.$$

The proof can be done by induction on $|\alpha|$ as for the case $d = 1$ (exercise).

4.1.2 Topologies on the spaces of regular functions with compact support

Let Ω be an open set of $\mathbb{R}^d$. The aim of this section is to describe the topologies of the spaces of regular functions that have compact support Ω, and in particular $C_0^\infty(\Omega)$.

Let us start with the simplest of situations. Let K be a compact of Ω and $k \in \mathbb{N}$. We denote by $C_K^k(\Omega)$ the set of functions of class C^k on Ω with support included in K. Then for $u \in C_K^k(\Omega)$ we set

$$\|\phi\|_{C_K^k(\Omega)} = \sum_{|\alpha| \leqslant k} \|\partial^\alpha \phi\|_{L^\infty(K)} . \tag{4.5}$$

This defines a norm on $C_K^k(\Omega)$, and $C_K^k(\Omega)$ is complete for this norm.

The situation is not that simple for the space $C_K^\infty(\Omega)$ of C^∞ functions defined on Ω and with support included in K. Obviously, one can not simply replace k by $+\infty$ in the definition (4.5). Each norm of (4.5) for $k \in \mathbb{N}$ is a norm for $C_K^\infty(\Omega)$, but $C_K^\infty(\Omega)$ is not complete for any of these norms (a sequence of very regular functions can converge towards a limit that is not that regular). To obtain a complete space, we need to consider a topology that takes into account all the derivatives of the functions. There is no such norm on $C_K^\infty(\Omega)$, but we can endow $C_K^\infty(\Omega)$ with a Frechet space structure from every norms of (4.5), for any $k \in \mathbb{N}$.

For $\phi, \psi \in C_K^\infty(\Omega)$ we set

$$d_K(\phi, \psi) = \sum_{k=0}^{+\infty} \frac{1}{2^k} \min\big(1, \|\phi - \psi\|_{C_K^k(\Omega)}\big). \tag{4.6}$$

This distance is not given by a norm, but it satisfies the important properties that we need for applications.

Proposition 4.3. *d_K is a distance on $C_K^\infty(\Omega)$, and $C_K^\infty(\Omega)$ is complete for this distance.*

Let us recall the basic properties of the topology associated to the distance d_K.

Proposition 4.4. (i) *Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of elements of $C_K^\infty(\Omega)$ and $\phi \in C_K^\infty(\Omega)$. Then ϕ_n tends to ϕ in $C_K^\infty(\Omega)$ if and only if*

$$\forall k \in \mathbb{N}, \quad \|\phi_n - \phi\|_{C_K^k(\Omega)} \xrightarrow[n\to+\infty]{} 0.$$

(ii) *A linear form T on $C_K^\infty(\Omega)$ if and only if there exists $k \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in C_K^\infty(\Omega)$ we have*

$$|T(\phi)| \leqslant C \|\phi\|_{C_K^k(\Omega)} .$$

More generally, if $(E, \|\cdot\|_E)$ *is a normed vector space, then a linear map* $T : C_K^\infty(\Omega) \to E$ *is continuous if there exists* $k \in \mathbb{N}$ *and* $C > 0$ *such that for any* $\phi \in C_K^\infty(\Omega)$ *we have*

$$\|T(\phi)\|_E \leqslant C \|\phi\|_{C_K^\infty(\Omega)} .$$

(iii) *A linear map* $T : C_K^\infty(\Omega) \to C_K^\infty(\Omega)$ *is continuous if for any* $j \in \mathbb{N}$ *there exists* $k \in \mathbb{N}$ *and* $C > 0$ *such that for any* $\phi \in C_K^\infty(\Omega)$ *we have*

$$\|T(\phi)\|_{C_K^j(\Omega)} \leqslant C \|\phi\|_{C_K^k(\Omega)} .$$

Now we turn to $C_0^\infty(\Omega)$. The difference compared to $C_K^\infty(\Omega)$ is that the functions of $C_0^\infty(\Omega)$ are not supported in the same compact. We note in particular that $C_K^\infty(\Omega) \subset C_0^\infty(\Omega)$ for any compact K of Ω. However there is no compact K of Ω such that $C_0^\infty(\Omega)$ is included in $C_K^\infty(\Omega)$.

We cannot endow $C_0^\infty(\Omega)$ with a distance analoguous to (4.6) or with norms analoguous to (4.5) where K would be replaced by Ω, since for the corresponding topology a sequence of functions compactly supported in Ω could converge to a function whose support is Ω.

To ensure that the limit of a convergent sequence has compact support, we need a topology defined in such a way that if $(\phi_n)_{n\in\mathbb{N}}$ is a convergent sequence in $C_0^\infty(\Omega)$ then the support of the functions ϕ_n, $n \in \mathbb{N}$, are included in a common compact K of Ω. Once this is done, to also ensure the regularity of the limit, we impose that the sequence $(\phi_n)_{n\in\mathbb{N}}$ is convergent in $C_K^\infty(\Omega)$ (this is now meaningful, since ϕ_n belongs to $C_K^\infty(\Omega)$ for all n).

Such a topology is complicated, but it does exist. We state without proof the following theorem.

Theorem 4.5. *There exists a topology on* $C_0^\infty(\Omega)$ *which satisfies the following properties:*

(i) *A sequence* $(\phi_n)_{n\in\mathbb{N}}$ *of* $C_0^\infty(\Omega)$ *converges to* $\phi \in C_0^\infty(\Omega)$ *if and only if*

- *there exists a compact* K *of* Ω *such that* $\mathrm{supp}(\phi_n) \subset K$ *for any* $n \in \mathbb{N}$,
- $\partial^\alpha \phi_n$ *goes uniformly to* $\partial^\alpha \phi$ *for any* $\alpha \in \mathbb{N}^d$.

(ii) *A linear form* T *on* $C_0^\infty(\Omega)$ *is continuous if and only if for any compact* K *of* Ω *there exist* $m \in \mathbb{N}$ *and* $C > 0$ *such that*

$$\forall \phi \in C_K^\infty(\Omega), \quad |T(\phi)| \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|_\infty . \tag{4.7}$$

4.1.3 Distributions

Now that we have described the topology of the space of test functions $C_0^\infty(\Omega)$, we can define the notion of distribution.

Definition 4.6. Let Ω be an open set of $\mathbb{R}^d$. A distribution on Ω is a continuous linear form on $C_0^\infty(\Omega)$. We denote by $\mathcal{D}'(\Omega)$ the set of distributions on Ω.

In general, we denote by $\langle T, \phi \rangle$ or $\langle T, \phi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$ instead of $T(\phi)$ for the image of the test function $\phi \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$ by the distribution $T \in \mathcal{D}'(\Omega)$.

We will give many examples of distributions in Section 4.2 (they can be consulted right now).

As the set of continuous of linear forms on a topological vector space, $\mathcal{D}'(\Omega)$ is naturally endowed with a structure of topological vector space. Thus if T and S are two distributions, the sum $T + S$ is defined by

$$\forall \phi \in C_0^\infty(\Omega), \quad \langle T + S, \phi \rangle = \langle T, \phi \rangle + \langle S, \phi \rangle,$$

and for $\lambda \in \mathbb{K}$ we define λT by

$$\forall \phi \in C_0^\infty(\Omega), \quad \langle \lambda T, \phi \rangle = \lambda \langle T, \phi \rangle.$$

We can check that $T + S$ and λS are indeed distributions on Ω and that $\mathcal{D}'(\Omega)$ endowed with these two operations is vector space. We then endow $\mathcal{D}'(\Omega)$ with the weak-$*$ topology.

Definition 4.7. Let Ω be an open set of $\mathbb{R}^d$. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of distributions on Ω. We say that T_n goes to T in $\mathcal{D}'(\Omega)$ if

$$\forall \phi \in C_0^\infty(\Omega), \quad \langle T_n, \phi \rangle \xrightarrow[n\to+\infty]{} \langle T, \phi \rangle.$$

By the Banach-Steinhaus Theorem, we have the following result:

Proposition 4.8. *Let Ω be an open set of $\mathbb{R}^d$ and $(T_n)_{n\in\mathbb{N}}$ be a sequence of distributions on Ω. We suppose that for any $\phi \in C^\infty(\Omega)$ the sequence $(\langle T_n, \phi \rangle)_{n\in\mathbb{N}}$ is convergent. Then the sequence $(T_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{D}'(\Omega)$.*

4.1.4 Finite order distributions

The parameters m and C can depend on the compact K in (4.7). However, it may happen (and it will actually often be the case) that we can choose the same integer m for any K.

Definition 4.9. Let Ω be an open set of $\mathbb{R}^d$ and T be a distribution on Ω. We say that T is a distribution of finite order if there exists $m \in \mathbb{N}$ such that for any compact K of Ω we can find $C > 0$ which satisfies

$$\forall \phi \in C_K^\infty(\Omega), \quad |T(\phi)| \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|_\infty.$$

In this case the order of T is the smallest integer m which satisfies this property. Otherwise, T is said to be of infinite order.

We will see at section 4.2 that many usual distributions are of finite order.

Remark 4.10 (This remark can be omitted). If T is a distribution of order $m \in \mathbb{N}$ on Ω, we only need to control a finite number of derivatives to ensure that if ϕ_n tends to ϕ then $T(\phi_n)$ tends to $T(\phi)$. Thus, a distribution of order m can be seen as a continuous linear form on the space $C_0^m(\Omega)$ of compactly supported functions of class C^m on Ω. More precisely, $C_0^m(\Omega)$ is endowed with a topology similar to the one described in theorem 4.5. A sequence $(\phi_n)_{n\in\mathbb{N}}$ in $C_0^m(\Omega)$ tends to $\phi \in C_0^m(\Omega)$ if and only if

(i) There exists a compact K of Ω such that $\operatorname{supp}(\phi_n) \subset K$ for any $n \in \mathbb{N}$,

(ii) $\partial^\alpha \phi_n$ tends uniformly to $\partial^\alpha \phi$ for any $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leqslant m$.

We can then verify that any distribution of order m on Ω extends into a continuous linear form on $C_0^m(\Omega)$ and conversely, a continuous linear form on $C_0^m(\Omega)$ defines by restriction to $C_0^\infty(\Omega)$ a distribution of order m on Ω.

4.1.5 Multiplication of a distribution by a regular function

The purpose of distributions is to generalize the notion of function. In order for this to be useful, it will be necessary to be able to generalize the operations that we usually perform on functions to this framework and in a suitable sense. We have already seen that we can naturally add distributions and multiply them by a scalar. To go further, the mechanism will essentially always be the same. The operation in question is carried over to the test function. In particular, the operation in question must preserve the space of test functions $C_0^\infty(\Omega)$. We illustrate this idea on a first example, the multiplication by a regular function.

Proposition-Definition 4.11. *Let Ω be an open set of $\mathbb{R}^d$ and T be a distribution on Ω. Let $f \in C^\infty(\Omega)$. For $\phi \in C_0^\infty(\Omega)$ we set*

$$\langle fT, \phi \rangle = \langle T, f\phi \rangle .$$

This defines a distribution fT on Ω.

Two points have to be checked in this definition. First, the expression $\langle T, f\phi \rangle$ has to be meaningful. For this, the functions $f\phi$ on the right has to be an element of $C_0^\infty(\Omega)$. This is clear here, but it will not always be the case. And then we must show that the map $\phi \mapsto \langle T, f\phi \rangle = T(f\phi)$ is indeed a continuous linear form on $C_0^\infty(\Omega)$. And in general, the continuity is not obvious.

Proof. • For $\phi \in C_0^\infty(\Omega)$ we have $f\phi \in C_0^\infty(\Omega)$, and the map $\phi \mapsto f\phi$ is linear on $C_0^\infty(\Omega)$. Then, by composition, fT is a linear form on $C_0^\infty(\Omega)$.

• Let K be a compact of Ω. There exists $m \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in C_K^\infty(\Omega)$ we have

$$|\langle T, \phi \rangle| \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|_\infty .$$

The function f and its derivatives are bounded on the compact K. By the Leibniz rule, we have

$$|\langle T, f\phi \rangle| \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha (f\phi)\|_\infty \leqslant C \sum_{|\alpha| \leqslant m} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \|\partial^{\alpha-\beta} f\|_{L^\infty(K)} \|\partial^\beta \phi\|_\infty .$$

Thus, there exists a constant $\tilde{C} > 0$ independant of ϕ such that

$$|\langle T, f\phi \rangle| \leqslant \tilde{C} \sum_{|\beta| \leqslant m} \|\partial^\beta \phi\|_\infty .$$

This proves that fT is continuous on $C_0^\infty(\Omega)$. □

Remark 4.12. We observe that if T is of finite order m then fT is of order at most m. Indeed, in this case, the integer m does not depend on K in the computations of the previous proof, and the last inequality shows that fT is of order not greater than m. In fact, if f is not identically zero, then fT has exactly the same order m as T.

Remark 4.13. Given a certain $f \in C^\infty(\Omega)$ the map $T \mapsto fT$ is continuous $\mathcal{D}'(\Omega)$. In other words, if $(T_n)_{n\in\mathbb{N}}$ is a sequence of distributions that converges to $T \in \mathcal{D}'(\Omega)$ (in the sense given by Definition 4.7), then fT_n converges to fT in $\mathcal{D}'(\Omega)$.

An important remark to finish this section. There is no reasonable definition for the product of two distributions ! One should just forget this idea.

4.2 Important examples of distributions

In this section we give some classical examples of distributions. Other examples will then appear in the exercises.

4.2.1 Locally integrable functions

One of the motivations for introducing the notion of distribution is that it must generalize the notion of function. It is therefore necessary that the set of distributions contain the set of functions in a reasonable sense. We said in the introduction that we could replace the evaluation of a function f at each point of Ω by the evaluation of mean weights of the form

$$\int_\Omega f\phi \,\mathrm{d}x.$$

This defines precisely a distribution, and it is indeed with this distribution that we will identify the function f. In order for all this to make sense, we must nevertheless restrict ourselves to the case of locally integrable functions. The constraint is reasonable.

Proposition-Definition 4.14. *Let Ω be an open set of $\mathbb{R}^d$ and $f \in L^1_{\mathrm{loc}}(\Omega)$. Then the map*

$$T_f : \phi \mapsto \int_\Omega f(x)\phi(x)\,\mathrm{d}x$$

is a distribution (of order 0) on Ω.

Proof. The map T_f is well defined, and it is linear on $C_0^\infty(\Omega)$ by linearity of the integral. Let K be a compact set of Ω. Then f is integrable on K and for $\phi \in C_K^\infty(\Omega)$ we have

$$\left|\int_\Omega f\phi\right| \leqslant \|\phi\|_\infty \int_K |f(x)|\,\mathrm{d}x.$$

This proves that T_f is a distribution of order 0 on Ω. □

We recall that we cannot multiply two distributions. On the other hand, the product of two locally integrable functions is not necessarily locally integrable. But we can multiply a locally integrable function by a regular function, and we have defined the product of a distribution by a regular function. We check that in the case of a distribution associated to a function, these two multiplications coincide. More precisely, for $f \in L^1_{\mathrm{loc}}(\Omega)$ and $g \in C^\infty(\Omega)$ we have

$$gT_f = T_{gf}.$$

Indeed, for any test function $\phi \in C_0^\infty(\Omega)$ we have

$$\langle gT_f, \phi\rangle = \langle T_f, g\phi\rangle = \int_\Omega f(x)(g\phi)(x)\,\mathrm{d}x = \int_\Omega (gf)(x)\phi(x)\,\mathrm{d}x = \langle T_{gf}, \phi\rangle.$$

We have said that we want to identify the (locally integrable) functions to distributions. For this we have to be sure that two different functions are not associated with the same distribution. This means that no information is lost by considering the set of weighted mean values of f instead of the sets of their values up to equality almost everywhere (it is clear that if f and g are two locally integrable functions equal almost everywhere on Ω then we have $T_f = T_g$).

