Chapter 3

Second order elliptic equations

In this chapter we discuss on some open subset Ω of $\mathbb{R}^d$ an equation of the form

$$Pu = f, \tag{3.1}$$

where f is some given function, u is the unknown and P is a so-called *elliptic* operator. The model of an elliptic operator is the Laplace operator $P = -\Delta$ (in this case (3.1) is refered to as the Poisson equation, see Section 3.5 below). When $\Omega \neq \mathbb{R}^d$, we will have to add boundary conditions to the equation to get a well posed problem (see Section 3.3).

We will only consider second order equations. This means that P will be a partial differential operator of second order:

$$P = -\operatorname{div} A(x)\nabla + B(x)\nabla + c(x) = -\sum_{j,k=1}^{d} \partial_j a_{j,k}(x)\partial_k + \sum_{k=1}^{d} b_k(x)\partial_j + c(x), \tag{3.2}$$

where $A = (a_{j,k})_{1\leqslant j,k\leqslant d}$, $B = (b_k)_{1\leqslant k\leqslant d}$ and c are bounded real-valued functions on Ω.

We will always assume that A is symmetric:

$$\forall j,k \in [\![1,d]\!], \quad a_{j,k}(x) = a_{k,j}(x). \tag{3.3}$$

We will also assume that the operator P is elliptic.

Définition 3.1. We say that the differential operator P defined by (3.2) is (uniformly) elliptic if there exists $\alpha > 0$ such that for almost all $x \in \Omega$ and for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ we have

$$A\xi \cdot \xi = \sum_{j,k=1}^{d} a_{j,k}(x)\xi_j\xi_k \geqslant \alpha \left|\xi\right|^2. \tag{3.4}$$

This means that the real symmetric matrix $A(x)$ is uniformly definite postive, with smallest eigenvalue greater than or equal to $\alpha > 0$.

All these assumptions are in particular satisfied for the Poisson equation ($A(x) = \mathrm{Id}$, $B = 0$, $c = 0$).

Since the equation (3.1) is linear and has real coefficients, it is enough to consider a real valued source term f, and we look for a real valued solution u.

3.1 Maximum Principle

In this paragraph we discuss the maximum principle. Let Ω be an open bounded subset of $\mathbb{R}^d$. We recall that if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta u = 0$ on Ω, then for $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$ we have

$$u(x) = \frac{1}{|S(x,r)|} \int_{S(x,r)} u(y)\, d\sigma(y).$$

To see this we compute

$$\begin{aligned} \frac{d}{dr} \frac{1}{|S(x,r)|} \int_{S(x,r)} u(y)\, d\sigma(y) &= \frac{d}{dr} \frac{1}{|S(0,1)|} \int_{S(0,1)} u(x+ry)\, d\sigma(y) \\ &= \frac{1}{|S(0,1)|} \int_{S(0,1)} \partial_r u(x+ry)\, d\sigma(y) \\ &= \frac{1}{|S(x,r)|} \int_{\partial B(x,r)} \partial_\nu u(y)\, d\sigma(y) \\ &= 0 \end{aligned}$$

This proves in particular that u cannot reach a strict maximum at x, and that if u atteins a maximum at x then u is constant on a neighborhood of x. On the other hand, u is continuous on the compact sets $\overline{\Omega}$ and on $\partial\Omega$, so it has a maximum. We get

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

And moreover, if Ω is connected and u reaches a maximum on Ω, then u is constant on Ω.

These facts are already known for holomorphic functions, which are particular cases of harmonic functions (that is solutions in dimension 2 of $\Delta u = 0$). It is already known that the maximum principle has many important consequences in that case. Our purpose in this section is to generalise these observations to more general settings. In dimension 1 it is not difficult to see that if $-u'' \leqslant 0$ on some interval $[a,b]$, then $u(x) \leqslant \max\big(u(a), u(b)\big)$, with equality if and only if u is constant on $[a,b]$.

Theorem 3.2. *Let Ω be an open bounded subset of $\mathbb{R}^d$. Let P be defined by (3.2) with $c = 0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be such that*

$$Pu \leqslant 0 \quad \text{on } \Omega.$$

(i) *We have*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(ii) *If moreover Ω is connected and u atteins its maximum at an interior point, then u is constant.*

The first statement is refered to as the weak maximum principle. The second statement is the strong maximum principle.

The idea for the weak maximum principle is the following. Consider the particular case $-\Delta u < 0$ on Ω. If u reaches a maximum at $x_0 \in \Omega$, then in particular $\partial_j^2 u(x_0) \leqslant 0$ for all $j \in [\![1, d]\!]$, which gives a contradiction. In the first step we generalize this idea to the general setting $Pu < 0$, and then we deduce the case $Pu \leqslant 0$.

Proof of the weak maximum principle. • We first consider the case where $Pu < 0$ in Ω. For $h \in \mathbb{R}^d \setminus \{0\}$ we denote by $\partial_h^2 u(x)$ the second derivative of $t \mapsto u(x+th)$ at $t=0$. Assume by contradiction that there exists $x_0 \in \Omega$ such that $u(x_0) = \max u$. Then we have $\nabla u(x_0) = 0$ and $\partial_h^2 u(x_0) \leqslant 0$ for any $h \in \mathbb{R}^d$. Since $A(x_0)$ is symmetric and definite positive, there exist an orthogonal matrix O and a diagonal matrix $D = \mathsf{diag}(\lambda_1, \dots, \lambda_d)$ with positive coefficients such that $A(x_0) = ODO^\mathsf{T}$. For $j \in [\![1,d]\!]$ we set $\tilde{e}_j = Oe_j$. This defines a new basis of $\mathbb{R}^d$. Then we have

$$(Pu)(x_0) = -\operatorname{div}(A(x_0)\nabla u)(x_0) = -\operatorname{div} ODO^\mathsf{T}\nabla u(x_0) = -\sum_{\ell=1}^d \lambda_j \partial_{\tilde{e}_\ell}^2 u(x_0) \geqslant 0.$$

This gives a contradiction and proves the weak maximum principle when $Pu < 0$ on Ω.
• Then we consider the general case $Pu \leqslant 0$. We can rewrite P as

$$P = -\sum_{j,k=1}^d a_{j,k}\partial_j\partial_k + \sum_{k=1}^d \tilde{b}_k \partial_k, \tag{3.5}$$

where for $k \in [\![1,d]\!]$ we have set $\tilde{b}_k = b_k + \sum_{j=1}^d \partial_j a_{j,k}$. For $\varepsilon > 0$ and $x = (x_1, \dots, x_d) \in \Omega$ we set

$$u_\varepsilon(x) = u(x) + \varepsilon e^{\beta x_1},$$

for some $\beta > 0$ to be fixed large enough. For $x \in \Omega$ we have

$$Pu_\varepsilon(x) = Pu(x) + \varepsilon e^{\beta x_1}\big(-\beta^2 a_{11}(x) + \beta\tilde{b}_1(x)\big) \leqslant \varepsilon e^{\beta x_1}\big(-\beta^2\alpha + \beta\|\tilde{b}_1\|_{L^\infty(\Omega)}\big).$$

This is negative if β was chosen large enough. By the first case we have

$$\forall \varepsilon > 0, \quad \max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon.$$

We conclude by taking the limit $\varepsilon \to 0$.
• Now we turn to the proof of the strong maximum principle. Let

$$F = \left\{ x \in \Omega \,:\, u(x) = \max_{\overline{\Omega}} u \right\}, \quad \omega = \Omega \setminus F.$$

F is closed in Ω and ω is open. Assume by contradiction that $F \neq \varnothing$ and $F \neq \Omega$. We denote by $\overline{\omega}$ the closure of ω in Ω. Since Ω is connected, we can consider x_1 in $\overline{\omega} \cap F \neq \varnothing$. Near x_1 we can find $x_c \in \omega$ such that $\mathsf{dist}(x_c, F) < \mathsf{dist}(x_c, \partial\Omega)$. Then we set $r = \mathsf{dist}(x_c, F)$ and we consider $x_0 \in F$ such that $|x_0 - x_c| = r$. We have $B(x_c, r) \subset \omega$ and, since $x_0 \in F$, we have

$$\nabla u(x_0) = 0.$$

For $x \in B(x_c, r)$ we set

$$v(x) = e^{-\beta|x-x_c|^2} - e^{-\beta r^2} \geqslant 0,$$

for some $\beta > 0$ to be chosen large enough below. For $x \in B(x_c, r)$ we have with the notation (3.5)

$$\begin{aligned} Pv(x) &= -\sum_{j,k=1}^d a_{j,k}(x)\partial_j\partial_k v(x) + \sum_{k=1}^d \tilde{b}_k(x)\partial_k v(x) \\ &= e^{-\beta|x|^2}\left(-4\beta^2 \sum_{j,k=1}^d a_{j,k}(x)(x_j - x_{c,j})(x_k - x_{c,k}) + 2\beta\sum_{j=1}^d a_{j,j}(x) + 2\beta\sum_{k=1}^d \tilde{b}_k(x)(x_k - x_{c,k})\right) \\ &\leqslant e^{-\beta|x|^2}\left(-4\alpha\beta^2\,|x - x_c|^2 + 2\beta\mathsf{Tr}(A) + 2\beta\|\tilde{b}\|_\infty\,|x - x_c|\right). \end{aligned}$$

If β is large enough then on $\mathcal{C} = B(x_c, r) \backslash \overline{B}\big(x_c, \frac{r}{2}\big)$ we have

$$Pv \leqslant 0.$$

There exists $\varepsilon > 0$ such that for all $x \in S\big(x_c, \frac{r}{2}\big)$ we have

$$u(x_0) \geqslant u(x) + \varepsilon v(x).$$

This also holds on $S(x_c, r)$ where v vanishes. We set $w(x) = u(x) + \varepsilon v(x) - u(x_0)$. Then $w \leqslant 0$ on $\partial\mathcal{C}$ and $Pw \leqslant 0$ on $\mathcal{C}$. By the weak maximum principle, we have $w \leqslant 0$ on $\mathcal{C}$, and in particular $\nabla w(x_0) \cdot (x_0 - x_c) \geqslant 0$. This gives

$$\nabla u(x_0) \cdot (x_0 - x_c) \geqslant -\varepsilon \nabla v(x_0) \cdot (x_0 - x_c) = 2\varepsilon\beta r^2 e^{-\beta r^2} > 0.$$

This gives a contradiction with $\nabla u(x_0) = 0$, and proves that $F = \varnothing$ or $F = \Omega$. □

3.2 Variational method

In this section we discuss the variational method used to solve second order elliptic equations. We illustrate the method on the simplest problem. We consider on $\mathbb{R}^d$ the equation

$$-\Delta u + u = f. \tag{3.6}$$

Before trying to solve this problem, we have to be explicit about what will be called a solution of (3.6). Since two derivatives of the unknown u are involved, it is natural to look for twice differentiable solutions.

