Chapter 1

Convolution and application to regularization

We introduce in this chapter the notion of convolution, essentially in $\mathbb{R}^d$ endowed with the Lebesgue measure. The convolution appears in a natural way in many contexts. Typically, and it is one of our main motivation, convolution is intimately linked to the regular product via the Fourier transform, that we will introduce later. Another important application, that we will devellop in the second part of this chapter, is the regularization of functions.

Without necessarily having mentioned it, we have already met convolution and the links it has with the usual product. Let us have a quick and non exhaustive overview before we get to the heart of the matter.

First, let us consider two real or complex sequences $(a_j)_{j\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ that are eventually vanishing (there exist $N, M \in \mathbb{N}$ such that $a_j = 0$ for $j > N$ and $b_k = 0$ for $k > M$). We note P and Q the corresponding polynomials:

$$P = \sum_{j=0}^{N} a_j X^j \quad \text{and} \quad Q = \sum_{k=0}^{M} b_k X^k.$$

The usual product of these polynomials is

$$PQ = \sum_{n=0}^{N+M} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) X^n.$$

This polynomial is associated to the eventually vanishing sequence $(c_n)_{n\in\mathbb{N}}$ defined by

$$\forall n \in \mathbb{N}, \quad c_n = \sum_{j=0}^{n} a_j b_{n-j}. \tag{1.1}$$

If we extend the sequences $(a_j)_{j\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ et $(c_n)_{n\in\mathbb{N}}$ by 0 to see them as $\mathbb{Z}$ indexed sequences, we get that the sequence $(c_n)_{n\in\mathbb{Z}}$ is exactly what we will define as the convolution of $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$. Thus, the usual product of polynomials is defined by convolution.

If we remove the hypothesis saying that the sequences $(a_j)_{j\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are eventually vanishing, the polynomials become power series, but the discussion remains the same. the product of $\sum_{j=0}^{+\infty} a_j z^j$ and $\sum_{k=0}^{+\infty} b_k z^k$ is the power series $\sum_{n=0}^{+\infty} c_n$ where c_n is again defined by (1.1).

This observation has, for example, applications in probabilities. If X and Y are two independant random variables with values in $\mathbb{N}$, then for any $n \in \mathbb{N}$ we have

$$P(X+Y=n) = \sum_{j=0}^{n} P\big(X=j \text{ and } Y=n-j\big) = \sum_{j=0}^{n} P(X=j)P(Y=n-j).$$

Thus, the distribution of $X+Y$ is given by the convolution of the distributions of X and Y. Thus it is relevant to associate to a random variable $X : \Omega \to \mathbb{N}$ the power series (called probability-generating function) defined by

$$G_X(z) = \sum_{j=0}^{+\infty} P(X=j) z^j.$$

We have $G_{X+Y} = G_X G_Y$ (usual power series product), and from G_{X+Y} we can then identify $X+Y$.

In the same spirit, but closer to the issues of this course, we consider the case of Fourier series. Let f and g be two locally integrable and 2π-perodic functions on $\mathbb{R}$. We note $(\alpha_n)_{n\in\mathbb{Z}}$ and $(\beta_n)_{n\in\mathbb{Z}}$ their Fourier coefficients. In $L^2(\mathbb{S})$ we have

$$f(x) = \sum_{n\in\mathbb{Z}} \alpha_n e^{inx} \quad \text{and} \quad g(x) = \sum_{n\in\mathbb{Z}} \beta_n e^{inx}.$$

The Fourier coefficients of fg are given by the convolution of the sequences of coefficients of f and g:

$$(fg)(x) = \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \alpha_k \beta_{n-k} \right) e^{inx}.$$

We observe the same phenomenon for the Fourier transform. More precisely, the Fourier transform of a product of function will be the convolution of the Fourier transforms of these functions. We will come back on this in the next chapter, but for this, we will need convolution for functions on $\mathbb{R}^d$. This is the object of this chapter.

When we will study the Fourier transform, we will see that its main interest, as for Fourier series, is its nice behaviour in the framework of differential equations. This suggests that the convolution will naturally appear when solving differential equtions. It is indeed the case, and can already be observed for the simplest cases. Consider $\alpha \in \mathbb{R}$ and f a continuous function from $\mathbb{R}_+$ to $\mathbb{R}$. Using variation of parameters, we know that the unique solution on $\mathbb{R}_+$ of the Cauchy problem

$$\begin{cases} y' + \alpha y = f(t), & t \geqslant 0, \\ y(0) = 0, \end{cases}$$

is the function

$$y : t \mapsto \int_0^t e^{-\alpha(t-s)} f(s)\, ds.$$

It is precisely the convolution of $s \mapsto e^{-\alpha s}$, solution of

$$\begin{cases} y' + \alpha y = 0, & t \geqslant 0, \\ y(0) = 1, \end{cases}$$

with the source term f (these two functions being extended by 0 on $\mathbb{R}_-^*$). This is not a coincidence, and the generalization of this remark will be one of the issues in the following chapters.

1.1 Convolution

$\mathbb{R}^d$ and $\mathbb{C}$ are endowed with their usual Borel algebras. On $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ we consider the Lebesgue measure.

1.1.1 Definitions and first properties

Let f and g be two measurable functions from $\mathbb{R}^d$ to $\mathbb{C}$. We observe once and for all that for any $x \in \mathbb{R}^d$ the function $y \mapsto f(x-y)g(y)$ is measurable from $\mathbb{R}^d$ to $\mathbb{C}$. We will no longer mention this point in the proofs of this chapter.

Definition 1.1. For $x \in \mathbb{R}^d$ such that the function $y \mapsto f(x-y)g(y)$ is integrable, we set

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy.$$

The function $(f * g)$ is called the *convolution* of f and g.

