Chapter 3

Green Formula

The aim of this chapter is to give a proof to the Stokes Formula. this is a $d \geqslant 2$ dimensional generalization of the fundamental theorem of calculus which makes the link between integrals and primitives in dimension 1. Our main motivation here is the Green formula that generalizes the integration by parts.

As often, to compute integrals in d-dimensionnal spaces we use the Fubini Theorem to compute integrales of dimension 1. For example, if we consider two functions u and v of class C^1 on $\mathbb{R}^2$, with u or v compactly supported, we can see that

$$\int_{\mathbb{R}^2} (\partial_{x_1} u)\, v \, dx = -\int_{\mathbb{R}^2} u\, (\partial_{x_1} v) \, dx.$$

Indeed, for $x_2 \in \mathbb{R}$ fixed, using the integration by parts on $\mathbb{R}$ (there is no boundary term since $(uv)(\cdot, x_2)$ vanishes outside a compact of $\mathbb{R}$)

$$\int_{\mathbb{R}} \partial_{x_1} u(x_1, x_2) v(x_1, x_2) \, dx_1 = -\int_{\mathbb{R}} u(x_1, x_2) \partial_{x_1} v(x_1, x_2) \, dx_1.$$

To conclude, we now have to integrate this equality with respect to $x_2 \in \mathbb{R}$. This is in fact valid in dimension d and for any of the partial derivatives.

Now we consider an open subset Ω of $\mathbb{R}^d$. The previous reasonning still holds for functions u and v of class C^1 with at least one having a compact support in Ω. Since u or v is null near the boundary of Ω, there is still no boundary terms in the integration by part. Finally, the following result seems accessible.

Proposition 3.1. *Let Ω be an open set of $\mathbb{R}^d$ and u, v be two fnctions of class C^1 on Ω, with u or v compactly supported. Then for $j \in [\![1, d]\!]$ we have*

$$\int_{\Omega} (\partial_{x_j} u)\, v \, dx = -\int_{\Omega} u\, (\partial_{x_j} v) \, dx.$$

The trouble begins when we consider functions u and v which do not vanish near the boundary. If we have a simple parametrization of the open set Ω, we can apply the same argument, with boundary terms. For example, if Ω is the open unit disc of $\mathbb{R}^2$, then for any $y \in]-1, 1[$ we compute

$$\begin{aligned}\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} &\partial_x u(x,y) v(x,y)\, dx \\ &= (uv)\big(\sqrt{1-y^2}, y\big) - (uv)\big(-\sqrt{1-y^2}, y\big) - \int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} u(x,y) \partial_x v(x,y)\, dx.\end{aligned}$$

Integrating with respect to $y \in]-1,1[$ we get

$$\iint_\Omega \partial_x u(x,y) v(x,y)\,dx\,dy$$
$$= \int_{y=-1}^{1} \Big((uv)\big(\sqrt{1-y^2}, y\big) - (uv)\big(-\sqrt{1-y^2}, y\big) \Big)\,dy - \iint_\Omega u(x,y)\partial_x v(x,y)\,dx\,dy.$$

The purpose of this chapter is to understand the first term of the right-hand side. As expected, it involves the values of u and v on the unit circle of $\mathbb{R}^2$, which is the boundary of Ω. Thus, it is an integral on a circle.

More generally, a formula of integration by parts on an open set Ω will necessarily involve an integral on the boundary $\partial\Omega$ of Ω. We will only consider the convenient case where $\partial\Omega$ is a sufficiently regular submanifold of $\mathbb{R}^d$. As any set, a submanifold of $\mathbb{R}^d$ can be endowed with a structure of measured space and the corresponding integral. In the first part of this chapter we will define the Lebesgue integral on submanifolds. The specifications are the same as in the Euclidean space. The Lebesgue measure of a curve should extend the notion of length, the Lebesgue measures of surfaces should extend the notion of area, etc.

Once this work is done, we will state the integration by parts formula, where the boundary term will be an integral on $\partial\Omega$ endowed with this Lebesgue measure.

For $k \in \mathbb{N} \cup \{\infty\}$ we denote by $C^\infty(\overline{\Omega})$ the set of restrictions to the open set Ω of C^∞ functions on $\mathbb{R}^d$. We denote by $e = (e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$.

3.1 Lebesgue measure on a submanifolds of $\mathbb{R}^d$

3.1.1 Hypersurfaces of $\mathbb{R}^d$

Definition 3.2. Let S be a subset of $\mathbb{R}^d$ and $k \in \mathbb{N}^* \cup \{\infty\}$. S is said to be an hypersurface of class C^k if for any $w \in S$ there exists a neighbourhood $\mathcal{V}$ of w in $\mathbb{R}^d$ and a map $F : \mathcal{V} \to \mathbb{R}$ of class C^k such that $\nabla F(w) \neq 0$ and $S \cap \mathcal{V} = F^{-1}(\{0\})$.

Examples 3.3.
- Let $\nu \in \mathbb{R}^d \setminus \{0\}$. Then the hyperplane $H = \mathsf{Vect}(\nu)^\perp$ is a hypersurface of class C^∞ in $\mathbb{R}^d$. For F we can consider the function which maps x to $x \cdot \nu$ (in the neighbourhood of any $w \in H$, the definition holds $\mathcal{V} = \mathbb{R}^d$).
- We consider the sphere
$$S^{d-1} = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d \,|\, x_1^2 + \dots + x_d^2 = 1 \right\}.$$
Then S^{d-1} is a hypersurface of class C^∞ in $\mathbb{R}^d$ (consider $F : (x_1, \dots, x_d) \mapsto x_1^2 + \dots + x_d^2 - 1$). For any $r > 0$ we can similarly consider the sphere of radius r:
$$S_r^{d-1} = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d \,|\, x_1^2 + \dots + x_d^2 = r^2 \right\}.$$
- Let $\mathcal{O}$ be an open set of $\mathbb{R}^{d-1}$ and φ be a function of class C^k ($k \in \mathbb{N}^* \cup \{\infty\}$) from $\mathcal{O}$ to $\mathbb{R}$. The graph of φ is then
$$\Gamma = \left\{ (x', \varphi(x')) \in \mathcal{O} \times \mathbb{R}, x' \in \mathcal{O} \right\}.$$
It is a hypersurface of class C^k in $\mathbb{R}^d$. To see this, consider
$$F : \left\{ \begin{array}{ccc} \mathcal{O} \times \mathbb{R} & \to & \mathbb{R}, \\ (x', x_d) & \mapsto & x_d - \varphi(x'). \end{array} \right.$$

The case of the graph will be particularly comfortable for the coming calculations. The Implicit Functions Theorem ensures that, locally in the neighborhood of each of its points, any hypersurface can in fact be seen as a graph (up to a change of basis).