Proposition 4.15. *Let Ω be an open set of $\mathbb{R}^d$. The map*

$$\begin{cases} L^1_{\text{loc}}(\Omega) & \to & \mathcal{D}'(\Omega) \\ f & \mapsto & T_f \end{cases}$$

is injective.

Proof. Let $f \in L^1_{\text{loc}}(\Omega)$ and assume that $\int_\Omega f\phi = 0$ for all $\phi \in C_0^\infty(\Omega)$. Soit K un compact de Ω. Pour $x \in \Omega$ on pose $g(x) = \text{sign}(f(x))\mathbb{1}_K(x)$. □

Proof. Let $f \in L^1_{\text{loc}}(\Omega)$ and assume that $\int_\Omega f\phi = 0$ for all $\phi \in C_0^\infty(\Omega)$.

Let K be a compact of Ω and let $N \in \mathbb{N}$. For $x \in \Omega$ we set

$$g_N(x) = \begin{cases} \overline{f(x)} & \text{if } x \in K \text{ and } |f(x)| \leqslant N, \\ 0 & \text{otherwise.} \end{cases}$$

We have $g \in L^1(\Omega)$ and g is 0 outside K. There exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ which converges to $\chi\overline{f}\mathbb{1}_{A_N}$ in $L^1(\Omega)$ and such that $\|\phi_n\|_\infty \leqslant N$ for all $n \in \mathbb{N}$ (the sequence constructed in $C_0^\infty(\mathbb{R}^d)$ belongs to $C_0^\infty(\Omega)$ for n large enough). After extracting a subsequence if necessary, we can assume that $\phi_n(x)$ goes to $g(x)$ for almost all $x \in \Omega$. Then by the dominated convergence theorem we have

$$0 = \int_\Omega f(x)\phi_n(x)\,\mathrm{d}x \xrightarrow[n\to\infty]{} \int_\Omega f(x)g_N(x)\,\mathrm{d}x = \int_K |f(x)|^2\,\mathbb{1}_{|f|\leqslant N}(x)\,\mathrm{d}x.$$

Par le théorème de convergence monotone on obtient à la limite $N \to \infty$

$$\int_K |f(x)|^2\,\mathrm{d}x = 0.$$

This prove that $f = 0$ almost everywhere on K. Since this holds for any compact $K \subset \Omega$, we deduce that $f = 0$ almost everywhere on Ω. □

Once we are used to distributions, we identify a distribution of the form T_f with the corresponding function f. Moreover, we can say that a distribution $T \in \mathcal{D}'(\Omega)$ is in $L^1_{\text{loc}}(\Omega)$ (or in $L^p(\Omega)$ for some $p \in [1, +\infty]$) if there exists $f \in L^1_{\text{loc}}(\Omega)$ ($f \in L^p(\Omega)$) such that $T = T_f$.

4.2.2 Dirac mass and other measures

The typical example of an object that we would like to consider as a function but which is not is the "Dirac function", mentioned in the introduction. What is usually meant by a Dirac function on $\mathbb{R}^d$ would be a positive-valued function, null outside $\{0\}$ and with an integral equal to 1. However, such a function cannot exist, since the definition of the integral implies in particular that a function which vanishes almost everywhere on $\mathbb{R}^d$ has an integral equal to 0. More precisely, we would like a function f which would satisfies

$$\forall \phi \in C_0^\infty(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} f\phi = \phi(0). \tag{4.8}$$

Proposition 4.16. *There is no function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ verifying (4.8).*

Proof. Assume by contradiction that there exists $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ which satisfies (4.8). Let $\phi \in C_0^\infty(\mathbb{R}^d, [0, 1])$ be supported in $B(0, 1)$ and such that $\phi(0) = 1$. For $n \in \mathbb{N}^*$ and

$x \in \mathbb{R}^d$ we set $\phi_n(x) = \phi(nx)$. Then ϕ_n is supported in $B\big(0, \frac{1}{n}\big)$, $\phi_n(x) \in [0,1]$ for all $x \in \mathbb{R}^d$, and $\phi_n(0) = 1$. With (4.8) we have for any $n \in \mathbb{N}^*$

$$1 = \phi_n(0) = \int_{\mathbb{R}^d} f(x)\phi(nx)\,\mathrm{d}x \leqslant \int_{B(0,\frac{1}{n})} |f(x)|\,\mathrm{d}x.$$

On the other hand, since f is integrable on $B(0,1)$, we have by the dominated convergence theorem

$$\int_{B(0,\frac{1}{n})} |f(x)|\,\mathrm{d}x \xrightarrow[n\to+\infty]{} 0.$$

This gives a contradiction. □

Thus we must definitely give up the idea to see the Dirac function as a function. But, on the other hand, the right-hand side of (4.8) does define a distribution. Thus the Dirac will be included in the theory of distributions.

Proposition-Definition 4.17. *Let $x_0 \in \mathbb{R}^d$. The map*

$$\delta_{x_0} : \phi \mapsto \phi(x_0)$$

is a distribution on $\mathbb{R}^d$ (of order 0), called Dirac distribution at x_0 (In general, when $x_0 = 0$, we simply write δ instead of δ_0).

Proof. The map δ_{x_0} is linear on $C_0^\infty(\Omega)$ and for any $\phi \in C_0^\infty(\Omega)$ we have

$$|\phi(x_0)| \leqslant \|\phi\|_\infty .$$

This proves that δ_{x_0} is a distribution of order 0 on Ω. □

Note that Proposition 4.16 can be adapted at any point x_0 to see that δ_{x_0} is not the distribution associated with a function $L^1_{\mathrm{loc}}(\Omega)$ (short version: δ_{x_0} is not in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$).

In the chapter about convolution, we have already discussed the fact that there is no Dirac function, and we had introduced the approximations to the Dirac mass to approximate the expected behavior of a Dirac function by regular functions. Given a sequence (ρ_n) of approximations of the unit on $\mathbb{R}^d$, we can then see that in $\mathcal{D}'(\mathbb{R}^d)$

$$T_{\rho_n} \xrightarrow[n\to+\infty]{} \delta.$$

Indeed, for $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\langle T_{\rho_n}, \phi\rangle = \int_{\mathbb{R}^d} \rho_n(x)\phi(x)\,\mathrm{d}x = (\rho_n * \phi)(0) \xrightarrow[n\to+\infty]{} \phi(0) = \langle \delta, \phi\rangle .$$

Before this, the notion of Dirac function was already made rigorous by the measure theory. Indeed, we defined the measure δ such that $\delta(\{0\}) = 1$ and $\delta(\mathbb{R}^d \setminus \{0\}) = 0$, which gives in particular

$$\forall \phi \in C_0^\infty(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \phi\,\mathrm{d}\delta = \phi(0).$$

In fact, measures are already a generalization of functions (with non-negative values, if we consider non-negative measures only). Indeed, if f is a locally integrable (and positive valued) function on $\mathbb{R}^d$, then the measure which maps $A \in \mathcal{B}(\mathbb{R}^d)$ to

$$\mu_f(A) = \int_A f\,\mathrm{d}\lambda$$

(where λ is the Lebesgue measure) is a locally finite measure (that is a measure which is finite on compact sets) on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. The notion of distribution generalizes the notion of (locally finite) measures.

Proposition 4.18. *Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then the map*

$$T_\mu : \phi \mapsto \int_\Omega \phi \, \mathrm{d}\mu$$

is a distribution on Ω.

Be careful, the fact that μ is locally finite is important to get the continuity of T_μ. Note that T_μ is a distribution of order 0, and that it is a positive distribution if μ is a positive measure (this means that $\langle T_\mu, \phi \rangle \geqslant 0$ if $\phi \geqslant 0$).

Remark 4.19. We can actually show that this gives all positive distributions of order 0 and this can give another way to define the Lebesgue measure on $\mathbb{R}$. Indeed, even if we only know the integral of continuous functions, we can see that the map

$$T : \phi \mapsto \int_{\mathbb{R}} \phi(x) \, \mathrm{d}x$$

is a distribution of order 0 on $\mathbb{R}$, and we can define the Lebesgue measure as the unique Radon measure λ (we do not detail this here) such that $T = T_\lambda$.

4.2.3 Principal value of $1/x$

The purpose of this paragraph is to define a distribution naturally associated with the function $x \mapsto 1/x$ on $\mathbb{R}$. Recall that this function is not in $L^1_{\mathrm{loc}}(\mathbb{R})$ since it is not integrable in a neighborhood of 0. Nevertheless, it is odd and the positive and negative parts compensate each other. We use this remark for the following definition (which may seem rather artificial, but which turs out to be relevant for applications).

Proposition 4.20. *The map*

$$\mathsf{p.v.}\left(\frac{1}{x}\right) : \phi \mapsto \lim_{\varepsilon \to 0} \int_{|x| \geqslant \varepsilon} \frac{\phi(x)}{x} \, \mathrm{d}x$$

is a distribution of order 1 on $\mathbb{R}$. It is called the principal value of $1/x$.

Proof. Let $R > 0$. We consider $\phi \in C_0^\infty(\mathbb{R})$ supported in $[-R, R]$. Let $\varepsilon \in]0, R[$. Since the function $x \mapsto \phi(0)/x$ is odd and integrable on $[-R, -\varepsilon] \cup [\varepsilon, R]$ we have

$$\int_{\varepsilon \leqslant |x| \leqslant R} \frac{\phi(x)}{x} \, \mathrm{d}x = \int_{\varepsilon \leqslant |x| \leqslant R} \frac{\phi(x) - \phi(0)}{x} \, \mathrm{d}x.$$

For $x \in [-R, R] \setminus \{0\}$ we have by the mean value theorem

$$\left| \frac{\phi(x) - \phi(0)}{x} \right| \leqslant \|\phi'\|_{L^\infty(-R,R)}.$$

Then, by the dominated convergence theorem, the limit

$$\lim_{\varepsilon \to 0} \int_{\varepsilon \leqslant |x| \leqslant R} \frac{\phi(x)}{x} \, \mathrm{d}x$$

exists and its modulus is not greater than $2R \|\phi'\|_{L^\infty(-R,R)}$. Moreover, this limit is linear with respect to ϕ, so it defines a distribution of order at most 1.

Finally, for $n \geqslant 3$ we consider $\phi_n \in C_0^\infty(\mathbb{R}, [0, 1])$ supported in $\left]\frac{1}{n}, 2\right[$ and equal to 1 on $\left[\frac{2}{n}, 1\right]$. Then $\|\phi_n\|_\infty = 1$ for any $n \geqslant 3$ and

$$\left\langle \mathsf{p.v.}\left(\frac{1}{x}\right), \phi_n \right\rangle \geqslant \int_{\frac{2}{n}}^1 \frac{1}{x} \, \mathrm{d}x \xrightarrow[n \to +\infty]{} +\infty.$$

This proves that the distribution $\mathsf{p.v.}\left(\frac{1}{x}\right)$ cannot be of order 0. It is then exactly of order 1. ☐

4.2.4 Exercises

***Exercise* 1.** For $x \in \mathbb{R}^*$ we set $f(x) = e^{\frac{1}{x}}$.
1. Prove that f belongs to $L^1_{\text{loc}}(\mathbb{R}^*)$. Deduce that f defines a distribution on $\mathbb{R}^*$.
2. Prove that f (well defined everywhere on $\mathbb{R}$) does not belong to $L^1_{\text{loc}}(\mathbb{R})$.
3. Prove that there is no distribution T on $\mathbb{R}$ such that

$$\forall \phi \in C_0^\infty(\mathbb{R}_+^*), \quad \langle T, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x.$$

***Exercise* 2.** Let $k \in \mathbb{N}$. Prove that the map which to $\phi \in C_0^\infty(\mathbb{R}) \mapsto \phi^{(k)}(0)$ is a distribution on $\mathbb{R}$ and give its order.

***Exercise* 3.** Prove that the map

$$T : \phi \in C_0^\infty(\mathbb{R}) \mapsto \langle T, \phi \rangle = \sum_{n=0}^{+\infty} \phi^{(n)}(n)$$

is a distribution of infinite order on $\mathbb{R}$.

***Exercise* 4.** In this exercise we give examples in the same spirit as the Dirac mass, since we have to integrate a function on a submanifolds of $\mathbb{R}^d$ of dimension strictly less than d. In other words, we integrate a function with respect to a measure which only "loads" a set of zero Lebesgue measure in $\mathbb{R}^d$.
1. Prove that the map

$$T : \phi \in C_0^\infty(\mathbb{R}) \mapsto \langle T, \phi \rangle = \int_{\mathbb{R}} \phi(0, x)\,\mathrm{d}x$$

is a distribution on $\mathbb{R}^2$ and give its order.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^1. Prove that the map

$$\phi \in C_0^\infty(\mathbb{R}^2) \mapsto \int_{\mathbb{R}} \phi\big(f(x), x\big)\sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

is a distribution sur $\mathbb{R}^2$ and give its order.
3. Let $f : \mathbb{R} \to \mathbb{R}$ and $\nu : \mathbb{R} \to \mathbb{R}^2$ be two functions of class C^1. Prove that the map

$$\phi \in C_0^\infty(\mathbb{R}^2) \mapsto \int_{\mathbb{R}} \nabla\phi\big(f(x), x\big) \cdot \nu(x)\,\mathrm{d}x$$

is a distribution on $\mathbb{R}^2$ and give its order.

***Exercise* 5.** We recall that for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$ we have denoted by $\tau_y f$ the translation of f ($\tau_y f : x \mapsto f(x - y)$). Let $T \in \mathcal{D}'(\mathbb{R}^d)$. For $y \in \mathbb{R}^d$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ we set

$$\langle \tau_y T, \phi \rangle = \langle T, \tau_{-y}\phi \rangle.$$

1. Prove that this defines a distribution $\tau_y T$ on $\mathbb{R}^d$.
2. Prove that for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ we have $\tau_y T_f = T_{\tau_y f}$.

4.3 Derivative of a distribution

We now turn to one of the main motivations for the notion of distribution, namely the extansion of the notion of derivability to functions that are not differentiable in the usual sense. The idea is to see any function as a distribution, i.e. as a linear form on $C_0^\infty(\Omega)$, and to transfer the derivation to the test function ϕ (which is differentiable).

If f is a function of class C^1 on $\mathbb{R}$ (in particular, it is locally integrable), then for any $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f'(x)\phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x. \tag{4.9}$$

With the notations of proposition 4.15, this can be written as

$$\langle T_{f'}, \phi \rangle = -\langle T_f, \phi' \rangle.$$

For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, the left-hand side in (4.9) has no meaning. However, the right-hand side does, and we notice that it defines a distribution. It is what we are going to define as the derivative of T_f. If this distribution is associated to a function $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, that is if there exists $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that

$$\forall \phi \in C_0^\infty(\mathbb{R}), \quad -\int_{\mathbb{R}} f(x)\phi'(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)\phi(x)\,\mathrm{d}x,$$

we will say that g is the derivative in the sense of distributions (or weak derivative) of f.