Définition 3.3. Assume that $f \in C^0(\mathbb{R}^d)$. Then a classical solution of (3.6) is a function $u \in C^2(\mathbb{R}^d)$ such that (3.6) holds in the usual sense.

We will see that this is not necessarily the best point of view to discuss this problem.

3.2.1 The Lax-Milgram Theorem

We recall in this paragraph the Lax-Milgram Theorem, which will be our main tool for the analysis of elliptic equations. We give different versions and different proofs.

Theorem 3.4 (Lax-Milgram's Theorem)**.** *Let $\mathcal{V}$ be a real Hilbert space. Let a be a bilinear form on $\mathcal{V}$. We assume that*

(i) *a is continuous: there exists $C > 0$ such that, for all $u, v \in \mathcal{V}$,*

$$|a(u,v)| \leqslant C \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}},$$

(ii) *a is coercive: there exists $\alpha > 0$ such that, for all $u \in \mathcal{V}$,*

$$a(u,u) \geqslant \alpha \|u\|_{\mathcal{V}}^2.$$

Then for any continuous linear form ℓ on $\mathcal{V}$ there exists a unique $u \in \mathcal{V}$ such that

$$\forall v \in \mathcal{V}, \quad a(u,v) = \ell(v). \tag{3.7}$$

Moreover

$$\|u\|_{\mathcal{V}} \leqslant \frac{\|\ell\|_{\mathcal{V}*}}{\alpha}.$$

This result is just a generalization of the Riesz representation theorem. If we add the assumption that the bilinear form a is symmetric, then it defines an inner product on $\mathcal{V}$, and the corresponding norm is equivalent to the original norm on $\mathcal{V}$. In particular, ℓ is still continuous if $\mathcal{V}$ is endowed with this new Hilbert structure. Then the result follows by the Riesz representation theorem.

In general, the bilinear form a is not symmetric, but we can still give a proof which relies on the Riesz theorem.

Proof. • Let $u \in \mathcal{V}$. The map $v \mapsto a(u,v)$ is a continuous linear form on $\mathcal{V}$, so by the Riesz representation theorem there exists an element of $\mathcal{V}$, which we denote by Au, such that

$$\forall v \in \mathcal{V}, \quad a(u,v) = \langle Au, v\rangle_{\mathcal{V}}.$$

This defines a map $A : \mathcal{V} \to \mathcal{V}$. Similarly, there exists $f \in \mathcal{V}$ such that $\ell(v) = \langle f, v\rangle_{\mathcal{V}}$ for all $v \in \mathcal{V}$, and $\|f\|_{\mathcal{H}} = \|\ell\|_{\mathcal{V}*}$. Then (3.7) holds if and only if $Au = f$.

• Let $u_1, u_2 \in \mathcal{V}$ and $\lambda \in \mathbb{R}$. For all $v \in \mathcal{V}$ we have

$$\begin{aligned}\langle A(u_1 + \lambda u_2), v\rangle_{\mathcal{V}} &= a(u_1 + \lambda u_2, v) = a(u_1, v) + \lambda a(u_2, v) = \langle Au_1, v\rangle_{\mathcal{V}} + \lambda \langle Au_2, v\rangle_{\mathcal{V}} \\ &= \langle Au_1 + \lambda Au_2, v\rangle .\end{aligned}$$

This proves that $A(u_1 + \lambda u_2) = Au_1 + \lambda Au_2$, and hence that the map $u \mapsto Au$ is linear. Moreover, for $u \in \mathcal{V}$ we have

$$\|Au\|_{\mathcal{V}}^2 = \langle Au, Au\rangle_{\mathcal{V}} = a(u, Au) \leqslant C \|u\|_{\mathcal{V}} \|Au\|_{\mathcal{V}},$$

so $\|Au\|_{\mathcal{V}} \leqslant C \|u\|_{\mathcal{V}}$. This proves that the operator A is bounded on $\mathcal{V}$.

• For $u \in \mathcal{V}$ we have

$$\alpha \|u\|_{\mathcal{V}}^2 \leqslant a(u,u) \leqslant \langle Au, u\rangle_{\mathcal{V}} \leqslant \|Au\|_{\mathcal{V}} \|u\|_{\mathcal{V}},$$

so

$$\|Au\|_{\mathcal{V}} \geqslant \alpha \|u\|_{\mathcal{V}}. \tag{3.8}$$

This proves in particular that A is injective. This also proves that the range of A is closed. Indeed, assume that the sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{V}$ is such that Av_n goes to some $w \in \mathcal{V}$ as n goes to $+\infty$. Then for $n, m \in \mathbb{N}$ we have

$$\|v_n - v_m\|_{\mathcal{V}} \leqslant \alpha^{-1} \|Av_n - Av_m\|_{\mathcal{V}} \xrightarrow[n,m\to+\infty]{} 0.$$

Since $\mathcal{V}$ is complete, the sequence $(v_n)_{n\in\mathbb{N}}$ has a limit $v \in \mathcal{V}$, and by continuity we have $w = Av \in \mathsf{Ran}(A)$.

Now let $w \in \mathsf{Ran}(A)^{\perp}$. Then in particular we have

$$0 = \langle Aw, w\rangle_{\mathcal{V}} = a(w,w) \geqslant \alpha \|w\|_{\mathcal{V}}^2,$$

so $w = 0$. Since $\mathsf{Ran}(A)$ is closed, this implies that $\mathsf{Ran}(A) = \mathcal{V}$. Thus A is bijective, so there exists a unique $u \in \mathcal{V}$ such that $Au = f$.

• Finally (3.8) gives

$$\|\ell\|_{\mathcal{V}*} = \|f\|_{\mathcal{V}} = \|Au\|_{\mathcal{V}} \geqslant \alpha \|u\|_{\mathcal{V}},$$

and the proof is complete. □

Corollary 3.5. *We keep the notation of Theorem 3.4 and assume that a is symmetric. For $u \in \mathcal{V}$ we set*

$$J(u) = \frac{a(u,u)}{2} - \ell(u).$$

Then J atteins a unique minimum, obtained for the solution u of (3.7).

Proof. Let u be given by Theorem 3.4. For $h \in \mathcal{V} \setminus \{0\}$ we have

$$J(u+h) = J(u) + a(u,h) - \ell(h) + \frac{a(h,h)}{2} = J(u) + \frac{a(h,h)}{2} > J(u),$$

so J has a strict minimum at point u. □

***Exercise* 30.** We use the notation of Theorem 3.4 and assume that a is symmetric. The purpose of this exercice is to give a new proof of Theorem 3.4 in this case, based on the analysis of the minima of the functional J defined in Corollary 3.5.
1. Prove that the function J is bounded from below.
2. Consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ of J. Prove that

$$\limsup_{n,m\to+\infty} a\left(\frac{u_n - u_m}{2}, \frac{u_n - u_m}{2}\right) \leqslant 0,$$

and deduce that this sequence has a limit u in $\mathcal{V}$.
3. Prove that J reaches a minimum at point u.
4. Prove that u solves the variational problem (3.7).
5. Prove that this minimum is strict, and hence unique.

***Exercise* 31.** We keep the notation of Theorem 3.4, and assume that $\mathcal{V}$ is separable. We consider a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of finite dimensional subspaces of $\mathcal{V}$ such that $\mathcal{V}_n \subset \mathcal{V}_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is dense in $\mathcal{V}$.
1. Prove that the problem (3.7) has at most one solution.
2. Prove that for all $n \in \mathbb{N}$ there exists a unique $u_n \in \mathcal{V}_n$ such that

$$\forall v \in \mathcal{V}_n, \quad a(u_n, v) = \ell(v).$$

3. Prove that the sequence $(u_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in $\mathcal{V}$. We denote by u the corresponding weak limit.
4. Prove that u is a solution of (3.7).
5. Prove that for $n \in \mathbb{N}$ and $v \in \mathcal{V}_n$ we have

$$\|u - u_n\|_{\mathcal{V}} \leqslant \frac{C}{\alpha} \|u - v\|_{\mathcal{V}}.$$

6. Prove that u_n goes to u strongly in $\mathcal{V}$.

In this chapter we will only consider problems on real Hilbert spaces. However, for many applications we also work in complex Hilbert spaces. All the results are easily adapted to this case, and in particular we have the following version of the Lax-Milgram theorem

Theorem 3.6 (Lax-Milgram's Theorem on a complex Hilbert space)**.** *Let $\mathcal{V}$ be a complex Hilbert space. Let a be a sesquilinear form on $\mathcal{V}$ (linear on the right and semi-linear on the left). We assume that a is continuous and that* $\operatorname{Re} a$ *is coercive: there exists $\alpha > 0$ such that for all $u \in \mathcal{V}$ we have*

$$\operatorname{Re}\big(a(u,u)\big) \geqslant \alpha \|u\|_{\mathcal{V}}^2 .$$

Then for any continuous linear form ℓ on $\mathcal{V}$ there exists a unique $u \in \mathcal{V}$ such that

$$\forall v \in \mathcal{V}, \quad a(u,v) = \ell(v).$$

Moreover

$$\|u\|_{\mathcal{V}} \leqslant \frac{\|\ell\|_{\mathcal{V}*}}{\alpha}.$$

***Exercise* 32.** Prove Theorem 3.6.

***Exercise* 33.** We consider the setting of Theorem 3.6, but instead of the coercivity we assume that there exist $\alpha > 0$ and two bounded linear operators Φ_1, Φ_2 on $\mathcal{V}$ such that, for every $u \in \mathcal{V}$,

$$|a(u,u)| + |a(\Phi_1(u),u)| \geqslant \alpha \|u\|_{\mathcal{V}}^2$$

and

$$|a(u,u)| + |a(u,\Phi_2(u))| \geqslant \alpha \|u\|_{\mathcal{V}}^2 .$$

Show that the first conclusion of Theorem 3.6 hold with this weaker assumption.

3.2.2 Weak solutions on the Euclidean space

Our purpose in this paragraph is to apply the Lax-Milgram Theorem to (3.6). For this we have to define a suitable notion of solution. The Lax-Milgram Theorem only applies in Hilbert spaces, so with this method we cannot work in $C^2(\Omega)$. Moreover, we do not want to restrict ourselves to the case $f \in C^0(\Omega)$.