Definition 3.4. Let Γ be a part of $\mathbb{R}^d$ and $k \in \mathbb{N}^* \cup \{\infty\}$. Γ is said to be a graph of class C^k if there exists an orthonormal basis $\beta = (\beta_1, \dots, \beta_d)$ of $\mathbb{R}^d$, an open set $\mathcal{O}$ of $\mathbb{R}^{d-1}$ and a map $\varphi : \mathcal{O} \to \mathbb{R}$ of class C^k such that Γ is the graph of φ in $\mathbb{R}^d$ endowed with the basis β. It means that Γ is the image of $\mathcal{O}$ by

$$\Phi : \begin{cases} \mathcal{O} & \to & \mathbb{R}^d, \\ (x_1, \dots, x_{d-1}) & \mapsto & x_1\beta_1 + \dots + x_{d-1}\beta_{d-1} + \varphi(x_1, \dots, x_{d-1})\beta_d. \end{cases}$$

Φ is then said to be a parametrization of Γ.

Proposition 3.5. *Let $k \in \mathbb{N}^* \cup \{\infty\}$. A graph of class C^k in $\mathbb{R}^d$ is an hypersurface of class C^k.*

Proof. With the notation of Definition 3.4 we set

$$\mathcal{V} = \left\{ \sum_{j=1}^{d} x_j\beta_j, (x_1, \dots, x_{d-1}) \in \mathcal{O}, x_d \in \mathbb{R} \right\},$$

and for $x \in \mathcal{V}$ and $x_1, \dots, x_d \in \mathbb{R}$ such that $x = \sum x_j\beta_j \in \mathcal{V}$ (the decomposition is unique) we set

$$F(x) = x_d - \varphi(x_1, \dots, x_{d-1}).$$

Then F satisfies all the conditions of Definition 3.2. $\square$

Proposition 3.6. *Let S a hypersurface of $\mathbb{R}^d$ and $w \in S$. There exists a neighbourhood $\mathcal{V}$ of w in $\mathbb{R}^d$ such that $S \cap \mathcal{V}$ is a graph.*

Proof. Let $\mathcal{V}$ be a neighbourhood of w and $F : \mathcal{V} \to \mathbb{R}$ as given by Definition 3.2. There exists $j \in [\![1, d]\!]$ such that $\partial_{x_j} F(w) \neq 0$. We construct an orthonormal basis $(\beta_1, \dots, \beta_d)$ of $\mathbb{R}^d$ by reordering the vectors of the canonical basis $(e_1, \dots, e_d)$ in such a way that $\beta_d = e_j$. We then denote by $(y_1, \dots, y_n)$ the coordinates of a point $y \in \mathbb{R}^d$ in this basis. Thus we have $\partial_{y_d} F(w) \neq 0$. By the Implicit Functions Theorem we get that, after having reduced $\mathcal{V}$ if necessary, $S \cap \mathcal{V}$ is indeed a graph of class C^k in $\mathbb{R}^d$. $\square$

Examples 3.7.

- We consider the sphere S^2 of $\mathbb{R}^3$. S^2 is not a graph, but each half of the sphere is a graph. For example, the half sphere

$$S^2 \cap \{x = (x_1, x_2, x_3) \,|\, x_3 > 0\} \tag{3.1}$$

is the graph of the map

$$\varphi_+ : \begin{cases} \mathbb{D} & \to & \mathbb{R} \\ (x_1, x_2) & \mapsto & 1 - x_1^2 - x_2^2 \end{cases} \tag{3.2}$$

where we set $\mathbb{D} = \left\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1^2 + x_2^2 < 1\right\}$. A graph is not necessarily the graph of a function expressing its last coordinate with respect to the others. We can also see the set

$$S_1^- = \left\{x \in S^2 \,|\, x_1 < 0\right\}$$

as a graph, since

$$S_1^- = \{\varphi_-(x_2, x_3)e_1 + x_2e_2 + x_3e_3, (x_2, x_3) \in \mathbb{D}\}$$

where
$$\varphi_-(x_2, x_3) = -(1 - x_2^2 - x_3^2).$$

To be consistent with the framework of Definition 3.4, it suffices to change the basis. Indeed, If we consider the basis $\beta = (\beta_1, \beta_2, \beta_3) = (e_2, e_3, e_1)$ then we also have
$$S_1^- = \{y_1\beta_1 + y_2\beta_2 + \varphi_-(y_1, y_2)\beta_3, (y_1, y_2) \in \mathbb{D}\}.$$

Proposition-Definition 3.8 (Tangent space and normal vector)**.** *Let Γ be a C^1 graph. Let β, $\mathcal{O}$, φ and Φ be as in Definition 3.4. Let $w \in \Gamma$ and $x' \in \mathcal{O}$ such that $w = \Phi(x')$.*

(i) *The tangent space $T_w\Gamma$ to Γ at w is the image of the differential $d_x\Phi$. It is a hyperplane of $\mathbb{R}^d$.*

(ii) *$\nu \in \mathbb{R}^d$ is said to be the normal vector to Γ at w if it is orthogonal to $T_w\Gamma$. In addition, ν is said to be a unit vector if $\|\nu\| = 1$.*

Proof. Let us prove that the definition of $T_w\Gamma$ does not depend of the choice of $(\beta, \mathcal{O}, \varphi)$. Assume that $\gamma = (\gamma_1, \dots, \gamma_d)$ is an orthonormal basis of $\mathbb{R}^d$, that $\tilde{\mathcal{O}}$ is an open set of $\mathbb{R}^{d-1}$ and that $\psi \in C^1(\tilde{\mathcal{O}}, \mathbb{R})$ is such that Γ is also the graph of ψ in the basis γ, that is the image of $\Psi : \tilde{\mathcal{O}} \to \mathbb{R}^d$, where for $y' = (y_1, \dots, y_{d-1}) \in \tilde{\mathcal{O}}$ we have set
$$\Psi(y') = \sum_{j=1}^{d-1} y_j\gamma_j + \psi(y')\gamma_d.$$

This map Ψ is of class C^1 on $\mathcal{O}'$ and realizes a bijection from $\mathcal{O}'$ to Γ. If we note Π the orthogonal projection of $\mathbb{R}^d$ on the hyperplane generated by $(\gamma_1, \dots, \gamma_{d-1})$ (this is a function of class C^∞), then the inverse of Φ is the restriction to Γ of Π. We note $\Theta = \Psi^{-1} \circ \Phi$. This defines a bijection from $\mathcal{O}$ to $\tilde{\mathcal{O}}$. Since we also have $\Theta = \Pi \circ \Phi$, Θ is of class C^1 on $\mathcal{O}$. Let $x' \in \mathcal{O}$, $w = \Phi(x')$ and $y' = \Theta(x') = \Psi^{-1}(w) \in \tilde{\mathcal{O}}$. We have
$$\mathrm{Im}(d_{x'}\Phi) = \mathrm{Im}\left(d_{x'}(\Psi \circ \Theta)\right) = \mathrm{Im}\left(d_{y'}\Psi \circ d_{x'}\Theta\right) \subset \mathrm{Im}(d_{y'}\Psi).$$