Example 4.21. For $\phi \in C_0^\infty(\mathbb{R})$ we have

$$-\int_{\mathbb{R}} |x|\,\phi'(x)\,\mathrm{d}x = \int_{-\infty}^{0} x\phi'(x)\,\mathrm{d}x - \int_{0}^{+\infty} x\phi'(x)\,\mathrm{d}x.$$

In each of these integrals we can do an integration by parts, this gives

$$\int_{\mathbb{R}} |x|\,\phi'(x)\,\mathrm{d}x = -\int_{-\infty}^{0} \phi(x)\,\mathrm{d}x + \int_{0}^{+\infty} \phi(x)\,\mathrm{d}x.$$

Thus, if we set

$$g(x) = \begin{cases} 1 & \text{if } x \geqslant 0, \\ -1 & \text{if } x < 0, \end{cases} \tag{4.10}$$

then for any $\phi \in C_0^\infty(\mathbb{R})$ we have

$$-\int_{\mathbb{R}} |x|\,\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)\phi(x)\,\mathrm{d}x.$$

This means that g is the derivative in the sense of distribution of the absolute value function. This was the expected result.

4.3.1 Definitions and first examples

We now give a precise definition for the derivatives of a distribution. As for usual derivatives, we use different notation in dimension 1 or in higher dimension.

Proposition-Definition 4.22. *Let Ω be an open set of $\mathbb{R}$ and $T \in \mathcal{D}(\Omega)$. The derivative T' of T is defined by*

$$\forall \phi \in C_0^\infty(\Omega), \quad \langle T', \phi \rangle = -\langle T, \phi' \rangle.$$

More generally, for $k \in \mathbb{N}^*$ we denote by $T^{(k)}$ the distribution defined by

$$\forall \phi \in C_0^\infty(\Omega), \quad \langle T^{(k)}, \phi \rangle = (-1)^k \langle T, \phi^{(k)} \rangle.$$

Proof. We prove that the map $\phi \mapsto -\langle T, \phi' \rangle$ is indeed a distribution on Ω. Let K be a compact set of Ω. There exist $m \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in C_K^\infty(\Omega)$ we have

$$|\langle T, \phi \rangle| \leqslant C \sum_{j=0}^{m} \|\phi^{(j)}\|_\infty.$$

Then for $\phi \in C_K^\infty(\Omega)$ we have

$$\left| -\left\langle T, \phi' \right\rangle \right| \leqslant C \sum_{j=0}^{m} \|(\phi')^{(j)}\|_\infty \leqslant C \sum_{j=0}^{m+1} \|\phi^{(j)}\|_\infty.$$

This proves that T' is a distribution on Ω. The general case follows by induction on the order of diffentiation. □

We observe that if T is a distribution of order m, then $T^{(k)}$ is a distribution of order $m + k$. The definition is analogous in any dimension.

Proposition-Definition 4.23. *Let Ω be an open set of $\mathbb{R}^d$ and $T \in \mathcal{D}'(\Omega)$. For $j \in [\![1, d]\!]$ we denote by $\partial_{x_j} T$ the partial derivative of the distribution T with respect to the j-th variable. It is defined by*

$$\forall \phi \in C_0^\infty(\Omega), \quad \left\langle \partial_{x_j} T, \phi \right\rangle = -\left\langle T, \partial_{x_j} \phi \right\rangle.$$

More generally, for $\alpha \in \mathbb{N}^d$ we define $\partial^\alpha T$ by

$$\forall \phi \in C_0^\infty(\Omega), \quad \left\langle \partial^\alpha T, \phi \right\rangle = (-1)^{|\alpha|} \left\langle T, \partial^\alpha \phi \right\rangle.$$

We now give some examples. In the specification of the definition, the derivative of the distribution associated to a differentiable function has to be the distribution associated with its derivative. This is indeed the case according to the equality (4.9) on which the definition of T' was based.

Example 4.24. If f is a function of class C^1 on $\mathbb{R}$, then $T_f' = T_{f'}$.

This is also the case in any dimension and for any order of deviation.

Example 4.25. Let f be a function of class C^k on an open set Ω of $\mathbb{R}^d$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. We have

$$\partial^\alpha T_f = T_{\partial^\alpha f}.$$

We now rewrite Example 4.21 in terms of differentiation in the sense of distributions.

Example 4.26. If we denote by f the absolute value function on $\mathbb{R}$, then we have $T_f' = T_g$, where g is defined by (4.10).

We now give the derivative of a function that is not continuous.

Example 4.27. The Heaviside function is defined on $\mathbb{R}$ by

$$H(x) = \begin{cases} 1 & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{4.11}$$

Then we have $T_H' = \delta$. Indeed, for $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\left\langle T_H', \phi \right\rangle = -\left\langle T_H, \phi' \right\rangle = -\int_0^{+\infty} \phi'(x)\,\mathrm{d}x = \phi(0) = \langle \delta, \phi \rangle.$$

We can give an example of derivative of a distribution which is not associated with a function.

Example 4.28. The derivative δ' of the Dirac distribution δ on $\mathbb{R}$ is given by

$$\forall \phi \in C_0^\infty(\mathbb{R}), \quad \left\langle \delta', \phi \right\rangle = -\left\langle \delta, \phi' \right\rangle = -\phi'(0).$$

Finally, we give an example of a locally integrable function whose derivative is a function which is not locally integrable.

Example 4.29. The function $f : x \mapsto \ln(|x|)$ is in $L^1_{\mathrm{loc}}(\mathbb{R})$. It is differentiable on $\mathbb{R}^*$ and its derivative is the function $x \mapsto 1/x$, which is not locally integrable. Let $\phi \in C_0^\infty(\mathbb{R})$. Since f is locally integrable we have by the dominated convergence theorem

$$-\int_{\mathbb{R}} \ln(|x|)\phi'(x)\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} I_\varepsilon,$$

where we have set

$$I_\varepsilon = -\int_{-\infty}^{-\varepsilon} \ln(|x|)\phi'(x)\,\mathrm{d}x - \int_{\varepsilon}^{+\infty} \ln(|x|)\phi'(x)\,\mathrm{d}x.$$

Since we have removed a neighbourhood of 0, we can do an integration by parts in both terms. This gives

$$\begin{aligned} I_\varepsilon &= -\ln(\varepsilon)\phi(-\varepsilon) + \int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x} + \ln(\varepsilon)\phi(\varepsilon) + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x} \\ &= \int_{|x| \geqslant \varepsilon} \frac{\phi(x)}{x} + \big(\phi(\varepsilon) - \phi(-\varepsilon)\big)\ln(\varepsilon). \end{aligned}$$

Since $\phi(\varepsilon) - \phi(-\varepsilon) = \mathcal{O}(\varepsilon)$ we get

$$I_\varepsilon \xrightarrow[\varepsilon \to 0]{} \left\langle \mathsf{p.v.}\left(\frac{1}{x}\right), \phi \right\rangle.$$

This proves that

$$T_f' = \mathsf{p.v.}\left(\frac{1}{x}\right).$$

Notice the importance of the domain on which we consider the distributions, as is already the case for the derivation in the usual sense. Let us consider for example on the open $\mathbb{R}^*$ the function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then f is derivable in the usual sense on $\mathbb{R}^*$ and its derivative is null. If we see f as a function on $\mathbb{R}$, then for any value of f at 0 we obtain a function which is not differentiable, because it is not differentiable at 0.

When we comput the derivative in the sense of distributions, we no longer evaluate the function at every point, but this distinction remains. If we see f as a function on $\mathbb{R}^*$, its derivative in the sense of distributions is 0 (in other words, $T_f' = 0$). On $\mathbb{R}$, the distribution T_f has a derivative as any distribution, and this derivative is $T_f' = \delta$.

In the sense of distributions we identify two functions that are equal almost everywhere, but it is not because a function is differentiable at almost every point of $\mathbb{R}$ with zero derivative that it is differentiable with zero derivative on $\mathbb{R}$. In the sense of distributions, the problem does not come from the differentiability at the point 0, but if we consider a test function which is not 0 around 0 then we can see the jump at 0 and T_f'

cannot be 0 (and cannot even be identified with a function).

We note that the derivative of a distribution of order m is a distribution of order at most $m+1$. Its order is not necessarily equal to $m+1$. For example if T is the distribution associated to a function of class C^1 then T and T' are both of order 0.

The following proposition is a generalization of the Leibniz formula. We can similarly give a result in higher dimension (generalization of proposition 4.2 given for functions).

Proposition 4.30. *Let I be an interval of $\mathbb{R}$, $f \in C^\infty(I)$ and $T \in \mathcal{D}'(I)$. Then we have*

$$(fT)' = f'T + fT'.$$

More generally, for $n \in \mathbb{N}$ we have

$$(fT)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} T^{(n-k)}.$$

Proof. We prove the first statement. The general case follows by induction on n as for functions. For $\phi \in C_0^\infty(I)$ we have

$$\begin{aligned}\left\langle (fT)', \phi \right\rangle &= -\left\langle fT, \phi' \right\rangle = -\left\langle T, f\phi' \right\rangle = -\left\langle T, (f\phi)' \right\rangle + \left\langle T, f'\phi \right\rangle = \left\langle T', f\phi \right\rangle + \left\langle T, f'\phi \right\rangle \\ &= \left\langle fT' + f'T, \phi \right\rangle.\end{aligned}$$

This proves that $(fT)' = f'T + fT'$. $\square$

4.3.2 Jumps formula in dimension 1

In this section, we compute the derivative in the sense of distributions of a piecewise C^1 function in dimension 1.

Let f be a piecewise C^1 function on an open interval I of $\mathbb{R}$. To simplify the notation, we assume that f has only a finite number N of discontinuities (but we can do the same if f has an infinite –necessarily countable– number of discontinuities). Thus there exist $a_1, \dots, a_N \in I$ such that $a_1 < \dots < a_N$, f has for all $j \in [\![1, N]\!]$ left and right limits at a_j (that we will respectively denote by $f(a_j^-)$ and $f(a_j^+)$), f is differentiable on $I \backslash \{a_1, \dots, a_N\}$ and its derivative (denoted by $[f']$) also has left and right limits at any point. In particular, $[f']$ defines a locally integrable function on I.

Proposition 4.31. *Let f be as described above. Then we have*

$$T_f' = T_{[f']} + \sum_{j=1}^{N} \big(f(a_j^+) - f(a_j^-) \big) \delta_{a_j}.$$

Proof. We set $a_0 = \inf(I) \in [-\infty, a_1[$ and $a_{N+1} = \sup(I) \in]a_N, +\infty]$. For $\phi \in C_0^\infty(I)$ we have

$$\begin{aligned}-\int_I f(x)\phi'(x)\,\mathrm{d}x &= -\sum_{j=0}^{N} \int_{a_j}^{a_{j+1}} f(x)\phi'(x)\,\mathrm{d}x \\ &= \sum_{j=0}^{N} \int_{a_j}^{a_{j+1}} f'(x)\phi(x)\,\mathrm{d}x + \sum_{j=0}^{N} \big(f(a_{j+1}^-)\phi(a_{j+1}) - f(a_j^+)\phi(a_j) \big) \\ &= \int_I [f'](x)\phi(x)\,\mathrm{d}x + \sum_{j=1}^{N} \big(f(a_j^+) - f(a_j^-) \big)\phi(a_j).\end{aligned}$$

For the last equality we used the fact that ϕ vanishes in a neighbourhood of the (possibly infinite) boundary points of I. $\square$

With this proposition we recover the derivatives of Examples 4.24, 4.26 and 4.27.

4.3.3 Examples in higher dimensions

Example 4.32. We consider on $\mathbb{R}^2$ the function f which to (x_1, x_2) associates

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 > 0, \\ 0 & \text{if } x_1 \leqslant 0. \end{cases}$$

This defines a function in $L^1_{\text{loc}}(\mathbb{R}^2)$. For $\phi \in C_0^\infty(\mathbb{R}^2)$ using the Fubini theorem and an integration by parts we have

$$-\int_{\mathbb{R}^2} f(x)\partial_{x_1}\phi(x)\,\mathrm{d}x = -\int_{x_2\in\mathbb{R}} \left(\int_{x_1=0}^{+\infty} \partial_{x_1}\phi(x_1, x_2)\,\mathrm{d}x_1\right)\mathrm{d}x_2 = \int_{\mathbb{R}} \phi(0, x_2)\,\mathrm{d}x_2,$$

hence $\partial_{x_1}T_f$ is the distribution seen in the exercise 4. On the other hand

$$-\int_{\mathbb{R}^2} f(x)\partial_{x_2}\phi(x)\,\mathrm{d}x = 0,$$

so $\partial_{x_2}T_f = 0$.

Example 4.33. Let $\alpha \in]-\infty, d-1[$. We consider the unit ball $B(1)$ of $\mathbb{R}^d$ and the function $f \mapsto |x|^{-\alpha}$. This defines an integrable function on $B(1)$. Moreover, it is of class C^∞ on $B(1)\setminus\{0\}$ and for $x \in B(1)\setminus\{0\}$ we have

$$\nabla f(x) = -\alpha\,|x|^{-\alpha-2}\,x.$$

Let $\phi \in C_0^\infty(B(1))$. Since f is integrable we have by the dominated convergence theorem

$$-\int_{B(1)} f\nabla\phi\,\mathrm{d}x = -\lim_{\varepsilon\to 0}\int_{B(1)\setminus B(\varepsilon)} f\nabla\phi\,\mathrm{d}x.$$

By the Green Formula we have for any $\varepsilon \in]0, 1[$

$$-\int_{B(1)\setminus B(\varepsilon)} f\nabla\phi\,\mathrm{d}x = -\int_{S(\varepsilon)} \varepsilon^{-\alpha}\phi\nu\,\mathrm{d}x - \alpha\int_{B(1)\setminus B(\varepsilon)} |x|^{-\alpha-2}\,x\phi\,\mathrm{d}x,$$

where $S(\varepsilon)$ is the sphere of radius ε and ν is the normal unit vector to $S(\varepsilon)$ directed towards $B(\varepsilon)$. We have on the one hand

$$\left|\int_{S(\varepsilon)} \varepsilon^{-\alpha}\phi\nu\,\mathrm{d}x\right| \leqslant |S(1)|\,\varepsilon^{d-1-\alpha}\,\|\phi\|_\infty \xrightarrow[\varepsilon\to 0]{} 0.$$

On the other hand, using the dominated convergence theorem, we have

$$-\alpha\int_{B(1)\setminus B(\varepsilon)} |x|^{-\alpha-2}\,x\phi\,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} -\alpha\int_{B(1)} |x|^{-\alpha-2}\,x\phi\,\mathrm{d}x$$

This proves that the gradient of T_f is the distribution associated to the integrable function $x \mapsto -\alpha\,|x|^{-\alpha-2}\,x$. In other words, for any $j \in [\![1, d]\!]$, the distribution $\partial_j T_f$ is the distribution associated to the function $x \mapsto -\alpha\,|x|^{-\alpha-2}\,x_j$.

4.3.4 First examples of differential equations

Now that we have introduced the derivatives of a distribution, we can try to solve differential equations in the space of distributions. As for the functions, the question is to find the set of distributions T such that some relations between T and its derivatives are satisfied.

We begin by a simple problem. The first question is to find all the distributions with a zero derivative in open interval of $\mathbb{R}$. Of course, the distributions associated to constant functions are solutions. They are the only solutions in the space of differentiable functions, but since we have enlarged the space where we look for a solution, we could have new solutions that are not distributions associated to derivable functions. This is actually not the case.

Proposition 4.34. *Let I be an open interval of $\mathbb{R}$. Let $T \in \mathcal{D}'(I)$. Then $T' = 0$ if and only if T is constant (that is if T can be identified to a constant function).*

For the proof we will use the following lemma. Notice that a function $\phi \in C_0^\infty(I)$ has primitives on I. They are necessarily of class C^∞, but in general they are not compactly supported.