The Sobolev spaces have been designed to be suitable this kind of analysis. With $p = 2$ they are Hilbert spaces, and the corresponding topologies take into account the derivatives of a function. This suggests the following definition.

Définition 3.7. Let $f \in L^2(\mathbb{R})$. A strong solution of (3.6) is a function $u \in H^2(\mathbb{R}^d)$ such that (3.6) holds in the sense of distributions (this is then an equality in $L^2(\mathbb{R}^d)$).

Thus we look for a function u such that

$$\forall \phi \in C_0^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} u\big(-\Delta\phi + \phi\big)\, dx = \int_{\mathbb{R}^d} f\phi\, dx. \tag{3.9}$$

Then we try to apply the Lax-Milgram Theorem. We denote by $a(u,\phi)$ and $\ell(\phi)$ the left-hand side and right-hand side of (3.9), respectively. This defines a bilinear form a and a linear form ℓ.

We cannot apply Theorem 3.4 with the topology of $L^2(\mathbb{R}^d)$, since then a is not continuous (there are too many derivatives in a), and we cannot work in $H^2(\mathbb{R}^d)$ since in this case a is not coercive (there are now too many derivatives in the definition of the norm $\|\cdot\|_{H^2(\mathbb{R}^d)}$).

The solution is to chose the intermediate situation, which will "equalize" the number of derivatives on u and ϕ. To write (3.9) we have transfered the two derivatives on the test function. A better choice is to transfer one derivative on the test function and to keep one on u. This gives this new definition.

Définition 3.8. Let $f \in L^2(\mathbb{R}^d)$. We say that $u \in H^1(\mathbb{R}^d)$ is a weak solution of (3.6) if for all $v \in H^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \nabla u \nabla v \, dx + \int_{\mathbb{R}^d} uv \, dx = \int_\Omega fv \, dx.$$

With this notion of solution, it is now easy to see that by the Lax-Milgram theorem applied with $\varphi : v \mapsto \int_{\mathbb{R}^d} fv$ the problem (3.6) is well-posed.

Proposition 3.9. *For $u, v \in H^1(\mathbb{R}^d)$ we set*

$$a(u,v) = \langle \nabla u, \nabla v \rangle_{L^2(\mathbb{R}^d)} + \langle u, v \rangle_{L^2(\mathbb{R}^d)}.$$

Let $\varphi \in H^1(\mathbb{R}^d)*$. There exists a unique $u \in H^1(\mathbb{R}^d)$ such that

$$\forall v \in H^1(\mathbb{R}^d), \quad a(u,v) = \varphi(v). \tag{3.10}$$

Moreover we have

$$\|u\|_{H^1(\mathbb{R}^d)} \leqslant \|\varphi\|_{H^1(\mathbb{R}^d)*}.$$

Proof. It is clear that a is a continuous bilinear form on $H^1(\mathbb{R}^d)$. Moreover for $u \in H^1(\mathbb{R}^d)$ we have

$$a(u,u) = \|u\|_{H^1}^2,$$

so the coercivity is also clear in $H^1(\mathbb{R}^d)$. The conclusions follow from Theorem 3.4. □

3.2.3 Regularity of the weak solution

We have seen that the Lax-Milgram Theorem gives existence and uniqueness of a weak solution with continuity of this solution with respect to f. However, this notion of weak solution which was precisely designed to be adapted to the Lax-Milgram Theorem is not so natural.

Moreover, in this particular case, on $\mathbb{R}^d$ and with constant coefficients, it is not difficult to prove with the Fourier transform that (3.6) has in fact a unique *strong* solution. Our purpose here is to recover this fact without the Fourier transform. For this we prove that the weak solution given by Proposition 3.9 belongs in fact to $H^2(\mathbb{R}^d)$ and is in fact a strong solution. The interest of this new method is that it will apply in situations where we can no longer use the Fourier transform. Since a strong solution is necessarily a weak solution, we already have uniqueness of a strong solution.

Proposition 3.10. *Let $f \in L^2(\mathbb{R}^d)$ and let $u \in H^1(\mathbb{R}^d)$ be the unique weak solution of (3.6) given by Proposition 3.9. Then $u \in H^2(\mathbb{R}^d)$, the equality (3.6) holds in $L^2(\mathbb{R}^d)$ and there exists $C > 0$ independant of f such that*

$$\|u\|_{H^2(\mathbb{R}^d)} \leqslant C \|f\|_{L^2(\mathbb{R}^d)}.$$

If moreover $f \in H^k(\mathbb{R}^d)$ for some $k \in \mathbb{N}$ then $u \in H^{k+2}(\mathbb{R}^d)$ and there exists $C_k > 0$ independant of f such that

$$\|u\|_{H^{k+2}(\mathbb{R}^d)} \leqslant C \|f\|_{H^k(\mathbb{R}^d)}.$$

In $\mathbb{R}^d$ it is natural to prove Proposition 3.10 with the Fourier transform.

Proof 1. Let $\phi \in C_0^\infty(\mathbb{R}^d)$. Since $u \in H^1(\mathbb{R}^d)$ we can write

$$-\int_{\mathbb{R}^d} u\Delta\phi\,dx = \int_{\mathbb{R}^d} \nabla u \cdot \nabla\phi\,dx = \int_{\mathbb{R}^d} (f-u)\phi\,dx.$$

This proves that in the sense of distributions we have $-\Delta u = f - u$. Then $\Delta u \in L^2(\mathbb{R}^d)$ and we have $-\Delta u + u = f$ in $L^2(\mathbb{R}^d)$. By Remark 2.28 we also have $u \in H^2(\mathbb{R}^d)$.

We prove by induction on $k \in \mathbb{N}$ that if $f \in H^k(\mathbb{R}^d)$ we have $u \in H^{k+2}(\mathbb{R}^d)$. We have proved the case $k = 0$. Assume that the result is proved up to $k-1$ for some $k \in \mathbb{N}^*$ and let $f \in H^k(\mathbb{R}^d)$. Since $f \in H^{k-1}(\mathbb{R}^d)$ we have $u \in H^{k+1}(\mathbb{R}^d)$. Then $\Delta u = u - f \in H^k(\mathbb{R}^d)$. We deduce that $u \in H^{k+2}(\mathbb{R}^d)$. Finally we have

$$\|u\|_{H^{k+2}(\mathbb{R}^d)} \lesssim \left\|(1+|\xi|^2)^{\frac{k+2}{2}}\hat{u}\right\|_{L^2} = \left\|(1+|\xi|^2)^{\frac{k}{2}}\hat{f}\right\|_{L^2} \lesssim \|f\|_{H^k(\mathbb{R}^d)}\,. \qquad \square$$

We provide another proof based on the difference quotients. The interest is that we will be able to apply the same strategy on the half-space in the next section.

Proof 2. Let $h \in \mathbb{R}^d \setminus \{0\}$. By (3.10) applied with $v = D_{-h}(D_h u) \in H^1(\mathbb{R}^d)$, (2.4), (2.5) and Proposition 2.38 we have

$$\begin{aligned}
\|D_h u\|_{H^1(\mathbb{R}^d)}^2 &= \|\nabla D_h u\|_{L^2(\mathbb{R}^d)}^2 + \|D_h u\|_{L^2(\mathbb{R}^d)}^2 \\
&= \langle \nabla u, \nabla v\rangle_{L^2(\mathbb{R}^d)} + \langle u, v\rangle_{L^2(\mathbb{R}^d)} \\
&= \langle f, v\rangle_{L^2(\mathbb{R}^d)} \\
&\leqslant \|f\|_{L^2(\mathbb{R}^d)} \|D_h u\|_{H^1(\mathbb{R}^d)}\,,
\end{aligned}$$

so

$$\|D_h u\|_{H^1(\mathbb{R}^d)} \leqslant \|f\|_{L^2(\mathbb{R}^d)}\,.$$

In particular, for all $j \in [\![1,d]\!]$ we have

$$\|D_h \partial_j u\|_{L^2(\mathbb{R}^d)} \leqslant \|f\|_{L^2(\mathbb{R})}\,.$$

By Proposition 2.12, this proves that $\partial_j u \in H^1(\mathbb{R}^d)$ with $\|\partial_j u\|_{H^1(\mathbb{R}^d)} \leqslant \|f\|_{L^2(\mathbb{R}^d)}$ for all $j \in [\![1,d]\!]$. Therefore $u \in H^2(\mathbb{R}^d)$ with $\|u\|_{H^2(\mathbb{R}^d)} \leqslant C\|f\|_{L^2(\mathbb{R}^d)}$ for some constant $C > 0$ independant of f.

We prove the higher regularity result by induction on $k \in \mathbb{N}$. We have proved the case $k = 0$, and we assume that the result is proved up to the case $k-1$ for some $k \in \mathbb{N}^*$. Assume that $f \in H^k(\mathbb{R}^d)$. By induction, since $f \in H^{k-1}(\mathbb{R}^d)$, we already know that $u \in H^{k+1}(\mathbb{R}^d)$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant k$. Then $\partial^\alpha u \in H^1(\mathbb{R}^d)$ is the weak solution of

$$-\Delta(\partial^\alpha u) + \partial^\alpha u = \partial^\alpha f.$$

For this we write for any $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}
\int_{\mathbb{R}^d} \nabla\partial^\alpha u \cdot \nabla\phi\,dx + \int_{\mathbb{R}^d} \partial^\alpha u\,\phi\,dx &= (-1)^{|\alpha|}\int_{\mathbb{R}^d} \nabla u \cdot \nabla\partial^\alpha\phi\,dx + (-1)^{|\alpha|}\int_{\mathbb{R}^d} \partial^\alpha u\,\phi\,dx \\
&= (-1)^{|\alpha|}\int_{\mathbb{R}^d} f\partial^\alpha\phi\,dx \\
&= \int_{\mathbb{R}^d} \partial^\alpha f\,\phi\,dx.
\end{aligned}$$

This proves that $\partial^\alpha u \in H^2(\mathbb{R}^d)$, and hence that $u \in H^{k+2}(\mathbb{R}^d)$. Moreover

$$\|\partial^\alpha u\|_{H^2(\mathbb{R}^d)} \lesssim \|\partial^\alpha f\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{H^k(\mathbb{R}^d)} .$$

Finally $u \in H^{k+2}(\mathbb{R}^d)$ and $\|u\|_{H^{k+2}(\mathbb{R}^d)} \lesssim \|f\|_{H^k(\mathbb{R}^d)}$. □

Notice that for this second proof we have also used the fact that the problem (3.6) is posed on $\mathbb{R}^d$. In the following section we will see how this method is adapted for a problem posed on an open subset $\Omega \neq \mathbb{R}^d$.