We similarly see that $\mathrm{Im}(d_{y'}\Psi) \subset \mathrm{Im}(d_{x'}\Phi)$. Hence $\mathrm{Im}(d_{x'}\Phi) = \mathrm{Im}(d_{y'}\Psi)$, and the definition of $T_w\Gamma$ does not depend of the choice of a parametrization. Moreover, the subspace $\mathrm{Im}(d_{x'}\Phi)$ is generated by the vectors
$$\frac{\partial\Phi}{\partial x_j}(x') = \beta_j + \frac{\partial\varphi}{\partial x_j}(x')\beta_d, \quad 1 \leqslant j \leqslant d-1,$$

which are linearly independant, so $T_w\Gamma$ has dimension $d-1$. □

Remark 3.9. The two normal unit vectors to Γ at w are given by
$$\nu(w) = \frac{1}{\sqrt{1 + \|\nabla\varphi(x')\|^2}}\left(\sum_{j=1}^{d-1} \frac{\partial\varphi}{\partial x_j}(x')\beta_j - \beta_d\right) \tag{3.3}$$

and its opposite $-\nu(w)$.

Remark 3.10. We can prove that with the previous notation we have
$$\det(d_{x'}\Theta) = \det\left(\partial_{x_1}\Phi(x'), \dots, \partial_{x_{d-1}}\Phi(x'), \gamma_d\right). \tag{3.4}$$

For $j \in [\![1, d]\!]$ there exist $a_{j,1}, \dots, a_{j,d} \in \mathbb{R}$ such that $\beta_j = \sum_{k=1}^d a_{j,k}\gamma_k$. Then we have
$$\partial_{x_j}\Phi(x') = \beta_j + \partial_{x_j}\varphi(x')\beta_d = \sum_{k=1}^d \left(a_{j,k} + \partial_{x_j}\varphi(x')a_{d,k}\right)\gamma_k,$$

Hence

$$\det_\gamma\big(\partial_{x_1}\Phi(x'),\dots,\partial_{x_{d-1}}\Phi(x'),\gamma_d\big)$$
$$= \begin{vmatrix} a_{1,1}+a_{d,1}\partial_{x_1}\varphi(x') & \dots & a_{d-1,1}+a_{d,1}\partial_{x_{d-1}}\varphi(x') & 0 \\ \vdots & & \vdots & \vdots \\ a_{1,d-1}+\alpha_{d,d-1}\partial_{x_1}\varphi(x') & \dots & a_{d-1,d-1}+\alpha_{d,d-1}\partial_{x_{d-1}}\varphi(x') & 0 \\ a_{1,d}+a_{d,d}\partial_{x_1}\varphi(x') & \dots & a_{d-1,d}+a_{d,d}\partial_{x_{d-1}}\varphi(x') & 1 \end{vmatrix} \tag{3.5}$$

On the other hand for $x' \in \mathcal{O}$ we have

$$\begin{aligned}\Phi(x') &= \sum_{j=1}^{d-1} x_j\beta_j + \varphi(x')\beta_d \\ &= \sum_{j=1}^{d-1} x_j \sum_{k=1}^{d} a_{j,k}\gamma_k + \varphi(x')\sum_{k=1}^{d} a_{d,k}\gamma_k \\ &= \sum_{k=1}^{d}\left(\sum_{j=1}^{d-1} x_j a_{j,k} + \varphi(x')a_{d,k}\right)\gamma_k,\end{aligned}$$

So if we note $y' = \Theta(x')$ then for $k \in [\![1, d-1]\!]$ we have

$$y_k = \sum_{j=1}^{d-1} x_j a_{j,k} + \varphi(x')a_{d,k}.$$

Hence

$$\det(d_{x'}\Theta) = \begin{vmatrix} a_{1,1}+a_{d,1}\partial_{x_1}\varphi(x') & \dots & a_{d-1,1}+a_{d,1}\partial_{x_{d-1}} \\ \vdots & & \vdots \\ a_{1,d-1}+a_{d,d-1}\partial_{x_1}\varphi(x') & \dots & a_{d-1,d-1}+a_{d,d-1}\partial_{x_{d-1}} \end{vmatrix}. \tag{3.6}$$

Expanding (3.5) with respect to the last column, we get that (3.5) and (3.6) coincide, which proves (3.4).

3.1.2 Lebesgue measure on a hypersurface

In this section we define the Lebesgue measure of a hypersurface of $\mathbb{R}^d$. This generalizes the notion of lenghth for a curve in $\mathbb{R}^2$ and of are for a surface in $\mathbb{R}^3$. We could similarly define the Lebesgue measure for a submanifolds of $\mathbb{R}^d$ of any dimension (for example, the length of a curve in $\mathbb{R}^3$).

***Exercise* 1.** Prove that the Lebesgue measure of a hypersurface of $\mathbb{R}^d$ is always 0.

We begin with the definition of the Borel sigma algebra of a hypersurface. We consider on S the topology induced by the usual topology of $\mathbb{R}^d$. It means that the open sets of S can be written as $S \cap \mathcal{O}$ where $\mathcal{O}$ is an open set of $\mathbb{R}^d$. We can then endow S with the corresponding Borel sigma algebra $\mathcal{B}(S)$. Then we can check (exercise) that

$$\mathcal{B}(S) = \left\{S \cap B, B \in \mathcal{B}(\mathbb{R}^d)\right\}.$$

The following proposition implies that we also have

$$\mathcal{B}(S) = \left\{B \in \mathcal{B}(\mathbb{R}^d) \mid B \subset S\right\}.$$

Proposition 3.11. *Let S be a hypersurface of class C^1 in $\mathbb{R}^d$. Then S is a Borel set of $\mathbb{R}^d$.*

Proof. Let $B = \overline{S} \backslash S$. If $B = \varnothing$, then S is closed. In particular, it is a Borel set. If $B \neq \varnothing$ then for $n \in \mathbb{N}^*$ we set

$$S_n = \left\{ x \in S \,|\, \mathsf{dist}(x, B) \geqslant \frac{1}{n} \right\}.$$

Then S_n is closed in $\mathbb{R}^d$ for any $n \in \mathbb{N}^*$. Since $S = \bigcup_{n \in \mathbb{N}^*} S_n$, we get that S is a Borel set of $\mathbb{R}^d$. □

Let us now define the measure of a graph. Using proposition 3.6, we will then extend this to more general hypersurfaces.