Lemma 4.35. *Let I be an open interval of $\mathbb{R}$ and $\phi \in C_0^\infty(I)$. Then there exists $\psi \in C_0^\infty(I)$ such that $\psi' = \phi$ if and only if $\int_I \phi = 0$.*

Proof. Assume that there exists $\psi \in C_0^\infty(I)$ such that $\psi' = \phi$. Then we have

$$\int_I \phi = \int_I \psi' = 0.$$

Conversely, assume that $\int_I \phi = 0$. For $x \in I$ we set $\psi(x) = \int_a^x \phi(t)\,\mathrm{d}t$, with $a = \inf(I)$. Then $\psi \in C_0^\infty(I)$ and $\psi' = \phi$. □

Proof of proposition 4.34. We know that the derivative in the sense of distributions of a constant function is 0. Conversely, assume that $T \in \mathcal{D}'(I)$ is such that $T' = 0$.

Let $\phi_0 \in C_0^\infty(I)$ such that $\int_I \phi_0 = 1$. We set $\alpha = \langle T, \phi_0 \rangle$. Let $\phi \in C_0^\infty(I)$. We have

$$\int_I \left(\phi - \phi_0 \int_I \phi \right) = 0,$$

So by Lemma 4.35 there exists $\psi \in C_0^\infty(I)$ such that $\psi' = \phi - \phi_0 \int_I \phi$. Then we have

$$\left\langle T, \phi - \phi_0 \int_I \phi \right\rangle = \langle T, \psi' \rangle = -\langle T', \psi \rangle = 0,$$

and on the other hand

$$\left\langle T, \phi - \phi_0 \int_I \phi \right\rangle = \langle T, \phi \rangle - \alpha \int_I \phi,$$

hence

$$\langle T, \phi \rangle = \alpha \int_I \phi.$$

This proves that T is the distribution associated to the constant function equal to α. We note that the definition of α does not depend of the choice of ϕ_0. □

The following generalizations are left as exercises for the reader.

Corollary 4.36. *Let I be an open interval of $\mathbb{R}$ and $k \in \mathbb{N}^*$. Let $T \in \mathcal{D}'(I)$. Then $T^{(k)} = 0$ if and only if T is a polynomials of degree at most $k-1$.*

Corollary 4.37. *Let I be an open interval of $\mathbb{R}$, $k \in \mathbb{N}^*$, $a_0, \dots, a_k \in \mathbb{C}$ and f a continuous function of I. The solutions $T \in \mathcal{D}'(I)$ of the equation*

$$a_k T^{(k)} + \cdots + a_0 T = f$$

are exactly the solutions of the same problems set in $C^k(I)$.

Example 4.38. Let $\alpha \in \mathbb{R}$. We consider the equation

$$T' - \alpha T = \delta, \tag{4.12}$$

of unknown $T \in \mathcal{D}'(\mathbb{R})$. As for the usual case, since it is an affine problem, it is enough to find a particular solution and to add the solutions of the homogeneous problem. To find a particular solution, we can use... the variation of the parameter. Let $T \in \mathcal{D}'(I)$ and $S = e^{-\alpha x}T$. Then T is solution of (4.12) if and only if $e^{\alpha x}S = \delta$, or $S' = e^{-\alpha x}\delta = \delta$. According to Example 4.27 we can take the Heaviside function H and we get a particular solution $T_0 = e^{\alpha x}H$. As for the case of functions, even if we are not convinced with the previous computation, we can check a posteriori (with Proposition 4.30 or 4.31) that T_0 is indeed a solution. Thus, the set of solutions of (4.12) is the set of distributions associated to functions of the form

$$x \mapsto e^{\alpha x}\big(H(x) + c\big),$$

where c is a constant.

Remark 4.39. We recall that if f is a continuous function with compact support on $\mathbb{R}$ then the solutions of the equation

$$y' - \alpha y = f$$

are functions of the form

$$t \mapsto Ce^{\alpha t} + \int_{-\infty}^{t} e^{\alpha(t-s)} f(s)\,\mathrm{d}s.$$

Notice that these solutions are precisely convolutions of functions that are solutions of (4.12) with f. This is not a coincidence, and we will generalize this remark later on.

We observe that as in the case of functions, the difficulty to solve an equation like (4.12) is to identify the primitive of a given distribution. As for continuous functions, we can ensure that any distribution on an interval of $\mathbb{R}$ has a primitive (this is Exercise 9). In particular, any function in $L^1_{\mathrm{loc}}(\mathbb{R})$ has a primitive in the sense of distributions.

The following proposition is left as an exercise fo the reader.

Proposition 4.40. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. For $x \in \mathbb{R}$ we set*

$$F(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Then F is in $L^1_{\mathrm{loc}}(\mathbb{R})$ and its derivative in the sense of distributions is f (in other words, $T_F' = T_f$).

4.3.5 Exercises

***Exercise* 6.** Let $T \in \mathcal{D}(\mathbb{R})$, $f \in C^\infty(\mathbb{R})$ and $k \in \mathbb{N}$. Prove that

$$(fT)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(k-j)} T^{(j)}.$$

***Exercise* 7.** **1.** Let $\alpha \in \mathbb{C}$. Determine the set of distributions $T \in \mathcal{D}'(\mathbb{R})$ such that

$$T' = \alpha T.$$

2. Solve in $\mathcal{D}'(\mathbb{R})$ the equation

$$T' - \alpha T = H(x),$$

where H is the Heaviside function.

***Exercise* 8.** For $\phi \in C_0^\infty(\mathbb{R}^2)$ we set

$$\langle T, \phi \rangle = \int_{\mathbb{R}} \phi(x, -x) \, \mathrm{d}x.$$

1. Prove that it defines a distribution T on $\mathbb{R}^2$.
2. Prove that T cannot be seen as a $L^1_{\text{loc}}(\mathbb{R}^2)$ function.
3. Compute $\partial_1 T - \partial_2 T$.

***Exercise* 9.** Let I be an interval of $\mathbb{R}$ and let T be a distribution on I.
1. Prove that T has a primitive $S \in \mathcal{D}'(I)$.
2. Use S to describe the set of primitives of T.

***Exercise* 10.** **1.** Determine the set of primitives of δ on $\mathbb{R}$. Verify in particular that the primitives of δ can be identified to functions of $L^\infty(\mathbb{R})$.
2. Let $f \in L^1(\mathbb{R})$. Prove that the set of primitives of f (in the sense of distributions) is the set of functions of the form $(G * f)$, where G is a primitive of δ.

***Exercise* 11.** Determine the set of solutions in $\mathcal{D}'(\mathbb{R})$ of the equation

$$-T'' + T = \delta.$$

***Exercise* 12.** Let $\Omega = \big\{(x_1, x_2) \in \mathbb{R}^2 \,:\, |x_1| < 1, |x_2| < 1\big\}$. For $(x_1, x_2) \in \Omega$ we set

$$u(x) = \begin{cases} 1 - x_1 & \text{if } |x_2| < x_1, \\ 1 + x_1 & \text{if } |x_2| < -x_1, \\ 1 - x_2 & \text{if } |x_1| < x_2, \\ 1 + x_2 & \text{if } |x_1| < -x_2. \end{cases}$$

Determine the derivatives of u in the sense of distributions on Ω.

***Exercise* 13.** For $(t, x) \in \mathbb{R}^2$ we set

$$G(t, x) = \begin{cases} \frac{1}{2} & \text{if } t - |x| > 0, \\ 0 & \text{otherwise} \end{cases}$$

Prove that $G \in L^1_{\text{loc}}(\mathbb{R}^2)$ and compute $(\partial_{tt} - \partial_{xx})G$ in the sense of distributions.

***Exercise* 14.** For $(t,x) \in \mathbb{R}^2$ we set

$$G(t,x) = H(t)\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}.$$

Prove that $G \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and compute $(\partial_t - \partial_{xx})G$ in the sense of distributions.

***Exercise* 15.** **1.** Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For $x \in \mathbb{R}$ we set $f(x) = \mu(]-\infty, x[)$.

a. Prove that f is a nondecreasing function, tends to 0 at $-\infty$ and tends to 1 at $+\infty$.

b. Prove that in the sense of distributions, we have $f' = \mu$.

2. Let f be a nondecreasing function on $\mathbb{R}$. Suppose that f tends to 0 at $-\infty$ and tends to 1 at $+\infty$. Prove that there exists a probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that, in the sense of distributions, $f' = \mu$.

4.4 Compactly supported distributions

In this section we focus on the case of compactly supported distributions, that is distributions that vanish outside a compact subset of Ω.

4.4.1 Restriction of a distribution - Support

Proposition 4.41 (Restriction of a distribution)**.** *Let Ω and ω be two open sets of $\mathbb{R}^d$ such that $\omega \subset \Omega$. Let T be a distribution on Ω. Then the restriction T_ω of T to $C_0^\infty(\omega)$ (we identify a function $\phi \in C_0^\infty(\omega)$ to its extension by 0 on Ω) defines a distribution on ω.*

Proof. T defines a linear map on $C_0^\infty(\omega)$. Let K be a compact set of ω. This is also a compact set of Ω, so there exist $m \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in C_K^\infty(\omega) \simeq C_K^\infty(\Omega)$ we have

$$|T_\omega(\phi)| = |T(\phi)| \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|_\infty .$$

This proves that T_ω is indeed a distribution on ω. □

Definition 4.42. Let Ω be an open set of $\mathbb{R}^d$ and T be a distribution on Ω. We say that T vanishes on ω if its restriction to ω is zero, that is if $\langle T, \phi \rangle = 0$ for any $\phi \in C_0^\infty(\Omega)$ supported in ω.

Lemma 4.43. *Let Ω be an open set of $\mathbb{R}^d$ and T be a distribution on Ω. We denote by $\mathcal{O}$ the union of all the open sets of Ω on which T vanishes. Then T vanishes on $\mathcal{O}$ (in particular, $\mathcal{O}$ is then the biggest open set on which T vanishes).*

Proof. Let $\phi \in C_0^\infty(\mathcal{O})$. Let K be the support of ϕ. Since K is compact, there exist $n \in \mathbb{N}$ and open sets $\omega_1, \dots, \omega_n \subset \Omega$ such that $K \subset \bigcup_{j=1}^n \omega_j$ and T vanishes on ω_j for any $j \in [\![1, n]\!]$. Let $\chi_1, \dots, \chi_n \in C_0^\infty(\mathcal{O})$ be an associated partition of unity ($\mathrm{supp}(\chi_j) \subset \omega_j$ for any $j \in [\![1, n]\!]$ and $\sum_{j=1}^n \chi_j = 1$ on K). Then $\mathrm{supp}(\chi_j \phi) \subset \omega_j$ for any $j \in [\![1, n]\!]$ and

$$\langle T, \phi \rangle = \sum_{j=1}^n \langle T, \chi_j \phi \rangle = 0.$$

This proves that T is vanishes on $\mathcal{O}$. □

Definition 4.44. Let Ω be an open set of $\mathbb{R}^d$ and let T be a distribution on Ω. The support of T, denoted by $\mathrm{supp}(T)$, is the complementary set in Ω of the biggest open set on which T vanishes.

Example 4.45.
- If f is a continuous function on Ω then $\operatorname{supp}(T_f) = \operatorname{supp}(f)$.
- $\operatorname{supp}(\delta) = \{0\}$.
- Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and ω be an open set of Ω. Then the restriction of T_f to ω is the distribution associated to the restriction on ω of f. In particular T_f vanishes on ω if and only if f is almost everywhere 0 on ω. Thus $\operatorname{supp}(T_f)$ is the complementary set of the biggest open set of Ω on which f is almost everywhere 0.

Proposition 4.46. *Let Ω be an open set of $\mathbb{R}^d$ and T a distribution on Ω.*

(i) $\operatorname{supp}(T)$ *is a closed set of* Ω

(ii) *Let $\phi \in C_0^\infty(\Omega)$ such that* $\operatorname{supp}(T) \cap \operatorname{supp}(\phi) = \varnothing$. *Then* $\langle T, \phi \rangle = 0$.

(iii) *For any $\alpha \in \mathbb{N}^d$, we have* $\operatorname{supp}(\partial^\alpha T) \subset \operatorname{supp}(T)$.

Remark 4.47. Let T be a distribution on Ω. There can exist $\phi \in C_0^\infty(\Omega)$ such that $\phi = 0$ on $\operatorname{supp}(T)$ but $\langle T, \phi \rangle \neq 0$. For example, we have $\operatorname{supp}(\delta') = \{0\}$ and for $\phi \in C_0^\infty(\mathbb{R})$ we have

$$\left\langle \delta', \phi \right\rangle = -\phi'(0).$$

But we can have $\phi(0) = 0$ and $\phi'(0) \neq 0$. In that case $\phi = 0$ on $\operatorname{supp}(T)$ but $\langle T, \phi \rangle \neq 0$. However, 0 belongs to the support of ϕ so $\operatorname{supp}(T) \cap \operatorname{supp}(\phi) \neq \varnothing$.

4.4.2 Compactly supported distributions

We now consider distributions on Ω whose support is a compact subset of Ω. We denote by $\mathcal{E}'(\Omega)$ the set of compactly supported distributions on Ω.

Since a distribution has "locally a finite order" it is not surprising that compactly supported distributions are of finite order.

Lemma 4.48. *Let Ω be an open set of $\mathbb{R}^d$ and $T \in \mathcal{E}'(\Omega)$. Let $\chi \in C_0^\infty(\Omega)$ equal to 1 on a neighbourhood*[1] *of* $\operatorname{supp}(T)$. *Then for any $\phi \in C_0^\infty(\Omega)$ we have*

$$\langle T, \phi \rangle = \langle T, \chi\phi \rangle.$$

Proof. Let $\phi \in C_0^\infty(\Omega)$. Since $\operatorname{supp}(T) \cap \operatorname{supp}((1-\chi)\phi) = \varnothing$ we have $\langle T, (1-\chi)\phi \rangle = 0$, hence $\langle T, \phi \rangle = \langle T, \chi\phi \rangle$. □

Proposition 4.49. *A compactly supported distribution is of finite order. More precisely, if K is a compact neighbourhood of* $\operatorname{supp}(T)$ *in Ω, then there exist $m \in \mathbb{N}$ et $C > 0$ such that for any $\phi \in C_0^\infty(\Omega)$ we have*

$$|\langle T, \phi \rangle| \leqslant C \sum_{|\alpha| \leqslant m} \sup_{x \in K} |\partial^\alpha \phi(x)|. \tag{4.13}$$

Proof. Let Ω be an open set of $\mathbb{R}^d$ and $T \in \mathcal{E}'(\Omega)$. By Proposition 1.22 there exists $\chi \in C_0^\infty(\Omega, [0,1])$ supported in K such that $\chi = 1$ in a neighbourhood of $\operatorname{supp}(T)$. There exist $m \in \mathbb{N}$ and $C_K > 0$ such that

$$\forall \phi \in C_K^\infty(\Omega), \quad |\langle T, \phi \rangle| \leqslant C_K \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|_\infty.$$

[1] As for Remark 4.47, it is not enough to assume that χ is equal to 1 on $\operatorname{supp}(T)$.

Then, by Lemma 4.48 and the Leibniz rule, we have for any $\phi \in C_0^\infty(\Omega)$

$$|\langle T, \phi \rangle| = |\langle T, \chi\phi \rangle| \leqslant C_K \sum_{|\alpha| \leqslant m} \|\partial^\alpha(\chi\phi)\|_\infty \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|_\infty ,$$

for some constant C which does not depend of ϕ. This proves (4.13) and in particular T is of finite order. $\square$

We observe that another consequence of lemma 4.48 is that a compactly supported distribution can be extended to a linear form on $C^\infty(\Omega)$. Indeed, if $\phi \in C^\infty(\Omega)$ has a non compact support, we can define $\langle T, \phi \rangle$ as $\langle T, \chi\phi \rangle$, with χ equal to 1 in a neighbourhood of the support of T. This definition does not depend of the choice of χ, and this new map is a linear form on $C^\infty(\Omega)$.