***Exercise* 34.** Let $\lambda > 0$ and $f \in L^2(\mathbb{R}^d)$. We consider on $\mathbb{R}^d$ the equation

$$-\Delta u + \lambda u = f.$$

1. Prove that this problem has a unique weak solution u (in a suitable sense to be defined).
2. Prove that this solution belongs to $H^2(\mathbb{R}^d)$ and give an estimate of $\|u\|_{H^2(\mathbb{R}^d)}$ with respect to $\|f\|_{L^2(\mathbb{R}^d)}$ and $\lambda > 0$.
3. What happens if $\lambda \leqslant 0$?

3.3 Boundary conditions

In the previous section, we have described the variational method for elliptic equations with the example of a problem on $\mathbb{R}^d$. We will apply the same global strategy on a general open subset Ω of $\mathbb{R}^d$, but some arguments have to be adapted. We first observe that, in general, the solution of (3.6) or the variational version (3.10) is not unique. This is easy to see in dimension 1. For instance, on $\Omega =]-1, 1[$ any function of the form

$$u(x) = Ae^x + Be^{-x}$$

is in $H^2(\Omega)$ and satisfies $-u'' + u = 0$. Similarly, on the unbouded open set $\Omega =]0, +\infty[$, the same applies to the functions of the form $x \mapsto Be^{-x}$.

One possibility to recover a well posed problem is to add boundary conditions. This choice is physically relevent since what happens at the boundary can be controled or at least measured. For instance, for $f \in L^2(-1, 1)$ the following problems on $]-1, 1[$ have at most one solution $u \in H^2([-1, 1])$:

$$\begin{cases} -u'' + u = f, \\ u(-1) = u(1) = 0, \end{cases} \qquad \begin{cases} -u'' + u = f, \\ u'(-1) = u'(1) = 0. \end{cases} \tag{3.11}$$

***Exercise* 35.** Give explicitely the solutions of the two problems (3.11).

We can write the corresponding problems in any dimension. If we add the condition that the solution vanishes at the boundary, we obtain the so-called boundary value problem with Dirichlet boundary condition:

$$\begin{cases} -\Delta u + u = f & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

As in (3.11), we can for instance solve the problem with the additional condition that the normal derivative of the solution vanishes at the boundary. This is the corresponding Neumann boundary problem:

$$\begin{cases} -\Delta u + u = f & \text{on } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

These are not the only possibilities, but we will focus on these two model cases in this course.

***Exercise* 36.** Solve the following problems:

$$\begin{cases} -u'' + u = 1, \\ u(-1) = 0, \quad u'(1) = 0, \end{cases} \qquad \begin{cases} -u'' + u = 1, \\ u'(-1) = 0, \quad u'(1) = u(1). \end{cases}$$

If f is continuous on Ω, then a classical solution of the Dirichlet problem (3.12) is a function $u \in C^2(\overline{\Omega})$ which satisfies (3.12) in the usual sense. We similarly define a classical solution of the Neumann problem (3.13).

As in the previous section, we try to solve these two problems with the Lax-Milgram Theorem. For this we need a suitable variational formulation (or, equivalently, a good definition for a weak solution).

3.3.1 Dirichlet boundary conditions

We begin with the Dirichlet problem. To take into account the condition $u = 0$, it is natural to try to work in $H_0^1(\Omega)$. If $u \in C^2(\overline{\Omega}) \cap H^2(\Omega)$ is a classical solution of (3.12) and $v \in H_0^1(\Omega)$, we have by the Green Formula (Theorem 2.54)

$$\int_\Omega (-\Delta u + u) v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx.$$

This suggests the following definition.

Définition 3.11. Let Ω be an open subset of $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. We say that $u \in H_0^1(\Omega)$ is a weak solution of (3.12) if

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} uv \, dx = \int_\Omega fv \, dx. \tag{3.14}$$

With this notion we can apply the Lax-Milgram theorem and (3.12) is well posed.

Proposition 3.12. *Let Ω be an open subset of $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. There exists a unique solution $u \in H_0^1(\Omega)$ of (3.14). Moreover $\|u\|_{H^1(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$.*

Proof. For $u, v \in H_0^1(\Omega)$ we set

$$a(u,v) = \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} + \langle u, v \rangle_{L^2(\Omega)}, \quad \ell(v) = \langle f, v \rangle_{L^2(\Omega)}.$$

This defines a continuous bilinear form a and a continuous linear form ℓ on $H_0^1(\Omega)$. Moreover, for $u \in H_0^1(\Omega)$ we have

$$a(u,u) = \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 = \|u\|_{H_1^0(\Omega)}^2,$$

so a is coercive. The conclusions follow from Theorem 3.4. $\square$

As in $\mathbb{R}^d$, we have worked with the H^1 regularity to be able to apply the Lax-Milgram theorem, but we would like to have a solution at least in H^2.

Here we prove this result for $\Omega = \mathbb{R}^d_+$. The case where the boundary of Ω is not flat will be discussed with more generality in the following section.

The proof of the following regularity result is divided into two steps. We first check that u belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^d_+)$ (interior regularity), and for this we localize the solution far from the boundary and apply the result known on the Euclidean space. Then we look at the regularity near the boundary. For this, we adapt the proof of Proposition 3.10 with difference quotients. We recall that for $\Omega \neq \mathbb{R}^d$, the fact that u and Δu belong to $L^2(\Omega)$ does not imply that $u \in H^2(\Omega)$.

Proposition 3.13. *Let $f \in L^2(\mathbb{R}^d_+)$ and let $u \in H^1_0(\mathbb{R}^d_+)$ be the weak solution of (3.12). Then $u \in H^2(\mathbb{R}^d_+)$, we have $-\Delta u + u = f$ almost everywhere and there exists $C > 0$ independant of f such that*

$$\|u\|_{H^2(\mathbb{R}^d_+)} \leqslant C \|f\|_{L^2(\Omega)} .$$

Proof. By Proposition 3.12 we already know that $u \in H^1_0(\mathbb{R}^d_+)$ and with $\|u\|_{H^1(\mathbb{R}^d_+)} \leqslant \|f\|_{L^2(\mathbb{R}^d_+)}$.

- Let ω be an open bounded subset of $\mathbb{R}^d_+$ such that $\overline{\omega} \subset \mathbb{R}^d_+$. Let $\chi \in C_0^\infty(\mathbb{R}^d_+, [0,1])$ be equal to 1 on a neighborhood of $\overline{\omega}$. Then χu belongs to $H^1_0(\mathbb{R}^d_+)$, it can be extended by 0 to a function $\tilde{u} \in H^1(\mathbb{R}^d)$, and $\tilde{u}$ is a weak solution on $\mathbb{R}^d$ for the problem

$$-\Delta \tilde{u} + \tilde{u} = \chi f - 2\nabla\chi \cdot \nabla u - u\Delta\chi, \tag{3.15}$$

where the right-hand side has been extended by 0 on $\mathbb{R}^d$. Since the right-hand side belongs to $L^2(\mathbb{R}^d)$ we obtain by Proposition 3.10 that $\tilde{u}$ belongs to $H^2(\mathbb{R}^d)$ and that the equality (3.15) holds in $L^2(\mathbb{R}^d)$. This proves that $u \in H^2(\omega)$ and $-\Delta u + u = f$ in $L^2(\omega)$. Since this holds for any ω, u belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^d_+)$ and the equality $-\Delta u + u = f$ holds in $L^2_{\mathrm{loc}}(\mathbb{R}^d_+)$. This implies

$$-\Delta u = f - u \in L^2(\mathbb{R}^d_+). \tag{3.16}$$

- Let $j \in [\![2, d]\!]$ and $t \neq 0$. By Proposition 2.38 we have

$$\left\|D_{te_j} u\right\|_{L^2(\mathbb{R}^d_+)} \leqslant \|\nabla u\|_{L^2(\mathbb{R}^d_+)} .$$

As in the proof of Proposition 3.10, we apply (3.14) with $v = D_{-te_j}(D_{te_j} u) \in H^1_0(\mathbb{R}^d_+)$ and we similarly obtain

$$\left\|D_{te_j} u\right\|_{H^1(\mathbb{R}^d_+)} \leqslant \|f\|_{L^2(\mathbb{R}^d_+)} .$$

In particular, for $k \in [\![1, d]\!]$ we have

$$\left\|D_{te_j} \partial_k u\right\|_{L^2(\mathbb{R}^d_+)} \leqslant \|f\|_{L^2(\mathbb{R}^d_+)}$$

so, by Proposition 2.12, $\partial_j \partial_k u \in L^2$ and

$$\|\partial_j \partial_k u\|_{L^2(\mathbb{R}^d_+)} \leqslant \|f\|_{L^2(\mathbb{R}^d_+)} .$$

- It remains to consider the second derivative $\partial_1^2 u$. By (3.16) we have

$$\partial_1^2 u = -\sum_{j=2}^d \partial_j^2 u - f - u \in L^2(\mathbb{R}^d_+),$$

and hence $u \in H^2(\mathbb{R}^d_+)$ with $\|u\|_{H^2(\mathbb{R}^d_+)} \leqslant C \|f\|_{L^2(\mathbb{R}^d_+)}$ for some $C > 0$ independant of f. $\square$

We finish this paragraph with the higher regularity result.

Proposition 3.14. *Let $k \in \mathbb{N}$ and $f \in H^k(\mathbb{R}^d_+)$. Let u be the weak solution of (3.12). Then $u \in H^{k+2}(\mathbb{R}^d_+)$.*

Proof. We prove the result by induction on $k \in \mathbb{N}$. The case $k = 0$ is Proposition 3.13. We assume that for some $k \in \mathbb{N}^*$ the result is proved up to order $k-1$. Let $f \in H^k(\mathbb{R}^d_+)$. Since $f \in H^{k-1}(\mathbb{R}^d_+)$ we already know by the inductive assumption that $u \in H^{k+1}(\mathbb{R}^d_+)$.

- Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^*$ such that $|\alpha| = k$ and $\alpha_1 = 0$. Then $\partial^\alpha u \in H^1_0(\mathbb{R}^d_+)$ is the weak solution of (3.12) with f replaced by $\partial^\alpha f \in L^2(\Omega)$. By Proposition 3.13, this proves that $\partial^\alpha u \in H^2(\mathbb{R}^d_+)$. Thus, for any $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ with $|\beta| \leqslant k+2$ and $\beta_1 \leqslant 2$ we have $\partial^\beta u \in L^2(\mathbb{R}^d_+)$.
- Now we prove by induction on $m \in [\![0, k+2]\!]$ that for $\beta \in \mathbb{N}^d$ with $|\beta| \leqslant k+2$ and $\beta_1 \leqslant m$ we have $\partial^\beta u \in L^2(\mathbb{R}^d_+)$. We already have the cases $m \leqslant 2$. Assume that for some $m \in [\![3, k+2]\!]$ we have prove this statement up to order $m-1$ and consider $\beta \in \mathbb{N}^d$ with $|\beta| \leqslant k+2$ and $\beta_1 = m$. Let $\tilde\beta = (\beta_1 - 2, \beta_2, \dots, \beta_d)$. We have

$$\partial^\beta u = \partial^{\tilde\beta}(\partial_1^2 u) = -\partial^{\tilde\beta}\left(f + \sum_{j=2}^d \partial_j^2 u\right) \in L^2(\mathbb{R}^d_+).$$

The conclusion follows by (double) induction. □

3.3.2 Neumann boundary conditions

Now we turn to the Neumann problem (3.13). Contrary to the Dirichlet problem, we cannot encode the boundary condition $\partial_\nu u = 0$ in the variational space $\mathcal{V}$. The normal derivative does not even have a sense in $H^1(\Omega)$.

It turns out that the solution of (3.13) will be given by the variational problem posed in the full space $H^1(\Omega)$, without any condition at the boundary.

This is not an obvious guess. But we have not used the boundary condition in the proof of Proposition 3.12, so it is a natural to wonder what happens if we replace $H^1_0(\Omega)$ by $H^1(\Omega)$ in the results of the previous paragraph (notice that on $\mathbb{R}^d$ we have $H^1 = H^1_0$, so this distinction was irrelevant in that case).

Définition 3.15. Let Ω be an open subset of $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. We say that $u \in H^1(\Omega)$ is a weak solution of (3.13) if

$$\forall v \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx. \tag{3.17}$$

Exactly as for Proposition 3.12, we have the following well-posedness result in the weak sense.

Proposition 3.16. *Let Ω be an open subset of $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. There exists a unique solution $u \in H^1(\Omega)$ of (3.17). Moreover $\|u\|_{H^1(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$.*

Now we prove the regularity of this weak solution when $\Omega = \mathbb{R}^d_+$. Compared to the Dirichlet case, the Neumann boundary condition is not explicit in the definition of the weak solution and can only be stated once we have the H^2 regularity.

Proposition 3.17. *Let $f \in L^2(\mathbb{R}^d_+)$ and let $u \in H^1(\mathbb{R}^d_+)$ be the weak solution of (3.12). Then $u \in H^2(\mathbb{R}^d_+)$ and $\partial_\nu u = 0$ on $\partial\mathbb{R}^d_+$.*

Recall that by $\partial_\nu u = 0$ we mean $\gamma_1(u) = 0$ in $L^2(\partial\mathbb{R}^d_+)$, where $\gamma_1 : H^2(\mathbb{R}^d_+) \to L^2(\mathbb{R}^d_+)$ is the normal trace (introduced in Paragraph 2.6.2).

Proof. As in the proof of Proposition 3.13 we see that $u \in H^2(\mathbb{R}^d_+)$ and, in $L^2(\mathbb{R}^d_+)$,

$$-\Delta u + u = f.$$

By the Green formula (see Theorem 2.55), we have for all $v \in H^1(\mathbb{R}^d_+)$

$$\begin{aligned}\int fv\,dx &= \int(-\Delta u + u)v\,dx\\ &= \int \nabla u\cdot\nabla v\,dx - \int_{\partial\mathbb{R}^d_+} \partial_\nu u\,v\,dx' + \int uv\,dx\\ &= \int fv\,dx - \int_{\partial\mathbb{R}^d_+} \partial_\nu u\,v\,dx'.\end{aligned}$$

This means that for all $v \in H^1(\mathbb{R}^d_+)$ we have

$$\int_{\partial\mathbb{R}^d_+} \partial_\nu u\,v\,dx' = 0.$$

Since the range of the trace operator γ_0 is dense in $L^2(\mathbb{R}^{d-1})$, this proves that in $L^2(\mathbb{R}^{d-1})$ we have

$$\partial_\nu u = 0. \qquad \square$$

3.3.3 Inhomogeneous boundary conditions

So far we have considered homogeneous Dirichlet boundary conditions ($u = 0$) or homogeneous Neumann boundary conditions ($\partial_\nu u = 0$). Now we introduce a problem with an inhomogeneous boundary condition. For simplicity we continue with the equation $-\Delta u+u = f$ on the half-space $\mathbb{R}^d_+$, and we only consider the case of a Dirichlet boundary condition. Given $g \in L^2(\partial\Omega)$ we consider the problem

$$\begin{cases}-\Delta u + u = f & \text{on } \mathbb{R}^d_+,\\ u = g, & \text{on } \partial\mathbb{R}^d_+.\end{cases} \tag{3.18}$$

The boundary condition makes sense as soon as u belongs to $H^1(\mathbb{R}^d_+)$ and $g \in L^2(\partial\mathbb{R}^d_+)$. It means

$$\gamma_0(u) = g.$$

We recall that the trace operator $\gamma_0 : H^1(\mathbb{R}^d_+) \to L^2(\partial\mathbb{R}^d_+)$ is not surjective. And it is clear that if g is not in the range $H^{1/2}(\partial\mathbb{R}^d_+)$, then the problem (3.18) cannot have a solution.

Now we assume that g belongs to $H^{1/2}(\partial\mathbb{R}^d_+)$ and we consider $w \in H^1(\mathbb{R}^d_+)$ such that $\gamma_0(w) = g$. Then u is solution of (3.18) if and only if $\tilde u = u - w$ is a solution of

$$\begin{cases}-\Delta\tilde u + \tilde u = f + \Delta w - w & \text{on } \mathbb{R}^d_+,\\ u = 0, & \text{on } \partial\mathbb{R}^d_+,\end{cases} \tag{3.19}$$

where Δu is seen as an element in $H^1(\mathbb{R}^d_+)^*$. Thus the right-hand side $f + \Delta w - w$ is not necessarily in $L^2(\mathbb{R}^d_+)$, but it is at least in $H^1(\mathbb{R}^d_+)^*$. Then the Lax-Milgram Theorem gives a unique weak solution $\tilde{u} \in H^1_0(\mathbb{R}^d_+)$ of (3.19). Setting $u = \tilde{u} + w$ we have

$$-\Delta u + u = f \in L^2(\mathbb{R}^d_+)$$

and

$$\gamma_0(u) = \gamma_0(w) = g.$$

Remark 3.18. Notice that $\Delta u \in L^2(\mathbb{R}^d_+)$ but u is not necessarily in $H^2(\mathbb{R}^d_+)$ (if $u \in H^2(\mathbb{R}^d_+)$ then $g = \gamma_0(u) \in H^{3/2}(\partial\mathbb{R}^d_+)$, which is not necessarily the case).

***Exercise* 37.** In this exercise we discuss the problem

$$\begin{cases} -\Delta u + u = f, & \text{on } \mathbb{R}^d_+, \\ \partial_\nu u = g, & \text{on } \partial\mathbb{R}^d_+, \end{cases} \tag{3.20}$$

where $f \in L^2(\mathbb{R}^d_+)$.
1. Discuss the problem when $g \in H^{1/2}(\partial\mathbb{R}^d_+)$.
2. Now we consider the case $g \in H^{-1/2}(\partial\mathbb{R}^d_+)$.
a. Prove that there exists a unique $u \in H^1(\mathbb{R}^d_+)$ such that

$$\forall v \in H^1(\mathbb{R}^d_+), \quad \int_{\mathbb{R}^d_+} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d_+} uv \, dx = \int_{\mathbb{R}^d_+} fv \, dx + \langle g, \gamma_0(v) \rangle_{H^{-1/2}(\partial\mathbb{R}^d_+), H^{1/2}(\partial\mathbb{R}^d_+)}.$$

b. Does Δu belong to $L^2(\mathbb{R}^d_+)$? Does u belong to $H^2(\mathbb{R}^d_+)$? What can we say about $\partial_\nu u$?

3.4 More general settings

In this section we discuss on a general bounded open subset Ω the general elliptic second order equation (3.1), with the operator P introduced in (3.2). We assume that A is symmetric and P is uniformly elliptic, see (3.3) and (3.4). We have to add boundary conditions. Here, we only consider the case of the Dirichlet boundary condition:

$$\begin{cases} Pu = f, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.21}$$

As above, we first solve a corresponding varitional problem and prove existence and uniqueness of a weak solution in $H^1_0(\Omega)$. Then we will prove the regularity of this weak solution to get a solution in $H^2(\Omega) \cap H^1_0(\Omega)$.

3.4.1 Weak solution for a general second order elliptic equation

Following the previous cases, we define the notion of weak solution in $H^1_0(\Omega)$ by transfering a derivative on the test function by a formal integration by parts.

Définition 3.19. Let Ω be an open subset of $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. We say that $u \in H_0^1(\Omega)$ is a weak solution of (3.21) if for all $v \in H_0^1(\Omega)$ we have

$$a_P(u,v) = \int_\Omega fv, \tag{3.22}$$

where we have set

$$\begin{aligned} a_P(u,v) &= \langle A\nabla u, \nabla u\rangle_{L^2(\Omega)} + \langle B\cdot\nabla u, v\rangle_{L^2(\Omega)} + \langle cu, v\rangle_{L^2(\Omega)} \\ &= \sum_{j,k=1}^d \int_\Omega a_{j,k}\partial_k u \partial_j v\,dx + \sum_{k=1}^d \int_\Omega b_k \partial_k u\, v\,dx + \int_{\mathbb{R}^d} cuv\,dx. \end{aligned} \tag{3.23}$$

As above, it is easy to check that a is a continuous bilinear form on $H_0^1(\Omega)$. However, it is not necessarily coercive. For instance, for $P = -\Delta$ on $\mathbb{R}^d$ we have

$$a(u,u) = \|\nabla u\|^2_{L^2(\mathbb{R}^d)}, \tag{3.24}$$

which is not coercive in $H^1(\mathbb{R}^d)$. But for any $\lambda > 0$ the bilinear form defined by

$$a_\lambda(u,u) = a(u,u) + \lambda\,\|u\|_{L^2(\mathbb{R}^d)} \tag{3.25}$$

is coercive. In other words, we have a bilinear form which does not control the square of the H^1 norm, but it controls at least the square of the norm of the gradient, so it is enough to add a multiple of the square of the L^2 norm (which corresponds to adding to the operator a multiple of the identity) to get coercivity. This is why we considered the operator $-\Delta + \mathrm{Id}$ instead of $-\Delta$ in the first example in Section 3.2.