Proposition-Definition 3.12. *Let Γ be a graph of class C^1 in $\mathbb{R}^d$. Let β, $\mathcal{O}$, φ and Φ be as in Definition 3.4.*

(i) *Let B be a Borel set of Γ and $B' = \Phi^{-1}(B)$. We set*

$$\sigma(B) = \int_{B'} \sqrt{1 + \|\nabla \varphi(x')\|^2}\, dx'.$$

This defines a measure σ on Γ that does not depend of the choice of β, $\mathcal{O}$, φ and Φ.

(ii) *Let f be a measurable function from $S \cap \mathcal{V}$ to $[0, +\infty]$. The integral of f with respect to the measure σ is given by*

$$\int_{S \cap \mathcal{V}} f \, d\sigma = \int_{\mathcal{O}} f\big(x'\beta' + \varphi(x')\beta_d\big) \sqrt{1 + \|\nabla \varphi(x')\|^2}\, dx', \tag{3.7}$$

where for $x' = (x_1, \ldots, x_{d-1})$ we set

$$x'\beta' = x_1\beta_1 + \cdots + x_{d-1}\beta_{d-1}.$$

(iii) *If $f : S \cap \mathcal{V} \to \mathbb{C}$ is integrable, then its integrale is also defined by (3.7).*

Proof. We leave as an exercise the fact that σ is a measure on $(\Gamma, \mathcal{B}(\Gamma))$ and we prove that it does not depend of the choice of β, $\mathcal{O}$, φ and Φ. We consider γ, $\tilde{\mathcal{O}}$, ψ and Ψ as in the proof of Proposition-Definition 3.8. Let B be a Borel set of Γ, $B' = \Phi^{-1}(B)$ and $\tilde{B}' = \Psi^{-1}(B)$. Applying Remark 3.9 with ψ we have

$$\int_{\tilde{B}'} \sqrt{1 + \|\nabla \psi(y')\|^2}\, dy' = \int_{\tilde{B}'} \frac{1}{|\nu(\Psi(y')) \cdot \gamma_d|}\, dy'.$$

We us the change of variables $y' = \Theta(x')$, where Θ is defined as in Proposition-Definition 3.8. Now using (3.4), this gives

$$\int_{\tilde{B}'} \sqrt{1 + \|\nabla \psi(y')\|^2}\, dy' = \int_{B'} \frac{1}{|\nu(\Phi(x')) \cdot \gamma_d|} \left|\det\big(\partial_{x_1}\Phi(x'), \ldots, \partial_{x_{d-1}}\Phi(x'), \gamma_d\big)\right| dx'.$$

Since $\nu(\Phi(x'))$ is orthogonal to the hyperplan generated by the vectors $\partial_{x_j}\Phi(x')$, $1 \leqslant j \leqslant d-1$, the properties of the determinant give

$$\begin{aligned} &\left|\det\big(\partial_{x_1}\Phi(x'), \ldots, \partial_{x_{d-1}}\Phi(x'), \gamma_d\big)\right| \\ &\qquad = \left|\nu(\Phi(x')) \cdot \gamma_d\right| \left|\det\big(\partial_{x_1}\Phi(x'), \ldots, \partial_{x_{d-1}}\Phi(x'), \nu(\Phi(x'))\big)\right|. \end{aligned}$$

We have an analoguous property with γ_d replaced by β_d. Applying successively these two equalities we get

$$\int_{\tilde{B}'} \sqrt{1+\|\nabla\psi(y')\|^2}\, dy' = \int_{B'} \frac{1}{|\nu(\Phi(x'))\cdot\beta_d|} \left|\det\left(\partial_{x_1}\Phi(x'),\dots,\partial_{x_{d-1}}\Phi(x'),\beta_d\right)\right| dx'.$$

Observing that this last determinant is just 1, and using Remark 3.9 one more time, we finally get that

$$\int_{\tilde{B}'} \sqrt{1+\|\nabla\psi(y')\|^2}\, dy' = \int_{B'} \sqrt{1+\|\nabla\varphi(x')\|^2}\, dx'.$$

This ensures that the definition of $\sigma(B)$ does not depend of the choice of the representation of Γ. $\square$

Proposition-Definition 3.13. *Let S be a hypersurface of class C^1 in $\mathbb{R}^d$ and K a compact part of S (if S is compact, we can choose $K = S$). Let $N \in \mathbb{N}$ and $\mathcal{V}_1,\dots,\mathcal{V}_N$ be open sets of $\mathbb{R}^d$ such that $K \subset \bigcup_{n=1}^N \mathcal{V}_n$ and $S \cap \mathcal{V}_n$ is a graph of class C^1 for any $n \in [\![1,N]\!]$. We consider $\chi_1,\dots,\chi_N \in C_0^\infty(\mathbb{R}^d,[0,1])$ such that $\sum_{n=1}^N \chi_n = 1$ on K and $\operatorname{supp}(\chi_n) \subset \mathcal{V}_n$ for any $n \in [\![1,N]\!]$. Let f be a measurable function from K to $\mathbb{R}$. We assume that f takes positive values or that*

$$\sum_{n=1}^N \int_{K\cap\mathcal{V}_n} \chi_n\,|f|\, d\sigma < +\infty.$$

Then we set

$$\int_K f\, d\sigma = \sum_{n=1}^N \int_{K\cap\mathcal{V}_n} \chi_n f\, d\sigma.$$

Taking for f the characteristic function of a Borel set of S included in K, this defines in particular a measure on K.

We can check that the definition of the measure σ on K does not depend of the choice of the open sets $\mathcal{V}_n$, $1 \leqslant n \leqslant N$, or of the partition of unity $(\chi_n)_{1\leqslant n\leqslant N}$.

Proposition-Definition 3.14. *Let S be a hypersurface of class C^1 in $\mathbb{R}^d$. There exists a non decreasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact parts of S (for the inclusion) such that $S = \bigcup_{n\in\mathbb{N}} K_n$. For $n \in \mathbb{N}$ we denote by σ_n the measure on K_n as defined in Proposition 3.13. Then for $B \in \mathcal{B}(S)$ we set*

$$\sigma(B) = \lim_{n\to+\infty} \sigma_n(B\cap K_n).$$

This defines a measure on σ sur S, called Lebesgue measure on S.

We can check if this definition is licit, it does not depend of the choice of the sequence $(K_n)_{n\in\mathbb{N}}$, and it indeed defines a measure on $(S,\mathcal{B}(S))$.