Example 1.2. • For any f measurable on $\mathbb{R}^d$ we have $f * 0 = 0$.
• If f is integrable on $\mathbb{R}^d$ we have

$$(1 * f)(x) = \int_{\mathbb{R}^d} f(y)\,dy.$$

• For $x \in \mathbb{R}$ we have

$$(\mathbb{1}_{[0,1]} * \mathbb{1}_{[0,1]})(x) = \int_0^1 \mathbb{1}_{[0,1]}(x-y)\,dy = \begin{cases} x & \text{if } x \in [0,1], \\ 2-x & \text{if } x \in [1,2], \\ 0 & \text{if } x \notin [0,2]. \end{cases}$$

• Let f be the characteristic function of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Let $g \in \mathcal{L}^1_{\text{loc}}(\mathbb{R})$. For $x \in \mathbb{R}$ we have

$$(f * g)(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} g(y)\,dy.$$

Thus, considering the convolution of g with f, at the point x we replace $g(x)$ (that has no meaning since we consider almost everywhere equal functions) by the mean of g on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (that always has a meaning, even for g in $L^1_{\text{loc}}(\mathbb{R})$). We observe that we obtain a continuous function $(f * g)$. Moreover, if g is of class C^k for some $k \in \mathbb{N}$, then $(f * g)$ is of class C^{k+1}.
• Let $\alpha \in \mathbb{R}$. For $n \in \mathbb{N}$ we now consider the function $g_n : x \mapsto \alpha + \cos(nx)$. Thus, for $x \in \mathbb{R}$ we have

$$(f * g_n)(x) = \alpha + \frac{\sin\left(n(x+\frac{1}{2})\right) - \sin\left(n(x-\frac{1}{2})\right)}{n}.$$

The function g_n is of class C^∞ for any $n \in \mathbb{N}$, but it oscillates with higher and higher frequency when n grows. The convolution erases these high frequency oscillations and $(f * g_n)$ is closer and closer to its mean value α.

These first examples show the regularizing effect of the convolution mentioned in the introduction. We will come back on this aspect in the second part of this chapter. For now, we begin by describing some general properties of the convolution.

Proposition 1.3. *Let $x \in \mathbb{R}^d$. The convolution $(f * g)(x)$ is well defined if and only if $(g * f)(x)$ is, and in that case their values coincide.*

Proof. Applying the change of variables $\eta = x - y$ we get

$$\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy = \int_{\mathbb{R}^d} |f(\eta)|\,|g(x-\eta)|\;d\eta.$$

This proves that $(f*g)(x)$ is defined if and only if $(g*f)(x)$ is defined. in that case, the same computation considering f and g instead of $|f|$ and $|g|$ shows that $(f*g)(x) = (g*f)(x)$. □

Proposition 1.4. *Suppose there exist $R_1, R_2 > 0$ such that f vanishes outside the ball $B(R_1)$ and g is null outside the ball $B(R_2)$. Then for $x \in \mathbb{R}^d$ such that $|x| > R_1 + R_2$ the convolution $(f*g)$ is well defined at x and $(f*g)(x) = 0$.*

Proof. Let $x \in \mathbb{R}^d$ such that $|x| > R_1 + R_2$. Let $y \in \mathbb{R}^d$. If $|y| \leqslant R_2$ then $|x-y| > R_1$ so $f(x-y) = 0$, while if $|y| > R_2$ we have $g(y) = 0$. In both cases we have $f(x-y)g(y) = 0$, so the function $y \mapsto f(x-y)g(y)$ is integrable with integral 0. □

1.1.2 Convolution in $L^p(\mathbb{R}^d)$ spaces

The purpose of this section is, just like the Hölder inequality for the usual product, to give necessary conditions which ensure that $(f*g)$ is some Lebesgue space if f and g are. As for the usual product, there are some obvious cases.

Proposition 1.5. (i) *Assume that $f \in L^1(\mathbb{R}^d)$ and $g \in L^\infty(\mathbb{R}^d)$. Then $(f*g)(x)$ is well defined for any $x \in \mathbb{R}^d$ and $(f*g) \in L^\infty(\mathbb{R}^d)$ with*

$$\|f*g\|_\infty \leqslant \|f\|_1 \|g\|_\infty .$$

(ii) *We suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $g \in L^\infty(\mathbb{R}^d)$ vanishes outside a compact of $\mathbb{R}^d$. Then $(f*g)(x)$ is well defined for any $x \in \mathbb{R}^d$.*

Of course, we have analoguous properties if we switch the roles of f and g.

Proof. For this first proof, we emphasize the distinction between $\mathcal{L}^1(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$ or $\mathcal{L}^\infty(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. We will not do this in the rest of the chapter.

So let $f \in \mathcal{L}^1(\mathbb{R}^d)$ and $g \in \mathcal{L}^\infty(\mathbb{R}^d)$. Let $x \in \mathbb{R}^d$. For almost every $y \in \mathbb{R}^d$ we have $|g(y)| \leqslant \|g\|_\infty$, so

$$\int_{y\in\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy \leqslant \|g\|_\infty \int_{y\in\mathbb{R}^d} |f(x-y)|\;dy.$$

Using the affine change of variables $\eta = x - y$, $d\eta = dy$, we get

$$\int_{y\in\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy \leqslant \|g\|_\infty \int_{\eta\in\mathbb{R}^d} |f(\eta)|\;d\eta = \|f\|_1 \|g\|_\infty .$$

This proves that $(f*g)(x)$ is well defined and

$$|(f*g)(x)| \leqslant \int_{y\in\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy \leqslant \|f\|_1 \|g\|_\infty .$$

Hence $\|(f*g)\|_\infty \leqslant \|f\|_1 \|g\|_\infty$.