The same applies in the more general setting of this section. The ellipticity assumption (3.4) ensures that $a_P(u,u)$ controls at least $\|\nabla u\|^2_{L^2(\mathbb{R}^d)}$, and hence it will be possible to apply the Lax-Milgram Theorem to the equation $Pu + \gamma u = f$ for $\gamma \geqslant 0$ large enough.

Lemma 3.20. *Let a_P be defined by (3.23), with A satisfying (3.3) and (3.4). Let $\alpha_0 \in]0, \alpha[$. There exists $\gamma_0 \in \mathbb{R}$ such that for all $u \in H_0^1(\Omega)$ we have*

$$a_P(u,u) \geqslant \alpha_0\,\|\nabla u\|^2_{L^2(\Omega)} - \gamma_0\,\|u\|^2_{L^2(\Omega)}.$$

Proof. Let $u \in H_0^1(\Omega)$. By (3.4) we have

$$\langle A\nabla u, \nabla u\rangle_{L^2(\Omega)} \geqslant \alpha\,\|\nabla u\|^2_{L^2(\Omega)}.$$

Let $\varepsilon = \alpha - \alpha_0 > 0$. We have

$$\left|\langle B\cdot\nabla u, u\rangle_{L^2(\Omega)}\right| \leqslant \|B\|_{L^\infty(\Omega)}\,\|\nabla u\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)} \leqslant \varepsilon\,\|\nabla u\|^2_{L^2(\Omega)} + \frac{\|B\|^2_{L^\infty(\Omega)}\,\|u\|^2_{L^2(\Omega)}}{4\varepsilon},$$

so the conclusion follows with

$$\gamma_0 = \frac{\|B\|^2_{L^\infty(\Omega)}}{4\varepsilon} - \inf_{x\in\Omega} c(x). \qquad \square$$

Thus, instead of (3.21), we consider for $\gamma > \gamma_0$ the problem

$$\begin{cases} Pu + \gamma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.26}$$

A weak solution of (3.26) is a function $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad a_P(u,v) + \gamma \langle u, v\rangle_{L^2(\Omega)}^2 = \int_\Omega f v \, dx. \tag{3.27}$$

By Lemma 3.20, the left-hand side defines a coercive bilinear form, so by the Lax-Milgram Theorem 3.4 we have the following result.

Proposition 3.21. *Let $\alpha_0 \in]0, \alpha[$ and let $\gamma_0 \in \mathbb{R}$ be given by Lemma 3.20. Then for $\gamma > \gamma_0$ and $f \in L^2(\Omega)$ the problem (3.26) has a unique weak solution in $H_0^1(\Omega)$. Moreover there exists $C_\gamma > 0$ independant of f such that*

$$\|u\|_{H^1(\Omega)} \leqslant C_\gamma \|f\|_{L^2(\Omega)}. \tag{3.28}$$

3.4.2 Regularity of a weak solution

Now we prove the regularity of a weak solution for the problem (3.21). Since we can replace c by $c+\gamma$, this also gives the regularity for a weak solution of (3.26).

Proposition 3.22. *Let Ω be a bounded open subset of $\mathbb{R}^d$ of class C^2. Let P be as above with $A \in C^1(\overline{\Omega})$ and $b, c \in L^\infty(\Omega)$. There exists $C > 0$ such that if $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ is a weak solution of (3.21), then $u \in H^2(\Omega)$ and*

$$\|u\|_{H^2(\Omega)} \leqslant C\big(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\big), \tag{3.29}$$

and (3.21) holds in $L^2(\Omega)$. If moreover Ω is of class C^{k+2}, $A, B, c \in C^{k+1}(\overline{\Omega})$ and $f \in H^k(\Omega)$ for some $k \in \mathbb{N}$, then $u \in H^{k+2}(\Omega)$.

With little more effort, we can we fact replace $\|u\|_{H^1(\Omega)}$ by $\|u\|_{L^2(\Omega)}$ in the right-hand side of (3.29). Prove it as an exercice. Notice that in the context of Proposition 3.21 we can apply (3.28) and have an estimate which depends on f only.

Proof. • We begin with the case $\Omega = \mathbb{R}^d$ and assume that the derivatives of A are bounded on $\mathbb{R}^d$. For all $v \in H^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} A\nabla u \cdot \nabla v = \int_{\mathbb{R}^d} \tilde{f} v,$$

where we have set

$$\tilde{f} = f - B \cdot \nabla u - cu \in L^2(\mathbb{R}^d).$$

There exists $C_1 > 0$ which only depends on B and c such that

$$\|\tilde{f}\|_{L^2(\mathbb{R}^d)}^2 \leqslant C_1\big(\|f\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{H^1(\mathbb{R}^d)}^2\big).$$

Let $h \in \mathbb{R}^d \setminus \{0\}$. As in the proof of Proposition 3.10 we apply (3.27) with $v = D_{-h}(D_h u) \in H^1(\mathbb{R}^d)$. For $\varepsilon > 0$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} A\nabla u \cdot \nabla\big(D_{-h}(D_h u)\big)\, dx \\
&\quad = \int_{\mathbb{R}^d} A\nabla(D_h u) \cdot \nabla(D_h u)\, dx + \int_{\mathbb{R}^d} (D_h A)\nabla u \cdot \nabla(D_h u)\, dx \\
&\quad \geqslant \alpha \|\nabla(D_h u)\|_{L^2(\mathbb{R}^d)}^2 - \varepsilon \|\nabla(D_h u)\|_{L^2(\mathbb{R}^d)}^2 - \frac{\|D_h A\|_{L^\infty(\mathbb{R}^d)}^2}{4\varepsilon} \|u\|_{H^1(\mathbb{R}^d)}^2
\end{aligned}$$

and

$$\int_{\mathbb{R}^d} \tilde{f} v \, dx \leqslant \varepsilon \|v\|_{L^2(\mathbb{R}^d)}^2 + \frac{\|\tilde{f}\|_{L^2(\mathbb{R}^d)}^2}{4\varepsilon} \leqslant \varepsilon \|\nabla(D_h u)\|_{L^2(\mathbb{R}^d)}^2 + \frac{C_1}{4\varepsilon}\big(\|f\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{H^1(\mathbb{R}^d)}^2 \big).$$

With $\varepsilon = \frac{\alpha}{4}$ we obtain

$$\frac{\alpha}{2} \|D_h(\nabla u)\|_{L^2(\mathbb{R}^d)}^2 \leqslant C_2\big(\|f\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{H^1(\mathbb{R}^d)}^2 \big), \quad C_2 = \frac{1}{\alpha}\big(C_1 + \|A\|_{C^1(\mathbb{R}^d)}\big).$$

This proves that $u \in H^2(\mathbb{R}^d)$ and, for some $C > 0$ which only depends on A, B and c,

$$\|u\|_{H^2(\mathbb{R}^d)} \leqslant C\big(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \big).$$

- The case $\Omega = \mathbb{R}^d_+$ is proved similarly by taking h parallel to $\partial\mathbb{R}^d_+$ as in the proof of Proposition 3.13. This proves that for $j \in [\![2, d]\!]$ and $k \in [\![1, d]\!]$ we have $\partial_j \partial_k u \in L^2(\mathbb{R}^d_+)$, with $\|\partial_j\partial_k u\|_{L^2(\mathbb{R}^d_+)} \leqslant C_{j,k}\big(\|f\|_{L^2(\mathbb{R}^d_+)} + \|u\|_{H^1(\mathbb{R}^d_+)} \big)$ for some $C_{j,k} > 0$ independant of f and u. Then we observe that the ellipticity (3.4) applied with $\xi = (1, 0, \dots, 0)$ shows that $a_{1,1}(x) \geqslant \alpha > 0$. Then

$$\begin{aligned}
\partial_1^2 u &= \frac{1}{a_{1,1}}\big(\partial_1 a_{1,1}\partial_1 u - (\partial_1 a_{1,1})\partial_1 u\big) \\
&= \frac{1}{a_{1,1}} \left(-f - \sum_{\substack{1 \leqslant j,k \leqslant d \\ (j,k) \neq (1,1)}} \partial_j a_{j,k} \partial_k u + B \cdot \nabla u + cu - (\partial_1 a_{1,1})\partial_1 u \right) \in L^2(\mathbb{R}^d_+),
\end{aligned}$$

with

$$\left\|\partial_1^2 u\right\|_{L^2(\mathbb{R}^d_+)} \leqslant C_{1,1}\left(\|f\|_{L^2(\mathbb{R}^d_+)} + \|u\|_{H^1(\mathbb{R}^d_+)} \right),$$

for some $C_{1,1} > 0$ independant of f and u. The conclusion follows in this case.

- We consider the case where Ω is a bounded subset of $\mathbb{R}^d$ of class C^2. As is now usual we will use a partition of unity and changes of variables as described in Paragraph 2.4.1. The regularity of a solution compactly supported in Ω is proved as in the proof of Proposition 3.13. Now we consider an open subset $\mathcal{U}$ of $\mathbb{R}^d$ such that $\partial\Omega \cap \mathcal{U}$ is a graph of class C^2, a diffeomorphism Φ of class C^2 from $\mathcal{U}$ to an open subset $\mathcal{W}$ such that $\Phi(\mathcal{U} \cap \Omega) = \mathcal{W} \cap \mathbb{R}^d_+$, and we assume that the solution $u \in H^1_0(\Omega)$ is supported in $\mathcal{U} \cap \overline{\Omega}$. We set $\Psi = \Phi^{-1}$ and we denote by $J\Phi$ and $J\Psi$ the jacobian matrices of Φ and Ψ, respectively. We also write $|J\Psi|$ for $|\det(J\Psi)|$.