Example 3.15. We consider in $\mathbb{R}^2$ the circle C_R of center 0 and radius $R > 0$. We look for the measure (here the length) of the right half circle

$$C_R^+ = \left\{\left(\sqrt{R^2-y^2},y\right), y \in]-R,R[\right\}.$$

For $y \in]-R,R[$ we set $\varphi(y) = \sqrt{R^2-y^2}$. Then φ is of class C^∞ and for $y \in]-R,R[$ we have

$$\varphi'(y) = -\frac{y}{\sqrt{R^2-y^2}}.$$

Then we have

$$\sigma(C_R^+) = \int_{-R}^{R} \sqrt{1 + \frac{y^2}{R^2 - y^2}}\, dy = \int_{-R}^{R} \frac{R}{\sqrt{R^2 - y^2}}\, dy = \int_{-R}^{R} \frac{1}{\sqrt{1 - \left(\frac{y}{R}\right)^2}}\, dy$$
$$= R \int_{-1}^{1} \frac{1}{\sqrt{1 - \eta^2}}\, d\eta = R\left[\arcsin(\eta)\right]_{-1}^{1} = \pi R.$$

Example 3.16. We consider the sphere S_R of radius $R > 0$ in $\mathbb{R}^3$. We compute the area of the top half of the sphere defined as in (3.1). It is the graph of the function φ_R^+ defined by

$$\varphi_R^+(x_1, x_2) = \sqrt{R^2 - x_1^2 - x_2^2},$$

for (x_1, x_2) in the disc $D(0, R)$ centered at 0 and of radius R in $\mathbb{R}^2$. For $(x_1, x_2) \in D(0, R)$ we have

$$\left\|\nabla \varphi_R^+(x_1, x_2)\right\|^2 = \frac{x_1^2 + x_2^2}{R^2 - (x_1^2 + x_2^2)},$$

hence, using the polar coordinate system,

$$\sigma(S_R^+) = \int_{D(0,R)} \sqrt{1 + \frac{x_1^2 + x_2^2}{R^2 - (x_1^2 + x_2^2)}}\, dx_1\, dx_2 = 2\pi \int_0^R \sqrt{1 + \frac{r^2}{R^2 - r^2}}\, r\, dr$$
$$= 2\pi \int_0^R r\sqrt{\frac{R^2}{R^2 - r^2}}\, dr = 2\pi R\left[-\sqrt{R^2 - r^2}\right]_0^R$$
$$= 2\pi R^2.$$

***Exercise* 2.** Prove that the Lebesgue measure of the sphere of radius R is $4\pi R^2$.

We could give a general change of variables theorem between submanifolds, but we only discuss an important example. For $r > 0$ we denote by B_r and S_r the open ball and the sphere centered at 0 and with radius r, and by σ_r the Lebesgue measure σ_r.

Proposition 3.17. *Let $r > 0$ and let f be an integrable function on S_r. Then the function $y \mapsto f(ry)$ is integrable on S_1 and*

$$\int_{x \in S_r} f(x)\, d\sigma_r(x) = r^{d-1} \int_{y \in S_1} f(ry)\, d\sigma_1(y).$$

Proof. Assume for instance that f vanishes outside

$$S_r^+ = \{x = (x_1, \dots, x_d) \in S_r \,:\, x_d > 0\}.$$

We can similarly consider the case where f vanishes outside any half-sphere and deduce the general case. The interest of this assumption is that S_r^+ is a graph. With the notation of Definition 3.4 we can take

$$\mathcal{O}_r = \left\{x' = (x_1, \dots, x_{d-1}) \,:\, |x'| < r\right\}$$

and $\varphi_r : x' \mapsto \sqrt{r^2 - |x'|^2}$. Then we denote by $\Phi_r : \mathcal{O}_r \to S_r^+$ the corresponding parametrization. We similarly define $\mathcal{O}_1$, φ_1 et Φ_1. Then we have $\mathcal{O}_r = r\mathcal{O}_1$ and for $x' \in \mathcal{O}_r$ we have $\varphi_r(x') = r\varphi_1(x'/r)$ and $\Phi_r(x') = r\Phi_1(x'/r)$. By the change of variables

$x' = ry'$, $dx' = r^{d-1}dy'$, we have

$$\begin{aligned}
\int_{S_r} f(x)\,d\sigma_r(x) &= \int_{\mathcal{O}_r} f(\Phi_r(x'))\sqrt{1+|\nabla\varphi_r(x')|^2}\,dx' \\
&= \int_{\mathcal{O}_r} f\left(r\Phi\left(\frac{x'}{r}\right)\right)\sqrt{1+\left|\nabla\varphi_1\left(\frac{x'}{r}\right)\right|^2}\,dx' \\
&= r^{d-1}\int_{\mathcal{O}_1} f\left(r\Phi(y')\right)\sqrt{1+|\nabla_1\varphi(y')|^2}\,dy' \\
&= r^{d-1}\int_{S_1} f(ry)\,d\sigma_1(y).
\end{aligned}$$

The conclusion follows. □

We now discuss an analog of polar coordinates in any dimension.

Proposition 3.18. *Let $R > 0$ and let f be an integrable function on B_R. Then the function $\omega \mapsto f(r\omega)$ is integrable on S_1 for almost all $r \in]0, R[$ and*

$$\int_{B_R} f(x)\,dx = \int_{r=0}^{R} r^{d-1}\int_{\omega\in S_1} f(r\omega)\,d\sigma_1(\omega)\,dr = \int_{r=0}^{R}\int_{x\in S_r} f(x)\,d\sigma_r(x).$$

Proof. As in the previous proof we consider the case where f vanishes outside

$$B_R^+ = \{x = (x_1,\dots,x_d) \in B_R \,:\, x_d > 0\}.$$

We use the notation S_1^+, $\mathcal{O}_1$ and Φ_1 of the previous proof. For $r \in]0, R[$ and $y' \in \mathcal{O}_1$ we set $\Psi(r, y') = r\Phi(y')$. This defines a bijection Ψ of class C^1 from $]0, R[\times\mathcal{O}_1$ to B_R^+, and for $(r, y') \in]0, R[\times\mathcal{O}_1$ we have

$$\left|\operatorname{Jac}\Psi(r, y')\right| = \frac{r^{d-1}}{\sqrt{1-|y'|^2}} = r^{d-1}\sqrt{1+\|\nabla\varphi_1(y')\|^2}.$$