We now consider $f \in L^1(\mathbb{R}^d)$ and $g \in L^\infty(\mathbb{R}^d)$. Let $f_1, f_2 \in \mathcal{L}^1(\mathbb{R}^d)$ be two elements of the equivalence class of f, and $g_1, g_2 \in \mathcal{L}^\infty(\mathbb{R}^d)$ be two elements of the equivalence class of g. Let $x \in \mathbb{R}^d$. The functions $y \mapsto f_1(x-y)g_1(y)$ and $y \mapsto f_2(x-y)g_2(y)$ are integrable on $\mathbb{R}^d$. Since $f_1(x-y)g_1(y)$ coincide for almost every $y \in \mathbb{R}^d$ with $f_2(x-y)g_2(y)$, their integrals also coincide, so $(f_1 * g_1)(x) = (f_2 * g_2)(x)$. We denote by $(f*g)(x)$ this common value. Moreover we have

$$\|f * g\|_\infty = \|f_1 * g_1\|_\infty \leqslant \|f_1\|_1 \|g_1\|_\infty = \|f\|_1 \|g\|_\infty .$$

This gives the first property.

For the second, we do not give so many details, we just identify f and g with two of their equivalence class representatives in $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\mathcal{L}^\infty(\mathbb{R}^d)$, with g vanishing almost everywhere outside $B(0,R)$ for some $R > 0$. Let $x \in \mathbb{R}^d$. As above, we have

$$\begin{aligned}\int_{y\in\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy &= \int_{y\in B(0,R)} |f(x-y)|\,|g(y)|\;dy \\ &\leqslant \|g\|_\infty \int_{y\in B(0,R)} |f(x-y)|\;dy \\ &\leqslant \|g\|_\infty \int_{\eta\in B(x,R)} |f|\;d\eta.\end{aligned}$$

Since f is integrable on $B(x,R)$, this proves that $(f*g)(x)$ is well defined. □

Now we turn to the first important result about convolution. We prove that the convolution of two integrable functions is an integrable function, and furthermore the norm of the convolution is not greater than the product of the norms of the two factors. From this perspective, convolution is more convenient than the usual product.

Proposition 1.6. *Let $f, g \in L^1(\mathbb{R}^d)$. The function $y \mapsto f(x-y)g(y)$ is integrable (and thus $(f*g)(x)$ is well defined) for almost every $x \in \mathbb{R}^d$. The function $(f*g)$ thus defined is integrable on $\mathbb{R}^d$ and*

$$\|f*g\|_1 \leqslant \|f\|_1 \|g\|_1 .$$

Proof. We identify f and g to representatives of their equivalence class in $\mathcal{L}^1(\mathbb{R}^d)$ (and we check that everything is independant of the choice of the representative). The map $(x,y) \mapsto |f(x-y)|\,|g(y)|$ is measurable from $\mathbb{R}^d \times \mathbb{R}^d$ to $[0,+\infty]$. Using Fubini-Tonelli theorem we get that the function

$$x \mapsto \int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy$$

is well defined and measurable from $\mathbb{R}^d$ to $[0,+\infty]$. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy\right) dx &= \int_{\mathbb{R}^d} |g(y)| \left(\int_{\mathbb{R}^d} |f(x-y)|\;dx\right) dy \\ &= \int_{\mathbb{R}^d} |g(y)| \left(\int_{\mathbb{R}^d} |f(\eta)|\;d\eta\right) dy \\ &= \|f\|_1 \int_{\mathbb{R}^d} |g(y)|\;dy \\ &= \|f\|_1 \|g\|_1 .\end{aligned} \tag{1.2}$$

For the third equality we used the change of variables $\eta = x - y$, $d\eta = dx$, in the x-variable integral. This proves that for almost every $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy < +\infty.$$

Thus $(f*g)(x)$ is well defined for almost every $x \in \mathbb{R}^d$ and

$$\int_{\mathbb{R}^d} |(f*g)(x)|\;dx \leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy\right) dx = \|f\|_1 \|g\|_1 .$$

The proposition is proved. □

The aim of this section is to give more results of this kind, that is, prove that if f and g are in suitable Lebesgue spaces, their convolution is well defined and satisfies some useful properties.

The following result generalizes the first property of Proposition 1.5 and Proposition 1.6. If a function is in $L^p(\mathbb{R}^d)$ for some p, then so is its convolution with an integrable function.

Proposition 1.7. *Let $p \in [1, +\infty]$, $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Then the convolution $(f * g)(x)$ is well defined for almost every $x \in \mathbb{R}^d$, we have $(f * g) \in L^p(\mathbb{R}^d)$ and*

$$\|f * g\|_p \leqslant \|f\|_1 \|g\|_p .$$

Using the commutativity of convolution, we have an analoguous result for $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$.

Proof. We identify f et g to representatives in $\mathcal{L}^1(\mathbb{R}^d)$ and $\mathcal{L}^p(\mathbb{R}^d)$. The result is already known if $p = +\infty$, so we can assume that $p \in [1, +\infty[$. Let $q = \frac{p}{p-1}$ be the conjugate exponent of p. For $x \in \mathbb{R}^d$ we have by the Hölder inequality

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\,dy &= \int_{\mathbb{R}^d} |f(x-y)|^{\frac{1}{q}}\,|f(x-y)|^{\frac{1}{p}}\,|g(y)|\,dy \\ &\leqslant \|f\|_1^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|^p\,dy \right)^{\frac{1}{p}} .\end{aligned}$$

By the Fubini-Tonelli Theorem we get

$$\begin{aligned}\||f| * |g|\|_p^p &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\,dy \right|^p dx \\ &\leqslant \|f\|_1^{\frac{p}{q}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|^p\,dy \right) dx \\ &\leqslant \|f\|_1^{\frac{p}{q}} \int_{\mathbb{R}^d} |g(y)|^p \left(\int_{\mathbb{R}^d} |f(x-y)|\,dx \right) dy \\ &\leqslant \|f\|_1^p \|g\|_p^p .\end{aligned}$$

We deduce that $(|f| * |g|) \in \mathcal{L}^p(\mathbb{R}^d)$ with $\||f| * |g|\|_p \leqslant \|f\|_1 \|g\|_p$. In particular the function $y \mapsto f(x-y)g(y)$ is integrable for almost every $x \in \mathbb{R}^d$, and

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)g(y)\,dy \right|^p dx \leqslant \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\,dy \right|^p dx \leqslant \|f\|_1^p \|g\|_p^p .$$

Hence $(f * g) \in \mathcal{L}^p(\mathbb{R}^d)$ and $\|f * g\|_p \leqslant \|f\|_1 \|g\|_p$. □

Remark 1.8. For $p \in [1, +\infty]$ et $f \in L^1(\mathbb{R}^d)$, we have proved that the map $F : g \mapsto f * g$ is a continuous linear map on $L^p(\mathbb{R}^d)$, with $\|F\|_{\mathcal{L}(L^p(X))} \leqslant \|f\|_1$.