Let us now define a topology on $C^\infty(\Omega)$, to see a compactly supported distribution as a continuous linear map on $C^\infty(\Omega)$. For $n \in \mathbb{N}$ we set

$$K_n = \left\{ x \in \Omega \,|\, |x| \leqslant n \text{ and } \mathsf{dist}(x, \mathbb{R}^d \backslash \Omega) \geqslant 2^{-n} \right\}.$$

This defines a sequence $(K_n)_{n\in\mathbb{N}}$ of compacts of Ω such that

$$\Omega = \bigcup_{n\in\mathbb{N}} K_n.$$

Moreover, if K is a compact set of Ω then there exists $n \in \mathbb{N}$ such that $K \subset K_n$.

For $u, v \in C^\infty(\Omega)$ we set

$$d_\infty(u, v) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \min\left(1, \|u - v\|_{C^n_{K_n}(\Omega)}\right).$$

This defines a distance on $C^\infty(\Omega)$, and $(C^\infty(\Omega), d_\infty)$ is a complete metric space.

With this topology, a sequence $(u_n)_{n\in\mathbb{N}}$ of $C^\infty(\Omega)$ converges to $u \in C^\infty(\Omega)$ if and only if for any $\alpha \in \mathbb{N}^d$ and any compact K of Ω the sequence $(\partial^\alpha u_n)_{n\in\mathbb{N}}$ converges to $\partial^\alpha u$ uniformly on K.

We can now check that if T is a compactly supported distribution on Ω, extended to a linear form on $C^\infty(\Omega)$, then T is in fact continuous on $C^\infty(\Omega)$.

4.4.3 Point-supported distributions

Among the compactly supported distributions, we now consider those supported on a point. We know that the support of a function cannot be a singleton. However, it is the case for the Dirac distribution and its derivatives. We prove that there is no other possibility.

Proposition 4.50. *Let T be a distribution on $\mathbb{R}^d$ such that* $\operatorname{supp}(T) = \{0\}$. *Then there exist $m \in \mathbb{N}$ and constants $c_\alpha \in \mathbb{C}$, $|\alpha| \leqslant m$, such that*

$$T = \sum_{|\alpha| \leqslant m} c_\alpha \delta^{(\alpha)}.$$

Proof. • By the previous proposition, T is of finite order, so there exist $m \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$|\langle T, \phi \rangle| \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|_\infty .$$

Let $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ be equal to 1 in a neighbourhood of 0 and supported in $B(0,1)$. For $\varepsilon \in]0,1]$ and $x \in \mathbb{R}^d$ we set $\chi_\varepsilon(x) = \chi\left(\frac{x}{\varepsilon}\right)$. Then χ_ε is supported in $B(0,\varepsilon)$ and there exists $C_\chi > 0$ such that for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant m$ we have

$$\|\partial^\alpha \chi_\varepsilon\|_\infty \leqslant C_\chi \varepsilon^{-|\alpha|}.$$

- Let $\phi \in C_0^\infty(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ we set

$$\psi(x) = \chi(x) \left(\phi(x) - \sum_{|\alpha| \leqslant m} \frac{\partial^\alpha \phi(0)}{\alpha!} x^\alpha \right).$$

Then $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\partial^\alpha \psi(0) = 0$ for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant m$. By the Taylor formula, there exists $C_\phi > 0$ such that for $|\alpha| \leqslant m$, $\varepsilon \in]0,1]$ and $x \in B(0,\varepsilon)$ we have

$$|\partial^\alpha \psi(x)| \leqslant C_\phi \varepsilon^{m+1-|\alpha|}.$$

Then by the Leibniz formula we have, for any $|\alpha| \leqslant m$,

$$\|\partial^\alpha(\chi_\varepsilon \psi)\| = \sup_{x \in B(0,\varepsilon)} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \left|\partial^{\alpha-\beta}\psi(x)\right| \left|\partial^\beta \chi_\varepsilon(x)\right| = \underset{\varepsilon \to 0}{O}(\varepsilon^{m+1-|\alpha|}) = \underset{\varepsilon \to 0}{O}(\varepsilon).$$

By Lemma 4.48 we have

$$\langle T, \psi \rangle = \langle T, \chi_\varepsilon \psi \rangle = \underset{\varepsilon \to 0}{O}(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0.$$

This proves that $\langle T, \psi \rangle = 0$. Thus,

$$\langle T, \phi \rangle = \langle T, \chi\phi \rangle = \sum_{|\alpha| \leqslant m} \partial^\alpha \phi(0) \left\langle T, \frac{x^\alpha \chi}{\alpha!} \right\rangle = \sum_{|\alpha| \leqslant n} c_\alpha \langle \partial^\alpha \delta, \phi \rangle,$$

where for $|\alpha| \leqslant m$ we have set [2]

$$c_\alpha = (-1)^{|\alpha|} \left\langle T, \frac{x^\alpha \chi}{\alpha!} \right\rangle. \qquad \square$$

This proposition is useful when we want to show that a distribution is a Dirac distribution. We can first show that its support is reduced to a point, which reduces the possibilities. It only remains to check that there cannot be a term involving a derivative of the Dirac distribution.

4.5 Convolution

In this section, we generalize the convolution product to distributions. This cannot be done in complete generality, but we can go further than what will be discussed here.

4.5.1 Derivation and integration under the bracket

We begin by generalizing to the distributions the theorems of derivation under the integral sign and the Fubini Theorem.

[2] Check that this definition of c_α does not depend of the choice of χ

Theorem 4.51 (Derivation under the bracket)**.** *Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\Phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^\nu)$. We assume that there exists a compact set K of $\mathbb{R}^d$ such that* $\operatorname{supp}(T) \subset K$ *or* $\operatorname{supp}(\Phi(\cdot,\lambda)) \subset K$ *for all $\lambda \in \mathbb{R}^\nu$. For $\lambda \in \mathbb{R}^\nu$ we set*

$$F(\lambda) = \langle T, \Phi(\cdot,\lambda)\rangle_{\mathcal{D}'(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d)}\,.$$

Then F is of class C^∞ on $\mathbb{R}^\nu$ and for $\alpha \in \mathbb{N}^\nu$ and $\lambda \in \mathbb{R}^\nu$ we have

$$\partial^\alpha F(\lambda) = \langle T, \partial_\lambda^\alpha \Phi(\cdot,\lambda)\rangle\,.$$

Proof. • Let $\lambda_0 \in \mathbb{R}^\nu$. Let $\tilde{K}$ be a compact neighbourhood of K in $\mathbb{R}^d$. There exist $m \in \mathbb{N}$ and $C > 0$ such that for all $\lambda \in \mathbb{R}^\nu$ we have

$$|F(\lambda) - F(\lambda_0)| = |\langle T, \Phi(\cdot,\lambda) - \Phi(\cdot,\lambda_0)\rangle| \leqslant C \sum_{|\alpha| \leqslant m} \sup_{x \in \tilde{K}} \left|\partial_x^\alpha \big(\Phi(x,\lambda) - \Phi(x,\lambda_0)\big)\right|.$$

Let $r > |\lambda_0|$. By the Mean Value Inequality we have for any $\lambda \in B(r)$

$$|F(\lambda) - F(\lambda_0)| \leqslant C \sum_{|\alpha| \leqslant m, |\beta| \leqslant 1} \sup_{x \in \tilde{K}} \sup_{\lambda_1 \in B(r)} \left|\partial_x^\alpha \partial_\lambda^\beta \Phi(x,\lambda_1)\right| |\lambda - \lambda_0| \xrightarrow[\lambda \to \lambda_0]{} 0.$$

This proves that F is continuous on λ_0, and then on $\mathbb{R}^\nu$.
• We denote by $(e_1,\dots,e_\nu)$ the canonical basis of $\mathbb{R}^\nu$. Let $\lambda \in \mathbb{R}^\nu$ and $j \in [\![1,\nu]\!]$. For $h \in \mathbb{R}^*$ we have

$$\frac{F(\lambda + he_j) - F(\lambda)}{h} - \left\langle T, \partial_{\lambda_j}\Phi(\cdot,\lambda)\right\rangle = \langle T, \Psi_h(\cdot,\lambda)\rangle\,,$$

where

$$\Psi_h(x,\lambda) = \frac{\Phi(x,\lambda + he_j) - \Phi(x,\lambda)}{h} - \partial_{\lambda_j}\Phi(x,\lambda).$$

For any $\lambda \in \mathbb{R}^\nu$ we have $\operatorname{supp}(\Psi_h(\cdot,\lambda)) \subset K$ if $\operatorname{supp}(\Phi(\cdot,\lambda)) \subset K$. Moreover, $\partial^\alpha \Psi_h$ converges uniformly to 0 as h tends to 0 for any $\alpha \in \mathbb{N}^d$, so we get as above that the derivative of F with respect to λ_j exists and is given by $\left\langle T, \partial_{\lambda_j}\Phi(\cdot,\lambda)\right\rangle$. Thus, F is continuous, its partial derivatives exist and are as given by the proposition. We conclude by applying this result to the successive derivatives of Φ with respect to λ. $\square$

Proposition 4.52. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\Phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^\nu)$. We suppose that there exists a compact set K of $\mathbb{R}^d$ such that* $\operatorname{supp}(T) \subset K$ *or* $\operatorname{supp}(\Phi(\cdot,\lambda)) \subset K$ *for any $\lambda \in \mathbb{R}^\nu$. Let $P = \prod_{j=1}^\nu [a_j,b_j]$ be a rectangular cuboid of $\mathbb{R}^\nu$. Then we have*

$$\int_P \langle T, \Phi(\cdot,\lambda)\rangle \,\mathrm{d}\lambda = \left\langle T, \int_P \Phi(\cdot,\lambda)\,\mathrm{d}\lambda \right\rangle.$$

Proof. Let $P' = \prod_{j=2}^\nu [a_j,b_j]$. For $\lambda_1 \in [a_1,b_1]$ we set

$$F(\lambda_1) = \left\langle T, \int_{a_1}^{\lambda_1} \int_{P'} \Phi(\cdot; s_1, \lambda')\,\mathrm{d}\lambda'\,\mathrm{d}s_1 \right\rangle.$$

By the theorem of derivation under the bracket, F is of class C^1 on $[a_1,b_1]$ and for $\lambda_1 \in [a_1,b_1]$ we have

$$F'(\lambda_1) = \left\langle T, \int_{P'} \Phi(\cdot; \lambda_1, \lambda')\,\mathrm{d}\lambda' \right\rangle.$$

Hence

$$\int_{a_1}^{b_1} \left\langle T, \int_{P'} \Phi(\cdot;\lambda_1,\lambda')\,\mathrm{d}\lambda' \right\rangle \mathrm{d}\lambda_1 = F(b_1) - F(a_1) = \left\langle T, \int_{a_1}^{b_1} \int_{P'} \Phi(\cdot;\lambda_1,\lambda')\,\mathrm{d}\lambda'\,\mathrm{d}\lambda_1 \right\rangle.$$

We proceed similarly to "get out of the bracket" the integrals with respect to every other variables. $\square$

4.5.2 Convolution of a distribution with a function

In this paragraph we define the convolution product of a distribution and a regular function, one of them at least being compactly supported.

Recall that for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathbb{R}^d)$ (one at least being compactly supported) the convolution $(f * \phi)$ is defined by

$$(f * \phi)(x) = \int_{\mathbb{R}^d} f(y)\phi(x-y)\,\mathrm{d}y.$$

With the notations introduced for distributions this can also be written as

$$(f * \phi)(x) = \langle T_f, \phi(x - \cdot)\rangle\,.$$

We recall that we have set $\mathcal{P}g = \check{g}$ the function $g : y \mapsto g(-y)$ and $\tau_x g$ is the function $y \mapsto g(y-x)$. Thus we can also write

$$(f * \phi)(x) = \left\langle T_f, \tau_x\check{\phi}\right\rangle$$

Note that if ϕ is compactly supported, this is still the case for $\tau_x\check{\phi}$, so this can be generalized to a general distribution.

Definition 4.53. Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$, or $T \in \mathcal{E}'(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ we set

$$(T * \phi)(x) = \langle T, \phi(x - \cdot)\rangle = \left\langle T, \tau_x\check{\phi}\right\rangle.$$

This defines a function on $\mathbb{R}^d$, called the convolution of T and ϕ.

Example 4.54. As usual the specification for defining an operation on distributions is that it should coincide with the usual operation for a distribution associated with a function. Here the definition has effectively been chosen in such a way that for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathbb{R}^d)$, one at least being is compactly supported, we have

$$(T_f * \phi) = f * \phi.$$

Example 4.55. The convolution of functions had no unit element, because the Dirac mass is not a function. Now that we can define the convolution with a distribution, we observe that

$$\forall \psi \in C^\infty(\mathbb{R}^d), \quad (\delta * \psi) = \psi.$$

We now give some properties of this convolution product. We first prove that two distributions which have the same convolution product with test functions are equal.

Lemma 4.56. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$. We suppose that for $\phi \in C_0^\infty(\mathbb{R}^d)$ we have $T * \phi = 0$. Then $T = 0$.*

Proof. For any $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\langle T, \phi\rangle = (T * \check{\phi})(0) = 0,$$

so $T = 0$. □

Corollary 4.57. *If $T_1, T_2 \in \mathcal{D}'(\mathbb{R}^d)$ are such that $T_1 * \phi = T_2 * \phi$ for any $\phi \in C_0^\infty(\mathbb{R}^d)$ then $T_1 = T_2$.*

Now we generalize the fact that the convolution of two compactly supported functions is compactly supported.

Proposition 4.58. *If $T \in \mathcal{E}'(\mathbb{R}^d)$ is supported in $B(0,R_1)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ is supported in $B(0,R_2)$ with $R_1,R_2 > 0$, then $(T * \phi)$ vanishes outside $B(0,R_1+R_2)$.*

Proof. Let $x \in \mathbb{R}^d$ such that $|x| > R_1 + R_2$. Then the supports of T and $\phi(x - \cdot)$ are disjoint, so $(T * \phi)(x) = 0$. □

We have seen in Proposition 1.18 that if one of the term of the convolution is regular, then the product is also regular and the derivatives of the product can be obtained by differentiating this term. Here, we have an analoguous result, and we can put the derivative on any term.

Proposition 4.59. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$, or $T \in \mathcal{E}'(\mathbb{R}^d)$ et $\phi \in C^\infty(\mathbb{R}^d)$. Then $(T * \phi)$ is a function of class C^∞ on $\mathbb{R}^d$ and for any $\alpha \in \mathbb{N}^d$ we have*

$$\partial^\alpha(T * \phi) = T * (\partial^\alpha \phi) = (\partial^\alpha T) * \phi.$$

Proof. By the theorem of derivation under the bracket, $(T * \phi)$ is of class C^∞ and for any $\alpha \in \mathbb{N}^d$ we have

$$\partial^\alpha(T * \phi) = \left\langle T, \partial_x^\alpha\big(\phi(x - \cdot)\big)\right\rangle = \langle T, (\partial_x^\alpha \phi)(x - \cdot)\rangle = T * (\partial^\alpha \phi).$$

Moreover, for $x, y \in \mathbb{R}^d$ we have

$$\partial_x^\alpha \phi(x-y) = (-1)^{|\alpha|}\partial_y^\alpha \phi(x-y),$$

so we also have

$$\langle T, \partial_x^\alpha \phi(x - \cdot)\rangle = (-1)^{|\alpha|}\left\langle T, \partial^\alpha\big(\phi(x - \cdot)\big)\right\rangle = \langle \partial^\alpha T, \phi(x - \cdot)\rangle.$$ □

We now generalize to this context the good behavior of the convolution with translations, and we show the associativity of the convolution of a distribution with two functions.