By the Inverse Function Theorem we deduce that Ψ is a C^1-diffeomorphism. By the change of variables theorem and the Fubini Theorem we get

$$\begin{aligned}
\int_{B_R^+} f(x)\,dx &= \iint_{]0,R[\times\mathcal{O}_1} f(\Psi(r, y'))\left|\operatorname{Jac}(r, y')\right|\,d\lambda(r, y') \\
&= \int_{r=0}^{R} r^{d-1}\int_{\mathcal{O}_1} f(r\Phi(y'))\sqrt{1+\|\nabla\varphi_1(y')\|^2}\,dy'\,dr \\
&= \int_{r=0}^{R} r^{d-1}\int_{S_1} f(r\omega)\,d\sigma_1(\omega)\,dr.
\end{aligned}$$

This gives the first equality. The second is given by Proposition 3.17. □

3.2 Stokes Formula - Green Formula

3.2.1 Regular open sets of $\mathbb{R}^d$

Definition 3.19 (Open set of class C^k)**.** Let Ω be an open set of $\mathbb{R}^d$. Then Ω is said to be of class C^k if for any $w \in \partial\Omega$ there exists an orthonormal basis $\beta = (\beta_1, \dots, \beta_d)$, an open set $\mathcal{O}$ of $\mathbb{R}^{d-1}$, $a, b \in \mathbb{R}$ with $a < b$ and a map $\varphi : \mathcal{O} \to]a, b[$ of class C^k such that

$$\mathcal{V} = \left\{\sum_{j=1}^{d} x_j\beta_j, (x_1,\dots,x_{d-1}) \in \mathcal{O}, x_d \in]a, b[\right\}$$

is a neighbourhood of w in $\mathbb{R}^d$ and

$$\Omega \cap \mathcal{V} = \left\{ \sum_{j=1}^{d} x_j \beta_j, (x_1, \ldots, x_{d-1}) \in \mathcal{O}, x_d \in \left]\varphi(x_1, \ldots, x_{d-1}), b\right[\right\}.$$

We then have

$$\partial\Omega \cap \mathcal{V} = \left\{ \sum_{j=1}^{d} x_j \beta_j, (x_1, \ldots, x_{d-1}) \in \mathcal{O}, x_d = \varphi(x_1, \ldots, x_{d-1}) \right\}.$$

In particular, $\partial\Omega$ is a hypersurface of class C^k in $\mathbb{R}^d$. The outward normal unit vector on $\partial\Omega$ is the vector ν defined by (3.3).

Remark 3.20. An open set whose boundary is a hypersurface of class C^k is not necessarily an open set of class C^k. Consider for example in $\mathbb{R}^2$ the open set $\Omega = \mathbb{R}^2 \backslash H$ where $H = \{(x_1, 0), x_1 \in \mathbb{R}\}$. Then we have $\partial\Omega = H$, and H is a hypersurface of class C^∞, but Ω is not an open set of class C^∞ because Ω is on both sides of its boundary.

3.2.2 Vector fields - Divergence operator

Let Ω be an open set of $\mathbb{R}^d$.

Definition 3.21. Let $k \in \mathbb{N}^* \cup \{\infty\}$. A vector field of class C^k on Ω is a map $X : \Omega \to \mathbb{R}^d$ of class C^k. We call vector field of class C^k on $\overline{\Omega}$ the restriction to Ω of a vector field of class C^k on $\mathbb{R}^d$.

Definition 3.22. Let X be a vector field of class C^1 on $\overline{\Omega}$. We note $X_1, \ldots, X_d$ the coordinates of X In the canonical basis. The divergence of X is then the function

$$\operatorname{div}(X) = \sum_{j=1}^{d} \frac{\partial X_j}{\partial x_j} = \sum_{j=1}^{d} \nabla X_j \cdot e_j.$$

The following proposition proves that the divergence of a vector field does not depend of the choice of an orthonormal basis on $\mathbb{R}^d$.

Proposition 3.23. *Let $X = (X_1, \ldots, X_d)$ be a vector field of class C^1 on $\overline{\Omega}$ and $\beta = (\beta_1, \ldots, \beta_d)$ an orthonormal basis on $\mathbb{R}^d$. We denote by $Y_1, \ldots, Y_d$ the coordinates of X in the basis β. Then we have*

$$\operatorname{div}(X) = \sum_{k=1}^{d} \nabla Y_k \cdot \beta_k.$$

If we denote by $y = (y_1, \ldots, y_d)$ the coordinates of a point in the basis β this can also be written

$$\operatorname{div}(X) = \sum_{k=1}^{d} \frac{\partial Y_k}{\partial y_k}.$$

Proof. For $k \in [\![1, d]\!]$ there exist $\alpha_{1,k}, \ldots, \alpha_{d,k} \in \mathbb{R}$ such that $\beta_k = \sum_{j=1}^{d} \alpha_{j,k} e_j$. Then we have

$$X = \sum_{k=1}^{d} Y_k \beta_k = \sum_{k=1}^{d} \sum_{j=1}^{d} Y_k \alpha_{j,k} e_j = \sum_{j=1}^{d} \left(\sum_{k=1}^{d} \alpha_{j,k} Y_k \right) e_j,$$

so for $j \in [\![1, d]\!]$ we have

$$X_j = \sum_{k=1}^{d} \alpha_{j,k} Y_k.$$

We then have

$$\sum_{k=1}^{d} \nabla Y_k \cdot \beta_k = \sum_{k=1}^{d} \nabla Y_k \cdot \left(\sum_{j=1}^{d} \alpha_{j,k} e_j \right) = \sum_{j=1}^{d} \nabla \left(\sum_{k=1}^{d} \alpha_{j,k} Y_k \right) \cdot e_j = \sum_{j=1}^{d} \nabla X_j \cdot e_j.$$

□

3.2.3 Stokes formula

We are now in position to prove the Stokes formula, which is an analogue in dimension $d \geqslant 2$ of the fundamental theorem of calculus. Indeed, if we apply the following theorem for an interval $]a,b[$ and to the vector field $x \mapsto f(x)e_1$, where f is a function of class C^1 on $[a,b]$ and e_1 is the vector of the canonical base of $\mathbb{R}$ (of course we do not state it like this in dimension 1), we get[1]

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Theorem 3.24 (Stokes Formula)**.** *Let Ω be an open set of class C^1 in $\mathbb{R}^d$. We denote by ν the outward unit normal vector on $\partial\Omega$. Let X be a vector field of class C^1 with compact support in $\overline{\Omega}$. Then we have*

$$\int_\Omega \operatorname{div}(X)\,dx = \int_{\partial\Omega} (X \cdot \nu)\,d\sigma,$$

where σ is the Lebesgue measure on $\partial\Omega$.