Let $\tilde{u} = u \circ \Psi \in H^1_0(\mathcal{W} \cap \mathbb{R}^d_+)$. Then we check that $\tilde{u}$ is a weak solution on $\mathcal{W}$ of the equation

$$-\operatorname{div}(\tilde{A}\nabla\tilde{u}) + \tilde{B} \cdot \nabla\tilde{u} + \tilde{c}\tilde{u} = \tilde{f},$$

where for $\tilde{x} \in \mathcal{W} \cap \mathbb{R}^d_+$ and $x = \Psi(\tilde{x})$ we have set

$$\tilde{A}(\tilde{x}) = |J\Psi(\tilde{x})|\, J\Phi(x)A(x)J\Phi(x)^{\mathsf{T}}, \quad \tilde{B}(\tilde{x}) = |J\Psi(\tilde{x})|\, B(x)J\Phi(x)^{\mathsf{T}}, \quad \tilde{c}(\tilde{x}) = |J\Psi(\tilde{x})|\, c(x)$$

and $\tilde{f}(\tilde{x}) = |J\Psi(\tilde{x})|\, f(x)$. For instance, for $\tilde{v} \in H^1_0(\mathcal{W} \cap \mathbb{R}^d_+)$ and $v = \tilde{v} \circ \Phi$ we have by the change of variables $x = \Psi(\tilde{x})$

$$\begin{aligned}
-\int_{\mathcal{W}\cap\mathbb{R}^d_+} (\tilde{A}(\tilde{x})\nabla\tilde{u}(\tilde{x})) \cdot \nabla\tilde{v}(\tilde{x})\, d\tilde{x} &= -\int_{\mathcal{W}\cap\mathbb{R}^d_+} (A(\Psi(\tilde{x}))\nabla u(\Psi(\tilde{x}))) \cdot \nabla v(\Psi(\tilde{x}))\, |J\Psi(\tilde{x})|\, d\tilde{x} \\
&= -\int_{\mathcal{U}\cap\Omega} (A(x)\nabla u(x)) \cdot \nabla v(x)\, dx.
\end{aligned}$$

The matrix $\tilde{A}$ is symmetric, since A is. Now let $\xi \in \mathbb{R}^d$. For $\tilde{x} \in \mathcal{W} \cap \mathbb{R}^d_+$ and $x = \Psi(\tilde{x})$ we have

$$(\tilde{A}(\tilde{x})\xi) \cdot \xi = |J\Psi(\tilde{x})| \left(A(x)J\Phi(x)^\mathsf{T}\xi\right) \cdot \left(J\Phi(x)^\mathsf{T}\xi\right) \geqslant \alpha \,|J\Psi(\tilde{x})| \left|J\Phi(x)^\mathsf{T}\xi\right|^2 \geqslant \tilde{\alpha} \,|\xi|^2 ,$$

with

$$\tilde{\alpha} = \alpha \frac{\inf |J\Psi|}{\sup \|J\Psi\|} > 0.$$

Thus, according to the case $\Omega = \mathbb{R}^d_+$, we have $\tilde{u} \in H^2(\mathcal{W} \cap \mathbb{R}^d_+)$ with $\|\tilde{u}\|_{H^2(\mathcal{W}\cap\mathbb{R}^d_+)} \leqslant \tilde{C}\big(\|\tilde{f}\|_{L^2(\mathcal{W}\cap\mathbb{R}^d_+)} + \|\tilde{u}\|_{H^1(\mathcal{W}\cap\mathbb{R}^d_+)}\big)$, for some constant $C > 0$ independant of f or $\tilde{f}$. Going back to Ω, we deduce that $u \in H^2(\mathcal{U}\cap\Omega)$ with $\|u\|_{H^2(\mathcal{U}\cap\Omega)} \leqslant C\big(\|f\|_{L^2(\mathcal{U}\cap\Omega)} + \|u\|_{H^1(\mathcal{U}\cap\Omega)}\big)$, with $C > 0$ independant of f.

Now we use the notation of Paragraph 2.4.1. For all $j \in [\![0, N]\!]$ we have in the weak sense

$$P(\chi_j u) = f_j,$$

where, for some $C_j > 0$,

$$\|f_j\|_{L^2(\Omega)} \leqslant C_j\big(\, \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \,\big).$$

For $j \in [\![0, N]\!]$ we apply the above results to $\chi_j u$. Then we deduce that $u = \sum_{j=0}^N (\chi_j u)$ belongs to $H^2(\Omega)$, and

$$\begin{aligned}\|u\|_{H^2(\Omega)} \leqslant \sum_{j=0}^N \|\chi_j u\|_{H^2(\Omega)} \leqslant \sum_{j=0}^N \tilde{C}_j\big(\, \|f_j\|_{L^2(\Omega)} + \|\chi_j u\|_{H^1(\Omega))} \,\big)\\ \leqslant C\big(\, \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \,\big).\end{aligned}$$

This completes the proof. For the higher regularity, we check that under the stronger assumptions of the proposition we have higher regularity for $\tilde{u}$ and u at each step of the proofs. We omit the details. □

***Exercise* 38.** Assume that $A \in W^{1,\infty}(\Omega)$, $B \in L^\infty(\Omega)$ and $c \in L^\infty(\Omega)$. Prove that there exist $\gamma \geqslant 0$ and $C \geqslant 1$ such that for all $u \in H^2(\Omega) \cap H^1_0(\Omega)$ we have

$$C^{-1} \|u\|_{H^2(\Omega)} \leqslant \|Pu\|_{L^2(\Omega)} + \gamma \|u\|_{L^2(\Omega)} \leqslant C \|u\|_{H^2(\Omega)} .$$

3.5 The Poisson equation on a bounded domain

After the analysis of a quite general second order elliptic equation, we go back to the model case, namely the Poisson equation. We have already discussed in Exercise 34 and in Paragraph 3.4.1 (see (3.24)) the fact that the equation $-\Delta u = f$ is not well posed on $\mathbb{R}^d$.

Here we consider the same problem on a bounded open subset Ω of $\mathbb{R}^d$. Of course, as above, we will have to add an additional condition to get a well posed problem (otherwise, we see that if u is a solution, then $u + \beta$ is also a solution for any constant β).

We begin with the Poisson equation with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{on } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

The corresponding bilinear form is given by (3.24) as in $\mathbb{R}^d$. The important difference is the Poincaré inequality, according to which the H^1 norm is controled by the norm of the gradient on $H_0^1(\Omega)$ (see Theorem 2.57).

Proposition 3.23. *Let Ω be a bounded subset of $\mathbb{R}^d$ and let $f \in L^2(\Omega)$. There exists a unique $u \in H_0^1(\Omega)$ such that*

$$\forall v \in H_0^1(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx.$$

Proof. For $u, v \in H_0^1(\Omega)$ we set

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v.$$

This defines a continuous bilinear form on $H_0^1(\Omega)$. By the Poincaré Inequality, there exists $\alpha > 0$ such that for $u \in H_0^1(\Omega)$ we have

$$a(u,u) = \|\nabla u\|_{L^2(\Omega)}^2 \geqslant \alpha \|u\|_{H^1(\Omega)}^2 .$$

This gives the coercivity of a and the conclusion follows from the Lax-Milgram Theorem. □

Then, by Proposition 3.22, the weak solution of (3.30) given by Proposition 3.23 belongs to $H^2(\Omega)$ (and it is even in $H^{k+2}(\Omega)$ if $f \in H^k(\Omega)$ and Ω is of class C^{k+2} for some $k \in \mathbb{N}$).

We continue with the same problem with Neumann boundary condition:

$$\begin{cases} -\Delta u = f, & \text{on } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.31}$$

Compared to (3.13), we cannot give a analog of Proposition 3.23 with $H_0^1(\Omega)$ replaced by $H^1(\Omega)$. It is clear that the constant functions belong to $H^1(\Omega)$ and breaks the coercivity of the bilinear form a on $H^1(\Omega)$.

Thus, to recover some coercivity, we have to remove at least the constant functions from $H^1(\Omega)$. The Poincaré-Wiertinger inequality tells us that this is in fact enough, see Theorem 2.59.

Notice that if $u \in H^2(\Omega)$ solves (3.31), then by the Green formula (see Theorem 2.55)

$$\int_\Omega f = -\int_\Omega \Delta u = 0. \tag{3.32}$$

This gives a necessary condition for (3.31) to have a solution in $H^2(\Omega)$. Thus it is natural to introduce

$$\tilde{L}^2(\Omega) = \left\{ f \in L^2(\Omega) \, : \, \int_\Omega f = 0 \right\}.$$

Proposition 3.24. *Let Ω be a bounded, connected and open subset of $\mathbb{R}^d$. Let $f \in \tilde{L}^2(\Omega)$. There exists a unique $u \in \tilde{H}^1(\Omega)$ (see (2.26)) such that*

$$\forall v \in \tilde{H}^1(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx. \tag{3.33}$$

For the proof we check that $\tilde{H}^1(\Omega)$ is a Hilbert space, and then we follow the proof of Proposition 3.23, using Theorem 2.59 instead of Theorem 2.57.

Proposition 3.25. *Let $f \in \tilde{L}^2(\Omega)$ and let u be the weak solution of (3.31) given by Proposition 3.24. Then in the sense of distributions we have $-\Delta u = f$.*

Proof. Let $\phi \in C_0^\infty(\Omega)$. Then $\phi - \frac{1}{|\Omega|}\int_\Omega \phi\, dy$ belongs to $\widetilde{H}^1(\Omega)$, so we can write

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla \phi \, dx &= \int_\Omega \nabla u \cdot \nabla \left(\phi - \frac{1}{|\Omega|} \int_\Omega \phi \, dy \right) dx \\
&= \int_\Omega f \left(\phi - \frac{1}{|\Omega|} \int_\Omega \phi \, dy \right) dx \\
&= \int_\Omega f \phi \, dy - \int_\Omega \left(\frac{1}{|\Omega|} \int_\Omega f(x)\, dx \right) \phi \, dy \\
&= \int_\Omega f \phi \, dy.
\end{aligned}$$

This proves that $-\Delta u = f$ in the sense of distributions. $\square$

***Exercise* 39.** Let Ω be a bounded subset of $\mathbb{R}^d$. Show that there exists $C > 0$ such that for any $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ we have

$$\|u\|_{H^1(\Omega)} \leqslant C \|\Delta u\|_{L^2(\Omega)} .$$

***Exercise* 40.** Let Ω be an open bounded subset of $\mathbb{R}^d$. We consider the Poisson equation with inhomogeneous Neumann boundary condition. Given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ we consider the problem

$$\begin{cases} -\Delta u = f, & \text{on } \Omega, \\ \partial_\nu u = g, & \text{on } \partial\Omega. \end{cases}$$

1. Give a necessary condition on f and g analogous to (3.32) for this problem to have a solution $u \in H^2(\Omega)$.
2. Give a variational formulation adapted to this problem. Prove existence and uniqueness of a weak solution in a suitable space.
3. What can we say about the regularity of this solution ?
4. What happens if g is only in $H^{-1/2}(\partial\Omega)$?