Let us discuss another generalization of Proposition 1.5. We still arrive in $L^\infty(\mathbb{R}^d)$, but we now consider f and g in $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ for general conjugate exponents p and q. It is interesting to compare this result with Hölder inequality for the usual product of functions.

Proposition 1.9. *Let $p, q \in [1, +\infty]$ be two conjugate exponents. Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Then the convolution $(f * g)(x)$ is well defined for any $x \in \mathbb{R}^d$, $(f * g)$ is bounded on $\mathbb{R}^d$ and*

$$\|f * g\|_{L^\infty} \leqslant \|f\|_{L^p} \|g\|_{L^q} .$$

Moreover, $(f * g)$ is uniformly continuous and, if p and q are finite, then $(f * g)(x)$ tends to 0 when $|x|$ tends to $+\infty$.

Proof. We identify f and g to representatives of their class in $\mathcal{L}^p(\mathbb{R}^d)$ and $\mathcal{L}^q(\mathbb{R}^d)$.
- Let $x \in \mathbb{R}^d$. By the Hölder Inequality we have

$$\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|\;dy \leqslant \left(\int_{\mathbb{R}^d} |f(x-y)|^p\;dy\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^d} |g(y)|^q\;dy\right)^{\frac{1}{q}} = \|f\|_p\,\|g\|_q\,.$$

Hence $(f*g)(x)$ is well defined and

$$|(f*g)(x)| \leqslant \|f\|_p\,\|g\|_q\,.$$

This proves that $(f*g)$ is bounded with

$$\|f*g\|_\infty \leqslant \|f\|_p\,\|g\|_q\,.$$

- Let $h \in \mathbb{R}^d$. We suppose that $p < +\infty$. For $x \in \mathbb{R}^d$ we have, still using the Hölder Inequality,

$$\begin{aligned}|(f*g)(x+h)-(f*g)(x)| &\leqslant \int_{\mathbb{R}^d} |f(x+h-y)-f(x-y)|\,|g(y)|\;dy\\ &\leqslant \|g\|_q\left(\int_{\mathbb{R}^d}|f(x+h-y)-f(x-y)|^p\;dy\right)^{\frac{1}{p}}\\ &\leqslant \|\tau_{-h}f-f\|_p\,\|g\|_q\end{aligned}$$

(recall that $\tau_{-h}f$ is the function $\eta \mapsto f(\eta+h)$). By continuity of the translation in $L^p(\mathbb{R}^d)$,

$$\sup_{x\in\mathbb{R}^d}|(f*g)(x+h)-(f*g)(x)| \leqslant \|\tau_h f - f\|_p\,\|g\|_q \xrightarrow[h\to 0]{} 0,$$

so $(f*g)$ is uniformly continuous. If $p=+\infty$ then $q=1$ and we can similarly write, by commutativity of convolution,

$$\begin{aligned}|(f*g)(x+h)-(f*g)(x)| &\leqslant \int_{\mathbb{R}^d} |f(y)|\,|g(x+h-y)-g(x-y)|\;dy\\ &\leqslant \|f\|_p\,\|\tau_{-h}g-g\|_q\\ &\xrightarrow[h\to 0]{} 0.\end{aligned}$$

- Now assume that p and q are finite. Then, there exist sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ of compactly supported and continuous functions that tends to f and g in $L^p(\mathbb{R}^d)$ and in $L^q(\mathbb{R}^d)$, respectively. For any $n\in\mathbb{N}$, the function (f_n*g_n) has compact support (see proposition 1.4). By linearity of the convolution with respect to each factor, we can write

$$(f*g)-(f_n*g_n) = (f-f_n)*g + f_n*(g-g_n),$$

hence

$$\|(f*g)-(f_n*g_n)\|_\infty \leqslant \|f-f_n\|_p\,\|g\|_q + \|f\|_p\,\|g-g_n\|_q \xrightarrow[n\to\infty]{} 0.$$

Thus $(f*g)$ is the uniform limit of a sequence of compactly supported functions. This implies that it goes to 0 when x goes to infinity. □

We can give a completely general result about the existence of convolution in Lebesgue spaces.

Proposition 1.10. *Let $p, q, r \in [1, \infty]$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Then $(f * g)(x)$ is well defined for almost every $x \in \mathbb{R}^d$, $(f * g)$ belongs to $L^r(\mathbb{R}^d)$ and

$$\|f * g\|_r \leqslant \|f\|_p \|g\|_q .$$

Proof. This result is a consequence of Proposition 1.7 if $q = 1$ and $r = p$ (where $p = 1$ and $r = q$) and of Proposition 1.9 if $r = \infty$. Thus we can assume that $r < \infty$ and $p, q \in]1, +\infty[$. Notice that $\max(p, q) < r$. As above, we identify f and g to representatives in $\mathcal{L}^p(\mathbb{R}^d)$ and $\mathcal{L}^q(\mathbb{R}^d)$. The function $(|f| * |g|)$ is measurable from $\mathbb{R}^d$ to $[0, +\infty]$ and for $x \in \mathbb{R}^d$ we can write

$$(|f| * |g|)(x) = \int_{\mathbb{R}^d} \left(|f(x-y)|^{\frac{p}{r}} |g(y)|^{\frac{q}{r}} \right) |f(x-y)|^{1-\frac{p}{r}} |g(y)|^{1-\frac{q}{r}} \, dy.$$