Proof. • For any $w \in \partial\Omega$ we consider a neigbourhood $\mathcal{V}_w$ of w in $\mathbb{R}^d$ as in Definition 3.19. There exist $N \in \mathbb{N}$ and $w_1, \dots, w_N \in \partial\Omega$ such that

$$\operatorname{supp}(X) \subset \Omega \cup \bigcup_{k=1}^{N} \mathcal{V}_{w_k}.$$

We consider an associated partition of unity $\chi_0, \chi_1, \dots, \chi_N$, with $\operatorname{supp}(\chi_0) \subset \Omega$ and $\operatorname{supp}(\chi_k) \subset \mathcal{V}_{w_k}$ for all $k \in [\![1, N]\!]$. Assume that

$$\forall k \in [\![0, N]\!], \qquad \int_\Omega \operatorname{div}(\chi_k X)\,dx = \int_{\partial\Omega} (\chi_k X) \cdot \nu\,d\sigma. \tag{3.8}$$

Then we have

$$\int_\Omega \operatorname{div}(X)\,dx = \sum_{k=0}^{N} \int_\Omega \chi_k \operatorname{div}(X)\,dx = \sum_{k=0}^{N} \int_\Omega \operatorname{div}(\chi_k X)\,dx - \sum_{k=0}^{N} \int_\Omega \nabla\chi_k \cdot X\,dx.$$

On the one hand we have

$$\sum_{k=0}^{N} \int_\Omega \nabla\chi_k \cdot X\,dx = \int_\Omega \nabla \left(\sum_{k=0}^{N} \chi_k \right) \cdot X\,dx = 0,$$

and on the other hand, according to (3.8),

$$\sum_{k=0}^{N} \int_\Omega \operatorname{div}(\chi_k X)\,dx = \sum_{k=0}^{N} \int_{\partial\Omega} (\chi_k X) \cdot \nu\,d\sigma = \int_{\partial\Omega} X \cdot \nu\,d\sigma.$$

[1] in dimension 0 the Lebesgue measure coincides with the counting measure…

This gives the result. It remains to prove (3.8). For $k = 0$, since $\chi_0 X$ is compactly supported we can apply Proposition 3.1.

• Let $w \in \partial\Omega$. With the notation intoduced in Definition 3.19, we suppose that X has its support included in $\overline{\Omega} \cap \mathcal{V}$. We denote by $X_1, \dots, X_d : \mathcal{V} \to \mathbb{R}$ the coordinates of X in the basis β. Using the change of variables $(y', y_d) \in \mathcal{O} \times]a, b[\to y'\beta' + y_d\beta_d$ (whose determinant is 1) we have

$$\begin{aligned}\int_{\mathcal{V}\cap\Omega} \nabla X_d \cdot \beta_d \, dx &= \int_{y'\in\mathcal{O}} \int_{y_d=\varphi(y')}^{b} \frac{\partial}{\partial y_d} X_d(y'\beta' + y_d\beta_d) \, dy_d \, dy' \\ &= -\int_{\mathcal{O}} X_d(y'\beta' + \varphi(y')\beta_d) \, dy'.\end{aligned}$$

Now let $j \in [\![1, d-1]\!]$. We extend X by 0 on $\mathbb{R}^d$ and for $y' \in \mathcal{O}$ and $t \in]0, b[$ we set

$$h(y', t) = X_j\big(y'\beta' + (t + \varphi(y'))\beta_d\big).$$

This defines a function $h \in C^1\big(\overline{\mathcal{O} \times]0, b[}\big)$. For $y' \in \mathcal{O}$ and $t \in]0, b[$ we have

$$\partial_{y_j} h(y', t) = \nabla X_j\big(y'\beta' + (t + \varphi(y'))\beta_d\big) \cdot \beta_j + \partial_{y_j}\varphi(y') \nabla X_j\big(y'\beta' + (t + \varphi(y'))\beta_d\big) \cdot \beta_d.$$

We deduce that

$$\begin{aligned}\int_{\mathcal{V}\cap\Omega} \nabla X_j \cdot \beta_j \, dx &= \int_{\mathcal{O}} \int_0^b \nabla X_j\big(y'\beta' + (t + \varphi(y'))\beta_d\big) \cdot \beta_j \, dt \, dy' \\ &= \int_{\mathcal{O}} \int_0^b \partial_{y_j} h(y', t) \, dt \, dy' - \int_{\mathcal{O}} \int_0^b \partial_{y_j}\varphi(x') \nabla X_j\big(y'\beta' + (t + \varphi(y'))\beta_d\big) \cdot \beta_d \, dt \, dx' \\ &= \int_0^b \int_{\mathcal{O}} \partial_{x_j} h(x', t) \, dx' \, dt - \int_{\mathcal{O}} \partial_{x_j}\varphi(x') \left(\int_0^b \frac{d}{dt} X_j\big(y'\beta' + (t + \varphi(y'))\beta_d\big) \, dt \right) dx' \\ &= \int_{\mathcal{O}} X_j\big(y'\beta' + \varphi(y')\beta_d\big) \partial_{x_j}\varphi(x') \, dx'.\end{aligned}$$

Summing over $j \in [\![1, d]\!]$ we get with (3.3)

$$\int_{\mathcal{V}} \operatorname{div}(X) \, dx = \int_{\mathcal{O}} (X \cdot \nu)\big(y'\beta' + \varphi(y')\beta_d\big) \sqrt{1 + \|\nabla\varphi(x')\|^2} \, dx' = \int_{\partial\Omega} X \cdot \nu \, d\sigma.$$

This proves (3.8) and concludes the proof. □

3.2.4 Green Formula

We now deduce from the Stokes Formula the Green Formula, which is an analogue in dimension $d \geqslant 2$ of the integration by parts. In the following theorem we assume that one of the factors has compact support in $\overline{\Omega}$ to ensure that the integrals are well defined even in the case where Ω is not bounded, but this does not prevent the two functions from being nonzero in the neighborhood of $\partial\Omega$.

Theorem 3.25. *Let Ω be an open set of class C^1 and $u, v \in C^1(\overline{\Omega})$ with u or v compactly supported in $\overline{\Omega}$. For $j \in [\![1, d]\!]$ we have*

$$\int_{\Omega} \partial_j u \, v \, dx = -\int_{\Omega} u \, \partial_j v \, dx + \int_{\partial\Omega} uv \, \nu_j \, d\sigma,$$

where $\nu_j = \nu \cdot e_j$ is the j-th coordonnate of ν in the canonical basis of $\mathbb{R}^d$.