3.6 Spectral properties of elliptic operators

3.6.1 The Fredholm alternative

Now we apply the abstract Fredholm theory to our second order elliptic equations. We introduce the formal adjoint P^* of the operator P defined (3.2). It is defined by

$$P^* u = -\operatorname{div}(A\nabla u) - B \cdot \nabla u + \big(c - \operatorname{div} B\big) u.$$

In particular $P^* = P$ if $B = 0$ (this is not the case with complex coefficients). The corresponding bilinear form is defined on $H_0^1(\Omega)$ by

$$a_{P^*}(u, v) = a_P(v, u),$$

where a_P is defined by (3.23). In particular, $u \in H_0^1(\Omega)$ is a weak solution for the problem

$$\begin{cases} P^* u = 0, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.34}$$

if and only if $a_P(v,u) = 0$ for all $v \in H_0^1(\Omega)$. Moreover P^* is elliptic with the same coefficient $\alpha > 0$ as P (see (3.4)), and for $\alpha_0 \in]0, \alpha[$ Lemma 3.20 gives the same γ_0 as for P.

Theorem 3.26. *Let Ω be a bounded open subset of $\mathbb{R}^d$ and let P be defined by (3.2).*

(i) *The problem (3.21) has a unique weak solution for any $f \in L^2(\Omega)$ if and only if 0 is the only weak solution for the homogeneous problem*

$$\begin{cases} Pu = 0, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.35}$$

(ii) *The problem (3.21) has a weak solution if and only if f is orthogonal in $L^2(\Omega)$ to the set N^* of weak solutions of the problem (3.34). And in this case the set of weak solutions of (3.21) is a subspace of $H_0^1(\Omega)$ of dimension $\mathsf{dim}(N^*)$.*

Proof. • It is clear that if (3.21) has a unique weak solution for any $f \in L^2(\Omega)$ then in particular 0 is the unique weak solution for (3.35). Conversely, assume that 0 is the unique weak solution for (3.35). By linearity, a weak solution of (3.21) is necessarily unique. It remains to prove existence. Let $f \in L^2(\Omega)$.

Let γ_0 be given by Lemma 3.20 and let $\gamma > \gamma_0$. For $g \in L^2(\Omega)$ we denote by Rg the unique weak solution u of the problem

$$\begin{cases} Pu + \gamma u = g, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

By Proposition 3.21, this defines a continuous operator $R : L^2(\Omega) \to H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ by Theorem 2.49, we can see R as a compact operator on $L^2(\Omega)$.

A function $u \in H_0^1(\Omega)$ is a weak solution of (3.21) if and only if $u = R(f + \gamma u)$. If we set $K = \gamma R$, this is equivalent to

$$(\mathrm{Id} - K)u = Rf. \tag{3.36}$$

If $u \in \mathrm{Ker}(\mathrm{Id} - K)$ then $u \in H_0^1(\Omega)$ and u is a weak solution of (3.35), so $u = 0$. This proves that $\mathrm{Id} - K$ is injective. By Theorem A.8, it is also surjective so there exists a solution $u \in L^2(\Omega)$ of (3.36). Since $u = Rf + Ku \in H_0^1(\Omega)$, it is a weak solution of (3.21), which proves the first statement.

• We observe that the adjoint R^* of R maps $g \in L^2(\Omega)$ to the unique weak solution of

$$\begin{cases} P^* u + \gamma u = g, & \text{on } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Therefore v is a weak solution of (3.34) if and only if

$$(\mathrm{Id} - K^*)v = 0.$$

Then, by Theorem A.8 again,

$$N^* = \mathrm{Ker}(\mathrm{Id} - K^*) = \mathsf{Ran}(\mathrm{Id} - K)^\perp.$$

Then the problem (3.21) has a weak solution if and only if $Rf \in (N^*)^\perp$, that is if and only if

$$\forall v \in N^*, \quad \langle f, v\rangle_{\mathcal{H}} = \langle f, K^*v\rangle_{\mathcal{H}} = \langle Kf, v\rangle_{\mathcal{H}} = \gamma \langle Rf, v\rangle_{\mathcal{H}} = 0.$$

This gives the first part of the second statement. Finally, by Theorem A.8 we also have

$$\mathsf{dim}(\mathrm{Ker}(\mathrm{Id} - K)) = \mathsf{dim}(N^*),$$

and the proof is complete. □

Remark 3.27. The first statement of Theorem 3.26 is very important, and it does not hold in general. For instance, given $f \in L^2(\mathbb{R})$, then u is a weak solution of the Poisson equation

$$-u'' = f$$

if and only if $u \in H^2(\mathbb{R})$ and, for almost all $\xi \in \mathbb{R}$,

$$\xi^2 \hat{u}(\xi) = \hat{f}(\xi).$$

Then we see that $u = 0$ is the only solution when $f = 0$, but there is no solution for instance if $\hat{f} = 1$ on a neighborhood of 0.

***Exercise* 41.** Let Ω be a bounded open connected subset of $\mathbb{R}^d$ of class C^1.
1. For $f \in L^2(\Omega)$ we denote by $R(f) \in H^1(\Omega)$ the unique weak solution u of the problem

$$\begin{cases} -\Delta u + u = f, & \text{on } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

Prove that this defines a compact operator R on $L^2(\Omega)$.
2. Prove that $R^* = R$.
3. Prove that $\mathrm{Ker}(\mathrm{Id} - R)$ is the set $\langle 1\rangle$ of constant functions on Ω.
4. Let $f \in L^2(\Omega)$. Prove that (3.31) has a solution if and only if $Rf \in \langle 1\rangle^\perp$. Deduce that (3.31) has a solution if and only if f itself is orthogonal to $\langle 1\rangle$.
5. Let $f \in L^2(\Omega)$ such that (3.31) has a solution $u_0 \in H^1(\Omega)$. Prove that the set of solution is given by $u_0 + \mathrm{Ker}(\mathrm{Id} - R)$.
Compare all these results with the results of Section 3.5.

3.6.2 Spectrum of elliptic operators

Let Ω be an open bounded subset of $\mathbb{R}^d$. We consider a second order elliptic operator P as defined by (3.2), with the symmetry and ellipticity assumptions (3.37) and (3.4).

So far we have mostly discussed the variational version of the problem (3.21), given by the bilinear form (3.23) on $H_0^1(\Omega)$. This means that P was seen as a function from $H_0^1(\Omega)$ to its dual $H^{-1}(\Omega)$. However we have seen that a weak solution $u \in H_0^1(\Omega)$ belongs in fact to $H^2(\Omega)$, and (3.21) holds in the strong sense.

Now let us see P directly as a linear map from $H^2(\Omega) \cap H_0^1(\Omega)$ to $L^2(\Omega)$. This is then an operator on $L^2(\Omega)$ with domain $\mathcal{D}(P) = H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem 3.28. *The spectrum of P consists of a discrete set of eigenvalues with finite algebraic multiplicities.*

Proof. Let $\alpha_0 \in]0, \alpha[$ and let γ_0 be given by Lemma 3.20. Let $\gamma > \gamma_0$. For $f \in L^2(\Omega)$ the problem (3.26) has a unique weak solution $u \in H_0^1(\Omega)$. Moreover $u \in H^2(\Omega)$ and there exists $C > 0$ independant of f such that $\|u\|_{H^2(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}$. This proves that $P + \gamma : \mathcal{D}(P) \to L^2(\Omega)$ is bijective with bounded inverse. Thus $-\gamma \in \rho(P)$. Moreover, since the inclusion $H^2(\Omega) \subset L^2(\Omega)$ is compact, the inverse $(P+\gamma)^{-1}$ is compact. Then we conclude with Theorem A.13. □

In particular we recover the first statement of Theorem 3.26. We know that the sets of weak and strong solutions of (3.21) coincide. Then Theorem 3.28 says that P is bijective with bounded inverse (for all $f \in L^2(\Omega)$ the problem (3.21) has a unique strong solution u and $\|u\|_{H^2(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}$ for some $C > 0$) if and only if 0 is not an eigenvalue of P (0 is the unique solution if $f = 0$).

Now we assume that $B = 0$. Then P is formally symmetric, in the sense that

$$P^* = P, \tag{3.37}$$

where P^* is as in (2.18). In particular, in Lemma 3.20 we can take $\alpha_0 = \alpha$ and $\gamma_0 = -\inf c$.

Notice then that if $u \in H_0^1(\Omega) \setminus \{0\}$ and $\lambda \in \mathbb{R}$ are such that

$$\forall v \in H_0^1(\Omega), \quad a(u,v) = \lambda \langle u, v \rangle ,$$

then we necessarily have $\lambda > -\gamma_0$.

Theorem 3.29. *Assume that the operator P is symmetric and let γ_0 be as above.*

(i) The spectrum of P consists of a sequence of eigenvalues greater than $(-\gamma_0)$ and going to $+\infty$. The geometric and algebraic multiplicities of these eigenvalues coincide, and they are all finite.

(ii) There exists an orthonormal basis of $L^2(\Omega)$ which consists of eigenfunctions for the operator P.

If we denote by $(\lambda_n)_{n\in\mathbb{N}}$ the non-decreasing sequence of eigenvalues repeated according to multiplicities we have

$$-\gamma_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \xrightarrow[n\to+\infty]{} +\infty.$$

Then there exists an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}^*}$ such that for $n \in \mathbb{N}^*$ we have $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ and $P\varphi_n = \lambda_n \varphi_n$. Equivalently, $\varphi_n \in H_0^1(\Omega)$ is the unique weak solution for the problem

$$\begin{cases} P\varphi_n = \lambda_n \varphi_n & \text{in } \Omega, \\ \varphi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.38}$$

Proof. We apply Theorem 3.28. We have already said that the eigenvalues are greater than $(-\gamma_0)$. Let $\gamma > \gamma_0$. By Theorem A.12, there exists an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}}$ of eigenfunctions for $(P+\gamma)^{-1}$. The functions φ_n, $n \in \mathbb{N}$, are also eigenfunctions for P. Indeed, if $(P+\gamma)^{-1}\varphi_n = \mu_n\varphi_n$ then $\mu_n > 0$, $1-\gamma\mu_n \neq 0$ and $P\varphi_n = (1-\gamma\mu_n)^{-1}\mu_n^{-1}\varphi_n)$. This implies that geometric and algebraic multiplicities of all the eigenvalues coincide. Moreover these multiplicities are finite, so there is an infinite (countable) number of eigenvalues. Since the spectrum is discrete and included in $]-\gamma_0,+\infty[$, the sequence of eigenvalues goes to $+\infty$. $\square$