By the generalized Hölder Inequality applied with exponents r, $\frac{rp}{r-p}$ and $\frac{rq}{r-q}$ we get

$$(|f|*|g|)(x) \leqslant \left(\int_{\mathbb{R}^d} |f(x-y)|^p |g(y)|^q \, dy \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^d} |f(x-y)|^p \, dy \right)^{\frac{r-p}{rp}} \left(\int_{\mathbb{R}^d} |g(y)|^q \, dy \right)^{\frac{r-q}{rq}}.$$

We get

$$|(|f| * |g|)(x)|^r \leqslant \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}^d} |f(x-y)|^p |g(y)|^q \, dy.$$

Integrating with respect to $x \in \mathbb{R}^d$ we obtain by the Fubini Theorem that

$$\begin{aligned}
\||f| * |g|\|_r^r &\leqslant \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y)|^p |g(y)|^q \, dy \right) dx \\
&\leqslant \|f\|_p^{r-p} \|g\|_q^{r-q} \int_{\mathbb{R}^d} |g(y)|^q \left(\int_{\mathbb{R}^d} |f(x-y)|^p \, dx \right) dy \\
&\leqslant \|f\|_p^{r} \|g\|_q^{r-q} \int_{\mathbb{R}^d} |g(y)|^q \, dy \\
&\leqslant \|f\|_p^{r} \|g\|_q^{r} .
\end{aligned}$$

Hence $|f| * |g| \in L^r(\mathbb{R}^d)$ with $\||f| * |g|\|_r \leqslant \|f\|_p \|g\|_q$. We conclude as before. □

1.1.3 Other convolutions

The aim of this chapter is to study convolution on $\mathbb{R}^d$, but since it was discussed in the introduction, we briefly mention the standard convolution products in other contexts.

Let u and v be two sequences in $\mathbb{C}^{\mathbb{Z}}$. When it makes sense, we define the sequence $(u * v)$ by

$$(u * v)_n = \sum_{k \in \mathbb{Z}} u_{n-k} v_k. \tag{1.3}$$

We can also consider convolution on the space of 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. If f and g are two 2π-periodic functions we set

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y) \, dy.$$

We cannot define this convolution on any measured space. We see here that we need to have an additive group (to give sense to $f(x-y)$). Thus we cannot define a convolution on $\mathbb{N}$ or $\mathbb{R}_+$. However, we can extend by zeros a sequence indexed by $\mathbb{N}$ to a sequence indexed by $\mathbb{Z}$. Thus, if u and v are sequences in $\mathbb{C}^\mathbb{N}$ that we extend in $\mathbb{Z}$ indexed sequences setting $u_{-m} = v_{-m} = 0$ for $m \in \mathbb{N}$, then (1.3) gives

$$(u * v)_n = \begin{cases} 0 & \text{si } n < 0, \\ \sum_{k=0}^n u_{n-k} v_k & \text{if } n \geqslant 0. \end{cases}$$

We obtain again a sequence whose terms with negative indices vanish, and we can see $(u * v)$ as a sequence indexed by $\mathbb{N}$.

Likewise, if f and g are two functions on $\mathbb{R}_+$, we can extend them by 0 on $\mathbb{R}_-$ (we still note f and g the extended functions), and the convolution of f and g is given by

$$(f * g)(x) = \begin{cases} 0 & \text{si } x < 0, \\ \int_0^x f(x-y)g(y)\,dy & \text{if } x \geqslant 0. \end{cases}$$

As for sequences, we can consider $(f * g)$ as a function on $\mathbb{R}_+$.

1.2 Regularization of functions

Since we introduced them, we have worded a lot with functions in Lebesgue spaces, and these functions can be very singular. We have already seen that the set of compact support continuous functions is dense in $L^p(\mathbb{R}^d)$ for $p \in [1, +\infty[$. The aim of this section is to go further and to approximate integrable functions by smooth (and compactly supported) functions.

1.2.1 Approximation to the identity

We have seen that convolution defines a product on $L^1(\mathbb{R}^d)$. A natural question is wether this product has an identity element $\mathbf{1}$, such that for any $f \in L^1(\mathbb{R}^d)$

$$\mathbf{1} * f = f * \mathbf{1} = f.$$

A formal computation shows that the "Dirac function" (supported on $\{0\}$ and of integral 1) could play this role, but we know that such a function cannot exist.

Notice that we could introduce the convolution of a function and a measure. When it has a meaning, we could set for a function f and a measure μ

$$(\mu * f)(x) = (f * \mu)(x) = \int_{\mathbb{R}^d} f(x-y)\,d\mu(y).$$

With the Dirac measure, we would get $(\delta * f)(x) = f(x)$. This will be done in the more general cases of distributions.

For now, we only work with (equivalence classes of) functions. So there is no identity element for the convolution, but we can consider arbitrarily close functions, in the sense that their integral is equal to 1 and that most of their mass is close to 0. More precisely, we introduce the following notion.

Definition 1.11. We call approximation to the identity a sequence $(\rho_n)_{n\in\mathbb{N}}$ of measurable functions on $\mathbb{R}^d$ such that

(i) ρ_n has positive values [1] for any $n \in \mathbb{N}$,

[1] we could also consider approximation of the unit of variable signs, but we need to ensure that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is bounded in L^1. In practice, we use positive unit approximation.

(ii) $\int_{\mathbb{R}^d} \rho_n \, d\lambda = 1$ for any $n \in \mathbb{N}$,

(iii) for any $\varepsilon > 0$ we have

$$\int_{|x| \geqslant \varepsilon} \rho_n \, d\lambda \xrightarrow[n \to +\infty]{} 0.$$

We could consider approximations to the identity parametrized by a real number instead of natural numbers.

In general we use the following result to build the sequences of approximation of the unit.