Proposition 4.60. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$, or $T \in \mathcal{E}'(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathbb{R}^d)$. Then for $a \in \mathbb{R}^d$ we have*

$$\tau_a(T * \phi) = T * (\tau_a \phi).$$

Proof. At $x \in \mathbb{R}^d$, these two functions are equal to $\langle T, \phi(x - a - \cdot)\rangle$. □

Proposition 4.61. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d)$, or $T \in \mathcal{E}'(\mathbb{R}^d)$ and $\psi \in C^\infty(\mathbb{R}^d)$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. Then we have*

$$(T * \psi) * \phi = T * (\psi * \phi).$$

Proof. Let P be a rectangular cuboid of $\mathbb{R}^d$ containing the support of ϕ. For $x \in \mathbb{R}^d$ we have

$$\big((T * \psi) * \phi\big)(x) = \int_P (T * \psi)(x-y)\phi(y)\,\mathrm{d}y = \int_P \langle T, \psi(x - y - \cdot)\rangle \phi(y)\,\mathrm{d}y.$$

By the theorem of integration under the bracket we have

$$\big((T * \psi) * \phi\big)(x) = \left\langle T, \int_P \psi(x - y - \cdot)\phi(y)\,\mathrm{d}y\right\rangle = \langle T, (\psi * \phi)(x - \cdot)\rangle = \big(T * (\psi * \phi)\big)(x),$$

and the conclusion follows. □

In the next section, we will also use the following equality.

Proposition 4.62. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d)$, or $T \in \mathcal{E}'(\mathbb{R}^d)$ and $\psi \in C^\infty(\mathbb{R}^d)$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. Then we have*

$$\langle T * \psi, \phi \rangle = \langle T, \check{\psi} * \phi \rangle.$$

Proof. As in the previous proof we consider a rectangular cuboid P containing the support of ϕ and we apply the theorem of integration under the bracket to get

$$\begin{aligned}
\langle T * \psi, \phi \rangle = \int_P (T * \psi)(x)\phi(x)\,\mathrm{d}x &= \int_P \langle T, \psi(x - \cdot)\rangle \phi(x)\,\mathrm{d}x \\
&= \left\langle T, \int_P \psi(x - \cdot)\phi(x)\,\mathrm{d}x \right\rangle \\
&= \left\langle T, \int_P \check{\psi}(\cdot - x)\phi(x)\,\mathrm{d}x \right\rangle \\
&= \left\langle T, \check{\psi} * \phi \right\rangle .
\end{aligned}$$

□

4.5.3 Density of regular functions in the space of distributions

We used convolution to prove that $C_0^\infty(\mathbb{R}^d)$ is dense in the Lebesgue spaces $L^p(\mathbb{R}^d)$, $p \in [1, +\infty[$. In the same way, we can prove that $C_0^\infty(\mathbb{R}^d)$ (that can be seen as a part of $\mathcal{D}'(\mathbb{R}^d)$ since each $f \in C_0^\infty(\mathbb{R}^d)$ is identified as a distribution of $\mathbb{R}^d$) is dense in $\mathcal{D}'(\mathbb{R}^d)$. Be careful, this seems to be a strong result since $\mathcal{D}'(\mathbb{R}^d)$ is larger than $L^p(\mathbb{R}^d)$, but the notion of density is not the same, and it is "easier" to be dense for the topology of $\mathcal{D}'(\mathbb{R}^d)$ than for the topology of $L^p(\mathbb{R}^d)$. For example, we get in particular that we can approach any $f \in L^\infty(\mathbb{R}^d)$ by a sequence of functions in $C_0^\infty(\mathbb{R}^d)$ for the topology of $\mathcal{D}'(\mathbb{R}^d)$, which was not the case for the topology of $L^\infty(\mathbb{R}^d)$.

Proposition 4.63. *Let Ω be an open set of $\mathbb{R}^d$ and $T \in \mathcal{D}'(\Omega)$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $C_0^\infty(\Omega)$ which converges to T in the sense of distributions (in other words, T_{f_n} converges to T in $\mathcal{D}'(\Omega)$).*

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence (for the inclusion) of compacts sets in Ω such that $\bigcup_{n\in\mathbb{N}} K_n = \Omega$. For $n \in \mathbb{N}$ we consider $\chi_n \in C_0^\infty(\Omega)$ such that $\chi_n = 1$ in a neighbourhood of K_n and then ε_n such that $B(x, 2\varepsilon_n) \subset \Omega$ for all $x \in \operatorname{supp}(\chi_n)$. We can choose the ε_n, $n \in \mathbb{N}$, in such a way that the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ is non-increasing and goes to 0. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be an approximation of the unit. For $n \in \mathbb{N}$ we set

$$f_n = (\chi_n T) * \rho_{\varepsilon_n}.$$

Let $\phi \in \mathcal{D}(\Omega)$. For $n \in \mathbb{N}$ we have by Proposition 4.62

$$\langle f_n, \phi \rangle = \langle \chi_n T, \widecheck{\rho_{\varepsilon_n}} * \phi \rangle .$$

There exist a compact subset K of Ω and $N \in \mathbb{N}$ such that $\widecheck{\rho_{\varepsilon_n}} * \phi$ is compactly supported in K for any $n \geqslant N$. In addition for any $\alpha \in \mathbb{N}^d$, $\partial^\alpha(\widecheck{\rho_{\varepsilon_n}} * \phi) = \widecheck{\rho_{\varepsilon_n}} * \partial^\alpha \phi$ converges uniformly to $\partial^\alpha \phi$. Thus,

$$\langle T, \widecheck{\rho_{\varepsilon_n}} * \phi \rangle \xrightarrow[n\to+\infty]{} \langle T, \phi \rangle .$$

Choosing N larger if necessary, we can assume that $\chi_n = 1$ on K for any $n \geqslant N$. Then we have

$$\langle f_n, \phi \rangle = \langle T, \chi_n(\widecheck{\rho_{\varepsilon_n}} * \phi) \rangle = \langle T, \widecheck{\rho_{\varepsilon_n}} * \phi \rangle \xrightarrow[n\to+\infty]{} \langle T, \phi \rangle .$$

This means that f_n goes to T in the sense of distributions. □

4.5.4 Fundamental solution of a PDE

Let P be a differential operator with constants coefficients on $\mathbb{R}^d$. This means that there exists $m \in \mathbb{N}$ and constants b_α for $\alpha \in \mathbb{N}^d$, $|\alpha| \leqslant m$ such that

$$P = \sum_{|\alpha| \leqslant m} b_\alpha \partial^\alpha.$$

The formal adjoint of P is defined by

$$P^* \phi = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \partial^\alpha (b_\alpha \phi).$$

We consider the problem

$$Pu = f, \tag{4.14}$$

where f is a given function and u is the unknown.

Definition 4.64. A distribution G on $\mathbb{R}^d$ is said to be a fundamental solution of the equation (4.14) if we have, in the sense of distributions,

$$PG = \delta.$$

This means that for $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\langle G, P^* \phi \rangle_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)} = \phi(0).$$

The following proposition is a direct consequence of Proposition 4.59, according to which $P(G * f) = (PG) * f$ for any $f \in C_0^\infty(\mathbb{R}^d)$.

Proposition 4.65. *Assume that $G \in \mathcal{D}'(\mathbb{R}^d)$ is an fundamental solution of the equation (4.14). Then for any $f \in C_0^\infty(\mathbb{R}^d)$ the function $u = G * f$ is a solution of (4.14) in the sense of distributions.*

Example 4.66. Recall that $H' = \delta$, where H is the Heaviside function. For $f \in C_0^\infty(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$(H * f)(x) = \int_{-\infty}^{x} f(y) \, \mathrm{d}y.$$

The proposition says that $(H * f)' = f$. This is indeed the case.

Example 4.67. Let $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$. We consider on $\mathbb{R}$ the equation

$$-u'' - z^2 u = f.$$

For $z = i$, we have already mentioned this example in the end of the chapter about the Fourier Transform. In particular, we observed that this problem can be simply solved with the variation of parameters. This gives the particular solution

$$u : x \mapsto - \int_{\mathbb{R}} e^{iz|x-y|} 2iz f(y) \, dt,$$

which is exactly what is given by Proposition 4.65 with the fundamental solution (or Green function) G given by

$$G_z(x) = -\frac{e^{iz|x|}}{2iz}.$$

Example 4.68. We have seen in Example 3.30 that in dimension $d \geqslant 3$ the fundamental solution of the Poisson equation

$$-\Delta u = f. \tag{4.15}$$

is given by

$$G(x) = \frac{1}{(d-2)\sigma(S^{d-1})\,|x|^{d-2}}.$$

In general, we want to solve equations like (4.14) with source terms that are more general than $f \in C_0^\infty(\mathbb{R}^d)$. Under some conditions, we can define the convolution of two distributions, and we can state Proposition 4.65 for some $f \in \mathcal{D}'(\mathbb{R}^d)$. In the following section, we define for example the convolution of two distributions, one of which being compactly supported.

But instead of pushing further the general theory, we give results for some particular situations which will include some models we are interested in.

Proposition 4.69. *Let $p, q, r \in [1, +\infty]$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Let us suppose that $G \in L^p(\mathbb{R}^d)$ is a fundamental solution of (4.14). Then for $f \in L^q(\mathbb{R}^d)$ we have $u = G * f \in L^r(\mathbb{R}^d)$ and, in the sense of distributions, $Pu = f$.

Proof. The fact that u belongs to $L^r(\mathbb{R}^d)$ has been proved in Proposition 1.10. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. By the Fubini Theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} (G * f)(x)(P^*\phi)(x)\,\mathrm{d}x &= \int_{x\in\mathbb{R}^d}\int_{y\in\mathbb{R}^d} G(x-y)f(y)(P^*\phi)(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{y\in\mathbb{R}^d} f(y)\left(\int_{x\in\mathbb{R}^d} G(x-y)(P^*\phi)(x)\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{y\in\mathbb{R}^d} f(y)\left(\int_{x\in\mathbb{R}^d} G(x)(P^*\phi)(x+y)\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{y\in\mathbb{R}^d} f(y)\left(\int_{x\in\mathbb{R}^d} G(x)(P^*\phi_{-y})(x)\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{\mathbb{R}^d} f(y)\phi(y)\,\mathrm{d}y.
\end{aligned}$$

This proves that $P(G * f) = f$ in the sense of distributions. □

In Example 4.66 we have $H \in L^\infty(\mathbb{R})$, and the result is valid for any $f \in L^1(\mathbb{R})$. For Example 4.67 we have $G \in L^p(\mathbb{R})$ for any $p \in [1, +\infty]$. We now consider the Helmholtz equation in dimension 3.

Example 4.70. For $z \in \mathbb{C}$ such that $\mathrm{Im}(z) > 0$ we consider on $\mathbb{R}^3$ the equation

$$(-\Delta - z^2)u = f.$$

For $x \in \mathbb{R}^3 \setminus \{0\}$ we set

$$G_z(x) = \frac{e^{iz|x|}}{4\pi\,|x|}.$$

This defines a function $G_z \in L^1(\mathbb{R}^3)$.

On $\mathbb{R}^3 \setminus \{0\}$ we have

$$(-\Delta - z^2)G_z = 0.$$

Let $\phi \in C_0^\infty(\mathbb{R}^3)$. By the Green formula we have

$$\begin{aligned} -\int_{\mathbb{R}^3} G_z(\Delta + z^2)\phi \,\mathrm{d}x &= -\lim_{\varepsilon\to 0}\int_{|x|\geqslant\varepsilon} G_z(\Delta + z^2)\phi\,\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} G_z\,\partial_\nu\phi\,\mathrm{d}x + \lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}\partial_\nu G_z\,\phi\,\mathrm{d}x. \end{aligned}$$

We have

$$\int_{|x|=\varepsilon} G_z\,\partial_\nu\phi\,\mathrm{d}x = \int_{|x|=\varepsilon}\frac{e^{iz\varepsilon}}{4\pi\varepsilon}\,\partial_\nu\phi\,\mathrm{d}x = \mathcal{O}(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0,$$

and

$$\int_{|x|=\varepsilon}\partial_\nu G_z\,\phi\,\mathrm{d}x = \int_{|x|=\varepsilon}\frac{e^{iz\varepsilon}}{4\pi\varepsilon^2}(1 - iz\varepsilon)\,\phi\,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} \phi(0).$$

This proves that $(-\Delta - z^2)G_z = \delta$ in the sense of distributions. Thus, if for $f \in L^2(\mathbb{R}^3)$ we set

$$u(x) = \int_{\mathbb{R}^3}\frac{e^{iz|x-y|}}{4\pi\,|x-y|}f(y)\,\mathrm{d}y,$$

then $u \in L^2(\mathbb{R}^3)$ and in the sense of distributions we have $(-\Delta - z^2)u = f$.

Example 4.71. For $(t,x) \in \mathbb{R}^2$ we set

$$G(t,x) = H(t)\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}.$$

Then $G \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and in the sense of distributions we have

$$(\partial_t - \partial_{xx})G = \delta.$$

Indeed, for $\phi \in C_0^\infty(\mathbb{R}^2)$ we have

$$\begin{aligned} \langle(\partial_t - \partial_{xx})G, \phi\rangle &= \langle G, (-\partial_t - \partial_{xx})\phi\rangle \\ &= -\lim_{\varepsilon\to 0^+}\int_{t=\varepsilon}^{+\infty}\int_{x\in\mathbb{R}}\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}(\partial_t + \partial_{xx})\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

For $t > 0$ and $x \in \mathbb{R}$ we have

$$\partial_t G(t,x) = \partial_{xx}G(t,x) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi}}\left(-\frac{1}{2t^{\frac{3}{2}}} + \frac{x^2}{4t^{\frac{5}{2}}}\right).$$

Thus, after using an integration by part we get

$$\begin{aligned} \langle(\partial_t - \partial_{xx})G, \phi\rangle &= \lim_{\varepsilon\to 0^+}\int_{x\in\mathbb{R}}\frac{e^{-\frac{x^2}{4\varepsilon}}}{\sqrt{4\pi\varepsilon}}\phi(\varepsilon,x)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0^+}\int_{\eta\in\mathbb{R}}\frac{e^{-\eta^2}}{\sqrt{\pi}}\phi(\varepsilon, 2\sqrt{\varepsilon}\eta)\,\mathrm{d}\eta \\ &= \phi(0,0). \end{aligned}$$

For $t > 0$ the function $G(t) = G(t,\cdot)$ is integrable on $\mathbb{R}$, and we can see G as a continuous and bounded function from $\mathbb{R}_+^*$ to $L^1(\mathbb{R})$. Let us consider now a function f defined on $\mathbb{R}_+^* \times \mathbb{R}$ such that for any $t > 0$ the function $f(t) = f(t,\cdot)$ belongs to $L^p(\mathbb{R})$ for some $p \in [1,+\infty]$. Let us suppose that f is continuous from $\mathbb{R}_+^*$ to $L^p(\mathbb{R})$, with

$$\int_0^{+\infty}\|f(t)\|_{L^p(\mathbb{R})}\,\mathrm{d}t < +\infty.$$

We can consider the function u from $\mathbb{R}_+^*$ to $L^p(\mathbb{R})$ defined as

$$u(t) = \int_{s=0}^{t} G(t-s) * f(s)\,\mathrm{d}s.$$

This means that for $t > 0$ the function $u(t) \in L^p(\mathbb{R})$ is defined for $x \in \mathbb{R}$ by

$$u(t,x) = \int_{s=0}^{t} \int_{y\in\mathbb{R}} G(t-s, x-y) f(s,y)\,\mathrm{d}y\,\mathrm{d}s. \tag{4.16}$$

We can then check that in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}$ we have

$$\partial_t u - \partial_{xx} u = f.$$

We could go even further. Even if it is not included in the framework we have discussed so far, we could imagine a source term f supported at time $t = 0$. Formally, we take $f(t,x) = \delta(t)u_0(x)$, where $u_0 \in L^p(\mathbb{R})$. Concretely, we define f as the distribution

$$\phi \in C_0^\infty(\mathbb{R}^2) \mapsto \int_{x\in\mathbb{R}} u_0(x)\phi(0,x)\,\mathrm{d}x.$$

Formally, the expression (4.16) becomes

$$u(t,x) = \int_{\mathbb{R}} G(t, x-y) u_0(y)\,\mathrm{d}y = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\,\mathrm{d}y. \tag{4.17}$$

And even if we did not give any general result including this case, we can verify that the function u defined this way is bounded as a function from $\mathbb{R}_+^*$ to $L^p(\mathbb{R})$, it satisfies in the sense of distributions the heat equation

$$\partial_t u - \partial_{xx} u = 0, \quad \text{sur } \mathbb{R}_+^* \times \mathbb{R},$$

and it verifies the initial condition $u(0) = u_0$ in the sense that

$$\|u(t,\cdot) - u_0\|_{L^p(\mathbb{R})} \xrightarrow[t\to 0^+]{} 0.$$

4.5.5 Convolution of two distributions, one of which at least being compactly supported

Now we define the convolution of two distributions. This section can be omitted. One should simply remember that under certain conditions one can define the convolution of two distributions, and that this convolution verifies the good expected properties (in particular, the Dirac distribution is indeed a unit for this product on $\mathcal{E}'(\mathbb{R}^d)$).