Proof. For $x \in \overline{\Omega}$ we set $X(x) = u(x)v(x)e_j$. For any $x \in \Omega$ we have

$$\operatorname{div}(X)(x) = \partial_j(uv)(x) = \partial_j u(x)v(x) + u(x)\partial_j v(x).$$

Moreover, for $x \in \partial\Omega$ we have $X(x) \cdot \nu(x) = u(x)v(x)\nu_j(x)$. We conclude with the Stokes Theorem. □

For $u \in C^1(\overline{\Omega})$ and $x \in \partial\Omega$ we set

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \nu(x).$$

Theorem 3.26. *Let Ω be an open set of class C^1 and $u, v \in C^2(\overline{\Omega})$ with u or v compactly supported in $\overline{\Omega}$. Then we have*

$$-\int_\Omega \Delta u \, v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, v \, d\sigma.$$

Proof. By Theorem 3.25 we have, for any $j \in [\![1, d]\!]$,

$$-\int_\Omega \partial_j^2 u \, v \, dx = \int_\Omega \partial_j u \, \partial_j v \, dx - \int_{\partial\Omega} \partial_j u \, v \, \nu_j \, d\sigma.$$

We conclude by summing over $j \in [\![1, d]\!]$. □

Corollary 3.27. *Let Ω be an open set of class C^1 and $u, v \in C^2(\overline{\Omega})$ with u or v compactly supported in $\overline{\Omega}$. Then we have*

$$\int_\Omega (u \, \Delta v - \Delta u \, v) \, dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu} \, v \right) \, d\sigma.$$

Example 3.28. Let $u, v \in C^1(\overline{B_1})$. Then we have

$$\int_{B_1} \nabla u(x) v(x) \, dx = \int_{x \in S_1} u(x)v(x)x \, d\sigma(x) - \int_{B_1} u(x)\nabla u(x) \, dx.$$

If $u \in C^2(\overline{B_1})$ we also have

$$\int_{B_1} \Delta u(x)v(x) \, dx = \int_{x \in S_1} \partial_r u(x)v(x) \, d\sigma(x) - \int_{B_1} \nabla u(x) \cdot \nabla v(x) \, dx,$$

where $\partial_r u(x) = \nabla u(x) \cdot \frac{x}{|x|}$ is the radial derivative of u.

Example 3.29. For $\phi \in C_0^\infty(\mathbb{R}^d)$ and $j \in [\![1, d]\!]$ we have

$$-\int_{\mathbb{R}^d} |x| \, \partial_j \phi(x) \, dx = \int_{\mathbb{R}^d} \frac{x_j}{|x|} \phi(x) \, dx. \tag{3.9}$$

The function $x \mapsto |x|$ is of class C^∞ on $\mathbb{R}^d \setminus \{0\}$ and its derivative with respect to x_j is $\frac{x_j}{|x|}$. However, (3.9) is not a direct consequence of the Green Formula because of the lack of regularity at 0.

By the dominated convergence theorem we have

$$-\int_{\mathbb{R}^d} |x| \, \partial_j \phi(x) \, dx = -\lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} |x| \, \partial_j \phi(x) \, dx.$$

For $\varepsilon > 0$ we have by the Green Formula

$$-\int_{|x| > \varepsilon} |x| \, \partial_j \phi(x) \, dx = -\int_{x \in S_\varepsilon} |x| \, \phi(x) \nu_j(x) \, d\sigma(x) + \int_{|x| > \varepsilon} \frac{x_j}{|x|} \phi(x) \, dx,$$

where $\nu(x) = -\frac{x}{\varepsilon}$ is the outward normal unit vector to $\mathbb{R}^d \backslash \overline{B_\varepsilon}$. Then we have

$$\left| \int_{x \in S_\varepsilon} |x| \, \phi(x) \nu_j(x) \, d\sigma_\varepsilon(x) \right| \leqslant \varepsilon \sigma_\varepsilon(S_\varepsilon) \, \|\phi\|_\infty \xrightarrow[\varepsilon \to 0]{} 0.$$

On the other hand, by the dominated convergence theorem we have

$$\int_{|x| > \varepsilon} \frac{x_j}{|x|} \phi(x) \, dx \xrightarrow[\varepsilon \to 0]{} \int_{\mathbb{R}^d} \frac{x_j}{|x|} \phi(x) \, dx.$$

This proves (3.9).

The following computation will be useful to study in the next chapter the Poisson equation $-\Delta u = f$ in dimension $d \geqslant 3$.

Example 3.30. Assume that $d \geqslant 3$. We prove that for $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$-\int_{\mathbb{R}^d} |x|^{-(d-2)} \, \Delta\phi(x) \, dx = (d-2)\sigma(S_1)\phi(0). \tag{3.10}$$

The function $u : x \mapsto |x|^{-(d-2)}$ is of class C^∞ on $\mathbb{R}^d \backslash \{0\}$. We recall that the Laplacian of a radial function is given in spherical coordinates by

$$\Delta G(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial G(r)}{\partial r} \right).$$

Here we see that $\Delta u = 0$ on $\mathbb{R}^d \backslash \{0\}$. The function $x \mapsto |x|^{2-d}$ is locally integrable on $\mathbb{R}^d$. By the dominated convergence theorem and the change of variables $x = \varepsilon y$ we have

$$\int_{\mathbb{R}^d} \frac{1}{|x|^{d-2}} \Delta\phi(x) \, dx = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{1}{|x|^{d-2}} \Delta\phi(x) \, dx = \lim_{\varepsilon \to 0} \int_{|y| > 1} \frac{1}{|y|^{d-2}} \Delta\phi_\varepsilon(y) \, dx.$$

where for $\varepsilon > 0$ and $y \in \mathbb{R}^d$ we have set $\phi_\varepsilon(y) = \phi(\varepsilon y)$. By the Green Formula we get

$$\int_{\mathbb{R}^d} \frac{1}{|x|^{d-2}} \Delta\phi(x) \, dx = \lim_{\varepsilon \to 0} I_\varepsilon - \lim_{\varepsilon \to 0} J_\varepsilon,$$

where for $\varepsilon > 0$ we have set

$$I_\varepsilon = \int_{|y|=1} \frac{1}{|y|^{d-2}} \frac{\partial \phi_\varepsilon}{\partial \nu}(y) \, d\sigma \quad \text{et} \quad J_\varepsilon = \int_{|y|=1} \phi_\varepsilon(y) \frac{\partial}{\partial \nu} \frac{1}{|y|^{d-2}} \, d\sigma.$$

We have

$$I_\varepsilon = \varepsilon \int_{|y|=1} \frac{\partial \phi}{\partial \nu}(\varepsilon y) \, d\sigma \xrightarrow[\varepsilon \to 0]{} 0,$$

and by the dominated convergence theorem

$$J_\varepsilon = -(d-2) \int_{|y|=1} \frac{1}{|y|^{d-1}} \phi_\varepsilon(y) \, d\sigma = -(d-2) \int_{|y|=1} \phi_\varepsilon(y) \, d\sigma \xrightarrow[\varepsilon \to 0]{} -(d-2)\sigma(S_1)\phi(0).$$

The conclusion follows.