Proposition 1.12. *Let $\rho : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function such that $\int_{\mathbb{R}^d} \rho \, d\lambda = 1$. For $n \in \mathbb{N}^*$ et $x \in \mathbb{R}^d$ we set*

$$\rho_n(x) = n^d \rho(nx).$$

Then the sequence $(\rho_n)_{n \in \mathbb{N}^*}$ is an approximation of the unit on $\mathbb{R}^d$.

For example, we can consider the mollifiers defined by

$$\rho(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{d}{2}}}, \tag{1.4}$$

or, on $\mathbb{R}$,

$$\rho(x) = \frac{1}{\pi(1+x^2)}. \tag{1.5}$$

The proof of Proposition 1.12 simply uses the affine change of variables $y = nx$.

The aim of an approximation of the unit is to ... approach the unit. In particular, we expect that $(\rho_n * f)$ is close to f for n large. We begin with the most convenient case.

Proposition 1.13. *Let $(\rho_n)_{n \in \mathbb{N}}$ be an approximation to the unit. Let f be a bounded and continuous function on $\mathbb{R}^d$. Then $f * \rho_n$ converges uniformly to f as n goes to $+\infty$.*

Proof. Since $\int_{\mathbb{R}^d} \rho_n \, d\lambda = 1$ we have for all $x \in \mathbb{R}^d$

$$(f * \rho_n)(x) - f(x) = \int_{\mathbb{R}^d} \left(f(x-y) - f(x)\right) \rho_n(y) \, dy.$$

Let $\varepsilon > 0$. There exists $\delta > 0$ such that for $x, y \in \mathbb{R}^d$ with $|x - y| \leqslant \delta$ we have $|f(x) - f(y)| \leqslant \frac{\varepsilon}{2}$. Let $M > 0$ be a bound for $|f|$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have

$$\int_{|y| \geqslant \eta} \rho_n(y) \, dy \leqslant \frac{\varepsilon}{4M}.$$

Then for $x \in \mathbb{R}^d$ and $n \geqslant N$ we have

$$\begin{aligned}
&|(f * \rho_n)(x) - f(x)| \\
&\quad \leqslant \int_{y \in B(x,\delta)} |f(x-y) - f(x)| \, \rho_n(y) \, dy + \int_{y \in \mathbb{R}^d \setminus B(x,\delta)} |f(x-y) - f(x)| \, \rho_n(y) \, dy \\
&\quad \leqslant \frac{\varepsilon}{2} \int_{y \in B(x,\delta)} \rho_n(y) \, dy + 2M \int_{y \in \mathbb{R}^d \setminus B(x,\delta)} \rho_n(y) \, dy \\
&\quad \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The result follows. □

Now we show that $\rho_n * f$ is close to f for any $f \in L^1(\mathbb{R}^d)$, and even for $f \in L^p(\mathbb{R}^d)$ for $p \in [1, +\infty[$. Taking for example $\rho_n = 2n\mathbb{1}_{[-\frac{1}{n}, \frac{1}{n}]}$ for $n \in \mathbb{N}^*$ and $f = \mathbb{1}_{[-1,1]}$ we see that the following result cannot hold for $p = +\infty$.

Proposition 1.14. *Let $(\rho_n)_{n\in\mathbb{N}}$ be an approximation of the unit. Let $p \in [1, +\infty[$ and $f \in L^p(\mathbb{R}^d)$. We have*

$$\|(f * \rho_n) - f\|_p \xrightarrow[n\to\infty]{} 0.$$

Proof. We identify f a representative in $\mathcal{L}^p(\mathbb{R}^d)$. Let $n \in \mathbb{N}^*$. Since $\int_{\mathbb{R}^d} \rho_n \, d\lambda = 1$ we have for any $x \in \mathbb{R}^d$

$$\begin{aligned}(f * \rho_n)(x) - f(x) &= \int_{\mathbb{R}^d} (f(x-y) - f(x))\, \rho_n(y)\, dy \\ &= \int_{\mathbb{R}^d} (f(x-y) - f(x))\, \rho_n(y)^{\frac{1}{p}} \rho_n(y)^{\frac{1}{q}}\, dy,\end{aligned}$$

where q the conjugate exponent of p. By the Hölder Inequality we get

$$|(f * \rho_n)(x) - f(x)|^p \leqslant \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p \, \rho_n(y)\, dy\right) \left(\int_{\mathbb{R}^d} \rho_n(y)\, dy\right)^{\frac{p}{q}}.$$

The last integral is equal to 1. Integrating with respect to $x \in \mathbb{R}^d$ an using the Fubini-Tonelli Theorem we get

$$\|f * \rho_n - f\|_p^p \leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p \; dx\right) \rho_n(y)\, dy \leqslant \int_{\mathbb{R}^d} \|\tau_y f - f\|_p^p \, \rho_n(y)\, dy.$$

Let $\varepsilon > 0$. By Proposition 0.68 there exists $\eta > 0$ such that for any $y \in \mathbb{R}^d$ with $|y| \leqslant \eta$ we have

$$\|\tau_y f - f\|_p^p \leqslant \frac{\varepsilon}{2}.$$

On the other hand there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$ we have

$$2^p \, \|f\|_p^p \int_{|y|\geqslant\eta} \rho_n(y)\, dy \leqslant \frac{\varepsilon}{2}.$$

Then for $n \geqslant N$ we have

$$\|f * \rho_n - f\|_p^p \leqslant \int_{|y|\leqslant\eta} \|\tau_y f - f\|_p^p \, \rho_n(y)\, dy + \int_{|y|\geqslant\eta} \|\tau_y f - f\|_p^p \, \rho_n(y)\, dy \leqslant \varepsilon.$$

This proves that

$$\|f * \rho_n - f\|_p^p \xrightarrow[n\to+\infty]{} 0.$$

□

At the beginning of the chapter, we observed that convolution has a regularizing effect. So we can expect $\rho_n * f$ to be more regular than f. All the more so if ρ_n is itself regular. Hence, Proposition 1.14 will allow us to approach L^p by a sequence of regular functions. To do that, we will use a sequence of approximation of the unit made of regular functions for example by building it from regular kernels (1.4) or (1.5). If we want to approach f by regular functions that have compact support, we will need to use compact support regularizing functions. This is not the case for (1.4) or (1.5).