Proposition-Definition 4.72. *Let T and S be two distributions on $\mathbb{R}^d$, one of which at least being compactly supported. We denote by $T * S$ the unique distribution on $\mathbb{R}^d$ such that*

$$\forall \phi \in C_0^\infty(\mathbb{R}^d), \quad (T * S) * \phi = T * (S * \phi).$$

Proof. Uniqueness comes from Corollary 4.57. For the existence we consider on $C_0^\infty(\mathbb{R}^d)$ the linear form

$$L : \phi \mapsto (T * (S * (\mathcal{P}\phi)))(0).$$

If $S \in \mathcal{E}'(\mathbb{R}^d)$ then $S * (\mathcal{P}\phi) \in C_0^\infty(\mathbb{R}^d)$, so $T * (S * (\mathcal{P}\phi))$ is a function of class C^∞, whereas if $T \in \mathcal{E}'(\mathbb{R}^d)$ then T defines a linear map from $C^\infty(\mathbb{R}^d)$ to itself, and $S * (\mathcal{P}\phi)$ belongs to $C^\infty(\mathbb{R}^d)$. In any case, $L(\phi)$ is well defined.

Let K be a compact set of $\mathbb{R}^d$. Let $R > 0$ such that $K \subset B(0,R)$ and T or S is supported in $B(0,R)$.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be supported in K. If S is supported in $B(0,R)$ then $S * \phi$ has is supported in $B(0,2R)$. Otherwise T is supported in $B(0,R)$. In any case there exist $m_T \in \mathbb{N}$ and $C_T > 0$ which do not depend of T and R such that

$$|L(\phi)| \leqslant C_T \sum_{|\alpha| \leqslant m_T} \sup_{|x| \leqslant 2R} |\partial^\alpha (S * (\mathcal{P}\phi))(x)| = C_T \sum_{|\alpha| \leqslant m_T} \sup_{|x| \leqslant 2R} |(S * \partial^\alpha(\mathcal{P}\phi))(x)| .$$

But there exist $m_S \in \mathbb{N}$ and $C_S > 0$ such that for $\psi \in C_0^\infty(\mathbb{R}^d)$ supported in $B(0,R)$ and $x \in B(0,2R)$ we have

$$|(S * \psi)(x)| = |\langle S, \psi(x - \cdot)\rangle| \leqslant C_S \sum_{|\beta| \leqslant m_S} \left\| \partial^\beta \psi \right\|_\infty .$$

Applied with $\psi = \partial^\alpha \mathcal{P}\phi$ we finally get

$$|L(\phi)| \leqslant C_T C_S \sum_{|\beta| \leqslant m_T + m_S} \left\| \partial^\beta \phi \right\|_\infty .$$

This proves that the map L is a distribution on $\mathbb{R}^d$.

Let us prove that $L * \phi = T * (S * \phi)$ for any $\phi \in C_0^\infty(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ we have

$$\begin{aligned}(L * \phi)(x) = \langle L, \tau_x \mathcal{P}\phi\rangle &= (T * (S * (\mathcal{P}\tau_x \mathcal{P}\phi)))(0) = (T * (S * (\tau_{-x}\phi)))(0) \\ &= (T * (\tau_{-x}(S * \phi)))(0) = (\tau_{-x}(T * (S * \phi)))(0) \\ &= (T * (S * \phi))(x).\end{aligned}$$

Thus $T * S$ is defined as the distribution L. □

Examples 4.73.

- For $T \in \mathcal{D}'(\mathbb{R}^d)$ we have $T * \delta = \delta * T = T$.
- Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $f \in C_0^\infty(\mathbb{R}^d)$ or $T \in \mathcal{E}'(\mathbb{R}^d)$ and $f \in C^\infty(\mathbb{R}^d)$. Then we have $T * T_f = T_{T*f}$.

We prove the commutativity, the good behaviour of the derivation and the associativity for this convolution.

Proposition 4.74. *Let T and S be two distributions on $\mathbb{R}^d$, one of which at least being compactly supported. Then we have*

$$T * S = S * T.$$

Proof. By Corollary 4.57 it is enough to prove that for any $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$(T * S) * \phi = (S * T) * \phi.$$

For this it is enough to prove that for $\psi \in C_0^\infty(\mathbb{R}^d)$ we have

$$((T * S) * \phi) * \psi = ((S * T) * \phi) * \psi.$$

Let $\phi, \psi \in C_0^\infty(\mathbb{R}^d)$. By Proposition 4.61 (twice), the definition of $T * S$ and the commutativity of the convolution of functions we have

$$\begin{aligned}((T * S) * \phi) * \psi = (T * S) * (\phi * \psi) &= T * (S * (\phi * \psi)) \\ &= T * ((S * \phi) * \psi) = T * (\psi * (S * \phi)) \\ &= (T * \psi) * (S * \phi).\end{aligned}$$

Similarly,

$$\begin{aligned}((S * T) * \phi) * \psi &= (S * T) * (\phi * \psi) = (S * T) * (\psi * \phi)\\ &= S * (T * (\psi * \phi)) = S * ((T * \psi) * \phi)\\ &= S * (\phi * (T * \psi)) = (S * \phi) * (T * \psi)\\ &= (T * \psi) * (S * \phi).\end{aligned}$$

□

Proposition 4.75. *Let T and S be two distributions on $\mathbb{R}^d$, one of which at least being compactly supported. For $\alpha \in \mathbb{N}^d$ we have*

$$\partial^\alpha (T * S) = (\partial^\alpha T) * S = T * (\partial^\alpha S).$$

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. By Proposition 4.59 we have

$$(\partial^\alpha (T*S))*\phi = (T*S)*(\partial^\alpha \phi) = T*(S*(\partial^\alpha \phi)) = T*(\partial^\alpha (S*\phi)) = (\partial^\alpha T)*(S*\phi) = ((\partial^\alpha T)*S)*\phi$$

and

$$(\partial^\alpha (T * S)) * \phi = T * (S * (\partial^\alpha \phi)) = T * ((\partial^\alpha S) * \phi)) = (T * (\partial^\alpha S)) * \phi.$$

□

Proposition 4.76. *Let T, R and S be two distributions on $\mathbb{R}^d$, one of which at least being compactly supported. Then we have*

$$(T * S) * R = T * (S * R).$$

Proof. For $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}((T * S) * R) * \phi &= (T * S) * (R * \phi) = T * (S * (R * \phi)) = T * ((S * R) * \phi)\\ &= (T * (S * R)) * \phi.\end{aligned}$$

□

4.6 Fourier Transform

4.6.1 Tempered distributions

The Fourier transform has been naturally defined on $L^1(\mathbb{R}^d)$, then to its restriction on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ (or on $\mathcal{S}(\mathbb{R}^d)$) and then it has been extended to $L^2(\mathbb{R}^d)$. Now we want to extend the Fourier transform to distributions.

Most of the usual operations on functions (derivation, multiplication by a regular function, etc.) are extended to distributions by reporting the operation on the test functions to which we apply the distribution. It is reasonable to want to proceed in the same way for the Fourier transform.

Recall that in Section 2.4 we have chosen $\mathcal{S}(\mathbb{R}^d)$ instead of $C_0^\infty(\mathbb{R}^d)$ as the space in which we can derive and multiply by polynomial functions as much as we wish. The reason was Proposition 2.18 which shows that the Schwartz space has a nice behavior with respect to the Fourier transform. This is not the case for $C_0^\infty(\mathbb{R}^d)$ which is not even stable under Fourier transform. Even worse, we can check that a function $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\hat{\phi} \in C_0^\infty(\mathbb{R}^d)$ is 0 (for $d = 1$, the Fourier transform of ϕ can be extended to a holomorphic function on $\mathbb{C}$ whose restriction on $\mathbb{R}$ cannot be compactly supported if it is not 0). However, since $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, the Fourier transform of a function in $C_0^\infty(\mathbb{R}^d)$ at least belongs to $\mathcal{S}(\mathbb{R}^d)$.

Thus, the Fourier transform $\mathcal{F}\phi$ of a function $\phi \in C_0^\infty(\mathbb{R}^d)$ is not in $C_0^\infty(\mathbb{R}^d)$ in general, so we cannot apply a distribution to it. To generalize the Fourier transform to distributions, we have to ... change the definition of a distribution. Thus we will not extend the Fourier transform to continuous linear forms on $C_0^\infty(\mathbb{R}^d)$, but to continuous linear forms on $\mathcal{S}(\mathbb{R}^d)$.

The purpose of this paragraph is therefore to introduce the continuous linear forms on $\mathcal{S}(\mathbb{R}^d)$. They will be called tempered distributions. For this we first have to describe the topology of $\mathcal{S}(\mathbb{R}^d)$.

This is another advantage of the tempered distributions over the usual distributions, the topology of $\mathcal{S}(\mathbb{R}^d)$ is simpler to describe than the topology of $C_0^\infty(\mathbb{R}^d)$.

For $k \in \mathbb{N}$ et $\phi \in \mathcal{S}(\mathbb{R}^d)$ we set

$$\mathcal{N}_k(\phi) = \sup_{|\alpha|,|\beta| \leqslant k} \left\| x^\alpha \partial^\beta \phi \right\|_\infty .$$

We observe that this defines a norm on $\mathcal{S}(\mathbb{R}^d)$, but a norm for which $\mathcal{S}(\mathbb{R}^d)$ is not complete. As we did for $C_K^\infty(\Omega)$, we define on $\mathcal{S}(\mathbb{R}^d)$ a topology that is associated to every norm at the same time. It is given by the distance defined on $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$ by

$$d_{\mathcal{S}}(\phi, \psi) = \sum_{k \in \mathbb{N}} \frac{1}{2^k} \min\big(1, \mathcal{N}_k(\phi - \psi)\big).$$

Proposition 4.77. *$(\mathcal{S}(\mathbb{R}^d), d_{\mathcal{S}})$ is a complete metric space.*

Proposition 4.78. (i) *Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{S}(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then ϕ_n tends to ϕ in $\mathcal{S}(\mathbb{R}^d)$ if and only if*

$$\forall k \in \mathbb{N}, \quad \mathcal{N}_k(\phi_n - \phi) \xrightarrow[n \to +\infty]{} 0.$$

(ii) *A linear form T on $\mathcal{S}(\mathbb{R}^d)$ is continuous if and only if there exist $k \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$|T(\phi)| \leqslant C \mathcal{N}_k(\phi).$$

More generally, if $(E, \|\cdot\|_E)$ is a normed vector space, then a linear map $T : \mathcal{S}(\mathbb{R}^d) \to E$ is continuous if and only if there exist $k \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\|T(\phi)\|_E \leqslant C \mathcal{N}_k(\phi).$$

(iii) *A linear map $T : \mathcal{S} \to \mathcal{S}$ is continuous if for any $j \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$\mathcal{N}_j(T(\phi)) \leqslant C \mathcal{N}_k(\phi).$$

Now that we have defined the topology of $\mathcal{S}(\mathbb{R}^d)$, we can give the basic properties.

Proposition 4.79. (i) *Let $\alpha \in \mathbb{N}^d$. The map $\phi \mapsto x^\alpha \phi$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to itself. More generally, if f is a function of class C^∞ on $\mathbb{R}^d$ with slowly increasing derivatives (see definition 2.3), then the multiplication by f is a continuous map on $\mathcal{S}(\mathbb{R}^d)$.*

(ii) *Let $\alpha \in \mathbb{N}^d$. The map $\phi \mapsto \partial^\alpha \phi$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to itself.*

(iii) *Let $p \in [1, +\infty]$. The inclusion $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ is continuous.*

(iv) *The Fourier transform and its inverse are continuous functions on $\mathcal{S}(\mathbb{R}^d)$.*

Proof. The first two properties are left as exercises.

- For $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have $\|\phi\|_\infty = \mathcal{N}_0(\phi)$, so the inclusion $\mathcal{S}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ is continuous.

Now let $p \in [1, +\infty[$. Let $k \in \mathbb{N}$ such that $kp > d$. For $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\|\phi\|_p^p = \int_{\mathbb{R}^d} (1+|x|)^{-kp}(1+|x|)^{kp} |\phi(x)|^p \, \mathrm{d}x \leqslant C\mathcal{N}_k(\phi)^p$$

with

$$C = 2^k \int_{\mathbb{R}^d} \frac{1}{(1+|x|)^{kp}} \, \mathrm{d}x < +\infty.$$

This proves that the inclusion $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ is continuous.

- Let $\alpha, \beta \in \mathbb{N}^d$. Using propositions 2.18 and 2.11 we have for any $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$\sup_{\xi \in \mathbb{R}^d} \left| \xi^\alpha \partial^\beta \hat{\phi}(\xi) \right| = \sup_{x \in \mathbb{R}^d} \left| \widehat{\partial^\alpha x^\beta \phi}(\xi) \right| \leqslant \left\| \partial^\alpha x^\beta \phi \right\|_{L^1(\mathbb{R}^d)}.$$

Since multiplication by x^β, derivation ∂^α and inclusion in $L^1(\mathbb{R}^d)$ are continuous maps on $\mathcal{S}(\mathbb{R}^d)$, there exists $k \in \mathbb{N}$ and $C > 0$ independants of ϕ such that

$$\left\| \partial^\alpha x^\beta \phi \right\|_{L^1(\mathbb{R}^d)} \leqslant C\mathcal{N}_k(\phi).$$

This proves that the Fourier transform is continuous on $\mathcal{S}(\mathbb{R}^d)$. □

Proposition 4.80. (i) *$C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$.*

(ii) *The inclusion of $C_0^\infty(\mathbb{R}^d)$ in $\mathcal{S}(\mathbb{R}^d)$ is continuous.*

Proof. • Let $\chi \in C_0^\infty(\mathbb{R}^d, [0,1])$ supported in $B(0,2)$ and equal to 1 on $B(0,1)$. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}^d$ we set $\chi_n(x) = \chi\left(\frac{x}{n}\right)$.

Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. For any $n \in \mathbb{N}^*$ we have $\chi_n \phi \in C_0^\infty(\mathbb{R}^d)$. Let $\alpha, \beta \in \mathbb{N}^d$. By the Leibniz rule we have

$$\partial^\beta\big((1-\chi_n)\phi\big) = \sum_{\tilde{\beta} \leqslant \beta} \binom{\beta}{\tilde{\beta}} \partial^{\beta - \tilde{\beta}}(1-\chi_n)\partial^{\tilde{\beta}}\phi.$$

Since $\partial^{\beta-\tilde{\beta}}(1-\chi_n)$ is supported outside $B(n)$ we have

$$\left\| x^\alpha \partial^{\beta-\tilde{\beta}}(1-\chi_n)\partial^{\tilde{\beta}}\phi \right\|_{L^\infty(\mathbb{R}^d)} \leqslant \left\| |x|^{-1} \partial^{\beta-\tilde{\beta}}(1-\chi_n) \right\|_{L^\infty(\mathbb{R}^d)} \mathcal{N}_{|\alpha|+|\beta|+1}(\phi) \xrightarrow[n \to +\infty]{} 0.$$

Hence

$$\left\| x^\alpha \partial^\beta(\phi - \chi_n\phi) \right\|_{L^\infty(\mathbb{R}^d)} \xrightarrow[n \to +\infty]{} 0,$$

and so, for any $k \in \mathbb{N}$,

$$\mathcal{N}_k(\phi - \chi_n\phi) \xrightarrow[n \to +\infty]{} 0.$$

Thus the sequence $(\chi_n\phi)_{n \in \mathbb{N}^*}$ tends to ϕ in $\mathcal{S}(\mathbb{R}^d)$, which proves the first point.

- Let us now consider a sequence $(\phi_n)_{n \in \mathbb{N}}$ that converges to ϕ in $C_0^\infty(\mathbb{R}^d)$. There exists a compact set K of $\mathbb{R}^d$, $k \in \mathbb{N}$ and $C > 0$ such that ϕ_n is supported in K for any $n \in \mathbb{N}$ and $\partial^\alpha \phi_n$ converges uniformly to $\partial^\alpha \phi$ uniformly on $\mathbb{R}^d$. This proves that ϕ_n converges to ϕ in $\mathcal{S}(\mathbb{R}^d)$ and gives the second property. □

Now that the topology of $\mathcal{S}(\mathbb{R}^d)$ is well understood, we can consider the continuous linear forms theron.

Definition 4.81. A tempered distribution on $\mathbb{R}^d$ is a continuous linear form of $\mathcal{S}(\mathbb{R}^d)$. The space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

Since $C_0^\infty(\mathbb{R}^d)$ is continuously included and dense in $\mathcal{S}(\mathbb{R}^d)$, we have the following link with the usual distributions.

Proposition 4.82. (i) *By restriction, a continuous linear form on $\mathcal{S}(\mathbb{R}^d)$ induces a continuous linear form on $C_0^\infty(\mathbb{R}^d)$. Thus we have $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$.*

(ii) *Let $T \in \mathcal{D}'(\mathbb{R}^d)$. Assume that there exist $k \in \mathbb{N}$ and $C > 0$ such that for any $\phi \in C_0^\infty(\mathbb{R}^d)$ we have*

$$|\langle T, \phi \rangle| \leqslant C\mathcal{N}_k(\phi).$$

Then T can be uniquely extended to a tempered distribution.

Examples 4.83.
- Compactly supported distributions (for example, a Dirac mass at one point) can be extended into tempered distributions.
- If $f \in L^1(\mathbb{R}^d)$ then $T_f : \phi \mapsto \int_{\mathbb{R}^d} f\phi$ is a tempered distribution.
- If f is measurable and slowly increasing on $\mathbb{R}^d$ then T_f is a tempered distribution.
- The function e^x defines a distribution on $\mathbb{R}$ but not a tempered distribution.

Basic operations on tempered distributions are defined as for usual distributions.

Proposition-Definition 4.84. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$.*

(i) *Let $\alpha \in \mathbb{N}^d$. We define the tempered distribution $\partial^\alpha T$ by*

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle .$$

(ii) *Let f be a smooth function with slowly increasing derivatives. We define the tempered distribution fT by*

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle fT, \phi \rangle = \langle T, f\phi \rangle .$$

(iii) *Let $y \in \mathbb{R}^d$. We define the tempered distribution $\tau_y T$ by*

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle \tau_y T, \phi \rangle = \langle T, \tau_{-y}\phi \rangle .$$

Remark 4.85. Let $T \in \mathcal{S}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$. By restriction, T defines a distribution $\tilde{T} \in \mathcal{D}'(\mathbb{R}^d)$ on $\mathbb{R}^d$, and its derivative $\partial^\alpha \tilde{T} \in \mathcal{D}'(\mathbb{R}^d)$ can be extended into a tempered distribution. This extension is exactly the tempered distribution $\partial^\alpha T$ that we just defined.

Definition 4.86. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of tempered distributions. Then T_n tends to T in $\mathcal{S}'(\mathbb{R}^d)$ if

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle T_n, \phi \rangle \xrightarrow[n\to+\infty]{} \langle T, \phi \rangle .$$

4.6.2 Fourier transform of tempered distributions

Now that we have introduced the tempered distributions, we can define their Fourier transforms. As for the other operations, the definition of the Fourier transform for distributions should generalize the definition already known for functions. The definition is based on Corollary 2.22.

Proposition-Definition 4.87. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$. Then the map $\hat{T}$ defined by*

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$$

is a tempered distribution on $\mathbb{R}^d$, called the Fourier transform of T. It can also be denoted by $\mathcal{F}T$.

By definition, we have

$$\left\langle \widehat{T_f}, \phi \right\rangle = \left\langle T_{\hat{f}}, \phi \right\rangle$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. This is also the case for $f \in L^2(\mathbb{R}^d)$ by Remark 2.24, and it is easy to see that it holds for $f \in L^1(\mathbb{R}^d)$, using the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$ or observing that the computation we made in the corollary 2.22 is still valid for $f \in L^1(\mathbb{R}^d)$.

Let us now give an example of Fourier transform for a distribution which is not a function.

Example 4.88. For $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \int_{\mathbb{R}} \phi(x)\,\mathrm{d}x,$$

so

$$\hat{\delta} = 1.$$

Conversely, by the inversion formula for the Fourier transform we have

$$\langle 1, \hat{\phi} \rangle = \int_{\mathbb{R}^d} \hat{\phi}(\xi)\,\mathrm{d}\xi = (2\pi)^d \phi(0),$$

so

$$\hat{1} = (2\pi)^d \delta.$$

We now conclude this chapter by extending to distributions some usual properties of the Fourier transform.

Proposition 4.89. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$. We note e_a the function*[3] $y \mapsto e^{-iy\cdot a} = e^{-ia\cdot y}$*. We have*

$$\mathcal{F}(\tau_a T) = e_a \mathcal{F}T$$

and

$$\mathcal{F}(e_{-a} T) = \tau_a \mathcal{F}T.$$

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. By Proposition 2.12.(iii) we have $\mathcal{F}(\tau_a \phi) = e_a \mathcal{F}\phi$ and $\mathcal{F}(e_{-a}\phi) = \tau_a \mathcal{F}\phi$. We see that this can be extended to distributions by writing

$$\langle \mathcal{F}(\tau_a T), \phi \rangle = \langle \tau_a T, \mathcal{F}\phi \rangle = \langle T, \tau_{-a} \mathcal{F}\phi \rangle = \langle T, \mathcal{F}(e_a \phi) \rangle = \langle \mathcal{F}T, e_a \phi \rangle = \langle e_a \mathcal{F}T, \phi \rangle$$

and

$$\langle \mathcal{F}(e_{-a} T), \phi \rangle = \langle e_{-a} T, \mathcal{F}\phi \rangle = \langle T, e_{-a} \mathcal{F}\phi \rangle = \langle T, \mathcal{F}(\tau_{-a}\phi) \rangle = \langle \mathcal{F}T, \tau_{-a}\phi \rangle = \langle \tau_a \mathcal{F}T, \phi \rangle .$$

∎

[3] We give two expressions to emphisize the fact that a and y can both play the roles of x and ξ, if we refer to the notations of Proposition 2.12.(iii).

Now we extend to distributions the results of Propositions 2.16 and 2.17.

Proposition 4.90. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$. Then we have*

$$\mathcal{F}(\partial^\alpha T) = (i\xi)^\alpha \mathcal{F}(T)$$

and

$$\mathcal{F}(x^\alpha T) = (i\partial)^\alpha \mathcal{F}(T).$$

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. By Propositions 2.16 and 2.17 we have [4]

$$\langle \mathcal{F}(\partial^\alpha T), \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \mathcal{F}\phi \rangle = (-1)^{|\alpha|} \langle T, \mathcal{F}((-iy)^\alpha \phi) \rangle = \langle \mathcal{F}T, (iy)^\alpha \phi \rangle = \langle (iy)^\alpha T, \mathcal{F}\phi \rangle$$

and

$$\langle \mathcal{F}(y^\alpha T), \phi \rangle = \langle T, y^\alpha \mathcal{F}\phi \rangle = \langle T, \mathcal{F}((-i\partial)^\alpha \phi) \rangle = \langle \mathcal{F}T, (-i\partial)^\alpha \phi \rangle = \langle (i\partial)^\alpha \mathcal{F}T, \phi \rangle .$$

The conclusions follow. □

We now generalize the inversion formula of Proposition 2.14. Recall that we have defined on $L^1(\mathbb{R}^d)$ the operator $\overline{\mathcal{F}}$, analogous to the Fourier transform. This definition can be extended to a tempered distribution T by

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \left\langle \overline{\mathcal{F}}T, \phi \right\rangle = \left\langle T, \overline{\mathcal{F}}\phi \right\rangle .$$

The operator $\mathcal{P}$ can equally be extended to tempered distributions by the definition

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad \langle \mathcal{P}T, \phi \rangle = \langle T, \mathcal{P}\phi \rangle .$$

The equalities (2.6) are then still valid for a tempered distribution.

Proposition 4.91. *For $T \in \mathcal{S}'(\mathbb{R}^d)$ we have*

$$T = \overline{\mathcal{F}}\mathcal{F}T = \frac{1}{(2\pi)^d}\mathcal{P}\mathcal{F}\mathcal{F}T = \frac{1}{(2\pi)^d}\mathcal{F}\mathcal{F}\mathcal{P}T = \mathcal{F}\overline{\mathcal{F}}T.$$

The following result gives an expression analoguous to (2.4) for compactly supported distributions.

Proposition 4.92. *Let T be a compactly supported distribution on $\mathbb{R}^d$. Then $\hat{T}$ is (the distribution associated to) a C^∞ function with slowly increasing derivatives, and for $\xi \in \mathbb{R}^d$ we have*

$$\hat{T}(\xi) = \langle T, e_\xi \rangle .$$

Proof. • For $\xi \in \mathbb{R}^d$ we set $f(\xi) = \langle T, e_\xi \rangle$. By the theorem of derivation under the bracket, f is of class C^∞ on $\mathbb{R}^d$ and for any $\alpha \in \mathbb{N}^d$ and $\xi \in \mathbb{R}^d$ we have

$$\partial^\alpha f(\xi) = \langle T, (-ix)^\alpha e_\xi \rangle .$$

• Let K be a compact neighbourhood of the support of T. There exist $m \in \mathbb{N}$ and $C > 0$ such that for any $\xi \in \mathbb{R}^d$

$$|\partial^\alpha f(\xi)| \leqslant C \sum_{|\beta| \leqslant m} \sup_{x \in K} \left| \partial_x^\beta (x^\alpha e^{-ix\cdot\xi}) \right| .$$

[4] Everywhere we note y the dummy variable, it can play the role of the variables x or ξ.

Thus there exists $\tilde{C} > 0$ such that for any $\xi \in \mathbb{R}^d$ we have

$$|\partial^\alpha f(\xi)| \leqslant \tilde{C}(1 + |\xi|^m).$$

This proves that the derivatives of f are slowly increasing.
• Let $\phi \in C_0^\infty(\mathbb{R}^d)$. Let P be a rectangular cuboid of $\mathbb{R}^d$ containing the support of ϕ. By the theorem of integration under the bracket we have

$$\langle T_f, \phi \rangle = \int_P \langle T, e_y \rangle \phi(y) \, \mathrm{d}y = \left\langle T, \int_P \phi(y) e_y \, \mathrm{d}y \right\rangle = \langle T, \mathcal{F}\phi \rangle = \langle \mathcal{F}T, \phi \rangle,$$

and so $\mathcal{F}T = T_f$. □

If T and S are two compactly supported distributions, we have defined the convolution $T * S$, and the product $\hat{T}\hat{G}$ has a meaning since $\hat{F}$ and $\hat{G}$ are regular functions. In that case we can generalize proposition 2.13.

Proposition 4.93. (i) *Let $T \in \mathcal{E}'(\mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d)$. Then we have*

$$\mathcal{F}(T * \psi) = \mathcal{F}(T)\, \mathcal{F}(\psi).$$

(ii) *Let $T, S \in \mathcal{E}'(\mathbb{R}^d)$. Then we have*

$$\mathcal{F}(T * S) = \mathcal{F}(T)\, \mathcal{F}(S).$$

Proof. By proposition 4.92, all the Fourier transforms are functions.
(i) Let $\xi \in \mathbb{R}^d$. Since $(T * \psi) \in C_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
\mathcal{F}(T * \psi)(\xi) &= \int_{\mathbb{R}^d} e^{-ix\cdot\xi} (T * \psi)(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \langle T, \psi(x - \cdot) \rangle \, \mathrm{d}x \\
&= \left\langle T, \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \psi(x - \cdot) \, \mathrm{d}x \right\rangle \\
&= \left\langle T, \int_{\mathbb{R}^d} e^{-i(x+\cdot)\cdot\xi} \psi(x) \, \mathrm{d}x \right\rangle \\
&= \left\langle T, e_\xi \hat{\psi}(\xi) \right\rangle \\
&= \hat{T}(\xi)\hat{\psi}(\xi).
\end{aligned}$$

(ii) We have

$$\mathcal{F}(T*S)(\xi) = \langle T * S, e_\xi \rangle = ((T*S)*(\mathcal{P}e_\xi))(0) = (T*(S*(\mathcal{P}e_\xi)))(0) = \langle T, \mathcal{P}(S * (\mathcal{P}e_\xi)) \rangle.$$

But for $x \in \mathbb{R}^d$ we have

$$(S * (\mathcal{P}e_\xi))(-x) = \langle S, (\mathcal{P}e_\xi)(-x - \cdot) \rangle = \langle S, e_\xi(x + \cdot) \rangle = \hat{S}(\xi) e^{-ix\cdot\xi},$$

so

$$\mathcal{F}(T * S)(\xi) = \left\langle T, \hat{S}(\xi) e_\xi \right\rangle = (\mathcal{F}T)(\xi)(\mathcal{F}S)(\xi).$$ □