1.2.2 Localized and regular functions

Let Ω be an open set of $\mathbb{R}^d$. We begin by giving basic properties of the space of smooth and compactly supported functions on Ω. In particular, the fact that this set is not reduced to the zero function.

Definition 1.15. We denote by $C_0^\infty(\Omega)$, $C_c^\infty(\Omega)$ or $\mathcal{D}(\Omega)$ the set of smooth and compactly supported functions on Ω.

Before going any further, we check that $C_0^\infty(\mathbb{R}^d)$ contains non trivial functions.

Proposition 1.16. *The function f defined on $\mathbb{R}^d$ by*

$$f(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geqslant 1, \end{cases}$$

belongs in $C_0^\infty(\mathbb{R}^d)$. Its support is the closed ball $\overline{B}(0,1)$.

Proof. • The function f is positive on the unit ball $B(0,1)$ and vanishes on $\mathbb{R}\backslash B(0,1)$, so its support is $\overline{B}(0,1)$. Moreover, f is of class C^∞ on $B(0,1)$ and $\mathbb{R}\backslash\overline{B}(0,1)$.
• For $r > 0$ we set

$$g(r) = \begin{cases} \exp\left(-\frac{1}{1-r}\right) & \text{if } r < 1, \\ 0 & \text{if } r \geqslant 1. \end{cases}$$

g is of class C^∞ on $]0,1[$. We can check by induction on $n \in \mathbb{N}$ that there exists a polynomial $P_n \in \mathbb{R}[X]$ such that for any $r \in]0,1[$ we have

$$g^{(n)}(r) = P_n\left(\frac{1}{1-r}\right)\exp\left(-\frac{1}{1-r}\right).$$

Let $n \in \mathbb{N}$. The function $s \mapsto P_n(s)e^{-s}$ tends to 0 when s tends to $+\infty$, so

$$g^{(n)}(r) \xrightarrow[r\to 1]{} 0.$$

In particular, g is continuous at $x = 1$. In addition, the derivatives of g vanish on $]1,+\infty[$. By the theorem of C^1 extension (based on the mean value theorem) we get by induction that g is n times differentiable at $x = 1$ with $g^{(n)}(1) = 0$. Finaly g is of class C^∞ on $]0,+\infty[$.
• By composition of g with the smooth function $x \mapsto |x|^2 = x_1^2 + \cdots + x_d^2$, we get that f is of class C^∞ on $\mathbb{R}^d \backslash \{0\}$. Since it is also C^∞ on $B(0,1)$, it is of class C^∞ on $\mathbb{R}^d$. □

From this first non trivial example of $C_0^\infty(\mathbb{R}^d)$ function, we can build "peak functions", that we will use as approximations of the unit.

Proposition 1.17. *Let $\varepsilon > 0$. There exists $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_+)$ such that* $\operatorname{supp}(\rho_\varepsilon) \subset \overline{B}(0,\varepsilon)$ *and* $\int_{\mathbb{R}^d} \rho_\varepsilon = 1$.

Proof. Let f be the function given by proposition 1.16. It is continuous and has compact support thus it is integrable on $\mathbb{R}^d$. Moreover, it is non-negative and non identically zero, so its integral is positive. Thus for $x \in \mathbb{R}^d$ we can set

$$\rho(x) = \frac{f(x)}{\int_{\mathbb{R}^d} f(t)\,dt}.$$

This defines a smooth function $\rho : \mathbb{R}^d \to \mathbb{R}_+$ whose support is $\overline{B}(0,1)$ and whose integral is equal to 1. We conclude by setting, for any $x \in \mathbb{R}$,

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right).$$ □

1.2.3 Regularization of functions

In this paragraph we prove that the convolution of a general function with a $C_0^\infty(\mathbb{R}^d)$ function is well defined and regular. We will then deduce that $C_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for any $p \in [1, +\infty[$.

We first recall some notation for derivatives in any dimension. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^d$ we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_d,$$

and then, for $f \in C^\infty(\mathbb{R}^d)$,

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} f.$$

Recall that by the Schwarz Theorem, the order of differentiation is not important.

Proposition 1.18. *Let $\rho \in C_0^\infty(\mathbb{R}^d)$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we have $(\rho * f) \in C^\infty(\mathbb{R}^d)$ and for any $\alpha \in \mathbb{N}^d$ we have*

$$\frac{\partial^\alpha}{\partial x^\alpha}(\rho * f) = \left(\frac{\partial^\alpha \rho}{\partial x^\alpha}\right) * f. \tag{1.6}$$

Proof. The convolution is well defined by Proposition 1.5. Let $j \in [\![1, d]\!]$. There exists $R > 0$ and $M > 0$ such that for any $x \in \mathbb{R}^d$

$$|\partial_j \rho(x)| \leqslant M \mathbb{1}_{B(0,R)}(x).$$

Let $r > 0$. For any $y \in \mathbb{R}^d$ the map $x \mapsto \rho(x - y) f(y)$ is of class C^1 on $B(0, r)$ and

$$\left|\frac{\partial}{\partial x_j}\big(\rho(x-y) f(y)\big)\right| = \left|\frac{\partial}{\partial x_j}\rho(x-y)\right| |f(y)| \leqslant M \mathbb{1}_{B(0,R+r)}(y)\, |f(y)|\,.$$

By differentiation under the integral sign, the function $(\rho * f)$ is differentiable with respect to x_j on $B(0, r)$ and its derivative is

$$\frac{\partial}{\partial x_j}(\rho * f) = (\partial_j \rho) * f.$$

Since this holds for any $r > 0$, the proposition is proved for $|\alpha| = 1$. We conclude by induction, replacing ρ by its successive derivatives. □

Theorem 1.19. *Let $p \in [1, +\infty[$. Then $C_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.*

Proof. Let $f \in L^p(\mathbb{R}^d)$ and $\varepsilon > 0$. There exists $R > 0$ such that $g = \mathbb{1}_{B(R)} f$ satisfies $\|f - g\|_p \leqslant \frac{\varepsilon}{2}$. Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of approximation to the identity such that $\rho_n \in C_0^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. By Proposition 1.18, $(\rho_n * g)$ is of class C^∞ for all $n \in \mathbb{N}^*$. It also has compact support by Proposition 1.4. Finally, by Proposition 1.14, we have $\|g - (g * \rho_n)\|_p \leqslant \frac{\varepsilon}{2}$ for n large enough, so $\|f - (g * \rho_n)\|_p \leqslant \varepsilon$. □

Remark 1.20. Let $p, q \in [1, +\infty[$. Let $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Let $\varepsilon > 0$. In the previous proof we can chose R such that $\|f - g\|_p \leqslant \frac{\varepsilon}{2}$ and $\|f - g\|_q \leqslant \frac{\varepsilon}{2}$. Since $(g * \rho_n)$ tends to g in $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$, there exists $n \in \mathbb{N}$ such that $\|g - g_n\|_p \leqslant \frac{\varepsilon}{2}$ and $\|g - g_n\|_q \leqslant \frac{\varepsilon}{2}$. This proves that one can construct a sequence $(f_m)_{m \in \mathbb{N}}$ of functions in $C_0^\infty(\mathbb{R}^d)$ such that f_m tends to f both in $L^p(\mathbb{R}^d)$ and in $L^q(\mathbb{R}^d)$.

1.2.4 Partitions of unity

We end this chapter with partitions of unity. The purpose is to be able to localize a problem, considering a function f defined on an open set Ω as a sum of functions that are localized on smaller and simpler open subsets. We could do it by multiplying f by characteristic functions of a partition of Ω, but we often need to localize with regular functions.

Proposition 1.21. *Let K and $\mathcal{O}$ be compact and open subsets of $\mathbb{R}^d$ such that $K \subset \mathcal{O}$. Then there exists $\chi \in C_0^\infty(\mathbb{R}^d, [0,1])$ such that $\chi = 1$ in a neighbourhood of K and $\chi = 0$ in a neighbourhood of $\mathbb{R}^d \backslash \mathcal{O}$.*

Proof. There exists $\varepsilon > 0$ such that for any $x \in K$ we have $B(x, 4\varepsilon) \subset \mathcal{O}$. We consider a peak function ρ_ε as given by proposition 1.17 and we set

$$K_\varepsilon = \bigcup_{x \in K} B(x, 2\varepsilon),$$

so that $K \subset \mathring{K}_\varepsilon \subset K_\varepsilon \subset \mathcal{O}$. It only remains to regularize the characteristic function of K_ε. For this we set, for $x \in \mathbb{R}$,

$$\chi(x) = \int_{\mathbb{R}} \mathbb{1}_{K_\varepsilon}(x-y)\rho_\varepsilon(y)\, dy.$$

We check that χ verifies the expected properties. □

Proposition 1.22. *Let K be a compact set of $\mathbb{R}^d$. Let $n \in \mathbb{N}^*$ and $\omega_1, \dots, \omega_n$ be open sets of $\mathbb{R}^d$ such that*

$$K \subset \bigcup_{j=1}^n \omega_j.$$

Then there exist functions $\chi_1, \dots, \chi_n \in C_0^\infty(\mathbb{R}^d, [0,1])$ such that $\operatorname{supp}(\chi_j) \subset \omega_j$ for any $j \in [\![1, N]\!]$ and $\sum_{j=1}^n \chi_j$ is equal to 1 on K.

Proof. • Let $x \in K$. There exists $j(x) \in [\![1, n]\!]$ and $r(x) > 0$ such that $\overline{B}(x, r(x)) \subset \omega_{j(x)}$. The family of balls $B(x, r(x))$ for $x \in K$ is an open cover of K, so there exist $k \in \mathbb{N}^*$ and $x_1, \dots, x_k \in K$ such that

$$K \subset \mathcal{O} = \bigcup_{l=1}^k B(x_l, r(x_l)).$$

For $j \in [\![1, n]\!]$ we set

$$K_j = \bigcup_{\substack{1 \leqslant l \leqslant k \\ j(x_l) = j}} \overline{B}(x_l, r(x_l)).$$

Then K_j is a compact included in ω_j. In addition we have

$$K \subset \bigcup_{j=1}^n K_j.$$

• We consider $\phi_0 \in C^\infty(\mathbb{R}^d, [0,1])$ equal to 1 outside $\mathcal{O}$ and equal to 0 on K. For instance, we can set $\phi_0 = 1 - \chi$, where $\chi \in C_0^\infty(\mathbb{R}^d, [0,1])$ is given by Proposition 1.21. For $j \in [\![1, n]\!]$ we also consider $\phi_j \in C_0^\infty(\mathbb{R}^d, [0,1])$ such that $\phi_j = 1$ on K_j and $\operatorname{supp}(\phi_j) \subset \omega_j$. For any $x \in \mathbb{R}^d$ we have

$$\sum_{i=0}^n \phi_i > 0.$$

For $j \in [\![1, n]\!]$ we set

$$\chi_j = \frac{\phi_j}{\sum_{i=0}^{n} \phi_i}.$$

The χ_j, $0 \leqslant j \leqslant n$, are of class C^∞ and they take values in $[0, 1]$, their sum is 1 everywhere and χ_j has compact support for $j \neq 0$. Moreover, since χ_0 vanishes on K, we have $\sum_{j=1}^{n} \chi_j = 1$ on K. □